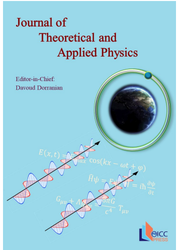

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Approximate solutions of the Dirac equation with deformed Woods-Saxon potential including a Hellmann-like tensor interation

Uduakobong S. Okorie[1]*, Akpan N. Ikot[2], Monday E. Udoh[3], Sylvester A. Ekong[1], Collins O. Edet[4], Ridha Horchani[5], Gaotsiwe J. Rampho[6]

[1]*Department of Physics, Akwa Ibom State University, Ikot Akpaden, Uyo, Nigeria.*
[2]*Department of Physics, Theoretical Physics Group, University of Port Harcourt, Choba, Nigeria.*
[3]*Department of Physics, University of Uyo, Uyo, Nigeria.*
[4]*Department of Physics, Cross River University of Technology, Calabar, Nigeria.*
[5]*Department of Physics, College of Science, Sultan Qaboos University, Al-Khod, Muscat, Sultanate of Oman.*
[6]*Department of Physics, University of South Africa, Florida, Johannesburg, South Africa.*

*__Corresponding author__: uduakobongokorie@aksu.edu.ng

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Abstract:
An Approximate bound state solutions of the Dirac equation under the spin and pseudospin symmetries for the deformed Woods-Saxon potential with a Hellmann-like tensor interaction was examined. With the help of the Nikiforov-Uvarov functional analysis (NUFA) method and an approximation scheme, the analytical and numerical energies of the combined potential were obtained for both symmetries, for different quantum numbers. Degeneracies were observed in the energy values in the absence of the tensor interaction and these degeneracies were removed with the help of the Hellmann-like tensor interaction. The variations of the energies for spin and pseudospin symmetries were studied for various values of the quantum numbers and deformation parameters. Our study shows that the relativistic energies obtained are very sensitive to the quantum numbers and the deformation parameter.

Keywords: Dirac equation; Bound state; Tensor interaction; Potential function; NUFA method

1. Introduction

Dirac equation is one of the wave equations which have received much attention by researchers, due to its relativistic background to spin $1/2$ particles [1]. It is a relativistic differential equation which describes spinor particles and relativistic behaviour of molecules and atoms under a strong potential field [2]. Dirac equation has been applied in various branches of physics such as nuclear physics and related areas. It has been recorded that Dirac Hamiltonian contains two symmetries: the spin and pseudospin symmetries, comprising of vector and scalar interaction term. In the Dirac theory, the concept of spin symmetry is obtained when the magnitude of the attractive Lorentz scalar potential $S(r)$ and the repulsive vector potential $V(r)$ are nearly equal but opposite in sign, i.e $S(r) \approx -V(r)$. On the other hand, the pseudospin symmetry is obtained when the sum of the vector and scalar potential term equal to a constant that is, $\Sigma(r) = V(r) + S(r) = C_{PS} = \text{const} \neq 0$ [3]. The spin symmetry case whose application is seen in the meson spectroscopy is achieved when the difference of the scalar $S(r)$ and $V(r)$ potentials are constant, i.e $\Delta(r) = V(r) - S(r) = C_S = \text{const} \neq 0$. In literature, the pseudospin symmetry is usually refers to as quasi-degeneracy of single nucleon doublets with non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n-1, l+2, j = l+3/2)$, where n, l, j represent the single nucleon radial, orbital and total angular momentum quantum numbers, respectively. The total angular momentum is defined as $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ is the pseudo-angular momentum and $\tilde{s}$ denotes the pseudospin angular momentum. Further study had shown that the pseudo-orbital angular momentum is the orbital angular momentum of the Dirac spinor lower component [4]. Different researchers have studied the Dirac equation with different potential models such as the Coulomb oscillator [5], Yukawa potential [6], Hellmann potential [7], Frost-Musulin potential [1], hyperbolic potentials [8], shifted Tietz-Wei potential [9], hyperbolic Poschl-Teller potential [10], generalized Morse potential [11], etc. In recent times, the studies of the Dirac equation with tensor interaction have attracted the attention of many researchers. Chenaghlou et al. [2] investigated the D-dimensional Dirac equation with Morse potential, using supersymmetric quantum mechanics (SUSYQM) approach [12]. Relativistic vibrational energies for CP and SiF^+ molecules were obtained at critical point. The relativistic energies of one-dimensional Dirac equation were obtained in the presence of Mathieu potential, using Fourier grid method [13]. Ahmadov et al. [14] studied the Dirac equation with Hulthen potential plus a class of Yukawa potential in the present of Coulomb tensor interaction using Nikiforov Uvarov (NU) method [15] and supersymmetry quantum mechanics (SUSYQM) method. In another development, Chenaghlou et al. [16] studied the Dirac equation with harmonic oscillator in the presence of magnetic field, using SUSYQM and asymptotic iteration method (AIM) [17]. The effect of constant magnetic field on the relativistic energy levels of Dirac particles was analyzed.

In this paper, we intend to investigate the deformed Woods-Saxon potential with a Hellmann-like tensor interaction in the Dirac theory. The Woods-Saxon potential generally, is known to be a short-range potential widely studied in different areas of Physics [18]. It is mostly preferred over the harmonic oscillator in both relativistic and nonrelativistic theories of mean-field shell model [19, 20]. The deformed Woods-Saxon potential is defined as [21]

$$V(r) = -\frac{V_1 e^{-\alpha r}}{(1+qe^{-\alpha r})} + \frac{V_2 e^{-2\alpha r}}{(1+qe^{-\alpha r})^2} \tag{1}$$

where r is the distance of separation of the potential, α is the screening parameter, q is the deformation parameter and V_1, V_2 are the potential strengths. On the other hand, the Hellmann-like tensor interaction is made up of the Coulomb and the Yukawa potentials, given as [22–25]

$$U(r) = -\frac{1}{r}(H_C + H_Y e^{-\alpha r}) \tag{2}$$

where H_C and H_Y are components of the Coulomb and Yukawa potentials, respectively.

The objectives of this work are to first review the Dirac equation theory with tensor coupling. Thereafter, the spin symmetry and pseudospin symmetry solutions of the Dirac equation will be obtained analytically by using the Nikiforov-Uvarov functional analysis (NUFA) method [26]. The analytical solutions will then be used to obtain the numerical energies of the deformed Woods-Saxon potential with the Hellmann-like tensor interaction. The dependence of the energies obtained on the potential parameters, screening parameters and quantum numbers is discussed appropriately.

2. Dirac equation with tensor coupling

The Dirac equation for fermionic massive spin-1/2 particles moving under an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$ and a tensor potential $U(r)(\hbar = c = 1)$ reads

$$[\mathbf{a}\cdot\mathbf{p} + b(M+S(r)) - ib\mathbf{a}\cdot\hat{r}U(r)]\psi(r) = [E - V(r)]\psi(r) \tag{3}$$

Here, E is the relativistic energy of the system, $\mathbf{p} = -i\boldsymbol{\nabla}$ is the 3-dimensional momentum operator, M is the mass of the fermionic particle, $\mathbf{a}$, b are the 4×4 Dirac matrices defined as

$$a = \begin{bmatrix} 0 & \boldsymbol{\delta}_i \\ \boldsymbol{\delta}_i & 0 \end{bmatrix},\ b = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{4}$$

where I is 2×2 unitary matrix and $\boldsymbol{\delta}_i$ being the three-vector Pauli spin matrices given as

$$\delta_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \delta_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \delta_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

The eigenvalues of the spin-orbit coupling operator are known to be $k = (j+1/2) > 0$, $k = -(j+1/2) < 0$; for unaligned spin $j = l-1/2$ and the aligned spin $j = l+1/2$, respectively.

The set (H^2, K, J^2, J_z) forms the complete set of conservative quantities with $\mathbf{J}$ being the total angular momentum operator and $\hat{K} = -b(\boldsymbol{\nabla}\cdot\mathbf{L}+1)$ is the spin-orbit where $\mathbf{L}$ is orbit angular momentum.

The spinors can be classified according to their angular momentum j, the spin-orbit quantum number k and the radial quantum number n.

The spinors can be written as

$$\psi_{nk}(r) = \frac{1}{r}\begin{bmatrix} W_{nk}(r) & Y^l_{jm}(\theta,\varphi) \\ iX_{nk}(r) & Y^{\tilde{l}}_{jm}(\theta,\varphi) \end{bmatrix} \tag{6}$$

Here, $W_{nk}(r)$, $X_{nk}(r)$ represent the upper and lower components of the Dirac spinors; $Y^l_{jm}(\theta,\varphi)$, $Y^{\tilde{l}}_{jm}(\theta,\varphi)$ represent the spin and pseudospin spherical harmonics and m is the projection on the z-axis.

With the help of the following identities [27];

$$\begin{aligned}
(\boldsymbol{\delta}\cdot\mathbf{A})(\boldsymbol{\delta}\cdot\mathbf{B}) &= \mathbf{A}\cdot\mathbf{B} + i\boldsymbol{\delta}\cdot(\mathbf{A}\times\mathbf{B}),\\
\boldsymbol{\delta}\cdot\mathbf{p} &= \boldsymbol{\delta}\cdot\hat{r}(\hat{r}\cdot\hat{p} + i\frac{\boldsymbol{\delta}\cdot\mathbf{L}}{r}),\\
(\boldsymbol{\delta}\cdot\mathbf{L})Y^{\tilde{l}}_{jm}(\theta,\varphi) &= (k-1)Y^{\tilde{l}}_{jm}(\theta,\varphi),\\
(\boldsymbol{\delta}\cdot\mathbf{L})Y^{l}_{jm}(\theta,\varphi) &= -(k+1)Y^{l}_{jm}(\theta,\varphi),\\
(\boldsymbol{\delta}\cdot\hat{r})Y^{l}_{jm}(\theta,\varphi) &= -Y^{\tilde{l}}_{jm}(\theta,\varphi),\\
(\boldsymbol{\delta}\cdot\hat{r})Y^{\tilde{l}}_{jm}(\theta,\varphi) &= -Y^{l}_{jm}(\theta,\varphi),
\end{aligned} \tag{7}$$

Eq. (3) become the following two coupled first-order Dirac equations of the form:

$$\big(\frac{d}{dr} + \frac{k}{r} - U(r)\big)W_{nk}(r) = (M + E_{nk} - \Delta(r))X_{nk}(r) \tag{8}$$

$$\big(\frac{d}{dr} - \frac{k}{r} + U(r)\big)X_{nk}(r) = (M - E_{nk} + \Sigma(r))W_{nk}(r) \tag{9}$$

where $\Delta(r) = V(r) - S(r)$; $\Sigma(r) = V(r) + S(r)$. Also, $\Delta(r)$ and $\Sigma(r)$ are the difference and sum potentials, respectively. By eliminating $W_{nk}(r)$ and $X_{nk}(r)$ in Eqs. (8) and (9), the following second-order Schrodinger-like equations are obtained:

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} + \frac{2kU(r)}{r} - \frac{dU(r)}{dr} - U^2(r) - (M+E_{nk}-\Delta(r))(M-E_{nk}+\Sigma(r)) + \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr}+\frac{k}{r}-U(r)\right)}{(M+E_{nk}-\Delta(r))}\right]W_{nk}(r) = 0 \tag{10}$$

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} + \frac{2kU(r)}{r} + \frac{dU(r)}{dr} - U^2(r) - (M+E_{nk}-\Delta(r))(M-E_{nk}+\Sigma(r)) - \frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr}-\frac{k}{r}+U(r)\right)}{(M-E_{nk}+\Sigma(r))}\right]X_{nk}(r) = 0 \tag{11}$$

where $k(k-1) = \tilde{l}(\tilde{l}+1)$, $k(k+1) = l(l+1)$.
For spin symmetry to occur, $d\Delta(r)/dr = 0$ and $\Delta(r)$ becomes a constant, C_S [28]. Hence, Eq. (11) becomes

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} + \frac{2kU(r)}{r} - \frac{dU(r)}{dr} - U^2(r) - (M + E_{nk} - C_S)\Sigma(r) + (E_{nk}^2 - M^2 + C_S(M-E_{nk}))\right]W_{nk}(r) = 0 \tag{12}$$

Here, $k = l$ for $k > 0$ and $k = -(l+1)$ for $k < 0$. The lower spinor component can be obtained from Eq. (8) as

$$X_{nk}(r) = \frac{1}{(M+E_{nk}(r)-C_S)}\left[\frac{d}{dr}+\frac{k}{r}-U(r)\right]W_{nk}(r) \tag{13}$$

There exist only real positive energy spectrum for exact spin symmetry where $E_{nk} \neq -M$ for $C_S = 0$.
In addition, pseudospin symmetry occurs when $d\Sigma(r)/dr = 0$ and $\Sigma(r)$ becomes a constant, C_{PS} [28]. Hence, Eq. (12) becomes

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} + \frac{2kU(r)}{r} + \frac{dU(r)}{dr} - U^2(r) - (M-E_{nk}+C_{PS})\Delta(r) - (M^2 - E_{nk}^2 + C_{PS}(M+E_{nk}))\right]X_{nk}(r) = 0 \tag{14}$$

Here, $k = -\tilde{l}$ for $k < 0$ and $k = \tilde{l}+1$ for $k > 0$. The *SU* (2) pseudospin symmetry can be obtained when $\tilde{l} \neq 0$, in which degenerate states are produced with $j = \tilde{l} \pm 1/2$. The upper spinor component can then be obtained from Eq. (9) as

$$W_{nk}(r) = \frac{1}{(M-E_{nk}+C_{PS})}\left[\frac{d}{dr}-\frac{k}{r}+U(r)\right]X_{nk}(r) \tag{15}$$

Here, there exist only real negative energy spectrum for exact pseudospin symmetry where $E_{nk} \neq M$ for $C_{PS} = 0$.

3. Analytical solutions using NUFA

3.1 Spin symmetry solution

First, for the spin symmetry consideration, we substitute the deformed Woods-Saxon potential of Eq. (1) and the Hellmann-like tensor interaction of Eq. (2) into Eq. (14) to obtain

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - \frac{2kH_C}{r} - \frac{2kH_Y e^{-\alpha r}}{r^2} - \frac{H_C}{r^2} - \frac{H_Y e^{-\alpha r}}{r^2} - \frac{\alpha H_Y e^{-\alpha r}}{r} - \frac{H_C}{r^2} - \frac{2H_C H_Y e^{-\alpha r}}{r^2} - \frac{H_Y^2 e^{-2\alpha r}}{r^2} + \frac{\gamma V_1 e^{-\alpha r}}{(1+qe^{-\alpha r})} - \frac{\gamma V_2 e^{-2\alpha r}}{(1+qe^{-\alpha r})^2} - \varepsilon_{nk}^2\right]W_{nk}(r) = 0 \tag{16}$$

Here, the sum potential $\Sigma(r)$ is taken as the deformed Woods-Saxon potential, the difference potential $\Delta(r)$ taken as constant C_S, the tensor potential $U(r)$ taken as the Hellmann-like tensor interaction and the following parameters γ and ε_{nk}^2 are also defined as

$$\gamma = (M + E_{nk} - C_S), \ \ \varepsilon_{nk}^2 = -(E_{nk}^2 - M^2 - C_S(M - E_{nk})) \tag{17}$$

Due to the presence of the centrifugal terms in Eq. (15), we employ the following approximation scheme [29]:

$$\frac{1}{r^2} \approx \frac{\alpha^2}{(1-qe^{-\alpha r})^2}; \frac{1}{r} \approx \frac{\alpha}{(1-qe^{-\alpha r})} \tag{18}$$

By substituting Eq. (17) and the coordinate transformation $z = -qe^{-\alpha r}$, we obtain

$$\frac{d^2W_{nk}(z)}{dz^2} + \frac{(1-z)}{z(1-z)}\frac{dW_{nk}(z)}{dz} + \frac{(-A_1z^2 + A_2z - A_3)}{(z(1-z))^2}W_{nk}(z) = 0 \tag{19}$$

where

$$\begin{aligned} A_1 &= \frac{\varepsilon_{nk}^2}{\alpha^2} + \frac{\gamma}{\alpha^2}\left(\frac{V_2}{q^2} - \frac{V_1}{q}\right) + \frac{H_Y}{q}\left(\frac{H_Y}{q} + 1\right), \\ A_2 &= \frac{2\varepsilon_{nk}^2}{\alpha^2} - \frac{\gamma V_1}{\alpha^2 q} + \frac{2H_Y}{q}(\beta_{kC}+1), \\ A_3 &= \frac{\varepsilon_{nk}^2}{\alpha^2} + H_C^2 + \beta_{kC}(\beta_{kC} + +1), \\ \beta_{kC} &= k + H_C \end{aligned} \tag{20}$$

By employing the NUFA method and proposing a wave function of the form

$$W_{nk}(z) = z^w(1-z)^\sigma f_{nk}(z) \tag{21}$$

where

$$\begin{aligned} w &= \sqrt{\frac{\varepsilon_{nk}^2}{\alpha^2} + H_C + \beta_{kC}(\beta_{kC}+1)} \\ \sigma &= \frac{1}{2}[1 + \sqrt{1 + 4(H_{01} + H_{02} + H_C^2)}] \end{aligned} \tag{22}$$

in which

$$H_{01} = \frac{\gamma V_2}{\alpha^2 q^2} + \frac{H_Y}{q}\left(\frac{H_Y}{q} + 1\right)$$

Table 1. Bound State Energies (in fm^{-1}) of the spin symmetry with $q = 0.5$.

l	$(n,k<0)$	$(l, j=l+1/2)$	$E_{n,k<0}$ $(H_C=0, H_Y=0)$	$E_{n,k<0}$ $(H_C=10, H_Y=10)$	$(n,k>0)$	$(l, j=l-1/2)$	$E_{n,k>0}$ $(H_C=0, H_Y=0)$	$E_{n,k>0}$ $(H_C=10, H_Y=10)$
1	0, - 2	$0p_{3/2}$	5.243963495	5.264159191	0, 1	$0p_{1/2}$	5.243963495	5.260085849
2	0, - 3	$0d_{5/2}$	5.249876782	5.266100796	0, 2	$0d_{3/2}$	5.249876782	5.259243556
3	0, - 4	$0f_{7/2}$	5.258204302	5.268343814	0, 3	$0f_{5/2}$	5.258204302	5.258612904
4	0, - 5	$0g_{9/2}$	5.268918742	5.270870104	0, 4	$0g_{7/2}$	5.268918742	5.258164665
1	1, - 2	$1p_{3/2}$	5.247260650	5.265905682	1, 1	$1p_{1/2}$	5.247260650	5.261019648
2	1, - 3	$1d_{5/2}$	5.255212106	5.268187439	1, 2	$1d_{3/2}$	5.255212106	5.259971612
3	1, - 4	$1f_{7/2}$	5.265364042	5.270803226	1, 3	$1f_{5/2}$	5.265364042	5.259163362
4	1, - 5	$1g_{9/2}$	5.277801514	5.273730145	1, 4	$1g_{7/2}$	5.277801514	5.258562407
1	2, - 2	$2p_{3/2}$	5.250014094	5.267980104	2, 1	$2p_{1/2}$	5.250014094	5.262166010
2	2, - 3	$2d_{5/2}$	5.260901366	5.270650114	2, 2	$2d_{3/2}$	5.260901366	5.260883351
3	2, - 4	$2f_{7/2}$	5.273061928	5.273691023	2, 3	$2f_{5/2}$	5.273061928	5.259872953
4	2, - 5	$2g_{9/2}$	5.287273502	5.277074367	2, 4	$2g_{7/2}$	5.287273502	5.259098632
1	3, - 2	$3p_{3/2}$	5.250013634	5.270435604	3, 1	$3p_{1/2}$	5.250013634	5.263558166
2	3, - 3	$3d_{5/2}$	5.266154759	5.273550040	3, 2	$3d_{3/2}$	5.266154759	5.262006826
3	3, - 4	$3f_{7/2}$	5.280826410	5.277077179	3, 3	$3f_{5/2}$	5.280826410	5.260765257
4	3, - 5	$3g_{9/2}$	5.296972930	5.280982132	3, 4	$3g_{7/2}$	5.296972930	5.259793111

$$H_{02} = (\beta_{kC}+1)\left(\beta_{kC} - \frac{2H_Y}{q}\right)$$

the approximate energy spectra of the deformed Woods-Saxon potential with Hellmann-like tensor interaction for the spin symmetry limit in closed form is obtained as

$$M^2 - E_{nk}^2 + C_S(M - E_{nk}) = \alpha^2\left[\left(\frac{T}{2(n+\sigma)} - \frac{(n+\sigma)}{2}\right)^2 - (H_C^2 + \beta_{kC}(\beta_{kC}+1))\right] \tag{23}$$

$$T = \frac{\gamma}{\alpha^2}\left(\frac{V_2}{q^2} - \frac{V_1}{q}\right) - \frac{H_Y}{q}\left(\frac{H_Y}{q}+1\right) - \beta_{kC}(\beta_{kC}+1) - H_C^2 \tag{24}$$

3.2 Pseudospin symmetry solution

Here, the deformed Woods-Saxon potential of Eq. (1) and the Hellmann-like tensor interaction of Eq. (2) are substituted into Eq. (14) to obtain

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - \frac{2kH_C}{r} - \frac{2kH_Y e^{-\alpha r}}{r^2} + \frac{H_C}{r^2} + \frac{H_Y e^{-\alpha r}}{r^2} + \frac{\alpha H_Y e^{-\alpha r}}{r} - \frac{H_C^2}{r^2} - \frac{2H_C H_Y e^{-\alpha r}}{r^2} - \frac{H_Y^2 e^{-2\alpha r}}{r^2} - \frac{\acute{\gamma} V_1 e^{-\alpha r}}{(1+qe^{-\alpha r})} + \frac{\acute{\gamma} V_2 e^{-2\alpha r}}{(1+qe^{-\alpha r})^2} + \zeta_{nk}^2\right] W_{nk}(r) = 0 \tag{25}$$

Here, the difference potential $\Delta(r)$ is taken as the deformed Woods-Saxon potential, the sum potential $\Sigma(r)$ taken as constant C_{PS}, the tensor potential $U(r)$ taken as the Hellmann-like tensor interaction and the following parameters $\acute{\gamma}$ and ζ_{nk}^2 are also defined as

$$\acute{\gamma} = (M - E_{nk} + C_{PS}), \zeta_{nk}^2 = -(M^2 - E_{nk}^2 + C_{PS}(M + E_{nk})) \tag{26}$$

Table 2. Bound State Energies (in fm^{-1}) of the spin symmetry with $q = 1.0$.

l	$(n,k<0)$	$(l, j=l+1/2)$	$E_{n,k<0}$ $(H_C=0, H_Y=0)$	$E_{n,k<0}$ $(H_C=10, H_Y=10)$	$(n,k>0)$	$(l, j=l-1/2)$	$E_{n,k>0}$ $(H_C=0, H_Y=0)$	$E_{n,k>0}$ $(H_C=10, H_Y=10)$
1	0, - 2	$0p_{3/2}$	5.244947839	5.511021869	0, 1	$0p_{1/2}$	5.244947839	5.573903795
2	0, - 3	$0d_{5/2}$	5.252525950	5.494827441	0, 2	$0d_{3/2}$	5.252525950	5.599398453
3	0, - 4	$0f_{7/2}$	5.263243600	5.481107564	0, 3	$0f_{5/2}$	5.263243600	5.627020410
4	0, - 5	$0g_{9/2}$	5.277040677	5.469905182	0, 4	$0g_{7/2}$	5.277040677	5.656674649
1	1, - 2	$1p_{3/2}$	5.248367355	5.531423181	1, 1	$1p_{1/2}$	5.248367355	5.596435754
2	1, - 3	$1d_{5/2}$	5.258296071	5.514476524	1, 2	$1d_{3/2}$	5.258296071	5.622559863
3	1, - 4	$1f_{7/2}$	5.271055540	5.500008769	1, 3	$1f_{5/2}$	5.271055540	5.650757701
4	1, - 5	$1g_{9/2}$	5.286743469	5.488080914	1, 4	$1g_{7/2}$	5.286743469	5.680929726
1	2, - 2	$2p_{3/2}$	5.250079247	5.551499363	2, 1	$2p_{1/2}$	5.250079247	5.618533317
2	2, - 3	$2d_{5/2}$	5.264048359	5.533823146	2, 2	$2d_{3/2}$	5.264048359	5.645242415
3	2, - 4	$2f_{7/2}$	5.279112000	5.518620573	2, 3	$2f_{5/2}$	5.279112000	5.673970220
4	2, - 5	$2g_{9/2}$	5.296754421	5.505969498	2, 4	$2g_{7/2}$	5.296754421	5.704613905
1	3, - 2	$3p_{3/2}$	5.251117102	5.571062566	3, 1	$3p_{1/2}$	5.251117102	5.640035203
2	3, - 3	$3d_{5/2}$	5.268531478	5.552671567	3, 2	$3d_{3/2}$	5.268531478	5.667294310
3	3, - 4	$3f_{7/2}$	5.286743553	5.536739914	3, 3	$3f_{5/2}$	5.286743553	5.696515754
4	3, - 5	$3g_{9/2}$	5.306580556	5.523361116	3, 4	$3g_{7/2}$	5.306580556	5.727594543

Table 3. Bound State Energies (in fm^{-1}) of the pseudospin symmetry with $q = 0.5$.

$\tilde{l}$	$(n,k<0)$	(l,j)	$E_{n,k<0}$ $(H_C=0,H_Y=0)$	$E_{n,k<0}$ $(H_C=10,H_Y=10)$	$(n-1,k>0)$	$(l+2,j+1)$	$E_{n,k>0}$ $(H_C=0,H_Y=0)$	$E_{n,k>0}$ $(H_C=10,H_Y=10)$
1	1, - 1	$1s_{1/2}$	-5.222276981	-4.971446730	0, 2	$0d_{3/2}$	-5.222276981	-4.910214325
2	1, - 2	$1p_{3/2}$	-5.210094761	-4.986359187	0, 3	$0f_{5/2}$	-5.210094761	-4.884719800
3	1, - 3	$1d_{5/2}$	-5.194414505	-4.998352560	0, 4	$0g_{7/2}$	-5.194414505	-4.856916198
4	1, - 4	$1f_{7/2}$	-5.175492897	-5.007351072	0, 5	$0h_{9/2}$	-5.175492897	-4.826950116
1	2, - 1	$2s_{1/2}$	-5.205190638	-4.933711122	1, 2	$1d_{3/2}$	-5.205190638	-4.870147901
2	2, - 2	$2p_{3/2}$	-5.191073952	-4.949475854	1, 3	$1f_{5/2}$	-5.191073952	-4.844006937
3	2, - 3	$2d_{5/2}$	-5.173158173	-4.962318754	1, 4	$1g_{7/2}$	-5.173158173	-4.815636998
4	2, - 4	$2f_{7/2}$	-5.152013873	-4.972134337	1, 5	$1h_{9/2}$	-5.152013873	-4.785187355
1	3, - 1	$3s_{1/2}$	-5.182173834	-4.895335351	2, 2	$2d_{3/2}$	-5.182173834	-4.829864759
2	3, - 2	$3p_{3/2}$	-5.167060348	-4.911820231	2, 3	$2f_{5/2}$	-5.167060348	-4.803213147
3	3, - 3	$3d_{5/2}$	-5.147694853	-4.925394316	2, 4	$2g_{7/2}$	-5.147694853	-4.774405279
4	3, - 4	$3f_{7/2}$	-5.124976086	-4.935928380	2, 5	$2h_{9/2}$	-5.124976086	-4.743591608
1	4, - 1	$4s_{1/2}$	-5.154345697	-4.856676918	3, 2	$3d_{3/2}$	-5.154345697	-4.789657443
2	4, - 2	$4p_{3/2}$	-5.138873539	-4.873767133	3, 3	$3f_{5/2}$	-5.138873539	-4.762607839
3	4, - 3	$4d_{5/2}$	-5.118721215	-4.887967640	3, 4	$3g_{7/2}$	-5.118721215	-4.733467652
4	4, - 4	$4f_{7/2}$	-5.095016169	-4.899130680	3, 5	$3h_{9/2}$	-5.095016169	-4.702387436

By employing the approximation scheme of Eq. (18) and the coordinate transformation $z = -qe^{-\alpha r}$, we obtain

$$\frac{d^2W_{nk}(z)}{dz^2}+\frac{(1-z)}{z(1-z)}\frac{dW_{nk}(z)}{dz} + \frac{(-B_1z^2+B_2z-B_3)}{(z(1-z))^2}W_{nk}(z)=0 \tag{27}$$

where

$$\begin{aligned} B_1 &= \frac{H_Y}{q}(\frac{H_Y}{q}-1)-\frac{\acute{\gamma}}{\alpha^2}(\frac{V_2}{q^2}-\frac{V_1}{q})-\frac{\zeta_{nk}^2}{\alpha^2}, \\ B_2 &= \frac{2H_Y}{q}(\beta_{kC}-1)+\frac{\acute{\gamma}V_1}{\alpha^2 q}-\frac{2\zeta_{nk}^2}{\alpha^2}, \\ A_3 &= \beta_{kC}(\beta_{kC}-1)+H_C^2-\frac{\zeta_{nk}^2}{\alpha^2}, \\ \beta_{kC} &= k+H_C \end{aligned} \tag{28}$$

By employing the NUFA method and proposing a wave function of the form

$$W_{nk}(z) = z^{\bar{w}}(1-z)^{\bar{\sigma}} f_{nk}(z) \tag{29}$$

where

$$\begin{aligned} \bar{w} &= \sqrt{\beta_{kC}(\beta_{kC}-1)+H_C^2-\frac{\zeta_{nk}^2}{\alpha^2}}, \\ \bar{\sigma} &= \frac{1}{2}[1+\sqrt{1+4(H_C^2+H_{00})}] \end{aligned} \tag{30}$$

in which

$$H_{00} = \frac{H_Y}{q}\left(\frac{H_Y}{q}-1\right)+(\beta_{kC}-1)\left(\beta_{kC}-\frac{2H_Y}{q}\right)-\frac{\acute{\gamma}V_2}{\alpha^2 q^2}$$

the approximate energy spectra of the deformed Woods-Saxon potential with Hellmann-like tensor interaction for the pseudospin symmetry limit in closed form is obtained

Table 4. Bound State Energies (in fm^{-1}) of the pseudospin symmetry with $q = 1.0$.

$\tilde{l}$	$(n,k<0)$	(l,j)	$E_{n,k<0}$ $(H_C=0,H_Y=0)$	$E_{n,k<0}$ $(H_C=10,H_Y=10)$	$(n-1,k>0)$	$(l+2,j+1)$	$E_{n,k>0}$ $(H_C=0,H_Y=0)$	$E_{n,k>0}$ $(H_C=10,H_Y=10)$
1	1, - 1	$1s_{1/2}$	-5.217343016	-4.885087768	0, 2	$0d_{3/2}$	-5.217343016	-4.801129602
2	1, - 2	$1p_{3/2}$	-5.201621309	-4.906599068	0, 3	$0f_{5/2}$	-5.201621309	-4.767376885
3	1, - 3	$1d_{5/2}$	-5.181313466	-4.924550297	0, 4	$0g_{7/2}$	-5.181313466	-4.731103143
4	1, - 4	$1f_{7/2}$	-5.156847899	-4.938781100	0, 5	$0h_{9/2}$	-5.156847899	-4.692519509
1	2, - 1	$2s_{1/2}$	-5.196094644	-4.844754470	1, 2	$1d_{3/2}$	-5.196094644	-4.759532519
2	2, - 2	$2p_{3/2}$	-5.178311723	-4.866832585	1, 3	$1f_{5/2}$	-5.178311723	-4.725535696
3	2, - 3	$2d_{5/2}$	-5.155618087	-4.885393942	1, 4	$1g_{7/2}$	-5.155618087	-4.689110625
4	2, - 4	$2f_{7/2}$	-5.128838855	-4.900252831	1, 5	$1h_{9/2}$	-5.128838855	-4.650465928
1	3, - 1	$3s_{1/2}$	-5.167959246	-4.804332555	2, 2	$2d_{3/2}$	-5.167959246	-4.718264511
2	3, - 2	$3p_{3/2}$	-5.149338733	-4.826839246	2, 3	$2f_{5/2}$	-5.149338733	-4.684150440
3	3, - 3	$3d_{5/2}$	-5.125325293	-4.845882412	2, 4	$2g_{7/2}$	-5.125325293	-4.647690766
4	3, - 4	$3f_{7/2}$	-5.097118557	-4.861257762	2, 5	$2h_{9/2}$	-5.097118557	-4.609090583
1	4, - 1	$4s_{1/2}$	-5.134697711	-4.764199417	3, 2	$3d_{3/2}$	-5.134697711	-4.677616969
2	4, - 2	$4p_{3/2}$	-5.116009349	-4.787021956	3, 3	$3f_{5/2}$	-5.116009349	-4.643483957
3	4, - 3	$4d_{5/2}$	-5.091507772	-4.806440572	3, 4	$3g_{7/2}$	-5.091507772	-4.607079065
4	4, - 4	$4f_{7/2}$	-5.062619248	-4.822237975	3, 5	$3h_{9/2}$	-5.062619248	-4.568603243

as

$$M^2 - E_{nk}^2 + C_{PS}(M + E_{nk}) = \alpha^2 \{ \left(\frac{\acute{T}}{2(n+\bar{\sigma})} - \frac{(n+\bar{\sigma})}{2} \right)^2 - (H_C^2 + \beta_{kC}(\beta_{kC} - 1)) \} \quad (31)$$

where

$$T = \frac{H_Y}{q}\left(\frac{H_Y}{q} - 1\right) - \frac{\acute{\gamma}}{\alpha^2}\left(\frac{V_2}{q^2} - \frac{V_1}{q}\right) - H_C^2 - \beta_{kC}(\beta_{kC} - 1) \quad (32)$$

4. Results and discussion

In this study, the numerical analysis of the energies obtained are carried out in the absence ($H_C = H_Y = 0$) and the presence ($H_C = H_Y = 10$) of the Hellmann-like tensor potential for various values of the quantum numbers n, l and k. In addition to the natural units employed, other parameters used in this study for convenience are given thus:
$V_1 = V_2 = C_S = 10$ fm^{-1}, $C_{PS} = -10$ fm^{-1}, $\alpha = 0.05$ fm^{-1}, $M = 4.76$ fm^{-1}. Different energy levels both for spin symmetry and pseudospin symmetry are presented Tables 1 – 4. In obtaining these energies, Eqs. (23) and (31) are employed respectively, for the spin and pseudospin symmetries.
We have observed also that the energies in both symmetries increase slightly as the deformation parameter increases, as seen in the computed tables. The computed energies increase with increase in quantum numbers n, l and $|k|$ for both spin symmetry and pseudospin symmetry conditions. In the absence of the Hellmann-like tensor, degeneracy is seen to occur. In the case of the spin symmetry, Dirac spin-doublet eigenstates are observed with the same n and l states. As the Hellmann-like tensor interaction occurs, the degeneracies disappear in the spin symmetry. In the case of pseudospin symmetry, the Dirac spin-doublets eigenstates are observed when n and l are different. The degeneracies also disappear in the presence of the Hellmann-like tensor interaction. In addition, we observe that $(np_{3/2}, np_{1/2})$, $(nd_{5/2}, nd_{3/2})$, $(nf_{7/2}, nf_{5/2})$ etc pair states degenerate in the case of spin symmetry. Conversely, $(ns_{1/2}, (n-1)d_{3/2})$, $(np_{3/2}(n-1)f_{5/2})$, $(nd_{5/2}, (n-1)g_{7/2})$, etc pair states degenerate in the case of pseudospin symmetry. The trend of our results is consistent with the results obtained in literatures [14, 30, 31].

5. Conclusion

In this work, we have employed the Nikiforov-Uvarov functional analysis (NUFA) method to solve Dirac equation with the deformed Woods-Saxon potential and the Hellmann-like tensor interaction with the help of an approximation scheme to the centrifugal term. Analytical and numerical energy results were obtained both in the absence and presence of the Hellmann-like tensor interaction, for various values of the quantum numbers. In the absence of the tensor interaction, the energies obtained for both spin and pseudospin symmetries are seen to be the same, hence degeneracy occurring. In the presence of the tensor interaction, there exists variance in energies at various quantum steps, hence degeneracy disappearing. In addition, the energies increase with increase in quantum numbers. Our results agree with the trend seen in literatures, as our study promises to be applicable in different areas of physics [32–35].

Conflict of interest statement:
The authors declare that they have no conflict of interest.

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