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A new form of Grad-Shafranov equation for a tokomak with an elongated cross section

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Abstract

Different forms of the well-known Grad-Shafranov (G-S) equations that define the equilibrium behavior of tokomak plasma have been already obtained. None of these equations contain explicitly the triangularity δ and the elongation ratio k_s. The aim of this work is to obtain a new form of G-S equation which includes both the triangularity and the elongation ratio. For verifying the correctness of the obtained equation, the triangularity is set to 0, which leads to a G-S equation for circular cross section. Also, the magnetic field and current density obtained from this new G-S equation is reduced to the quantities already derived for circular cross section.

Keywords: Elongation, Triangularity, G-S equation, Magnetic field, Current density

PACS: 52.25-Xz, 52.55-s, 28.52-s

Introduction

Tokomaks are axisymmetric devices with toroidal and equilibrium (also called poloidal) fields. There are four basic magnet systems in the tokomak: (1) the toroidal field coils, which produce the large toroidal field; (2) the ohmic transformer, which induces the toroidal plasma current required for equilibrium and ohmic heating; (3) the vertical field system, which is required for toroidal force balance; and (4) the shaping coils, which produce a noncircular cross section to improve the magnetohydrodynamic (MHD) stability limits and alleviate plasma-wall impurity problems.

In this device, the plasma current itself also produces the required poloidal field for equilibrium. Toroidal field is used besides the ohmic heating of plasma for suppressing MHD instabilities and will be strong enough to confine the hot plasma.

This toroidal field is produced by external electric currents flowing in coils wound around the tours. Superimposed on the toroidal field is a much weaker poloidal field generated by an electric current flowing in the plasma around the tours, so the plasma forms the secondary circuit of a transformer. Today, most tokomaks have an elongated cross section. The reason for using an elongated cross section comes out essentially from problems that are related to MHD instabilities like kink and ballooning instabilities [1-4]. In this paper, the main advantage of using an elongated D-shaped cross section is discussed.

In the section 'Derivation of G-S equation in the new approach', G-S equation including the effect of elongation ratio and triangularity is derived, and in the section 'Reduction of new G-S equation to the simple case of circular cross section', the reduction of the obtained equations to the case of circular cross section is shown as a justification. Finally, in the 'Conclusions' section, a brief conclusion will be given. A schematic view of a tokomak is shown in Figure 1.

Derivation of G-S equation in the new approach

First of all, using relations between geometrical parameters on an elongated tokomak, the partial differential equation between these parameters is obtained. Using these relations, the new G-S equation, magnetic field, and the current density are obtained. The basis of the paper relies on the local and cylindrical coordinates as follows [5,6]:

$$R = R_0 + rcos(\theta + \delta sin\theta) \tag{1}$$

and

$$Z = rk_s sin\theta \tag{2}$$

where R_0 and r are respectively the major and minor radii, and θ is the angle between r and the direction of the major radius.

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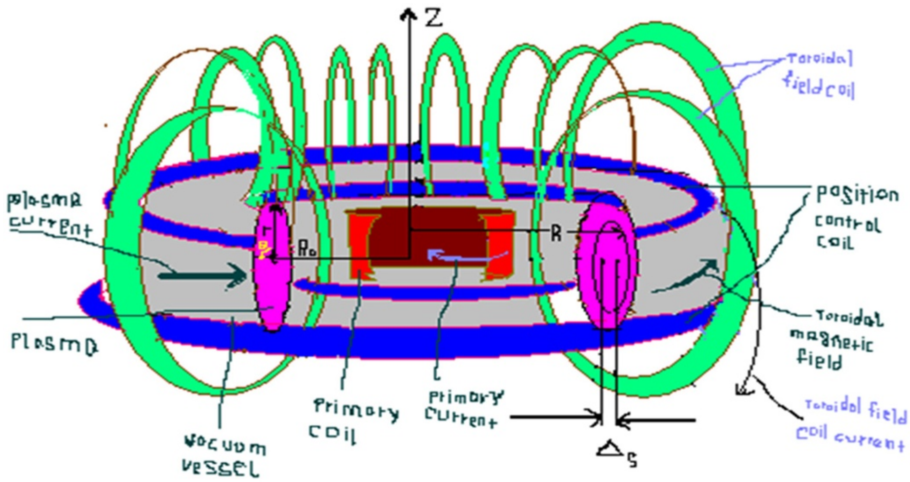

Figure 1 A schematic view of a tokomak.

Substituting $\theta = sin^{-1}\left(\frac{z}{rk_s}\right)$ from Equation 2 in Equation 1, one can get

$$R = R_0 + r\ \ cos\left(sin^{-1}\left(\frac{z}{rk_s}\right) + \frac{\delta z}{rk_s}\right) \tag{3}$$

Differentiating both sides of Equation 3, we have

$$\begin{aligned} dR = {} & cos\left(sin^{-1}\left(\frac{z}{rk_s}\right) + \frac{\delta z}{rk_s}\right) \\ & - r\left[\frac{d}{dr}\left(sin^{-1}\left(\frac{z}{rk_s}\right) - \frac{\delta z}{r^2 k_s}\right]sin\left(sin^{-1}\left(\frac{z}{rk_s}\right) + \frac{\delta z}{rk_s}\right)dr \end{aligned} \tag{4}$$

We use a new variable $t_r = sin^{-1}\left(\frac{z}{rk_s}\right)$, for which we will have

$$t'_r = \frac{d}{dr}\left(sin^{-1}\left(\frac{z}{rk_s}\right)\right) = -\frac{sin\theta}{r\ \ cos\theta} \tag{5}$$

Substituting Equations 5 and 2 in Equation 4, we get

$$\begin{aligned} \frac{\partial r}{\partial R} & = \frac{1}{cos(\theta + \delta sin\theta) + [tan\theta + \delta sin\theta] sin(\theta + \delta sin\theta)} \\ & = M_1(\theta) \end{aligned} \tag{6}$$

Equation 3 can be rearranged in the form of

$$F(r, z) = cos\left(sin^{-1}\left(\frac{z}{rk_s}\right) + \frac{\delta z}{rk_s}\right) - \frac{R - R_0}{r} = 0 \tag{7}$$

If we derivate Equation 7 with respect to r and use Equations 1, 2, 5, and 6, we find

$$F'_r = \frac{1}{rM_1(\theta)} \tag{8}$$

Similarly, by taking the derivative of Equation 7 with respect to Z, the following result will be achieved:

$$F'_z = -sin\left(sin^{-1}\left(\frac{Z}{rk_s}\right) + \frac{\delta\, Z}{rk_s}\right)\left[\frac{d}{dz}\left(sin^{-1}\left(\frac{Z}{rk_s}\right) + \frac{\delta}{rk_s}\right]\right. \tag{9}$$

Again, using $t_z = sin^{-1}\left(\frac{z}{rk_s}\right)$ and $t'_z = \frac{d}{dz}\left(sin^{-1}\left(\frac{Z}{rk_s}\right)\right)$ we arrived at:

$$\begin{aligned} F'_z & = -\frac{M_2(\theta)}{rk_s} \ \ where\ M_2(\theta) \\ & = \left(sin(\theta + \delta sin\theta)\left[\frac{1}{cos\theta} + \delta\right]\right) \end{aligned} \tag{10}$$

From Equations 8 and 10, one can get

$$\frac{\partial r}{\partial z} = \frac{M_1(\theta) M_2(\theta)}{k_s} \tag{11}$$

The following relation has been used in the past step: $\frac{dy}{dx} = -\frac{F'_x}{F'_y}$. For obtaining partial differentiations of θ with respect to R and Z, both sides of Equation 1 should be divided by Equation 2 so that we find

$$\frac{cos(\theta + sin\theta)}{sin\theta} = k_s\left(\frac{R - R_0}{Z}\right) \tag{12}$$

Differentiation of Equation 12 with respect to R results in

$$\frac{\partial\theta}{\partial R}=\frac{N_1(\theta)}{r}, where\ N_1(\theta) = -\frac{sin\theta}{\left(cos(\delta sin\theta)+\frac{\delta}{2}sin2\theta sin(\theta+\delta sin\theta)\right)} \tag{13}$$

In the same manner, differentiation of Equation 12 with respect to Z will give

$$\frac{\partial\theta}{\partial Z}=\frac{N_2(\theta)}{rk_s}, where\ N_2(\theta) = \frac{cos(\theta+\delta sin\theta)}{\left(cos(\delta sin\theta)+\frac{\delta}{2}sin2\theta sin(\theta+\delta sin\theta)\right)} \tag{14}$$

Using Equations 6, 11, 13, and 14, the partial differential equations for an elongated cross section can be written as follows:

$$\frac{\partial}{\partial R}=M_1(\theta)\frac{\partial}{\partial r}+\frac{N_1(\theta)}{r}\frac{\partial}{\partial\theta}=K_1(r,\theta);\frac{\partial}{\partial z} = \frac{M_1(\theta)M_2(\theta)}{k_s}\frac{\partial}{\partial r}+\frac{N_2(\theta)}{rk_s}\frac{\partial}{\partial\theta}=K_2(r,\theta) \tag{15}$$

where $K_1(r,\theta)$ and $K_2(r,\theta)$ are operators in terms of r and θ.

In the cylindrical coordinates, G-S equation, magnetic fields, and current densities are given (1,2) as

$$R\frac{\partial}{\partial r}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2}=-\mu_0R^2\frac{dP}{d\psi}-F\frac{dF}{d\psi} \tag{16}$$

$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial Z};B_z=\frac{1}{R}\frac{\partial\psi}{\partial R};B\phi=\left(\frac{\partial B_R}{\partial Z}-\frac{\partial B_Z}{\partial_R}\right) \tag{17}$$

$$J_R=-\frac{1}{\mu_0 R}\frac{\partial F}{\partial Z};J_Z=\frac{1}{\mu_0 R}\frac{\partial F}{\partial R};J\phi = \frac{1}{\mu_0}\left(\frac{\partial B_R}{\partial Z}-\frac{\partial B_Z}{\partial R}\right) \tag{18}$$

Substituting Equation 15 in Equation 16, the G-S equation for an elongated cross section is obtained where $L(r,\theta) = R = R_0 + r\ cos(\theta + \delta\ sin\ \theta)$:

$$L(r,\theta)K_1(r,\theta)\left(\frac{K_1(r,\theta)}{L(r,\theta)}\right)(\psi)+(K_2(r,\theta))^2(\psi) = -\mu_0L^2(r,\theta)\frac{dP}{d\psi}-\frac{1}{2}\frac{dF^2}{d\psi} \tag{19}$$

The relations between local and cylindrical coordinates for magnetic fields are given by

$$B_r=B_Rcos(\theta+\delta sin\theta)+B_Zsin(\theta+\delta sin\theta) \tag{20}$$

$$B_\theta=-B_Rsin(\theta+\delta sin\theta)+B_Zcos(\theta+\delta sin\theta) \tag{21}$$

Also in the cylindrical coordinates, the magnetic field components are defined as [5,6]

$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial Z};B_Z=\frac{1}{R}\frac{\partial\psi}{\partial R} \tag{22}$$

Using Equations 15, 17, 20, 21, and 22, it will be easy to find magnetic fields for an elongated cross section:

$$B_r=-\frac{cos(\theta+\delta sin\theta)}{L(r,\theta)}K_2(r,\theta)(\psi)+\frac{sin(\theta+\delta sin\theta)}{L(r,\theta)}K_1(r,\theta)(\psi) \tag{23}$$

$$B_\theta=\frac{sin(\theta+\delta sin\theta)}{L(r,\theta)}K_2(r,\theta)(\psi)+\frac{cos(\theta+\delta sin\theta)}{L(r,\theta)}K_1(r,\theta)(\psi) \tag{24}$$

The relations between local and cylindrical coordinates for current densities are

$$J_r=J_Rcos(\theta+\delta sin\theta)+J_Zsin(\theta+\delta sin\theta) \tag{25}$$

$$J_\theta=-J_Rsin(\theta+\delta sin\theta)+J_Zcos(\theta+\delta sin\theta) \tag{26}$$

Also in the cylindrical coordinates, the current density components are defined as [5,6]:

$$J_R=-\frac{1}{\mu_0}\frac{\partial B_\phi}{\partial Z};J_Z=\frac{1}{\mu_0 R}\frac{\partial RB_\phi}{\partial R} \tag{27}$$

Using Equations 15, 25, 26, and 27, one can find the current density components for an elongated cross section as follows:

$$J_r=\frac{1}{\mu_0L(r,\theta)}(sin\theta K_1(r,\theta)-cos\theta K_2(r,\theta))\left(L(r,\theta)B_\phi\right) \tag{28}$$

$$J_\theta=\frac{1}{\mu_0L(r,\theta)}(cos\theta K_1(r,\theta)+sin\theta K_2(r,\theta))\left(L(r,\theta)B_\phi\right) \tag{29}$$

The following step is taken to obtain the toroidal component of the current density:

$$B_R=B_rcos(\theta+\delta sin\theta)-B_\theta sin(\theta+\delta sin\theta) \tag{30}$$

$$B_Z=B_rsin(\theta+\delta sin\theta)+B_\theta cos(\theta+\delta sin\theta) \tag{31}$$

$$J_\phi=\frac{1}{\mu_0}\left(\frac{\partial B_R}{\partial Z}-\frac{\partial B_Z}{\partial R}\right) \tag{32}$$

Thus, using Equations 15, 30, 31, and 32 leads to the toroidal component of the current density for an elongated cross section:

$$J_\phi = \frac{1}{\mu_0}[K_1(r,\theta)(B_r sin(\theta+\delta sin\theta)+B_\theta cos(\theta+\delta sin\theta)) - K_2(r,\theta)(B_r cos(\theta+\delta sin\theta)-B_\theta sin(\theta+\delta sin\theta))] \quad (33)$$

Here, we summarize all equations governing tokomaks with an elongated cross section:

$$L(r,\theta)K_1(r,\theta)\left(\frac{K_1(r,\theta)}{L(r,\theta)}\right)(\psi)+(K_2(r,\theta))^2(\psi) = -\mu_0 L^2(r,\theta)\frac{dP}{d\psi}-\frac{1}{2}\frac{dF^2}{d\psi} \quad (34)$$

$$B_r = -\frac{cos(\theta+\delta sin\theta)}{L(r,\theta)}K_2(r,\theta)(\psi) + \frac{sin(\theta+\delta sin\theta)}{L(r,\theta)}K_1(r,\theta)(\psi) \quad (35)$$

$$B_\theta = \frac{sin(\theta+\delta sin\theta)}{L(r,\theta)}K_2(r,\theta)(\psi) + \frac{cos(\theta+\delta sin\theta)}{L(r,\theta)}K_1(r,\theta)(\psi) \quad (36)$$

$$J_r = \frac{1}{\mu_0 L(r,\theta)}(sin\theta K_1(r,\theta)-cos\theta K_2(r,\theta))\big(L(r,\theta)B_\phi\big) \quad (37)$$

$$J_\theta = \frac{1}{\mu_0 L(r,\theta)}(cos\theta K_1(r,\theta)+sin\theta K_2(r,\theta))\big(L(r,\theta)B_\phi\big) \quad (38)$$

$$J_\phi = \frac{1}{\mu_0}\Big[K_1(r,\theta)(B_r sin(\theta+\delta sin\theta)+B_\theta cos(\theta+\delta sin\theta)) - K_2(r,\theta)(B_r cos(\theta+\delta sin\theta)-B_\theta sin(\theta+\delta sin\theta))\Big] \quad (39)$$

$$\frac{\partial}{\partial R} = M_1(\theta)\frac{\partial}{\partial r}+\frac{N_1(\theta)}{r}\frac{\partial}{\partial\theta} = K_1(r,\theta) \quad (40)$$

$$\frac{\partial}{\partial Z} = \frac{M_1(\theta)M_2(\theta)}{k_s}\frac{\partial}{\partial r}+\frac{N_2(\theta)}{rk_s}\frac{\partial}{\partial\theta} = K_2(r,\theta) \quad (41)$$

$$M1(\theta) = \frac{1}{cos(\theta+\delta sin\theta)+[tan\theta+\delta sin\theta]sin(\theta+\delta sin\theta)} \quad (42)$$

$$M_2(\theta) = \left(sin(\theta+\delta sin\theta)\left[\frac{1}{cos}+\delta\right]\right) \quad (43)$$

$$N_1(\theta) = -\frac{sin\theta}{\left(cos(\delta sin\theta)+\frac{\delta}{2}sin2\theta sin(\theta+\delta sin\theta)\right)} \quad (44)$$

$$N_2(\theta) = \frac{cos(\theta+\delta sin\theta)}{\left(cos(\delta sin\theta)+\frac{\delta}{2}sin2\theta sin(\theta+\delta sin\theta)\right)} \quad (45)$$

Reduction of new G-S equation to the simple case of circular cross section

Now, we are in a stage where we can prove the correctness of Equations 34 to 39. For this, the triangularity is set to 0 (knowing that in this case the elongation ratio is unity) in Equations 34 to 39, which reduces the results to equations for circular cross section:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\psi) - \frac{1}{(R_0+r\ cos\theta)}\left(cos\theta\frac{\partial}{\partial r}-\frac{sin\theta}{r}\frac{\partial}{\partial\theta}\right)(\psi) = -\mu_0(R_0+r\ cos\theta)^2\frac{dP}{d\psi}-F\frac{dF}{d\psi} \quad (46)$$

$$B_r = -\frac{1}{Rr}\frac{\partial\psi}{\partial\theta}; B_\theta = \frac{1}{R}\frac{\partial\psi}{\partial r} \quad (47)$$

$$J_r = -\frac{1}{\mu_0 Rr}\frac{\partial RB_\phi}{\partial\theta}; J_\theta = \frac{1}{\mu_0 R}\frac{\partial RB_\phi}{\partial r}; \mu_0 J_\phi = \frac{1}{r}\frac{\partial(rB_\theta)}{\partial r}-\frac{1}{r}\frac{\partial B_r}{\partial\theta} \quad (48)$$

$$\frac{\partial}{\partial R} = cos\theta\frac{\partial}{\partial r}-\frac{sin\theta}{r}\frac{\partial}{\partial\theta} \quad (49)$$

$$\frac{\partial}{\partial Z} = sin\theta\frac{\partial}{\partial r}+\frac{cos\theta}{r}\frac{\partial}{\partial\theta} \quad (50)$$

Setting $\delta = 0$ and $k_S = 1$ in Equations 42 to 45 leads to the following equations:

$$M_1(\theta) = cos\theta$$
$$M_2(\theta) = tan\theta$$
$$N_1(\theta) = -sin\theta$$
$$N_2(\theta) = cos\theta$$

Substituting the obtained results in the partial differentiations of Equations 40 and 41 leads to

$$\frac{\partial}{\partial R} = cos\theta \frac{\partial}{\partial r} - \frac{sin\theta}{r}\frac{\partial}{\partial \theta} = K_1(r,\theta) \tag{51}$$

$$\frac{\partial}{\partial Z} = sin\theta \frac{\partial}{\partial r} + \frac{cos\theta}{r}\frac{\partial}{\partial \theta} = K_2(r,\theta) \tag{52}$$

Also, substituting Equations 51 and 52 in Equations 34 to 39 leads to

$$\begin{aligned}&\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\psi)\\ &\quad - \frac{1}{(R_0 + r\ \ cos\theta)}\left(cos\theta\frac{\partial}{\partial r} - \frac{sin\theta}{r}\frac{\partial}{\partial \theta}\right)(\psi)\\ &\qquad = -\mu_0(R_0 + r\ \ cos\theta)^2\frac{dP}{d\psi} - F\frac{dF}{d\psi}\end{aligned} \tag{53}$$

$$B_r = -\frac{1}{Rr}\frac{\partial \psi}{\partial \theta}; B_\theta = \frac{1}{R}\frac{\partial \psi}{\partial r} \tag{54}$$

$$\begin{aligned}J_r &= -\frac{1}{\mu_0 Rr}\frac{\partial RB_\phi}{\partial \theta}; J_\theta = \frac{1}{\mu_0 R}\frac{\partial RB_\phi}{\partial r}; \mu_0 J_\phi\\ &= \frac{1}{r}\frac{\partial (rB_\theta)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \theta}\end{aligned} \tag{55}$$

The comparison of Equations 46 to 50 with Equations 51 to 55 respectively shows the reduction of partial differential and G-S equations, magnetic fields, and current densities from an elongated cross section to a circular one.

Conclusions

In this paper, new formulae are obtained for partial differential and G-S equations, magnetic fields, and current densities that are special on elongated cross section. Thus, with these new formulations, one can easily understand the effect of the elongation ratio on tokomak parameters like magnetic fields and current densities. Furthermore, solving the G-S equation with this new formulation, the exact solution for the magnetic surfaces of the tokomak plasma with elongated cross section is obtained. Obviously, with this information one can easily plot the real magnetic surfaces of the elongated tokomak.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

ASKM contributed significantly to this study, and SS edited the manuscript and its correctness. Both authors read and approved the final manuscript.

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