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Exact solutions of Feinberg-Horodecki equation for time-dependent Deng-Fan molecular potential

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Abstract

The exact bound state solutions of the Feinberg-Horodecki equation with the rotating time-dependent Deng-Fan oscillator potential are presented within the framework of the generalized parametric Nikiforov-Uvarov method. It is shown that the solutions can be expressed in terms of Jacobi polynomials or the generalized hypergeometric functions. The energy eigenvalues and the corresponding wave functions are obtained in closed forms.

Keywords: Feinberg-Horodecki equation; Deng-Fan oscillator potential; NU method

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Background

The following space-like counterpart of the Schrödinger equation given by

$$\left[-\frac{\hbar^2}{2mc^2}\frac{d^2}{dt^2} + V(t)\right]\psi_n(t) = cP_n\psi_n(t) \qquad (1)$$

was derived by Horodecki [1] from the relativistic Feinberg equation [2]. Here, t is the space-like parameter x, $V(t)$ denotes the vector potential, m is the mass of a particle, c is the speed of light, and P_n is quantized momentum according to the quantum number $n = 0, 1, 2, \ldots, s$. The bound states of Equation 1 have not been considered yet, as they are difficult to interpret in terms of temporal vibrational motion. However, in the case of anharmonic vector potential, there are no bound states in the dissociation limit and the direction of temporal motion is consistent with the arrow of time (is not of the oscillatory type). In such circumstances, the space-like solutions of Equation 1 can be employed to test their relevance in different areas of science including physics, biology and medicine [3].

Molski constructed the space-like coherent states of a time-dependent Morse oscillator on the basis of the Feinberg-Horodecki (FH) quantal equation for minimizing the time-energy uncertainty relation and showed that the results are useful for interpreting the formation of the

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specific growth patterns during crystallization process and biological growth [3]. Moreover, Molski obtained Feinberg-Horodecki equation to demonstrate a possibility of describing the biological systems in terms of the space-like quantum supersymmetry for anharmonic oscillators [4]. Very recently, Bera and Sil found the exact solutions of the Feinberg-Horodecki equation for the time-dependent Wei-Hua oscillator and Manning-Rosen potentials by the Nikiforov-Uvarov (NU) method [5].

In 1957, Deng and Fan [6,7] proposed a simple potential model for diatomic molecules called the Deng-Fan oscillator potential. The Deng-Fan potential was called a general Morse potential [8,9] whose analytical expressions for energy levels and wave functions have been derived [6-9] and related to the Manning-Rosen potential [10,11] (also called Eckart potential by some authors [12-14]) is anharmonic potential. It has the correct physical boundary conditions at $t = 0$ and ∞. Here, the space-like DF potential takes the form:

$$V(t) = D\left[1 - \frac{b}{e^{\alpha t}-1}\right]^2, \quad b = e^{\alpha t_e}-1, \qquad (2)$$

where the three positive parameters D, t_e and α stand for the dissociation energy, the equilibrium time point and the range of the potential well, respectively. The space-like Deng-Fan potential (Equation 2) is qualitatively similar to the Morse potential but has the correct asymptotic behavior when the inter-nuclear distance approaches zero [6] and used to describe ro-vibrational energy

levels for the diatomic molecules and electromagnetic transitions [15-18].

The motivation of this work is to solve FH equation by the NU method for systems whose potentials have a certain time dependence like time-dependent Deng-Fan potential.

The present work is organized as follows: in the 'Methods' section, the generalized parametric NU method is briefly introduced; in the 'Exact solution of the FH equation for the time-dependent Deng-Fan potential' section, we find the exact solutions of Feinberg-Horodecki equation for the time-dependent Deng-Fan oscillator potential by the parametric generalization of the NU method. Finally, we end with our concluding remarks in the 'Conclusions' section.

Method

This powerful mathematical tool is usually used to solve second-order differential equations. It starts by considering the following differential equation [19]:

$$\frac{d^2\psi_n(s)}{ds^2} + \frac{\tilde{\tau}(s)}{\sigma(s)}\frac{d\psi_n(s)}{ds} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \qquad (3)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials with at most second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of solution. We present a short-cut for the method. Hence, firstly, we write the general form of the Schrödinger-like equation (Equation 3) in a rather more general form as [20-29]

$$\frac{d^2\psi_n(s)}{ds^2} + \frac{(c_1-c_2s)}{s(1-c_3s)}\frac{d\psi_n(s)}{ds} + \frac{(-p_2s^2 + p_1s-p_0)}{s^2(1-c_3s)^2}\psi_n(s) = 0, \qquad (4)$$

with the wave functions

$$\psi_n(s) = \phi(s)y_n(s). \qquad (5)$$

Secondly, we compare (Equation 4) with its counterpart (Equation 3) to obtain the following parameter values,

$$\tilde{\tau}(s) = c_1-c_2s, \quad \sigma(s) = s(1-c_3s), \quad \tilde{\sigma}(s) = -p_2s^2 + p_1s-p_0. \qquad (6)$$

Now, following the NU method [19], we obtain the following energy equation [20,21]:

$$c_2n-(2n + 1)c_5 + (2n + 1)(\sqrt{c_9} + c_3\sqrt{c_8}) \qquad (7)$$
$$+ n(n-1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0,$$

and the corresponding wave functions

$$\rho(s) = s^{c_{10}}(1-c_3s)^{c_{11}}, \quad \phi(s) = s^{c_{12}}(1-c_3s)^{c_{13}}, \quad c_{12} > 0, \; c_{13} > 0,$$

$$y_n(s) = P_n^{(c_{10},c_{11})}(1-2c_3s), \; c_{10} > -1, \; c_{11} > -1,$$

$$\psi_{n\kappa}(s) = N_{n\kappa}s^{c_{12}}(1-c_3s)^{c_{13}}P_n^{(c_{10},c_{11})}(1-2c_3s), \qquad (8)$$

where $P_n^{(\mu,\nu)}(x), \mu > -1, \nu > -1$, and $x \in [-1, 1]$ are Jacobi polynomials with the following constants:

$$c_4 = \frac{1}{2}(1-c_1), \qquad c_5 = \frac{1}{2}(c_2-2c_3),$$
$$c_6 = c_5^2 + p_2, \qquad c_7 = 2c_4c_5-p_1,$$
$$c_8 = c_4^2 + p_0, \qquad c_9 = c_3(c_7 + c_3c_8) + c_6,$$
$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}-1 > -1, \quad c_{11} = 1-c_1-2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1, c_3 \neq 0,$$
$$c_{12} = c_4 + \sqrt{c_8} > 0, \qquad c_{13} = -c_4 + \frac{1}{c_3}(\sqrt{c_9}-c_5) > 0, c_3 \neq 0, \qquad (9)$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

In the special case where $c_3 = 0$, the wave function (Equation 8) can be rewritten as

$$\lim_{c_3 \to 0} P_n^{(c_{10},c_{11})}(1-2c_3s) = L_n^{c_{10}}(c_{11}s),$$
$$\lim_{c_3 \to 0}(1-c_3s)^{c_{13}} = e^{c_{13}s}, \psi(s) = Ns^{c_{12}}e^{c_{13}s}L_n^{c_{10}}(c_{11}s), \qquad (10)$$

where $L_n^{\alpha}(x)$ are the Laguerre polynomials.

Exact solution of the FH equation for the time-dependent Deng-Fan potential

In this section, we present the exact solutions of FH equation for the time-dependent DF oscillator potential. In the substitution of Equation 2 into Equation 1, one obtains

$$\frac{d^2\psi_n(t)}{dt^2} + \left[\xi_n-d\left(1-\frac{b}{e^{at}-1}\right)^2\right]\psi_n(t) = 0, \qquad (11)$$

where

$$\xi_n = \frac{2mc^3}{\hbar^2}P_n \text{ and } d = \frac{2mc^2}{\hbar^2}D \qquad (12)$$

Now, making an appropriate change of variables $s = e^{-at}$, where $s \in (0, 1)$, we obtain

$$\frac{d^2\psi_n(s)}{ds^2} + \frac{1-s}{s(1-s)}\frac{d\psi_n(s)}{ds}$$
$$+ \frac{1}{\alpha^2s^2(1-s)^2}\left[(\xi_n-d)(1-s)^2 + 2dbs(1-s)-db^2s^2\right]\psi_n(s) = 0, \qquad (13)$$

satisfying the asymptotic behaviors where $\psi_n(s = 0) = 0$ and $\psi_n(s = 1) = 0$. Further, comparing Equation 13 with

Equation 4, we can easily obtain the coefficients c_i ($i = 1$, 2, 3) and analytical expressions p_j ($j = 0, 1, 2$) as follows:

$$c_1 = 1, \qquad p_0 = \frac{(d-\xi_n)}{\alpha^2},$$

$$c_2 = 1, \qquad p_1 = \frac{2d(b+1)-2\xi_n}{\alpha^2},$$

$$c_3 = 1, \qquad p_2 = \frac{d(b+1)^2-\xi_n}{\alpha^2}.$$

$$(14)$$

The values of the coefficients c_i ($i = 4, 5, ..., 13$) are also found from Equation 15 as below

$$c_4 = 0, \qquad\qquad c_5 = -\frac{1}{2},$$

$$c_6 = \frac{1}{4} + \frac{d(b+1)^2-\xi_n}{\alpha^2}, \qquad c_7 = \frac{2\xi_n-2d(b+1)}{\alpha^2},$$

$$c_8 = \frac{d-\xi_n}{\alpha^2}, \qquad\qquad c_9 = \frac{1}{4} + \frac{db^2}{\alpha^2},$$

$$c_{10} = 2\sqrt{d-\xi_n}, d \geq \xi_n, \qquad c_{11} = \sqrt{\frac{4db^2}{\alpha^2} + 1},$$

$$c_{12} = \sqrt{\frac{d-\xi_n}{\alpha^2}}, d \geq \xi_n, \qquad c_{13} = \frac{1}{2}\left(1 + \sqrt{\frac{4db^2}{\alpha^2} + 1}\right),$$

$$(15)$$

Using the energy equation (Equation 7), we can obtain the quantized momentum eigenvalues equation, in closed form, as

$$\left(n + \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4db^2}{\alpha^2}} + \sqrt{\frac{d-\xi_n}{\alpha^2}}\right)^2 = \frac{d(b+1)^2-\xi_n}{\alpha^2}.$$

$$(16)$$

After substituting Equation 12 into above energy equation and making some simplifications, we obtain the quantized momentum as

$$P_n = \frac{D}{c} - \frac{\hbar^2}{2mc^3}\left[\frac{(b^2+2b)d}{\left(2n+1+\sqrt{1+\frac{8mc^2}{\alpha^2\hbar^2}Db^2}\right)\alpha^2}\right.$$
$$\left. - \frac{1}{4}\left(2n+1+\sqrt{1+\frac{8mc^2}{\alpha^2\hbar^2}Db^2}\right)^2\right].$$

$$(17)$$

Let us find the corresponding wave functions. In reference to Equation 15 and expressions in Equation 8, we find the following useful functions

$$\rho(s) = s^{2\sqrt{(d-\xi_n)/\alpha^2}}(1-s)^{\sqrt{4db^2/\alpha^2+1}},$$

$$\phi(s) = s^{\sqrt{(d-\xi_n)/\alpha^2}}(1-s)^{\frac{1}{2}\left(1+\sqrt{4db^2/\alpha^2+1}\right)},$$

$$y_n(s) \sim P_n^{\left(2\sqrt{(d-\xi_n)/\alpha^2}, \sqrt{4db^2/\alpha^2+1}\right)}(1-2s).$$

$$(18)$$

Thus, using the relation (Equation 5), we can finally obtain the wave functions with the aid of Equation 8 as

$$\psi_n(s) = N_n s^{\sqrt{(d-\xi_n)/\alpha^2}}(1-s)^{\frac{1}{2}\left(1+\sqrt{4db^2/\alpha^2+1}\right)}$$
$$P_n^{\left(2\sqrt{(d-\xi_n)/\alpha^2}, \sqrt{4db^2/\alpha^2+1}\right)}(1-2s),$$

$$(19)$$

or equivalently

$$\psi_n(t) = N_n e^{-\alpha\sqrt{2mc^2(D-cP_n)/\alpha^2\hbar^2}\,t}(1-e^{-\alpha t})^{\frac{1}{2}\left(1+\sqrt{8mc^2b^2D/\alpha^2\hbar^2+1}\right)}$$
$$\times P_n^{\left(2\sqrt{2mc^2(D-cP_n)/\alpha^2\hbar^2}, \sqrt{8mc^2b^2D/\alpha^2\hbar^2+1}\right)}(1-2e^{-\alpha t}),$$

$$(20)$$

where N_n is the normalization constant. Obviously, the above wave function is finite at both $t = 0$ and $t \to \infty$.

Conclusions

In the framework of the generalized parametric NU method, we have presented a simple way to obtain the exact solutions of the FH equation for the time-dependent Deng-Fan oscillator potential. It is worthy mentioning that the parametric NU method is an elegant and powerful technique to provide closed forms for the energy eigenvalues and the corresponding wave functions. The results may find applications in some branches of physics and also biophysics [3,4].

Authors' contributions
MH, SMI, and MA carried out all the analyses, designed the study, and drafted the manuscript together. All authors read and approved the final manuscript.

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