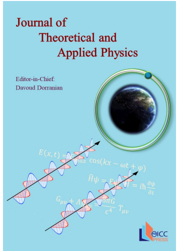

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Time dependency of the spin-orbit interaction and magnetic field in the long-range Ising model

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Abstract:
In this study, we have investigated a two-spin Ising model with added Dzyaloshinskii-Moriya (DM) interaction in an external magnetic field. We have considered two different initial states of the system and surveyed the time evolution of entanglement of the system for different parameters such as exchange coefficient, DM coefficient, and magnetic field. The Ising interaction has been considered as a function of distance between two spins. Moreover, both external magnetic field and DM interaction have been considered as a function of time. Under the time-dependent magnetic field and DM interaction, the entanglement of the two-spin system with long-range Ising model present the different dynamic behavior. This effect is strongly dependent on the initial state of the system.

Keywords: Entanglement dynamics; Ising model; Dzyaloshinskii-Moriya interaction; Two-spin system

1. Introduction

Quantum entangled states nowadays find important applications in quantum information processes [1–3]. Quantum teleportation can be implemented with help of quantum entanglement concept [4]. Moreover, super-dense coding [5], quantum key distribution [6], and quantum computing [7, 8] use the quantum entanglement to be technologically implemented. Although, it has been proved that entanglement is vital in quantum tasks, it is very fragile because the interaction between quantum system and environment. In fact, this noise can destroy the initial entanglement of the quantum system and fail the quantum processes. So, researchers investigate the various methods to protect the entanglement against the unwanted environmental noises. One of the practical methods is to use the proper interaction between particles of the quantum system. For instance, it has been found that the exchange interaction is useful to maintain the entanglement over the time [9–12]. Also, Dzyaloshinskii-Moriya (DM) interaction, an antisymmetric exchange interaction between the nearest neighbor spins, can control and improve the dynamics of the quantum entanglement [13]. Generally, the different interactions between qubits with each other and qubits with external fields are the physical basis of manipulating the entangled states. Various candidates have been proposed to serve as qubit in the entanglement technology. Some of those are spin of electrons, photons, or quantum dots [14–16]. Moreover, different proposals have been introduced to create the experimental setup in order to control and manipulate the entangled states [17]. Demand for controllable mechanisms leads to an innovative proposal that is to introduce the time-dependency of the exchange interactions [18, 19]. The exchange interaction can be controlled by time-dependence electrical field which has been studied by researchers in different areas of physics such as semiconductor quantum dots [20], ultracold atoms [21], strong correlated materials [22], and semiconductors Polluted with impurity spins [23]. Simulations have found that the time-dependence characters of the spin-spin coupling, DM interaction, and external magnetic field offer a longer entanglement for the system [18]. Similar to our study, reference [18], the quantum entanglement dynamics of two qubits Heisenberg-XYZ spin chain under a time dependent magnetic field effects, and considering the Dzyaloshinskii-Moriya (DM) interactions has been studied. They found that by tuning the strength of the DM coupling associated with a time varying magnetic field and a time varying spin-spin anisotropic coupling, the system can be better protected from unwanted effects of the environment and thus, entanglement can be preserved for a longer period of time. Reference [24] shows how entanglement can be tuned by both time-varying external magnetic field and time-dependence exchange coupling interactions of the spin system.

In this regard, the time-dependent external magnetic field can improve the entanglement of the system and protect the entangled system against the environmental noises [24]. In reference [25] a Heisenberg XYZ two- qubit model affected by the time-dependence magnetic field has been studied. It has been proved that it is possible to control the entanglement with the help of the time-varying uniform magnetic field.

In our previous research, the dynamics of the spin-1/2 Ising model with added DM interaction was studied. We assumed DM interaction is a function of the time. We have shown that the oscillation frequency of the DM interaction influences the fluctuations of the entanglement over the time [26]. Continuing the previous research, we consider the spin-1/2 long range Ising model with DM interaction under time-dependent external magnetic field.

In recent years, long range interactions attract much attention because they can produce interesting new phenomena [27–29]. Also, some studies have presented the experimental application of the long-range-interacting spin systems [30]. Indeed, the inverse-square, trigonometric and hyperbolic interacting particle systems [31, 32] and its spin generalizations [33–36] are important model of many-body systems due to their exact solvability and intimate connection to spin systems in condensed matter [37, 38]. References [39] has found that the long-range interactions introduce an effective attractive force between a pair of domain walls, so that confining them into a bound state.

The paper is organized as follows: in the next section, we introduce the model Hamiltonian and describe briefly the techniques used to obtain the results discussed in the subsequent sections. In section III, we present and discuss the numerical results of modeling. Finally, Section 4 contains the concluding remarks.

2. Model

Here, we consider a set of two localized spin-1/2 particles coupled through Ising exchange interaction J and DM interaction subjected to an external magnetic field of strength h:

$$H = J(R)S_1^xS_2^x + D(S_1^xS_2^y - S_1^yS_2^x) + h(t)(S_1^z + S_2^z) \qquad (1)$$

Where $J(r)$ is the exchange interaction parameter that varies with the distance between spins, r, as the $J(r) = J_0/r^2$, D is the strength of Dzyaloshinskii–Moryia interaction, h is the external magnetic field. Time dependency of D and h is as $\sin(\omega t)$.

In the base ket of S^z_{tot}, the Hamiltonian matrix is formed as:

$$H = \begin{bmatrix} h & 0 & 0 & \frac{J}{4} \\ 0 & 0 & \frac{J}{4} + i\frac{D}{2} & 0 \\ 0 & \frac{J}{4} - i\frac{D}{2} & 0 & 0 \\ \frac{J}{4} & 0 & 0 & -h \end{bmatrix} \qquad (2)$$

The eigenvalues and eigenstates are:

$$\begin{aligned} \varepsilon_1 &= \frac{+x}{4},\ |1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + \alpha|\downarrow\uparrow\rangle) \\ \varepsilon_2 &= \frac{-x}{4},\ |2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - \alpha|\downarrow\uparrow\rangle) \\ \varepsilon_3 &= \frac{+y}{4},\ |3\rangle = \frac{1}{\sqrt{1+\beta^2}}(|\beta\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \\ \varepsilon_4 &= \frac{-y}{4},\ |4\rangle = \frac{1}{\sqrt{1+\acute{\beta}^2}}(\acute{\beta}|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \end{aligned} \qquad (3)$$

where

$$x = \sqrt{J^2 + 4D^2},\ y = \sqrt{J^2 + 16h^2}, \\ \alpha = \frac{J - 2iD}{x},\ \beta = \frac{4h - y}{J},\ \acute{\beta} = \frac{4h + y}{J} \qquad (4)$$

On the other hand, to study the entanglement, one should obtain the density matrix of the system:

$$\rho(0) = |\psi(0)\rangle\langle\psi(0)| \qquad (5)$$

$|\psi(0)\rangle$ is the initial state of the system Hamiltonian. In our system the density matrix reduced to:

$$\rho = \begin{bmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{bmatrix} \qquad (6)$$

Using the eigenvalues of the partial transpose of the density matrix, one can be obtained the negativity of this model. Negativity is given by [40]:

$$N = \frac{\|\rho^T\| - 1}{2} \qquad (7)$$

Where $\|\rho^T\|$ is the trace norm or the sum of the absolute values of the operator ρ^T. In following, we try to describe the dynamics of entanglement of this system. The dynamic evolution operator $U(t) = \exp(-iHt)$ can be obtained as:

$$U(t) = \exp\big[-\frac{i}{\hbar}\int_0^t H(\acute{t})d\acute{t}\big] \qquad (8)$$

Can be obtained as:

$$U(t) = \exp\Big[-i\big[(\int_0^t J(R,\acute{t})d\acute{t})S_1^xS_2^x + (\int_0^t D(\acute{t})d\acute{t}) \\ \times (S_2^xS_1^y - S_1^yS_2^x) + (\int_0^t L(\acute{t})d\acute{t})(S_1^z + S_2^z)\big]\Big] \qquad (9)$$

By introducing:

$$\begin{aligned} J_0(R,t) &= \int_0^t J(R,\acute{t})d\acute{t} \\ D_0(t) &= \int_0^t D(\acute{t})d\acute{t} \\ h_0(t) &= \int_0^t h(\acute{t})d\acute{t} \end{aligned} \qquad (10)$$

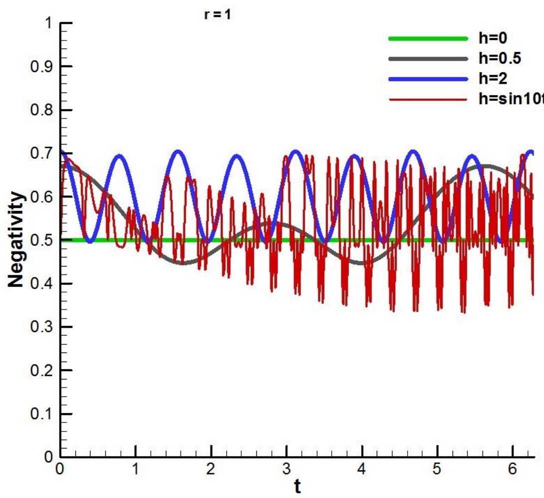

Figure 1. Time-dependency of entanglement of the Ising model under different magnetic fields.

Matrix form of the evolution operator is obtained as:

$$U = \begin{bmatrix} U_{11} & 0 & 0 & U_{14} \\ 0 & U_{22} & U_{23} & 0 \\ 0 & U_{32} & U_{33} & 0 \\ U_{41} & 0 & 0 & U_{44} \end{bmatrix} \tag{11}$$

where

$$U_{11} = \cos\frac{yt}{4} + \frac{4ih}{y}\sin\frac{yt}{4}, \ U_{14} = U_{41} = \frac{iJ}{y}\sin\frac{yt}{4},$$
$$U_{22} = U_{33} = \cos\frac{xt}{4}, \ U_{23} = -i\alpha\sin\frac{xt}{4}, \ U_{32} = -i\alpha^*\sin\frac{xt}{4},$$
$$U_{44} = \cos\frac{yt}{4} - \frac{4ih}{y}\sin\frac{yt}{4}, \ U_{14} = U_{41} = \frac{iJ}{y}\sin\frac{yt}{4} \tag{12}$$

by applying the evolution operator on the $|\psi_0\rangle$, we can obtain the physical state of the system at time t:

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \tag{13}$$

In the next section, we will choose a special initial state and then will investigate the effect of distance between two particles, r, and time-dependency of external magnetic field on the dynamics behavior of the entanglement in the system.

3. Results and discussion

In this study, we consider two different initial states of the system and survey the time evolution of entanglement of the system for different parameters such as J, D, and h. Also, we consider the J as a function of distance between two spins, r. Moreover, h and D are considered as a function of time, t.

3.1 Case 1

Suppose that at time $t = 0$ the qubits are entangled together and initial state given by:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \tag{14}$$

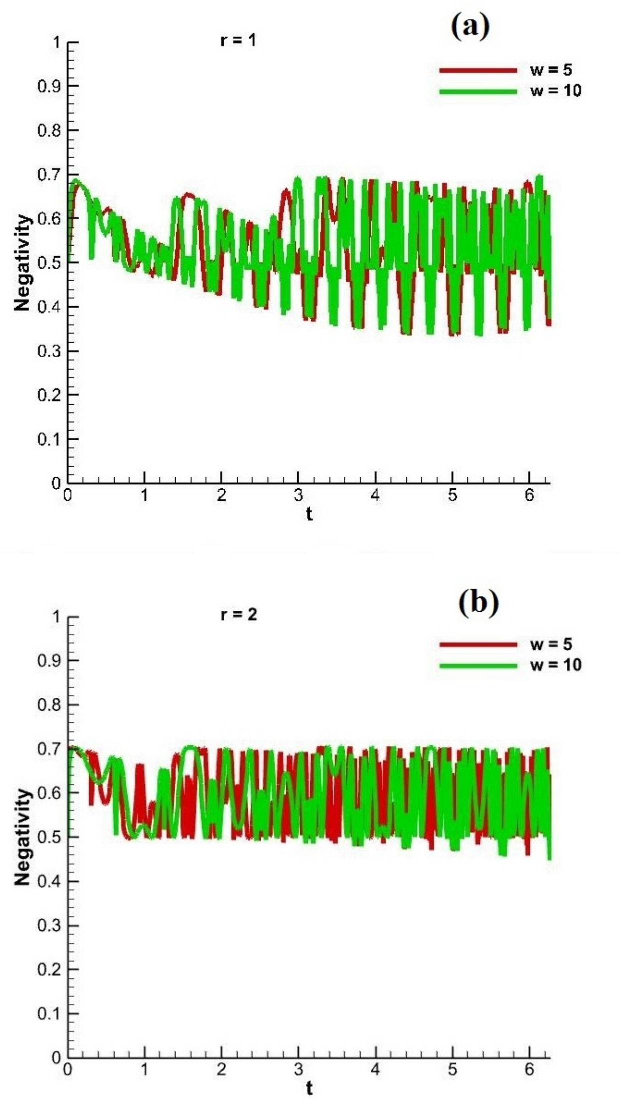

Figure 2. Time-dependency of entanglement of the model for different angular frequency. (a) $r = 1$, (b) $r = 2$.

Choosing this initial state makes the effect of the DM interaction removes from the dynamic behavior of the entanglement. By applying the instruction mentioned in previous section, we obtain the negativity of the system:

$$N = \frac{1}{2}\sqrt{(1 - \frac{32h^2}{y^2}\sin^2\frac{yt}{4})^2 + \frac{16h^2}{y^2}\sin^2\frac{yt}{2}} \tag{15}$$

Now, we begin studying the entanglement dynamics of the system under the different conditions.

Firstly, we study the effect of time dependency of magnetic field on the dynamic entanglement. The distance between two spins is considered constant. As observed in Figure 1, in the absence of external magnetic field, h, the negativity is constant over the time ($N = 0.5$). Although, the presence of the magnetic field increases the initial negativity, it creates the oscillatory behavior for entanglement dynamics. Raising h prevents the fluctuations' amplitude from less than the minimum value ($N_{min} = 0.5$). Moreover, a higher magnetic field makes a more regular oscillations of the negativity. As

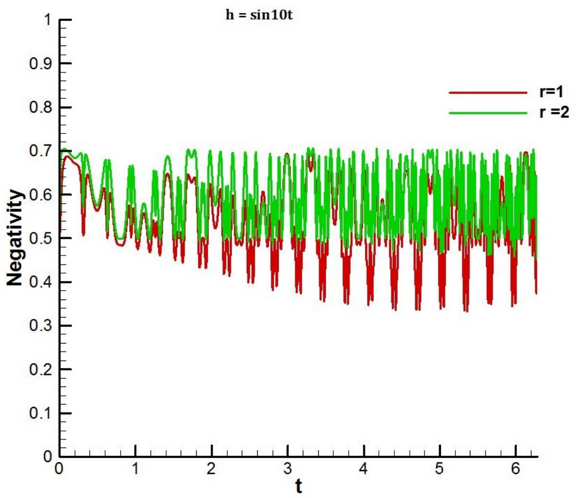

Figure 3. Time-dependency of entanglement of the model for different r.

a result, the magnetic field has a relatively positive effect on the entanglement dynamics of the system. If the magnetic field is a time function as $h = h_0 \sin(\omega t)$, the irregular oscillations with growing amplitude appear, as seen in Figure 1. Does time-dependence magnetic field improve or maintain the entanglement over the time? Compared to entanglement dynamics under the constant magnetic field, one can say that the model with constant magnetic field, especially higher values of h, performs better to improve the dynamic behavior of the entanglement.
Now, we study the effect of angular frequency of the time-dependence magnetic field on the entanglement dynamics. As seen in Figure 2(a), increasing the angular frequency of magnetic field does not have a positive effect on entanglement. Figure 2(b) shows that increasing the r, and therefore decreasing the Ising interaction between two spins, can remove the improper effects of magnetic field with high angular frequency on the entanglement dynamics.
Thus, if an oscillatory magnetic field with high angular frequency is applied to an entangled system, long range Ising interaction can leads to a more stable dynamics of entanglement.
Briefly, the comparison of above figures indicates the significant role of the magnetic field in dynamics of the entanglement. For low constant magnetic field, $h = 0.5$, the amplitude range of the entanglement fluctuations is $0.4-0.7$, approximately. For high constant magnetic field, $h = 2$, the amplitude range of the entanglement fluctuations is $0.5-0.7$. For oscillatory magnetic field, the amplitude range of the entanglement fluctuations is $0.3-0.7$. Therefore, one can conclude the constant strong magnetic field can better improve the dynamic behavior of entanglement.
In the next step, we investigate the role of distance between spins, r, in the dynamic behavior of the entanglement. In the absence of the magnetic field, the distance of the spins does not influence the strength of Ising interaction. In fact, for all values of r, the negativity remains $N = 0.5$. This behavior can be explained as follows. When $h = 0$, the fourth eigenstate of the system converts into: $|4\rangle = 1/\sqrt{2}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)(\acute{\beta} = 1)$ which is the same as the initial state of the system. In fact, in such conditions ($h = 0$), the initial state of the system is exactly in accordance with one of the eigenstates of the system and so the system will not have the dynamics behavior and will always remain in its eigenstate.
Figure 3 indicates the effect of r on the negativity behavior over the time. The magnetic field is supposed to be time-dependent. In this situation, increasing the distance between two spins shifts the fluctuations to higher values of the negativity. This is a desired behavior to gain an increased entanglement over the time.

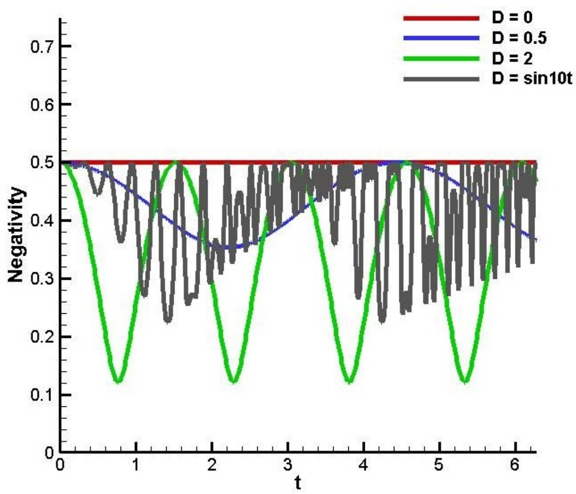

Figure 4. Time-dependency of entanglement for different DM interactions, here $r = 1$.

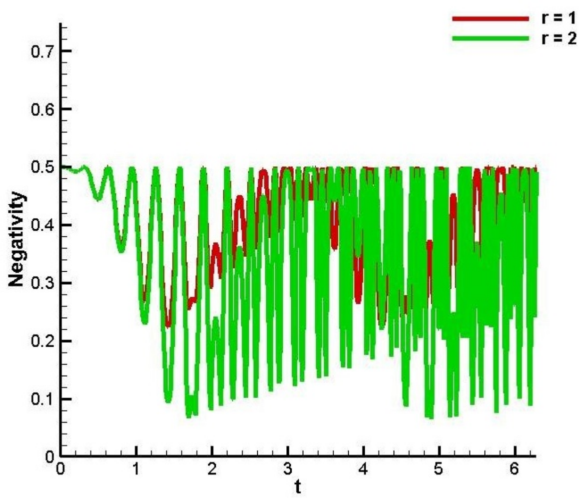

Figure 5. Time-dependency of entanglement for different r, here, $D = \sin 10t$.

3.2 Case 2

Now, suppose that at time $t = 0$ the qubits are entangled together and initial state given by:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{16}$$

By applying the evolution operator on the $|\psi_0\rangle$, we can obtain the physical state of the system at time t as:

$$\begin{aligned} |\psi(t)\rangle =& \frac{1}{\sqrt{2}}[(\cos\frac{xt}{4} - i\alpha\sin\frac{xt}{4})|\uparrow\downarrow\rangle \\ &+ (\cos\frac{xt}{4} - i\alpha^*\sin\frac{xt}{4})|\downarrow\uparrow\rangle] \end{aligned} \tag{17}$$

The interesting thing is that the selected initial state makes role of the magnetic field disappears in the time evolution of the entanglement. In fact, the change of h does not affect the entanglement dynamics of the system. Therefore, the negativity is written as:

$$N = \frac{\sqrt{1 - \frac{4D^2}{x^2}\sin^2\frac{xt}{2}}}{2} \tag{18}$$

In following, we study the dynamics behavior of entanglement under different DM interactions. Figure 4 shows the role of DM interaction in dynamic behavior of the entanglement for the two-spin Ising model. As figure indicates, raising DM interaction increases the entanglement fluctuations over the time. Thus, the DM interaction plays a destroying role in the entanglement dynamics of the system. If DM interaction is a sinusoidal function of time, the fluctuations of the entanglement are increased. It is worth noting that the upper limit of entanglement for all different values of D is the same and equal to its value for system without DM interaction.

In the paper, we try to study if long range exchange interaction compensates the negative role of the DM interaction, especially when DM is a time dependent parameter. Figure 5 demonstrates the effect of long-range Ising interaction on covering the negative effect of time- dependent DM interaction. Increasing the distance between two spins with Ising interaction reinforces the destroying effect of DM interaction. Unlike the Figure 3, in which increasing r and consequently reducing the strength of Ising interaction highlights the positive role of the magnetic field and improves the entanglement dynamics, Figure 5 depicted that increasing r downplays the effect of Ising interaction and thus the DM interaction dominates in the system. In result, the fluctuations of the negativity increases significantly.

4. Conclusion

In this study, we have investigated a two-spin system which is represented by the Ising model with DM interaction in an external magnetic field. We have considered two different initial states of the system and surveyed the time evolution of entanglement of the system for different parameters such as J, D, and h. The Ising interaction has been considered as a function of distance between two spins, r, and the external magnetic field and DM interaction have been considered as a function of time, t. For more detailed analysis, two different initial states have been defined for the system.

The first initial state has been given by: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$. The noteworthy point is that this choice has removed the role of the DM interaction in the entanglement dynamics of the system. Although, the presence of the constant magnetic field has increased the initial negativity, a higher magnetic field leads to more regular oscillations. Thus, the magnetic field has a relatively positive effect on the entanglement dynamics of the system. If the magnetic field is a time function as $h = h_0\sin(\omega t)$, the irregular oscillations with growing amplitude appear. Compared to entanglement dynamics under the constant magnetic field, one can say that the model with constant magnetic field, especially higher values of h, shows better performance to improve the dynamic behavior of entanglement.

Also, we have investigated the effect of distance between spins, r, on the dynamic behavior of the entanglement. In the absence of the magnetic field, the distance of the spins does not influence the strength of Ising interaction. In the presence of time-dependent magnetic field, increasing the distance between two spins shifts the fluctuations to higher values of the negativity. This is a desired behavior to gain an increased entanglement over the time.

The second initial state has been given by: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$. The interesting point is that this choice makes role of the magnetic field disappears in the time evolution of the entanglement. In this case, the DM interaction, whether constant or time-dependent, plays a destroying role in the entanglement dynamics of the system. For system with second initial state, increasing the distance between two spins reinforces the destroying effect of DM interaction.

Summary, in the conditions that the magnetic field and DM interaction are time-dependent, the entanglement of the two-spin system with long-range Ising model will present the different dynamic behavior. This effect is strongly dependent on the initial state of the system.

Conflict of interest statement:

The authors declare that they have no conflict of interest.

References

[1] K. S. Chou and J. Z. Blumoff. "Deterministic teleportation of a quantum gate between two logical qubits". *Nature*, **561**:368, 2018.

[2] J. G. Ren, P. Xu, H. L. Yong, L. Zhanga, S. K. Liao, J. Yin, W. Y. Liu, W. Q. Cai, M. Yang, L. Li, and et al. "Ground-to-satellite quantum teleportation". *Nature 549(7670), 70 (2017)*, **549**:70, 2017.

[3] R. L. Franco and G. Compagno. "Indistinguishability of elementary systems as a resource for quantum information processing". *Phys. Rev. Lett.*, **120**:240403, 2018.

[4] S. Takeda, T. Mizuta, M. Fuwa, P. Van Loock, and A. Furusawa. "Deterministic quantum teleportation of

photonic quantum bits by a hybrid technique". *Nature*, **500**:315, 2013.

[5] X. S. Liu, G. L. Long, D. M. Tong, and F. Li. "General scheme for superdense coding between multiparties". *Phys. Rev. A*, **65**:022304, 2002.

[6] F. Basso Basset, M. Valeri, E. Roccia, V. Muredda, D. Poderini, and J. Neuwirth. "Quantum key distribution with entangled photons generated on demand by a quantum dot". *Science Advances*, **7**:1, 2021.

[7] M. A. Zidan. "novel quantum computing model based on entanglement degree". *Modern Physics Letters B*, **34**:2050401, 2020.

[8] I. Cohen and K. Mølmer. "Deterministic quantum network for distributed entanglement and quantum computation". *Physical Review A*, **98**:030302, 2018.

[9] Y. Y. Gao, B. J. Lester, and R. J. Schoelkopf. "Entanglement of bosonic modes through an engineered exchange interaction". *Nature*, **566**:509, 2019.

[10] N. S. Ananikian, L. N. Ananikyan, L. A. Chakhmakhchyan, and O. Rojas. "Thermal entanglement of a spin-1/2 Ising–Heisenberg model on a symmetrical diamond chain". *Journal of Physics: Condensed Matter*, **24**:256001, 2012.

[11] A. V. Fedorova and M. A. Yurischev. "Quantum entanglement in the anisotropic Heisenberg model with multicomponent DM and KSEA interactions". *Quantum Information Processing*, **2**:1, 2021.

[12] S. Oh, M. Friesen, and X. Hu. "Even-odd effects of Heisenberg chains on long-range interaction and entanglement". *Physical Review B*, **82**:140403, 2010.

[13] M. Mahmoudi, S. Mahdavifar, and M. R. Soltani. "Effect of the Dzyaloshinskii–Moriya interaction on the propagation velocity of the entanglement". *Physica A: Statistical Mechanics and its Applications*, **456**:176, 2016.

[14] J. I. Latorre and A. Riera. "A short review on entanglement in quantum spin systems". *Journal of Physics a: Mathematical and Theoretical*, **42**:504002, 2009.

[15] X. L. Wang, L. K. Chen, W. Li, H. L. Huang, C. Liu, and C. Chen. "Experimental ten-photon entanglement". *Physical Review Letters*, **117**:210502, 2016.

[16] D. Kim, S. G. Carter, A. Greilich, A. S. Bracker, and D. Gammon. "Ultrafast optical control of entanglement between two quantum-dot spins". *Nature Physics*, **7**:223, 2011.

[17] M. Blaauboer and D. P. DiVincenzo. "Detecting Entanglement Using a Double-Quantum-Dot Turnstile". *Physical Review Letters*, **95**:160402, 2005.

[18] T. Martin and T. A. Giresse. "Entanglement Dynamics of a Two-qubit XYZ Spin Chain Under both Dzyaloshinskii-Moriya Interaction and Time-dependent Anisotropic Magnetic Field". *International Journal of Theoretical Physics*, **59**:2232, 2020.

[19] G. Sadiek, B. Alkurtass, and O. Aldossary. "Entanglement in a time-dependent coupled XY spin chain in an external magnetic field". *Physical Review A*, **82**:052337, 2010.

[20] J. H. Mentink, K. Balzer, and M. Eckstein. "Ultrafast and reversible control of the exchange interaction in Mott insulators". *Nature Communications*, **6**:1, 2015.

[21] L. M. Duan, E. Demler, and M. D. Lukin Aug. "Controlling spin exchange interactions of ultracold atoms in optical lattices". *Phys. Rev. Lett.*, **91**:090402, 2003.

[22] T. Li, A. Patza, L. Mouchliadis, J. Yan, T. A. Lograsso, I. E. Perakis, and J. Wang. "Femtosecond switching of magnetism via strongly correlated spin-charge quantum excitations". *Nature*, **496**:69, 2013.

[23] C. Piermarocchi and G. F. Quinteiro. "Coherent optical control of spin–spin interaction in doped semiconductors". *Phys. Rev. B*, **70**:235210, 2004.

[24] G. Sadiek, E. I. Lashin, and M. S. Abdalla. "Entanglement of a two-qubit system with anisotropic XYZ exchange coupling in a nonuniform time-dependent external magnetic field". *Physica B: Condensed Matter*, **404**:1719, 2009.

[25] A. Abliz, H. J. Gao, X. C. Xie, Y. S. Wu, and W. M. Liu. "Entanglement control in an anisotropic two-qubit Heisenberg X Y Z model with external magnetic fields". *Physical Review A*, **74**:052105, 2006.

[26] M. Mahmoudi and M. R. Soltani. "Entanglement Dynamics in the System of Two Spin-1/2 Particles with Ising Interaction and Time-Dependent Dzyaloshinskii-Moriya Interaction". *Journal of Superconductivity and Novel Magnetism*, **35**:1527, 2022.

[27] M. R. Soltani, S. Mahdavifar, and M. Mahmoudi. "Entanglement in a two-spin system with long-range interactions". *Chin. Phys. B*, **25**:087500, 2016.

[28] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempela, P. Zoller, and C. F. Roos. "Quasiparticle engineering and entanglement propagation in a quantum many-body system". *Nature*, **511**:202, 2014.

[29] J. Schachenmayer, B. P. Lanyon, C. F. Roos, and A. Daley. "Entanglement Growth in Quench Dynamics with Variable Range Interactions". *J. Phys. Rev. X*, **3**:031015, 2013.

[30] S. Hermes, J. Apollaro, S. Paganelli, and T. Macri. "Dimensionality-enhanced quantum state transfer in long-range-interacting spin systems". *Physical Review A*, **101**:053607, 2020.

[31] S. D. Han. “Entanglement in (1/2,1) Mixed-Spin XY Model with Long-Range Interaction”. *International Journal of Theoretical Physics*, **56**:1474, 2017.

[32] N. Zidan. “Entropic Uncertainty in Spin XY Model with Long-Range Interactions”. *Entropy*, **22**:837, 2020.

[33] J. Gibbons and T. Hermsen. “A generalisation of the Calogero-Moser system”. *Physica D*, **11**:337, 1984.

[34] S. Wojciechowski. “Integrable one-particle potentials related to the Neumann system and the Jacobi problem of geodesic motion on an ellipsoid”. *Phys. Lett. A.*, **111**:107, 1985.

[35] F. D. Haldane. “Exact Jastrow-Gutzwiller resonating-valence-bond ground state of the spin-(1/2) antiferromagnetic Heisenberg chain with 1/r 2 exchange”. *Phys. Rev. Lett.*, **60**:635, 1988.

[36] B. S. Shastry. “Exact solution of an S= 1/2 Heisenberg antiferromagnetic chain with long-ranged interactions”. *Phys. Rev. Lett.*, **60**:639, 1988.

[37] H. H. Tu and Y. Wu. “Exactly Solvable Quantum Impurity Model with Inverse-Square Interactions”. *Phys. Rev. Lett.*, **123**:066406, 2019.

[38] H. Fakhri and S.B. Seyedein Ardebili. “Exact solutions for the ferromagnetic and antiferromagnetic two-ring Ising chains of spin-1/2”. *Physica A: Statistical Mechanics and its Applications*, **523**:557, 2019.

[39] F. Liu, R. Lundgren, P. Titum, G. Pagano, J. Zhang, C. Monroe, and A. V. Gorshkov. “Confined quasiparticle dynamics in long-range interacting quantum spin chains”. *Physical review letters*, **122**:150601, 2019.

[40] G. Vidal and R. F. Werner. “Computable measure of entanglement”. *Physical Review A*, **65**:032314, 2002.