El-Nabulsi *Journal of Theoretical and Applied Physics* 2013, **7**:60
http://www.jtaphys.com/content/7/1/60

Journal of Theoretical and Applied Physics
a SpringerOpen Journal

RESEARCH **Open Access**

Some consequences of nonstandard Lagrangians with time-dependent coefficients in general relativity

Rami A El-Nabulsi

Abstract

Nonstandard Lagrangians entitled 'nonnaturals' by Arnold have recently gained increasing importance both in applied mathematics and in physical theories. These types of Lagrangians appear in some group of dissipative dynamical systems, and they play an important role in a number of field theories. However, the role of nonstandard Lagrangians in geometric theories like general relativity is still absent. In this communication, we would like to discuss the relevance of nonstandard Lagrangians in general relativity using the principles of calculus of variations. In fact, nonstandard Lagrangians came in different forms, features, and characteristics, depending on the nature of the dynamical problem under study. In this work, we will be concerned with time-dependent Lagrangians of the form $L^{1+\gamma(t)}$. After deriving the modified geodesic equation using the basic techniques of Riemannian differential geometry which will be used to axiomatize a large part of our work, we show that many interesting consequences will be raised accordingly when applied to FRW cosmology.

Keywords: Nonstandard Lagrangian; Modified geodesic equation; FRW cosmology

Background

In recent years, the topic of nonstandard Lagrangians (NSLs) has attracted attention due to its wide applications in different branches of applied mathematics and theoretical physics. In reality, NSL is not new and its origin dated back to 1978 when Arnold entitled them 'nonnatural' [1]. The topic was completely ignored until 1984 when Alekseev and Arbuzov used NSL to describe large-distance interactions in the field of applicability of classical theory within the framework of Yang-Mills field theory, a problem which is directly related to color confinement issue [2]. In fact, in years of progress, it was realized that NSL plays an important role in nonlinear differential equations like the nonlinear second-order Riccati equation [3] and the Lienard-type nonlinear differential equation [4,5]. Interest in NSL increases gradually with time, and more applications were discussed in dissipative dynamical systems [6-10], quantum field theory [11,12], and cosmology [13,14].

Although the topic of NSL was addressed generally in classical dynamical systems, its implications in differential geometry and astrophysics are still ignored, and to the best of our knowledge, the topic is completely missed. This paper aims to explore how NSL can help bridge the gap between NSL physics and applied differential geometry. Recently, we have discussed the problem of finding a nonstandard Lagrangian description for a large class of dynamical systems [15-17]. A good number of NSLs which have been overlooked in the existing literature were derived and solved. For the sake of clarity, it should be emphasized that in our approach the term NSL refers principally to any 'standard Lagrangian function that modify the Euler-Lagrange equations and accordingly Hamilton's equations of motion'. In fact, two different types of NSL were introduced: the exponential NSL and the power-law NSL. In this communication, we will deal with the power-law NSL (PNSL) with time-dependent coefficient, and we keep the second

Correspondence: nabulsiahmadrami@yahoo.fr
College of Mathematics and Information Science, Neijiang Normal University, Neijiang, Sichuan 641112, China

one for a future work. We will show that the presence of a time-dependent coefficient in PNSL is in fact connected to a time-dependent gravitational potential (TDGP). In reality, an assumption that in the universe there is a time-dependent gravitational potential was proposed in [18,19] and discussed in [20-23]. It is an interesting attempt as it explains the Hubble redshift and the anomalous acceleration from the spacecraft Pioneer 10 and 11 mainly when TDGP decreases linearly with time. A TDGP also has many astrophysical impacts as its change modify the planetary orbits which were proven to be not axially symmetric. Moreover, it was observed that if the gravitational potential is time-dependent, the distant galaxies' velocity is not extremely fast as it is currently considered. In fact, the change of the gravitational potential is most likely caused by the change of the dark energy density, but nothing is confirmed about it. One naturally expects a violation of the strong and weak equivalence principle of general relativity. This violation was discussed in many theoretical aspects [24-41]. It is noteworthy that general relativity has been systematically tested on solar system scales. Therefore, it is unsurprising if we try to correct the theory at galactic and cosmological scales. One interesting recent anthropic argument suggests that the equivalence principle is violated at a small level [42]. This approach motivates the demand for enhanced tests of the equivalence principle. Some interesting points concerning the violation of the equivalence principle and violation of diffeomorphism invariance in connection with emergent gravity were discussed in [43,44] and references therein.

The paper is organized as follows: in the section 'PNSL and the modified geodesic equation', PNSL with time-dependent coefficient is revised, and the corresponding modified geodesic equation is introduced. In the section 'A cosmological application of $G \to G + H/\rho a$', we discuss some main astrophysical and cosmological consequences. Finally in the last section, conclusions and perspectives are presented.

PNSL and the modified geodesic equation

The PNSL that will be considered in this paper is $L^{1+\gamma(t)}(t, \dot{q}(t), q(t))$ (γ is a time-dependent parameter), and the corresponding action functional is $S = \int_a^b L^{1+\gamma(t)}(t, \dot{q}(t), q(t))dt$. Here $(t, \dot{q}, q) \to L(t, \dot{q}, q)$ is assumed to be a C^2 function, $q \in C^1([a, b]; \mathbb{R}^n)$ is the generalized coordinate, $\dot{q} = dq/dt$, and $L(t, \dot{q}, q) \in C^2([a, b] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ is the Lagrangian. It is easy to check that any admissible function $q \in C^1[a, b]$ subject to given boundary conditions $q(a) = q_a$ and $q(b) = q_b$ for which the action functional has an extremum satisfies the following Euler-Lagrange equations [17]:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\gamma(t)}{L}\frac{\partial L}{\partial \dot{q}}\left(\frac{\partial L}{\partial t} + \dot{q}\frac{\partial L}{\partial q} + \ddot{q}\frac{\partial L}{\partial \dot{q}}\right) + \frac{1}{1+\gamma(t)}\frac{d\gamma(t)}{dt}\frac{\partial L}{\partial \dot{q}} \tag{1}$$

We would like to apply this result to compute the geodesics for a given metric. The mathematical translation from classical mechanics to differential geometry is discussed in any graduate textbook (see [42] for details). We use the Einstein summation convention in which we sum over repeated indices which occur as a subscript and superscript pair. In order to obtain the modified geodesic equation, we use naturally the variational principle. In the language of differential geometry, the variational principle states that any freely falling test particles follow a certain path between two fixed points in curved spacetime which extremize the proper time. This is one of the basic postulates of Einstein's general relativity (EGR). In what follows, we set for convenience τ the proper time. Besides, we work in units $\hbar = c = 1$ with a metric sign (−, +, +, +).

In EGR, the action is maximum for straight paths and the path length is $S = \int L d\tau$ where $L = (1/2)g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$, where $g_{\mu\nu}$ is the covariant spacetime metric tensor. The jump from classical mechanics to differential geometry is therefore realized by making the change $(t, \dot{q}, q) \to (\tau, \dot{x}^{\mu}, x^{\mu})$ where $\dot{x}^{\mu} = dx^{\mu}/d\tau$. In this context, we can rewrite Equation 1 as

$$\frac{\partial L}{\partial x^{\alpha}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\alpha}}\right) = \frac{\gamma(\tau)}{L}\frac{\partial L}{\partial \dot{x}^{\alpha}}\left(\frac{\partial L}{\partial \tau} + \dot{x}^{\alpha}\frac{\partial L}{\partial x^{\alpha}} + \ddot{x}^{\alpha}\frac{\partial L}{\partial \dot{x}^{\alpha}}\right) + \frac{1}{1+\gamma(\tau)}\frac{d\gamma(\tau)}{d\tau}\frac{\partial L}{\partial \dot{x}^{\alpha}} \tag{2}$$

After replacing $L = (1/2)g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ into Equation 2, we find after simple algebra

$$\underbrace{\ddot{x}^{\nu} + \Gamma^{\nu}_{\gamma\delta}\dot{x}^{\gamma}\dot{x}^{\delta}}_{\text{standard part}} \underbrace{+ \frac{2\gamma(\tau)}{g_{\mu\nu}\dot{x}^{\mu}}\left(\frac{1}{2}g_{\gamma\delta,\alpha}\dot{x}^{\gamma}\dot{x}^{\delta}\dot{x}^{\alpha} + \ddot{x}^{\alpha}g_{\gamma\alpha}\dot{x}^{\gamma}\right) + \frac{1}{1+\gamma(\tau)}\frac{d\gamma(\tau)}{d\tau}\dot{x}^{\nu}}_{\text{extra-part}} = 0 \tag{3}$$

Here $\Gamma^{\nu}_{\alpha\beta}$ are the Christoffel connection coefficients of the metric $g_{\alpha\beta}$ defined by [45]

$$\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2} g^{\nu\gamma}\left[g_{\alpha\gamma,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}\right] \tag{4}$$

where $g_{\alpha\beta,\gamma} = \partial g_{\alpha\beta}/\partial x^{\gamma}$ and so on.

Remark 1 The well-known reparametrization invariance procedure is unable to bring the equations of motion (3) into the standard form even on-shell. Therefore, we expect new physics to be obtained from the present approach by choosing a different action for particles in relativity.

The calculation is straightforward: in fact, from $L = \frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$, we get $\partial L/\partial \dot{x}^{\alpha} = g_{\gamma\alpha}\dot{x}^{\gamma}$ and $\partial L/\partial x^{\alpha} = g_{\gamma\delta,\alpha}\dot{x}^{\gamma}\dot{x}^{\delta}/2$. After replacing into Equation 2, we find

$$\begin{aligned} \frac{1}{2} g_{\gamma\delta,\alpha}\dot{x}^{\gamma}\dot{x}^{\delta} - \frac{d}{d\tau} g_{\gamma\alpha}\dot{x}^{\gamma} = {} & \frac{2\gamma(\tau)}{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} g_{\gamma\alpha}\dot{x}^{\gamma} \\ & \times\left(\frac{1}{2} g_{\gamma\delta,\alpha}\dot{x}^{\gamma}\dot{x}^{\delta}\dot{x}^{\alpha} + \ddot{x}^{\alpha} g_{\gamma\alpha}\dot{x}^{\gamma}\right) \\ & + \frac{1}{1+\gamma(\tau)}\frac{d\gamma(\tau)}{d\tau} g_{\gamma\alpha}\dot{x}^{\gamma} \end{aligned} \tag{5}$$

Using $dg_{\gamma\alpha}/d\tau = g_{\gamma\alpha,\delta}\dot{x}^{\delta}$, multiplying both sides of Equation 5 by $g^{\nu\alpha}$ and using $g^{\nu\alpha} g_{\gamma\alpha}\dot{x}^{\gamma} = \dot{x}^{\nu}$, we find

$$\begin{aligned} & \ddot{x}^{\nu} - g^{\nu\alpha}\left(\frac{1}{2} g_{\gamma\delta,\alpha} - g_{\gamma\alpha,\delta}\right)\dot{x}^{\gamma}\dot{x}^{\delta} \\ & + \frac{2\gamma(\tau)}{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}\dot{x}^{\nu}\left(\frac{1}{2} g_{\gamma\delta,\alpha}\dot{x}^{\gamma}\dot{x}^{\delta}\dot{x}^{\alpha} + \ddot{x}^{\alpha} g_{\gamma\alpha}\dot{x}^{\gamma}\right) \\ & + \frac{1}{1+\gamma(\tau)}\frac{d\gamma(\tau)}{d\tau}\dot{x}^{\nu} = 0 \end{aligned} \tag{6}$$

With the help of $g_{\gamma\alpha,\delta}\dot{x}^{\gamma}\dot{x}^{\delta} = g_{\alpha\delta,\gamma}\dot{x}^{\gamma}\dot{x}^{\delta}$ and Equation 4, we effortlessly find Equation 3 which may also be written in the following useful form:

$$\frac{dx^{\mu}}{d\tau}\left(\underbrace{\frac{d^2x^{\nu}}{d\tau^2} + \Gamma^{\nu}_{\gamma\delta}\frac{dx^{\gamma}}{d\tau}\frac{dx^{\delta}}{d\tau}}_{\text{standard part}}\right) + \underbrace{\frac{2\gamma(\tau)}{g_{\mu\nu}}\left(\frac{1}{2} g_{\gamma\delta,\alpha}\frac{dx^{\gamma}}{d\tau}\frac{dx^{\delta}}{d\tau}\frac{dx^{\alpha}}{d\tau} + g_{\gamma\alpha}\frac{dx^{\gamma}}{d\tau}\frac{d^2x^{\alpha}}{d\tau^2}\right) + \frac{1}{1+\gamma(\tau)}\frac{d\gamma(\tau)}{d\tau}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}}_{\text{extra part}} = 0 \tag{7}$$

It is obvious that for $\gamma = 0$, Equation 7 is reduced to the standard geodesic form. The importance of the last term in Equation 7 which concerns the variation of $\gamma(\tau)$ with time will be appreciated in the next subsection.

Remark 2 If we set $\gamma(\tau) \propto \tau^{-x}, x > 0$, then Equation 7 is reduced for a very large time to its standard form as $\lim_{\tau\to\infty} \gamma(\tau) \to 0$ and $\lim_{\substack{\tau\to\infty \\ x>0}} \frac{1}{1+\gamma(\tau)}\frac{d\gamma(\tau)}{d\tau} = \lim_{\substack{\tau\to\infty \\ x>0}} -\frac{x}{1+\tau^{x}}\frac{1}{\tau} \to 0.$

We consider the motion of a particle of mass m assumed to be slowly moving in a time-dependent stationary field. The limit of slow motion reduces the Christoffel symbol to:

$$\begin{aligned} \Gamma^{\nu}_{00} &= \frac{1}{2} g^{\nu\alpha}\left[g_{0\alpha,0} + g_{\alpha0,0} - g_{00,\alpha}\right] \\ &= -\frac{1}{2} g^{\nu i} g_{00,i} + \frac{1}{2} g^{\nu 0} g_{00,0} \end{aligned} \tag{8}$$

We follow the standard formalism and perform the perturbation $g_{\alpha\beta}(x^i, \tau) = \eta_{\alpha\beta} + h_{\alpha\beta}(x^i, \tau)$ (small deviation from a Minkowski flat spacetime) with $g^{\alpha\beta}(x^i, \tau) = \eta^{\alpha\beta} - h^{\alpha\beta}(x^i, \tau)$ since $g_{\delta\beta}g^{\alpha\beta} = \delta^{\alpha}_{\delta}$ with $|h_{\alpha\beta}(x^i, \tau)| < < 1$. The Minkowski metric η is constant, whereas $h_{\alpha\beta}(x^i, \tau)$ is time-dependent since the gravitational field is assumed to be nonstationary. To the first order, the Christoffel symbol takes the form

$$\Gamma^{\nu}_{00} = -\frac{1}{2}\eta^{\nu i} h_{00,i} + \frac{1}{2}\eta^{\nu 0}\hat{h}_{00,0} \tag{9}$$

In fact we assumed that $h_{00}(x^i, \tau) = h_{00}(x^i) + \hat{h}_{00}(\tau)$, where $\hat{h}_{00}(\tau)$ is expected to be the time-dependent gravitational potential correction to $h_{00}(x^i)$. We accordingly find $\Gamma^{0}_{00} = \frac{1}{2}\eta^{00}\hat{h}_{00,0}$ and $\Gamma^{j}_{00} = -\frac{1}{2}\eta^{ji} h_{00,i}$. For $\nu = 0$, since $dx^i/d\tau = O(\varepsilon)$, we get

$$\begin{aligned} & (1+2\gamma(\tau))\frac{d^2x^0}{d\tau^2} + \left(-\frac{1}{2} + \gamma(\tau)\frac{1}{h_{00}-1}\right)\hat{h}_{00,0}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} \\ & + \frac{1}{1+\gamma(\tau)}\frac{d\gamma(\tau)}{d\tau}\frac{dx^0}{d\tau} = 0 \end{aligned} \tag{10}$$

As in general $|h_{00}| < < 1$, we can approximate Equation 10 by

$$\frac{d^2x^0}{d\tau^2} - \left(\frac{1}{2}\hat{h}_{00,0}\frac{dx^0}{d\tau} - \frac{1}{(1+\gamma(\tau))(1+2\gamma(\tau))}\frac{d\gamma(\tau)}{d\tau}\right)\frac{dx^0}{d\tau} = 0 \quad (11)$$

For $\nu = j$ and then $i = j$, we get with $|h_{ii}| < < 1$

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}h_{00,i}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + 2\gamma(\tau)\left(\frac{1}{2}\hat{h}_{00,0}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} - \frac{d^2x^0}{d\tau^2}\right) + \frac{1}{1+\gamma(\tau)}\frac{d\gamma(\tau)}{d\tau}\frac{dx^0}{d\tau} = 0 \quad (12)$$

Using Equation 11, we can rewrite Equation 12 as

$$\frac{d^2x^i}{d\tau^2} - \left(\frac{1}{2}h_{00,i}\frac{dx^0}{d\tau} - \frac{4\gamma(\tau)+1}{(1+\gamma(\tau))(1+2\gamma(\tau))}\frac{d\gamma(\tau)}{d\tau}\right)\frac{dx^0}{d\tau} = 0 \quad (13)$$

Up to this stage, the situation seems complicated as we ignore the form of $\gamma(\tau)$ of which we are aware from the beginning that it may take a large number of forms. However, one particular class of solution may be extracted from Equation 11. If we set the factor inside the parenthesis equal to zero, then $\ddot{x}^0 \equiv d^2t/d\tau^2 = 0$, i.e., $\dot{x}^0 \equiv dt/d\tau = k =$ constant. In that case, we have

$$\frac{1}{(1+\gamma(\tau))(1+2\gamma(\tau))}\frac{d\gamma(\tau)}{d\tau} = \frac{k}{2}\hat{h}_{00,0} \quad (14)$$

and therefore, after simple integration, we get

$$\hat{h}_{00}(\tau) = \frac{2}{k}\log\frac{2\gamma(\tau)+1}{\gamma(\tau)+1} \quad (15)$$

where we assumed that when $\gamma(\tau) = 0$, $\hat{h}_{00} = 0$. This solution seems interesting as in that case Equation 13 is simplified directly to

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}h_{00,i} + (4\gamma(\tau)+1)\frac{k}{2}\hat{h}_{00,0} = 0 \quad (16)$$

This equation may be written in the following useful form

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\nabla_i h_{00} - (4\gamma(\tau)+1)\frac{k}{2}\frac{d\hat{h}_{00}}{d\tau} \equiv \frac{1}{2}\nabla_i h_{00} + \Delta_\tau h_{00} \quad (17)$$

where

$$\Delta_\tau h_{00} = -(4\gamma(\tau)+1)\frac{k}{2}\frac{d\hat{h}_{00}}{d\tau} \quad (18)$$

is the time-dependent gravitational potential correction to h_{00}. It is noticeable that if h_{00} is time-independent, then Equation 17 is reduced to its standard from. If now we expect that near the spherical body the gravitational potential in spherical symmetry is $\Phi(r) = -MG/r$, then we get $h_{00} = -2\Phi(r)$. As discussed in the introductory text of this work, the assumption that the time-dependent gravitational potential correction decreases linearly or almost linearly with time is very appealing due to what it may explain at both the astrophysical and cosmological levels. Motivated by this argument, we postulate that

$$\gamma(\tau) = -\frac{e^\tau - 1}{2e^\tau - 1} \quad (19)$$

which gives making use of Equation 16 $\hat{h}_{00}(\tau) = -2\tau/k$ which is a linearly decreasing time-dependent gravitational potential. In that case, Equation 17 is reduced to

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}h_{00,i} + \frac{2e^\tau - 3}{2e^\tau - 1}\frac{2}{k} = 0 \quad (20)$$

To find k, we follow the arguments of [15,16]: for a light starting from a certain star with a frequency $\nu_0(\tau = 0)$, then its frequency at time τ is given according to the principle of general relativity by $\nu(\tau) = \nu(\tau = 0) \times \left(1 + \tau(d\hat{h}_{00}/d\tau)\right)$. Now for $R = \tau$, we get $\nu(\tau) = \nu(\tau = 0)\left(1 + R(d\hat{h}_{00}/d\tau)\right)$. If we compare this equation with the Hubble law $\nu(\tau) = \nu(\tau = 0)(1 - RH))$ where $H \approx$ 60–70 km/s/Mpc is the Hubble constant, then we find $d\hat{h}_{00}(\tau)/d\tau = -2/k = -H$, i.e., $k = 2/H$. In that case, we can now write Equation 20 as

$$\frac{d^2x^i}{d\tau^2} + \nabla_i\Phi + \frac{2e^\tau - 3}{2e^\tau - 1}H = 0 \quad (21)$$

This equation represents the modified Newton's law in the PNSL approach with a time-dependent coefficient. We can define the total gravitational potential by

$$\Phi_{\text{total}}\left(x^i, \tau\right) = \Phi + \frac{2e^\tau - 3}{2e^\tau - 1}Hx^i \quad (22)$$

so that Equation 21 is reduced to

$$\frac{d^2x^i}{d\tau^2} + \nabla_i\Phi_{\text{total}} = 0 \quad (23)$$

At a very large time, $\Phi_{\text{total}}(x^i, \infty) = \Phi + Hx^i$, whereas at the origin of time, we find $\Phi_{\text{total}}(x^i, 0) = \Phi - Hx^i$. In units where $c \neq 1$, we find $\Phi_{\text{total}}(x^i, \infty) = \Phi + Hcx^i$ and $\Phi_{\text{total}}(x^i, 0) = \Phi - Hcx^i$. During the cosmological evolution of the universe, the effective gravitational potential can change sign.

Remark 3 In Cartesian coordinates, i.e., $x^i = (x, y, z)$, we can write in units $c \neq 1$

$$\Phi_{\text{total}}\left(x^i, \tau\right) = -\frac{MG}{x^i} + \frac{2e^\tau - 3}{2e^\tau - 1} Hcx^i = -\frac{M(G + G_{\text{effective}})}{x^i} \tag{24}$$

where

$$G_{\text{effective}} = -\frac{2e^\tau - 3}{2e^\tau - 1} cH \frac{x^i x^i}{M} \tag{25}$$

If we model the universe as a ball of radius α and mass $M \propto \rho a^3$, then $x^i x^i \propto a^2$ because the distance to an object that is receding with the expansion of the universe is proportional to the cosmic scale factor α. In that case and mainly at a very large time, $|G_{\text{effective}}| \propto Hca^2/\rho a^3 = Hc/\rho a$, i.e., $\rho|G_{\text{effective}}| \propto Hc/a$. This equation is interesting as it shows that the effective gravitational constant varies with time since it depends on the evolution of the scale factor and the density of the universe. A cosmological implication of the effective gravitational constant is discussed in the next section.

Remark 4 In fact, the complete dynamical description, i.e., the presence of matter and energy curves of the ambient spacetime is obtained making use of Einstein's field equation (EFE) which is nothing else but the general relativity analog of the Poisson equation. In other words, EFE reduces to Newton's law of gravity by using both the weak-field and the slow-motion approximations, i.e., time derivatives are much smaller than spatial derivatives and, consequently, the gravitational coupling constant appears in the EFE which takes the form [45]: $R_{\alpha\beta} - g_{\alpha\beta}R/2 = 8\pi G T_{\alpha\beta}$ where $R_{\alpha\beta}$ is the Riemann tensor, R is the Ricci tensor, $T_{\alpha\beta}$ is the stress energy tensor defined by $T_{\alpha\beta} = (p + \rho)u_\alpha u_\beta + pg_{\alpha\beta}$ for matter component, p and ρ are respectively the pressure of the perfect barotropic fluid, and μ_α is the fluid rest frame four-velocity. In our approach, EFE is not modified and keeps its standard form.

From an astrophysical point of view, it is recognizable that the Schwarzschild spacetime metric is a solution of EFE that describes a gravitational field exterior to an isolated sphere assumed to be at rest. In fact, this special solution has played an important role in general relativity. It should be stressed at this stage that if we assume the usual Schwarzschild spacetime characterized in symmetric spherical coordinates by the metric $ds^2_{\text{static}} = -e^{2m(r)}dt^2 + e^{2n(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, then the metric is time-independent, and from Equation 14, it follows that $\gamma(\tau)$ is constant. In such a case, the system results in the standard Schwarzschild solution. However, it was argued that a Schwarzschild spacetime with a time-dependent metric is suitable for the description of the interior black hole solution [46]. At the interior of a black hole Schwarzschild solution, nature changes its properties. A remarkable switch occurs between the external spatial radial and temporal coordinates [47-50]. That is the main reason an interior solution is time-dependent and far from being stationary. A singularity occurs at a space like a hypersurface at the origin of time, and accordingly, observing the physical effects of this singularity is somewhat not viable. Therefore, one expects that solutions obtained at this stage will be practical if one would like to describe a particular spacetime characterized by the metric $ds^2_{\text{time-dependent}} = -e^{2m(t)}dt^2 + e^{2n(t)}dr^2 + t^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ [46] which is suitable for the description of the interior black hole solution. It is not the aim of the present work to discuss the physics inside a black hole, yet it is an interesting topic that deserves a future analysis.

A cosmological application of $G \to G + H/\rho a$

To illustrate, we discuss briefly for completeness the gravitational relaxation on the flat Friedmann-Robertson-Walker (FRW) metric given by $ds^2 = -d\tau^2 + a^2(\tau)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$ where $\alpha(\tau)$ is the scale factor. Mainly, we will concentrate on the late-time dynamics. We stress that our aim is not to discuss the whole cosmological scenario but simply to present the basic consequences of $G \to G + H/\rho a$.

With the replacement $G \to G + H/\rho a$, the first of the Friedmann equation in the presence of a time-dependent cosmological constant $\Lambda(\tau)$ is [51]

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda(\tau)}{3} + \frac{8\pi H}{3a} \tag{26}$$

where $H = \dot{a}/a$ is the Hubble parameter, and ρ is the energy density of the (perfect) cosmological fluid. The dot represents the derivative with respect to τ. We shall assume that the equation of state for the cosmological matter is $p = \gamma\rho$, where γ is a constant and p is the corresponding pressure. To leading order, the corrections to the Friedmann equation can be parameterized $-H/a$. In what follows, we conjecture that the cosmological constant varies as $\Lambda = 8\pi H/a$. There exist in the literature lots of phenomenological ansatzs for the variation of the cosmological constant with time ([52] and references therein). However, to the best of our knowledge, the ansatz introduced here is completely new. The usual conservation of the energy-momentum tensor $T^{\alpha\beta}_{;\beta} = 0$ results in two useful relations: $\dot{\rho} + 3H(p + \rho) = 0$ and $\dot{\Lambda} + 8\pi\dot{G}\rho = 0$. The first equation gives $\rho(a) = \rho_0(a/a_0)^{-3(1+\gamma)}$ where $\rho_0 = \rho(a = a_0)$, whereas the second equation may be written after performing the following replacement:

$$\dot{G} \rightarrow -\frac{\dot{H}}{\rho a} + \frac{H\dot{\rho}}{\rho^2 a} - \frac{H\dot{a}}{\rho a^2}, \tag{27}$$

as

$$2\frac{\dot{H}}{H} - 2\frac{\dot{a}}{a} - \frac{\dot{\rho}}{\rho} = 0 \tag{28}$$

Equation 28 after simple integration gives $H^2 = C\rho a^2$ where C is an integration constant. The cosmological constant varies subsequently as $\Lambda = 8\pi H/a \propto \sqrt{\rho}$. Accordingly, Equation 26 takes the form

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} + A\sqrt{\rho} \tag{29}$$

where $A = 16\pi C/3$. This equation is interesting as it is similar to the Cardassian model proposed by Freese and Lewis to explain the current accelerated expansion of the universe [53]. The effective gravitational constant is then replaced by $G \rightarrow G + C\rho^{-1/2}$. Now with the help of $\rho(a) = \rho_0(a/a_0)^{-3(1+\gamma)}$, we can write Equation 29 as

$$\dot{a}^2 = Ba^{-(1+3\gamma)} + Aa^{(1-4\gamma)/2} \tag{30}$$

where $B = 8\pi G\rho_0(1/a_0)^{-3(1+\gamma)}$. We set for convenience $A = B = 1$. This equation seems interesting as for $\gamma = 0$ (pressureless matter), the solution of $\dot{a}^2 = a^{-1} + a^{1/2}$ with the initial condition $\alpha(0) = 0$ is $a(\tau) \propto \tau^{4/3}$ which corresponds to an accelerated universe without the need of the dark energy component. In general, one expects that the term $A\sqrt{\rho}$ is initially negligible, but it dominates the universe at redshift $z \approx O(1)$, as indicated by supernova observations [54,55]. Once it dominates, the universe starts it accelerated expansion, and in that case, we can neglect the first term on the RHS of Equation 29 with respect to $A\sqrt{\rho}$. Therefore, the scale factor evolves like $\dot{a}^2 \propto a^{(1-4\gamma)/2}$. For $\gamma = 0$, we find $a(t) \propto \tau^{4/3}$ as it is expected. Accordingly, the energy density decays like $\rho \propto a^{-3} = \tau^{-4}$, whereas the cosmological constant and the effective gravitational constant vary respectively as $\Lambda \propto \tau^{-2}$ and $G_{\text{effective}} \propto \tau^2$. In reality, a number of authors have argued in favor of the decaying law $\Lambda \propto \tau^{-2}$ ([52] and references therein). The time variation of the gravitational constant is not new and is well discussed largely in literature ([52] and references therein).

It is interesting to fall into the Cardassian model from nonstandard Lagrangians and mainly the PNSL with a time-dependent coefficient. This model has been confronted largely by cosmic observations, and for recent result, the authors are referred to [56].

Conclusions and perspectives

In this work, we have tried to prove that nonstandard Lagrangians are extremely important and deserve special attention. Starting from a power-law nonstandard Lagrangian characterized by a time-dependent coefficient, we have applied the basic machinery of the calculus of variations to differential geometry and in particular to general relativity. We have derived the modified geodesic equation and have proven that the presence of a time-dependent coefficient is directly connected to a time-dependent gravitational potential. As a TDGP is typical for the description of the interior black hole solution, we expect that our approach will have a large number of applications. We have proven as well that for the case of an expanding universe, the gravitational constant is replaced by $G \rightarrow G + H/\rho a$ and that the resulting FRW cosmological model is similar to the Cardassian cosmology without referring to quantum corrections or brane approach. We anticipate that our formalism can be applied effectively to a wide range of astrophysical and cosmological problems. It is noteworthy that nonstandard Lagrangians introduced in this manuscript as well as by other authors are simply generating functions of different equations of motions. One naturally asks if they have any obvious meaning, and therefore it seems that they are more interesting for mathematicians than for physicists. In fact, this is an open problem which is still in its infancy and much work is required. Obtaining most dynamical equations through generating Lagrangian functions have been shown by mathematicians, who have also demonstrated that there is an infinite number of such functions. In this work, we have just found one of them out of an infinite set.

It would be interesting to consider in a future work more generalized forms of NSL, to discuss their implications in different approaches in cosmology and astrophysics, and to compare them with new approaches [57-61]. Finally, it was observed that nonstandard Lagrangians result in the violation of the weak equivalence principle. It is notable at the end that in [62], it was argued that nonconservative gravitational field equations result in a cosmological model with a locally varying Einstein's lambda. A number of phenomenological theories do not hold this property; however, they are also motivating [63]. This problem should be addressed carefully as well in the future.

Competing interests
The author declares no competing interests.

Acknowledgements
We would like to thank the anonymous referees for their useful comments, remarks, and suggestions.

Received: 12 March 2013 Accepted: 8 November 2013
Published: 20 November 2013

References

1. Arnold, VI: Mathematical Methods of Classical Mechanics. Springer, New York (1978)
2. Alekseev, AI, Arbuzov, BA: Classical Yang-Mills field theory with nonstandard Lagrangian. Theor. Math. Phys. **59**, 372–378 (1984)
3. Carinena, JG, Ranada, MF, Santander, M: Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability. J. Math. Phys. **46**, 062703–062721 (2005)
4. Chandrasekar, VK, Pandey, SN, Senthilvelan, M, Lakshmanan, M: Simple and unified approach to identify integrable nonlinear oscillators and systems. J. Math. Phys. **47**, 023508–023545 (2006)
5. Chandrasekar, VK, Senthilvelan, M, Lakshmanan, M: On the Lagrangian and Hamiltonian description of the damped linear harmonic oscillator. Phys Rev **E72**, 066203–066211 (2005)
6. Musielak, ZE: Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients. J. Phys. A Math. Theor. **41**, 055205–055222 (2008)
7. Musielak, ZE: General conditions for the existence of non-standard Lagrangians for dissipative dynamical systems. Chaos Solitons Fractals **42**(42), 2645–2652 (2009)
8. Chandrasekar, VK, Senthilvelan, M, Lakshmanan, M: A nonlinear oscillator with unusual dynamical properties. In: Proceedings of the Third National Systems and Dynamics. University of Madras, Chennai (2006)
9. Lukkassem, D: Reiterated homogenization of non-standard Lagrangians, C. R. Acad. Sci. Paris **332**, 999–1004 (2001). Série I
10. El-Nabulsi, RA, Soulati, T, Rezazadeh, H: Non-standard complex Lagrangian dynamics. J. Adv. Res. Dyn. Contin. Syst **5**(1), 50–62 (2013)
11. El-Nabulsi, RA: Modified Proca equation and modified dispersion relation from a power-law Lagrangian functional. Indian J. Phys. Erratum Indian J. Phys **87**(5), 465–470 (2013). doi:10.1007/s12648-013-0357-6
12. El-Nabulsi, RA: Quantum field theory from an exponential action functional. Indian J. Phys. **87**(4), 379–383 (2013)
13. El-Nabulsi, RA: Nonstandard fractional exponential Lagrangians, fractional geodesic equation, complex general relativity, and discrete gravity. Can. J. Phys. **91**, 618–622 (2013)
14. Dimitrijevic, DD, Milosevic, M: About non-standard Lagrangians in cosmology, proceedings of the physics conference TIM - 11. AIP Confer Proc **1472**, 41–46 (2012)
15. El-Nabulsi, RA: Non-standard fractional Lagrangians. Nonlinear Dyn. **74**, 381–394 (2013)
16. El-Nabulsi, RA: Fractional oscillators from non-standard Lagrangians and time-dependent fractional exponent. Comp. Appl. Math (2013). doi:10.1007/s40314-013-0053-3
17. El-Nabulsi, RA: Nonlinear dynamics with non-standard Lagrangians. Qual. Theory Dyn. Syst. **13**, 273–291 (2013)
18. Trencevski, K: Time dependent gravitational potential in the universe and some consequences. Gen. Rel. Grav. **37**, 507–519 (2005)
19. Trencevski, K: Hubble red shift and the anomalous acceleration of Pioneer 10 and 11. Gen. Rel. Grav. **37**, 507–519 (2005)
20. Trencevski, K: Deformation of the planetary orbits caused by the time dependent gravitational potential in the universe. Gen. Rel. Grav. **37**, 507–519 (2005)
21. Williams, PE: New time dependent gravity displays dark matter and dark energy effects. Apeiron **15**, 294–313 (2008)
22. Cheng, DM, Wu, XP, Jiang, DR: Time delay effect on the cosmic background radiation from static gravitational potential of clusters. Astrophys. J. **544**, 1–5 (2000)
23. Seljak, U, Zaldarriaga, M: Direct signature of an evolving gravitational potential from the cosmic microwave background. Phys Rev **D60**, 043504–043508 (1999)
24. Cho, YM: Quantum violation of the equivalence principle in Brans-Dicke theory. Class. Quantum Grav. **14**, 2963–2970 (1997)
25. Cho, YM: Violation of equivalence principle in Brans-Dicke theory. Class. Quant. Grav. **14**, 2963–2970 (1997)
26. Wolf, C: The entropy production in a non-conservative gravitational theory. Astronomische Nachrichten **309**(3), 177–180 (1988)
27. Fuji, Y: Induced violation of weak equivalence principle in the Brans-Dicke theory. Mod. Phys. Lett. **A9**, 3685–3690 (1994)
28. Damour, T: Questioning the equivalence principle. **arXiv**, gr-qc/0109063 (2001)
29. Damour, T: Theoretical aspects of the equivalence principle. **arXiv**, 1202–631 (2012)
30. Halprin, A, Leung, CN, Pantaleone, J: Possible violation of the equivalence principle by neutrinos. Phys Rev **D53**, 5365–5376 (1996)
31. Schlamminger, S, Choi, KY, Wagner, TA, Gundlach, JH, Adelberger, EG: Test of the equivalence principle using a rotating torsion balance. Phys Rev Lett **100**, 041101–041105 (2008)
32. Gerard, JM: The strong equivalence principle from gravitational gauge structure. Class. Quantum Grav. **24**, 1867–1878 (2007)
33. Piazzese, F, Rizzi, G: Violation of the strong equivalence principle due to the self-force in a Kerr spacetime. Gen. Rel. Grav. **23**, 417–429 (1991)
34. Canuto, VM, Goldman, I: Astrophysical consequences of a violation of the strong equivalence principle. Int. J. Theor. Phys. **28**, 1019–1033 (1989)
35. Arzoumanian, Z: Improved bounds on violation of the strong equivalence principle. In: Matthew, B, Nice, DJ, Thorsett, SE (eds.) Radio Pulsars, ASP Conference Proceedings, 302nd edn, p. 69. Astronomical Society of the Pacific, San Francisco (2003)
36. Kaplan, DV, Wise, MB: Couplings of a light dilation and violations of the equivalence principle. J. High Energy Phys. **0008**, 037–049 (2000)
37. Chowdhury, P, Home, D, Majumdar, AS, Mousavi, SV, Mozaffari, MR, Sinha, S: Strong quantum violation of the gravitational weak equivalence principle by a non-Gaussian wave-packet. Class. Quantum Grav. **29**, 025010–025021 (2012)
38. Carroll, SM, Mantry, S, Ramsey-Musolf, MJ, Stubbs, CW: Dark-matter-induced violation of the weak equivalence principle. Phys Rev Lett **103**, 011301–011305 (2009)
39. Damour, T, Donoghue, JF: Phenomenology of the equivalence principle with light scalars. Class. Quantum Grav. **27**, 202001–202009 (2010)
40. Fujii, Y: Induced violation of weak equivalence principle in the Brans-Dicke theory. Mod. Phys. Lett. **A9**, 3685–3690 (1994)
41. Olmo, GZ: Violation of the equivalence principle in modified theories of gravity. Phys Rev Lett **98**, 061101–061105 (2007)
42. Damour, T, Donoghue, JF: Equivalence principle violations and couplings of a light dilaton. Phys Rev **D82**, 084033–8472 (2010)
43. Ander, MM, Aydemir, U, Donoghue, JF: Breaking diffeomorphism invariance and tests for the emergence of gravity. Phys Rev **D80**, 084059–084071 (2010)
44. Jenkins, A: Constraints on emergent gravity. Int. J. Mod. Phys. **D18**, 2249–2255 (2009)
45. Faber, RL: Differential Geometry and Relativity Theory: An Introduction. Marcel Dekker, New York (1983)
46. Doran, R, Lobo, FSN, Crawford, P: Interior of a Schwarzschild black hole revisited. Found. Phys. **38**, 160–187 (2008)
47. Penrose, R: Gravitational collapse: the role of general relativity. Riv. Nuovo Vimento Soc. Ital. Fis. **1**, 252–276 (1969)
48. Narlikar, JV, Padmanabhan, T: The Schwarzschild solution: some conceptual difficulties. Found. Phys. **18**, 659–668 (1988)
49. Dymnikova, I: Spherically symmetric space-time with the regular de Sitter center. Int. J. Mod. Phys. **D12**, 1015–1034 (2003)
50. Chapline, G, Hohlfeld, E, Laughlin, RB, Santiago, DI: Quantum phase transitions and the breakdown of classical general relativity. Int. J. Mod. Phys. **A18**, 3587–3590 (2003)
51. Weinberg, S: General Relativity and Gravitation. Wiley, New York (1972)
52. Singh, KP: On Robertson-Walker universe model with variable cosmological term and gravitational constant in cosmological relativity theory. Turk. J. Phys. **34**, 173–180 (2010)
53. Freese, K, Lewis, M: Cardassian expansion: a model in which the universe is flat, matter dominated, and accelerating. Phys. Lett. **B540**, 1–8 (2002)
54. Perlmutter, S, Aldering, G, Goldhaber, G, Knop, RA, Nugent, P, Castro, PG, Deustua, S, Fabbro, S, Goobar, A, Groom, DE, Hook, IM, Kim, AG, Kim, MY, Lee, JC, Nunes, NJ, Pain, R, Pennypacker, CR, Quimby, R, Lidman, C, Ellis, RS, Irwin, M, McMahon, RG, Ruiz-Lapuente, P, Walton, N, Schaefer, B, Boyle, BJ, Filippenko, AV, Matheson, T, Fruchter, AS, Panagia, N, et al.: (Supernova cosmology project), measurements of omega and lambda from 42 high-redshift supernovae. Astrophys. J **517**, 565–586 (1999)
55. Riess, AG, Filippenko, AV, Challis, P, Clocchiattia, A, Diercks, A, Garnavich, PM, Gilliland, RL, Hogan, CJ, Jha, S, Kirshner, RP, Leibundgut, B, Phillips, MM, Reiss, D, Schmidt, BP, Schommer, RA, Smith, RC, Spyromilio, J, Stubbs, C, Suntzeff, NB: Tonry, J: (High-z supernova team), observational evidence from supernovae for an accelerating universe and a cosmological constant. Astron. J **116**, 1009–1038 (1998)

56. Xu, L: Revisiting Cardassian model and cosmic constraint. Eur. Phys. J. **C72**, 2134–2140 (2010)
57. Jamil, J, Momeni, D, Rashid, MA: Fractional action cosmology with power law weight function. J. conf. Series **354**, 012008 (2012)
58. Pasqua, A, Chattopadhyay, S: On the dynamics of dark energy with higher time derivatives of Hubble parameter in El-Nabulsi fractional action cosmology. Int. J. Theor. Phys (2013). doi:10.1007/s10773-013-1828-4
59. Chattopadhyay, S, Pasqua, A: Reconstruction of modified holographic Ricci dark energy in El-Nabulsi fractional action cosmology. J. Theor. Appl. Phys. **7**, 22–28 (2013)
60. Debnath, U, Jamil, M, Chattopadhyay, S: Fractional action cosmology: emergent, logamediate, intermediate, power law scenarios of the universe and generalized second law of thermodynamics. Int. J. Theor. Phys. **51**, 812–837 (2012)
61. Pasqua, S, Chattopadhyay, S: Theoretical constraints on kinetic k-essence dark energy model in El-Nabulsi fractional action cosmology. Canad. J. Phys. **91**, 844–849 (2013)
62. Lindblom, L, Hiscock, WA: Criticism of some non-conservative gravitational theories. J Geophys Res **A15**, 1827–1830 (1982)
63. Mignani, R, Pessa, E, Resconi, G: Non-conservative gravitational equations. Gen. Rel. Grav. **19**(8), 1049–1073 (1997)

doi:10.1186/2251-7235-7-60
Cite this article as: El-Nabulsi: Some consequences of nonstandard Lagrangians with time-dependent coefficients in general relativity. *Journal of Theoretical and Applied Physics* 2013 **7**:60.