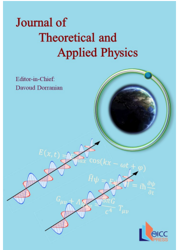

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Rational solutions to the KdV equation from Riemann theta functions

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Abstract:
Rational solutions to the KdV are constructed from the finite gap solutions of the KdV equation given in terms of abelian functions. For this we use a previous result giving the connection between Riemann theta functions and Fredholm determinants and also wronskians.
By choosing the parameters of these solutions according to a number intended to move towards zero, we obtain rational solutions when this number tends towards zero. So, we construct a hierarchy of rational solutions depending on multi real parameters and we give explicitly expressions for the first orders.

1. Introduction

Korteweg and de Vries [1] introduced the following equation

$$u_t = 6uu_x - u_{xxx}, \tag{1}$$

where the subscripts x and t denote partial derivatives, for the first time in 1895 to describe the propagation of waves with weak dispersion.
A lot of studies have been realized for this equation. A method of resolution was proposed by Gardner et al. [2] in 1967. Solutions were constructed with the bilinear method [3] by Hirota in 1971; Its and Matveev constructed solutions in terms of Riemann theta functions [4] in 1975. Other works can be quoted: for example Airault et al. in 1977 [5], Freeman and Nimmo in 1984 [6], Ma in 2004 [7].
We use a recent paper [8], in which we have degenerated the solutions to this KdV equation given in terms of Riemann theta functions. We have constructed solutions in terms of Fredholm determinants and wronskians. From this representations we construct rational solutions to the KdV equation by degenerating these solutions when parameters are chosen to tend to 0.

2. Solutions to the KdV equation in terms of Fredholm determinants and wronskians

2.1 Solutions to the KdV equation in terms of Fredholm determinants

We briefly recall the approach in terms of Riemann theta functions given in [4] in 1975. We consider the Riemann surface Γ of the algebraic curve defined by

$$\omega^2 = \prod_{j=1}^{2g+1} (z - E_j),$$

with $E_j \neq E_k,\ j \neq k$. Let D be some divisor $D = \sum_{j=1}^{g} P_j$, $P_j \in \Gamma$, then the finite gap solution of the KdV equation

$$u_t = 6uu_x - u_{xxx} \tag{2}$$

can be written in the form [4]

$$u(x,t) = -2\partial_x^2 \left[\ln \theta(xg + tv + l)\right] + C. \tag{3}$$

In (3), θ is the Riemann function defined by

$$\theta(z) = \sum_{k \in Z^g} \exp\{\pi i (Bk|k) + 2\pi i (k|z)\}, \tag{4}$$

constructed from the matrix of the B-periods of the surface Γ.
In [8], we have realized the degeneracy of these solutions following the ideas exposed for example in [9]). For E_j reals such that $E_m < E_j$ if $m < j$, we have evaluated the limits of all objects in formula (3) when E_{2m}, E_{2m+1} tends to $-\alpha_m$, $-\alpha_m = -\kappa_m^2$, $\kappa_m > 0$, for $1 \leq m \leq g$, and E_1 tends to 0.
All the details of the degeneracy of the components of the solution can be found in [8].
Different representations in terms of Fredhholm determinants have been given, in particular, we got the following

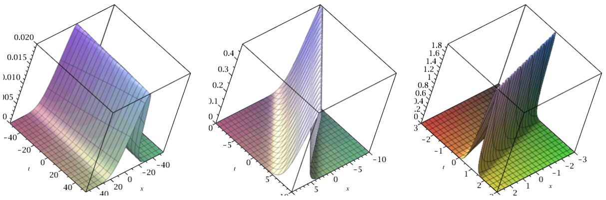

Figure 1. Solution of order 1 to KdV, on the left for $K_1 = 0,1$, $k_1 = 0,1$; in the center for $K_1 = 0,5$, $k_1 = 0,5$; on the right for $K_1 = 1$, $k_1 = 1$.

representation of the solutions to the KdV equation

Theorem 2.1 The function u defined by

$$u(x,t) = -2\partial_x^2 \ln(\det(I+D)), \tag{5}$$

with D the matrix defined by $D = (d_{jk})_{1\leq j,k\leq m}$

$$d_{jk} = (-1)^j \exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j k_j)\right] \prod_{l\neq j} \left|\frac{\kappa_l + \kappa_k}{\kappa_l - \kappa_j}\right|, \tag{6}$$

and κ_j, k_j arbitrary real parameters, is a solution to the KdV equation (1).

Proof: see [8].

2.2 Solutions to the KdV equation in terms of wronskians

In [8], we have given a connection between Fredholm determinants and wronskians. We use here these results.

We consider the following functions

$$\phi_j^a(x) = \sinh(\kappa_j x - 4\kappa_j^3 t + \kappa_j k_j + \frac{1}{2}\ln(\frac{z+i\kappa_j}{z-i\kappa_j})) = \sinh(\theta_j^a),$$
$$\phi_j^b(x) = \sinh(\kappa_j x - 4\kappa_j^3 t + \kappa_j k_j) = \sinh(\theta_j^b), \tag{7}$$

with k_j, K_j arbitrary parameters.

$W = W(\phi_j,\ldots,\phi_N)(x,t)$ is the classical wronskian $W = \det[(\partial_x^{j-1}\phi_i)_{i,\,j\in[1,\ldots,N]}]$.

We consider the matrix $D = (d_{jk})_{j,k\in[1,\ldots,N]}$ defined in (6)

$$d_{jk} = (-1)^j \exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j k_j)\right] \prod_{l\neq j} \left|\frac{\kappa_l + \kappa_k}{\kappa_l - \kappa_j}\right|.$$

Then we recall the result proven in [8]

Theorem 2.2

$$\det(I+D) = \frac{2^N(-1)^{\frac{N(N+1)}{2}} \exp(\sum_{j=1}^N \theta_j^b)}{\prod_{j=2}^N \prod_{i=1}^{j-1}(\kappa_j - \kappa_i)} W(\phi_1^b,\ldots,\phi_N^b)(x,t) \tag{8}$$

Proof: see [8].

2.3 Some examples

Example 2.1 Solution of order 1: the function u defined by

$$u(x,t) = 8\,\frac{\mathrm{e}^{-2K_1\left(4tK_1{}^2 - x - k_1\right)} K_1{}^2}{\left(-1 + \mathrm{e}^{-2K_1\left(4tK_1{}^2 - x - k_1\right)}\right)^2}$$

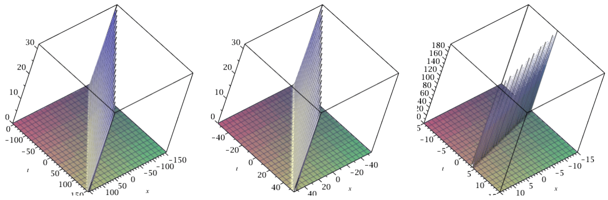

Figure 2. Solution of order 1 to KdV, on the left for $K_1 = 0,5$, $K_2 = 0$, $k_1 = 0,5$, $k_2 = 0$; in the center for $K_1 = 0$, $K_2 = 0,5$, $k_1 = 0$, $k_2 = 0,5$; on the right for $K_1 = 1$, $K_2 = 0$, $k_1 = 1$, $k_2 = 0$.

is a solution to the KdV equation (1). **Example 2.2 Solution of order 2**: the function u defined by

$$u(x,t) = \frac{n_u(x,t)}{d_u(x,t)}$$

$$\begin{aligned}
n_u(x,t) = {} & 16\,e^{2K_1x - 8K_1{}^3t + 2K_1k_1 + 2K_2x - 8K_2{}^3t + 2K_2k_2} K_1{}^4 \\
& - 8\,e^{-2K_1(4tK_1{}^2 - x - k_1)} K_1{}^4 \\
& - 8\,K_1{}^4 e^{2K_1x - 8K_1{}^3t + 2K_1k_1 + 4K_2x - 16K_2{}^3t + 4K_2k_2} \\
& + 8\,e^{-2K_2(4tK_2{}^2 - x - k_2)} K_2{}^2 K_1{}^2 \\
& + 8\,K_1{}^2 e^{-16K_1{}^3t + 4K_1x + 4K_1k_1 + 2K_2x - 8K_2{}^3t + 2K_2k_2} K_2{}^2 \\
& - 32\,K_1{}^2 e^{2K_1x - 8K_1{}^3t + 2K_1k_1 + 2K_2x - 8K_2{}^3t + 2K_2k_2} K_2{}^2 \\
& + 8\,K_1{}^2 e^{2K_1x - 8K_1{}^3t + 2K_1k_1 + 4K_2x - 16K_2{}^3t + 4K_2k_2} K_2{}^2 \\
& + 8\,e^{-2K_1(4tK_1{}^2 - x - k_1)} K_1{}^2 K_2{}^2 \\
& + 16\,e^{2K_1x - 8K_1{}^3t + 2K_1k_1 + 2K_2x - 8K_2{}^3t + 2K_2k_2} K_2{}^4 \\
& - 8\,e^{-2K_2(4tK_2{}^2 - x - k_2)} K_2{}^4 \\
& - 8\,e^{-16K_1{}^3t + 4K_1x + 4K_1k_1 + 2K_2x - 8K_2{}^3t + 2K_2k_2} K_2{}^4
\end{aligned}$$

and

$$\begin{aligned}
d_u(x,t) = {} & (-e^{-2K_2(4tK_2{}^2 - x - k_2)} K_1 + K_1 + e^{-2K_1(4tK_1{}^2 - x - k_1)} K_1 \\
& - K_1 e^{2K_1x - 8K_1{}^3t + 2K_1k_1 + 2K_2x - 8K_2{}^3t + 2K_2k_2} \\
& + e^{2K_1x - 8K_1{}^3t + 2K_1k_1 + 2K_2x - 8K_2{}^3t + 2K_2k_2} K_2 - K_2 \\
& - e^{-2K_2(4tK_2{}^2 - x - k_2)} K_2 + e^{-2K_1(4tK_1{}^2 - x - k_1)} K_2)^2
\end{aligned}$$

is a solution to the KdV equation (1) We could go on for greater orders, but even in the simple case of order 3, the explicit expression of the solution to the KdV equation takes more than 5 pages. For this reason, we cannot give it here.

3. Rational solutions to the KdV equation

Using Riemann's theta functions, the solutions to the KdV equation were constructed in terms of Fredholm and wronskiens determinants. By degenerating these solutions, we obtain rational solutions that are simpler and more suitable for use in physics. The simplest possible solutions for small orders have been built.

To obtain rational solutions to the KdV equation, we choose $K(j)$ and k_j as functions of e for each integer j and we perform a limit when the parameter e tends to 0.

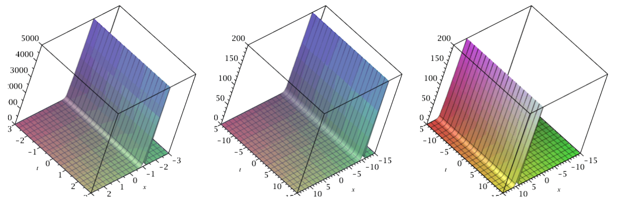

Figure 3. Rational solution of order 1 to KdV, on the left for $k_1 = 1$; in the center for $k_1 = 10$; on the right for $k_1 = -10$.

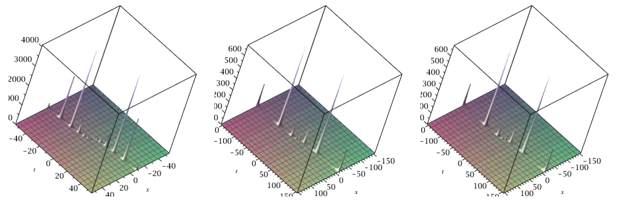

Figure 4. Rational solution of order 1 to KdV, on the left for $k_1 = 0$, $k_2 = 0$; in the center for $k_1 = 10$, $k_2 = 10$; on the right for $k_1 = 100$, $k_2 = 100$.

3.1 Rational solutions as a limit case

We get the following result:
Theorem 3.1 Let $\tilde{D}$ be the matrix defined by

$$\tilde{d}_{jk} = (-1)^j \exp\left[2(-4\tilde{\kappa}_j^3 t + \tilde{\kappa}_j \tilde{k}_j)\right] \prod_{l \neq j} \left| \frac{\tilde{\kappa}_l + \tilde{\kappa}_k}{\tilde{\kappa}_l - \tilde{\kappa}_j} \right|, \qquad (9)$$

then the function u defined by

$$u(x,t) = -2 \lim_{e \to 0} \partial_x^2 \ln(\det(I + \tilde{D})), \qquad (10)$$

is a rational solution to the KdV equation (1)

$$u_t = 6uu_x - u_{xxx} \qquad (11)$$

Proof: It is sufficient to perform a passage to the limit when e tends to 0, it is an obvious consequence of the previous result.
So a hierarchy of rational solutions to the KdV equation depending on the integer N is obtained.
In the following we give some examples of rational solutions.

3.2 First order rational solutions

We replace K_1 by $K_1 e$ and choose k_1 independent of e. We have the following result at order $N = 1$:
Proposition 3.1 The function u defined by

$$u(x) = \frac{2}{x^2 + 2k_1 x + {k_1}^2} \qquad (12)$$

is a solution to the KdV equation (1).

3.3 Second order rational solutions

Here we replace K_j by $K_j e$ and k_j by $k_j e^2$. Then we get:
Proposition 3.2 The function u defined by

$$u(x) = \frac{n_u(x,t)}{d_u(x,t)} \qquad (13)$$

with

$$n_u(x,t) = -6x(24{K_1}^4 t - 6{K_1}^2 k_1 - 48{K_2}^2{K_1}^2 t + 6{K_2}^2 k_1 + 24{K_2}^4 t - 6{K_2}^2 k_2 + 6{K_1}^2 k_2 - {K_1}^4 x^3 + 2{K_2}^2{K_1}^2 x^3 - {K_2}^4 x^3)$$

and

$$d_u(x,t) = 72{K_1}^2 t k_2 - 48{K_2}^2{K_1}^2 t x^3 - 6{K_1}^2 k_1 x^3 + 72{K_2}^2 k_1 t - 18k_1k_2 + 6{K_2}^2 k_1 x^3 + 6{K_1}^2 x^3 k_2 - 72{K_2}^2 t k_2 + 24{K_2}^4 t x^3 - 6{K_2}^2 k_2 x^3 - 72{K_1}^2 t k_1 + 24{K_1}^4 t x^3 - 288{K_2}^2{K_1}^2 t^2 + 144{K_1}^4 t^2 + 9{k_1}^2 + {K_1}^4 x^6 - 2{K_2}^2{K_1}^2 x^6 + 144{K_2}^4 t^2 + 9{k_2}^2 + {K_2}^4 x^6$$

is a solution to the KdV equation (1).
The structure of the solutions being very insensitive to the coefficients K, we choose to take $K_j = j$ in all the following figures.

3.4 Rational solutions of order three

We replace K_j by $K_j e$ and k_j by $k_j e$. Then we get the following rational solutions given by:
Proposition 3.3 The function u defined by

$$u(x) = \frac{n_u(x,t)}{d_u(x,t)},$$

with

$$n_u(x,t) = -6x(24{K_1}^4 t - 6{K_1}^2 k_1 - 48{K_2}^2{K_1}^2 t + 6{K_2}^2 k_1 + 24{K_2}^4 t - 6{K_2}^2 k_2 + 6{K_1}^2 k_2 - {K_1}^4 x^3 + 2{K_2}^2{K_1}^2 x^3 - {K_2}^4 x^3)$$

and

$$d_u(x,t) = 72{K_1}^2 t k_2 - 48{K_2}^2{K_1}^2 t x^3 - 6{K_1}^2 k_1 x^3 + 72{K_2}^2 k_1 t - 18k_1k_2 + 6{K_2}^2 k_1 x^3 + 6{K_1}^2 x^3 k_2 - 72{K_2}^2 t k_2 + 24{K_2}^4 t x^3 - 6{K_2}^2 k_2 x^3 - 72{K_1}^2 t k_1 + 24{K_1}^4 t x^3 - 288{K_2}^2{K_1}^2 t^2 + 144{K_1}^4 t^2 + 9{k_1}^2 + {K_1}^4 x^6 - 2{K_2}^2{K_1}^2 x^6 + 144{K_2}^4 t^2 + 9{k_2}^2 + {K_2}^4 x^6$$

is a solution to the KdV equation (1).
In this case the solution has the following structure: the numerator is polynomial of degree 0 in x, 0 in t; the denominator is polynomial of degree 2 in x, 0 in t.
The shape of the solutions depending very little on the K coefficients, we choose to take $K_j = j$ in all these following figures.

3.5 Quasi rational solutions of order four

We replace K_j by $K_j e$ and l_j by $l_j e^2$. We get the following rational solutions given by:
Proposition 3.4 The function u defined by

$$u(x) = \frac{n_u(x,t)}{d_u(x,t)},$$

with

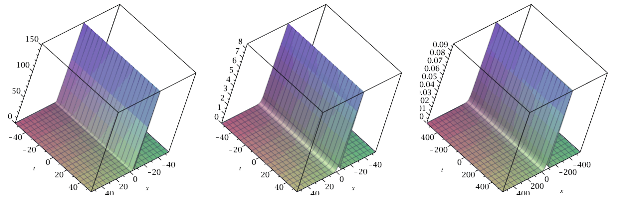

Figure 5. Rational solution of order 1 to KdV, on the left for $k_1 = 0$, $k_2 = 2$, $k_3 = 3$; in the center for $k_1 = 1$, $k_2 = 0$, $k_3 = 3$; on the right for $k_1 = 10$, $k_2 = 20$, $k_3 = 30$.

$$
\begin{aligned}
&n_u(x,t) = -6\,(K_1{}^4K_2{}^2k_3 - K_1{}^4K_2{}^2k_4 - K_1{}^4K_3{}^2k_2 \\
&+K_1{}^4K_3{}^2k_4 + K_1{}^4K_4{}^2k_2 - K_1{}^4K_4{}^2k_3 - K_1{}^2K_2{}^4k_3 \\
&+K_1{}^2K_2{}^4k_4 + K_1{}^2K_3{}^4k_2 - K_1{}^2K_3{}^4k_4 - K_1{}^2K_4{}^4k_2 \\
&+K_1{}^2K_4{}^4k_3 + K_2{}^4K_3{}^2k_1 - K_2{}^4K_3{}^2k_4 - K_2{}^4K_4{}^2k_1 \\
&+K_2{}^4K_4{}^2k_3 - K_2{}^2K_3{}^4k_1 + K_2{}^2K_3{}^4k_4 + K_2{}^2K_4{}^4k_1 \\
&-K_2{}^2K_4{}^4k_3 + K_3{}^4K_4{}^2k_1 - K_3{}^4K_4{}^2k_2 - K_4{}^4K_3{}^2k_1 \\
&+K_4{}^4K_3{}^2k_2)x(-x^3K_1{}^4K_2{}^2k_3 + x^3K_1{}^4K_2{}^2k_4 + x^3K_1{}^4K_3{}^2k_2 \\
&-x^3K_1{}^4K_3{}^2k_4 - x^3K_1{}^4K_4{}^2k_2 + x^3K_1{}^4K_4{}^2k_3 + x^3K_1{}^2K_2{}^4k_3 \\
&-x^3K_1{}^2K_2{}^4k_4 - x^3K_1{}^2K_3{}^4k_2 + x^3K_1{}^2K_3{}^4k_4 + x^3K_1{}^2K_4{}^4k_2 \\
&-x^3K_1{}^2K_4{}^4k_3 - x^3K_2{}^4K_3{}^2k_1 + x^3K_2{}^4K_3{}^2k_4 + x^3K_2{}^4K_4{}^2k_1 \\
&-x^3K_2{}^4K_4{}^2k_3 + x^3K_2{}^2K_3{}^4k_1 - x^3K_2{}^2K_3{}^4k_4 - x^3K_2{}^2K_4{}^4k_1 \\
&+x^3K_2{}^2K_4{}^4k_3 - x^3K_4{}^2K_3{}^4k_1 + x^3K_4{}^2K_3{}^4k_2 + x^3K_4{}^4K_3{}^2k_1 \\
&-x^3K_4{}^4K_3{}^2k_2 + 24\,tK_1{}^4K_2{}^2k_3 - 24\,tK_1{}^4K_2{}^2k_4 \\
&-24\,tK_1{}^4K_3{}^2k_2 + 24\,tK_1{}^4K_3{}^2k_4 + 24\,tK_1{}^4K_4{}^2k_2 \\
&-24\,tK_1{}^4K_4{}^2k_3 - 24\,K_1{}^2tK_2{}^4k_3 + 24\,K_1{}^2tK_2{}^4k_4 \\
&+24\,K_1{}^2tK_3{}^4k_2 - 24\,K_1{}^2tK_3{}^4k_4 - 24\,K_1{}^2tK_4{}^4k_2 \\
&+24\,K_1{}^2tK_4{}^4k_3 + 24\,K_2{}^4tK_3{}^2k_1 - 24\,tK_2{}^4K_3{}^2k_4 \\
&-24\,K_2{}^4tK_4{}^2k_1 + 24\,tK_2{}^4K_4{}^2k_3 - 24\,tK_2{}^2K_3{}^4k_1 \\
&+24\,K_2{}^2tK_3{}^4k_4 + 24\,tK_2{}^2K_4{}^4k_1 - 24\,K_2{}^2tK_4{}^4k_3 \\
&+24\,K_3{}^4tK_4{}^2k_1 - 24\,K_3{}^4tK_4{}^2k_2 - 24\,tK_4{}^4K_3{}^2k_1 \\
&+24\,tK_4{}^4K_3{}^2k_2 - 6\,K_1{}^2K_2{}^2k_1k_3 + 6\,K_1{}^2K_2{}^2k_1k_4 \\
&+6\,K_1{}^2K_2{}^2k_3k_2 - 6\,K_1{}^2K_2{}^2k_4k_2 + 6\,K_1{}^2K_3{}^2k_2k_1 \\
&-6\,K_1{}^2K_3{}^2k_1k_4 - 6\,K_1{}^2K_3{}^2k_3k_2 + 6\,K_1{}^2K_3{}^2k_3k_4 \\
&-6\,K_1{}^2K_4{}^2k_2k_1 + 6\,K_1{}^2K_4{}^2k_1k_3 + 6\,K_1{}^2K_4{}^2k_4k_2 \\
&-6\,K_1{}^2K_4{}^2k_3k_4 - 6\,K_2{}^2K_3{}^2k_2k_1 + 6\,K_2{}^2K_3{}^2k_1k_3 \\
&+6\,K_2{}^2K_3{}^2k_4k_2 - 6\,K_2{}^2K_3{}^2k_3k_4 + 6\,K_2{}^2K_4{}^2k_2k_1 \\
&-6\,K_2{}^2K_4{}^2k_1k_4 - 6\,K_2{}^2K_4{}^2k_3k_2 + 6\,K_2{}^2K_4{}^2k_3k_4 \\
&-6\,K_3{}^2K_4{}^2k_1k_3 + 6\,K_3{}^2K_4{}^2k_1k_4 + 6\,K_3{}^2K_4{}^2k_2k_3 \\
&-6\,K_3{}^2K_4{}^2k_2k_4)
\end{aligned}
$$

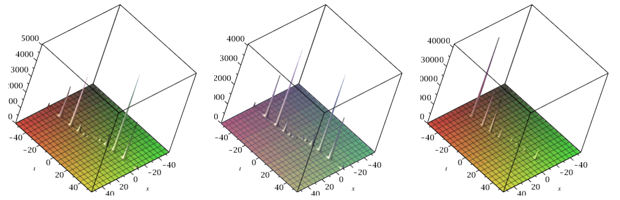

Figure 6. Rational solution of order 1 to KdV, on the left for $k_1 = 0$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$; in the center for $k_1 = 10$, $k_2 = 0$, $k_3 = 0$, $k_4 = 0$; on the right for $k_1 = 10$, $k_2 = 0$, $k_3 = 30$, $k_4 = 0$.

$$
\begin{aligned}
&d_u(x,t) = (x^3K_1{}^4K_2{}^2k_3 - x^3K_1{}^4K_2{}^2k_4 - x^3K_1{}^4K_3{}^2k_2 \\
&+x^3K_1{}^4K_3{}^2k_4 + x^3K_1{}^4K_4{}^2k_2 - x^3K_1{}^4K_4{}^2k_3 - x^3K_1{}^2K_2{}^4k_3 \\
&+x^3K_1{}^2K_2{}^4k_4 + x^3K_1{}^2K_3{}^4k_2 - x^3K_1{}^2K_3{}^4k_4 - x^3K_1{}^2K_4{}^4k_2 \\
&+x^3K_1{}^2K_4{}^4k_3 + x^3K_2{}^4K_3{}^2k_1 - x^3K_2{}^4K_3{}^2k_4 - x^3K_2{}^4K_4{}^2k_1 \\
&+x^3K_2{}^4K_4{}^2k_3 - x^3K_2{}^2K_3{}^4k_1 + x^3K_2{}^2K_3{}^4k_4 + x^3K_2{}^2K_4{}^4k_1 \\
&-x^3K_2{}^2K_4{}^4k_3 + x^3K_4{}^2K_3{}^4k_1 - x^3K_4{}^2K_3{}^4k_2 - x^3K_4{}^4K_3{}^2k_1 \\
&+x^3K_4{}^4K_3{}^2k_2 + 12\,tK_1{}^4K_2{}^2k_3 - 12\,tK_1{}^4K_2{}^2k_4 \\
&-12\,tK_1{}^4K_3{}^2k_2 + 12\,tK_1{}^4K_3{}^2k_4 + 12\,tK_1{}^4K_4{}^2k_2 \\
&-12\,tK_1{}^4K_4{}^2k_3 - 12\,K_1{}^2tK_2{}^4k_3 + 12\,K_1{}^2tK_2{}^4k_4 \\
&+12\,K_1{}^2tK_3{}^4k_2 - 12\,K_1{}^2tK_3{}^4k_4 - 12\,K_1{}^2tK_4{}^4k_2 \\
&+12\,K_1{}^2tK_4{}^4k_3 + 12\,K_2{}^4tK_3{}^2k_1 - 12\,tK_2{}^4K_3{}^2k_4 \\
&-12\,K_2{}^4tK_4{}^2k_1 + 12\,tK_2{}^4K_4{}^2k_3 - 12\,tK_2{}^2K_3{}^4k_1 \\
&+12\,K_2{}^2tK_3{}^4k_4 + 12\,tK_2{}^2K_4{}^4k_1 - 12\,K_2{}^2tK_4{}^4k_3 \\
&+12\,K_3{}^4tK_4{}^2k_1 - 12\,K_3{}^4tK_4{}^2k_2 - 12\,tK_4{}^4K_3{}^2k_1 \\
&+12\,tK_4{}^4K_3{}^2k_2 - 3\,K_1{}^2K_2{}^2k_1k_3 + 3\,K_1{}^2K_2{}^2k_1k_4 \\
&+3\,K_1{}^2K_2{}^2k_3k_2 - 3\,K_1{}^2K_2{}^2k_4k_2 + 3\,K_1{}^2K_3{}^2k_2k_1 \\
&-3\,K_1{}^2K_3{}^2k_1k_4 - 3\,K_1{}^2K_3{}^2k_3k_2 + 3\,K_1{}^2K_3{}^2k_3k_4 \\
&-3\,K_1{}^2K_4{}^2k_2k_1 + 3\,K_1{}^2K_4{}^2k_1k_3 + 3\,K_1{}^2K_4{}^2k_4k_2 \\
&-3\,K_1{}^2K_4{}^2k_3k_4 - 3\,K_2{}^2K_3{}^2k_2k_1 + 3\,K_2{}^2K_3{}^2k_1k_3 \\
&+3\,K_2{}^2K_3{}^2k_4k_2 - 3\,K_2{}^2K_3{}^2k_3k_4 + 3\,K_2{}^2K_4{}^2k_2k_1 \\
&-3\,K_2{}^2K_4{}^2k_1k_4 - 3\,K_2{}^2K_4{}^2k_3k_2 + 3\,K_2{}^2K_4{}^2k_3k_4 \\
&-3\,K_3{}^2K_4{}^2k_1k_3 + 3\,K_3{}^2K_4{}^2k_1k_4 + 3\,K_3{}^2K_4{}^2k_2k_3 \\
&-3\,K_3{}^2K_4{}^2k_2k_4)^2
\end{aligned}
$$

is a solution to the KdV equation (1).
In this case, the solution to the KdV equation has the following structure: the numerator is a polynomial of degree 4 in x, 1 in t; the denominator is a polynomial of degree 6 in x, 2 in t; the solution depends on six arbitrary parameters k_j and K_j for $1 \leq j \leq 3$.
The structure of the solutions depending very little on the K coefficients, we choose to take $K_j = j$ in all the following figures.

These simple solutions could be used in various fields, including in particular physics. These solutions are singular. These solutions are new and different from the previous ones built by the author [8, 10, 11]. For example, in this paper, in the case of order 3, the denominator of the so-

lution is a polynomial od degree 6 in x and 2 in t; in [8], the denominator of the solution of order 3 is a polynomial of degree 12 in x, 4 in t; in [10], the denominator of the solution of order 3 is a polynomial of degree 12 in x, 4 in t different from this of [8]; in [11], the denominator of the solution of order 3 is a polynomial of degree 12 in x, 4 in t different from the previous ones.
I must mention the article [12] in connection with my research which deals with equations such as the KdV equation and the representation of solutions as sum of solitons, and also the relationship of these solutions with Riemann's theta functions in particular.

4. Conclusion

From the degenerate θ solutions to the KdV equation expressed in terms of Fredholm determinants or wronskians, we succeeded to get rational solutions to the KdV equation. So we obtain an infinite hierarchy of multi-parametric families of rational solutions to the KdV equation as a quotient of a polynomials depending on real parameters.
The quasi-rational solutions to the KdV equation were obtained by the passage to the limit when one of the parameters tends towards zero. These solutions are not obtained uniformly as in the construction of the solutions for example to the non-linear Schrödinger equation [13]. In the quasi-rational solutions constructed, the parameters were chosen in such a way as to obtain quasi-rational solutions of maximum degree in x and in t. It would be relevant to continue this work for higher orders and to study the structure of polynomials defining these solutions.

Conflict of interest statement:
The authors declare that they have no conflict of interest.

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