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Torsion of space-time in $f(R)$ gravity

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Abstract

In this paper, we first review some aspects of the $f(R)$ gravity, and then the concept of the torsion of space-time due to metric-affine formalism in $f(R)$ gravity is studied. Within this formalism, in which the matter action is supposed to be dependent on the connection, we achieve interesting cases including nonzero torsion tensor. Then with the physical interpretation of the torsion of space-time in high energy limit, the modified expression of Mach's principle in a very strong gravitational region is obtained.

Keywords: $f(R)$ gravity; Metric-affine formalism; Torsion tensor

Background

General relativity (GR) is widely accepted as a fundamental theory to describe the geometric properties of space-time. In a homogeneous and isotropic space-time, the Einstein field equations give rise to the Friedmann equations that describe the evolution of the universe. In fact, the standard big-bang cosmology based on radiation and matter dominated epochs can be well described within the framework of GR.

However, the rapid development of observational cosmology which started from 1990s shows that the universe has undergone two phases of cosmic acceleration. The first one is called inflation, which is believed to have occurred prior to the radiation domination. This phase is required not only to solve the flatness and horizon problems plagued in the big-bang cosmology but also to explain a nearly flat spectrum of temperature anisotropies observed in cosmic microwave background (CMB). The second accelerating phase has started after the matter domination [1]. The unknown component giving rise to this late-time cosmic acceleration is called the dark energy [2]. The existence of the dark energy has been confirmed by a number of observations such as supernovae Ia, large scale structure, baryon acoustic oscillations, and CMB.

These two phases of cosmic acceleration cannot be explained by the presence of standard matter whose equation of the state $\omega = \frac{P}{\rho}$ satisfies the condition $\omega \geq 0$ (here P and ρ are the pressure and the energy density of matter, respectively). In fact, we further require some component of negative pressure, with $\omega < -\frac{1}{3}$, to realize the acceleration of the universe. The cosmological constant Λ is the simplest candidate of dark energy, which corresponds to $\omega = -1$. However, if the cosmological constant originates from a vacuum energy of particle physics, its energy scale is too large to be compatible with the dark energy density. Hence, we need to find some mechanism to obtain a small value of Λ consistent with observations. Since the accelerated expansion in the very early universe needs to end to connect to the radiation-dominated universe, the pure cosmological constant is not responsible for inflation. A scalar field ϕ with a slowly varying potential can be a candidate for inflation as well as for the dark energy.

Although many scalar-field potentials for inflation have been constructed in the framework of string theory and supergravity, the CMB observations still do not show particular evidence to favor one of such models. This situation is also similar in the context of dark energy; there is a degeneracy as for the potential of the scalar field (quintessence) due to the observational degeneracy to the dark energy equation of the state around $\omega = -1$. Moreover, it is generally difficult to construct viable quintessence potentials motivated from particle physics because the field mass responsible for cosmic acceleration today is very small ($m_\phi \simeq 10^{-33} eV$).

While scalar-field models of inflation and dark energy correspond to a modification of the energy-momentum tensor in Einstein equations, there is another approach to explain the acceleration of the universe. This corresponds to the modified gravity in which the gravitational theory

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is modified compared to GR. The Lagrangian density for GR is given by $f(R) = R - 2\Lambda$, where R is the Ricci scalar and Λ is the cosmological constant (corresponding to the equation of state $\omega = -1$). The presence of Λ gives rise to an exponential expansion of the universe, but we cannot use it for inflation because the inflationary period needs to connect to the radiation era. It is possible to use the cosmological constant for dark energy since the acceleration today does not need to end. However, if the cosmological constant originates from a vacuum energy of particle physics, its energy density would be enormously larger than today's dark energy density. While the Λ-cold dark matter (ΛCDM) model ($f(R) = R - 2\Lambda$) fits a number of observational data well, there is also a possibility for the time-varying equation of the state of dark energy.

One of the simplest modifications to GR is the $f(R)$ theories of gravity in which the Lagrangian density is supposed to be an arbitrary function of R [3,4]. The $f(R)$ theories of gravity come about by a straightforward generalization of the Lagrangian in the Einstein-Hilbert action,

$$S_{EH} = \frac{1}{2k}\int d^4x\sqrt{-g}R, \tag{1}$$

where $k \equiv 8\pi G$, G is the gravitational constant, g is the determinant of the metric $g_{\mu\nu}$, and R is the Ricci scalar ($c = \hbar = 1$) to become a general function of R, i.e.,

$$S = \frac{1}{2k}\int d^4x\sqrt{-g}f(R). \tag{2}$$

As can be found in many textbooks, see for example [5,6], there are actually two variational principles that one can apply to the Einstein-Hilbert action in order to derive Einstein's equations: the standard metric variation and a less standard variation dubbed Palatini variation [even though it was Einstein and not Palatini who introduced it [7]]. In the former, the metric is assumed to be independent variable but in the latter the metric and the connection are assumed to be independent variables, and one varies the action with respect to both of them, under the important assumption that the matter action does not depend on the connection. The choice of the variational principle is usually referred to as a formalism, so one can use the terms metric (or second order) formalism and Palatini (or first order) formalism. Therefore, it is intuitive that there will be two version of $f(R)$ gravity, according to which variational principle or formalism is used. Indeed this is the case: $f(R)$ gravity in the metric formalism is called metric $f(R)$ gravity, and $f(R)$ gravity in the Palatini formalism is called Palatini $f(R)$ gravity [8].

Finally, there is actually even a third version of $f(R)$ gravity: metric-affine $f(R)$ gravity [9,10]. This comes about if one uses the Palatini variation but abandons the assumption that the matter action is independent of the connection. Clearly, metric-affine $f(R)$ gravity is the most general of these theories and reduces to metric or Palatini $f(R)$ gravity if further assumptions are made.

In this paper, we first study the formalism of the modified gravity, i.e., $f(R)$ gravity. Then, we are going to express a modified form of Mach's principle with a closer look at the concepts of curvature and torsion. Therefore, in section "Metric-affine formalism of $f(R)$ gravity," we study the metric-affine formalism of the modified gravity in the detailed review. We take a closer look in the concept of torsion of space-time and correct expression of Mach's principle in section "Modified expression of Mach's principle." We bring a summary of the results in the final section.

Results and discussion

Metric-affine formalism of $f(R)$ gravity

As we pointed out, the Palatini formalism of $f(R)$ gravity relies on the crucial assumption that the matter action does not depend on the independent connection. This assumption relegates this connection to the role of some sort of auxiliary field, and the connection carrying the usual geometrical meaning, parallel transport, and definition of the covariant derivative remains the Levi-Civita connection of the metric [11]. But if we decided to be faithful to the geometrical interpretation of the independent connection $\Gamma^{\lambda}_{\mu\nu}$, then this would imply that we would define the covariant derivatives of the matter fields with this connection and, therefore, we would have

$$S_M = S_M\left(g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}, \psi\right), \tag{3}$$

where ψ collectively denotes the matter fields. The action of this theory, dubbed metric-affine $f(R)$ gravity [10], would then be

$$S_{ma} = \frac{1}{2k}\int d^4x\sqrt{-g}f(\Re) + S_M\left(g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}, \psi\right). \tag{4}$$

where $f(\Re)$ is a general function of $\Re$, and the Ricci scalar $\Re$ is constructed with the independent connection $\Gamma^{\lambda}_{\mu\nu}$.

Before going further and deriving the field equations from this action, certain issues need to be clarified. First, since now the matter action depends on the connection, we should define a quantity representing the variation of S_M with respect to the connection, which mimics the definition of the stress-energy tensor. We call this quantity the hyper-momentum and is defined as [12]

$$\Delta_{\lambda}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta\Gamma^{\lambda}_{\mu\nu}}. \tag{5}$$

Additionally, since the connection is now promoted to the role of a completely independent field, it is interesting to consider not placing any restrictions to it. Therefore, besides dropping the assumption that the connection is related to the metric, we will also drop the assumption that the connection is symmetric. Also, it is useful to define the Cartan torsion tensor

$$S_{\mu\nu}{}^{\lambda} \equiv \Gamma^{\lambda}{}_{[\mu\nu]}, \tag{6}$$

which is the anti-symmetric part of the connection. $[\mu\nu]$ denote anti-symmetrization over the indices μ and ν.

By allowing a non-vanishing Cartan torsion tensor, we are allowing the theory to naturally include the torsion. Even though this brings complications, it has been considered by some to be an advantage for a gravity theory since some matter fields, such as Dirac fields, can be coupled to gravity in a way which might be considered more natural [13]: one might expect that at some intermediate or high energy regime, the spin of particles might interact with the geometry (in the same sense that macroscopic angular momentum interacts with geometry) and torsion can naturally arise. Theories with torsion have a long history, probably starting with the Einstein-Cartan(Sciama-Kibble) theory [14,15]. In this theory, as well as in other theories with an independent connection, some part of the connection is still related to the metric (e.g., the non-metricity is set to zero). In our case, the connection is left completely unconstrained and is to be determined by the field equations. Metric-affine gravity with the linear version of the action (4) was initially proposed in the work of Hehl et al. [12], and the generalization to $f(\Re)$ actions was considered in the work of Sotiriou and Liberati [9,10].

The final form of the field equations is [11]:

$$f'(\Re)\Re_{(\mu\nu)} - \frac{1}{2}f(\Re)g_{\mu\nu} = kT_{\mu\nu}, \tag{7}$$

$$\frac{1}{\sqrt{-g}}\left[-\bar{\nabla}_{\lambda}\left(\sqrt{-g}f'(\Re)g^{\mu\nu}\right)+\bar{\nabla}_{\sigma}\left(\sqrt{-g}f'(\Re)g^{\mu\sigma}\right)\delta^{\nu}{}_{\lambda}\right] + 2f'(\Re)g^{\mu\sigma}S_{\sigma\lambda}{}^{\nu} = k(\Delta_{\lambda}{}^{\mu\nu} - \frac{2}{3}\Delta_{\sigma}{}^{\sigma[\nu}\delta^{\mu]}{}_{\lambda}, \tag{8}$$

$$S_{\mu\sigma}{}^{\sigma} = 0 \tag{9}$$

where $T_{\mu\nu}$ is the stress-energy tensor, prime denote the variation with respect to the metric, $\bar{\nabla}$ denotes the covariant derivative defined with the independent connection $\Gamma^{\lambda}_{\mu\nu}$, and $(\mu\nu)$ denote symmetrization over the indices μ and ν.

Next, we examine the role of $\Delta_{\lambda}{}^{\mu\nu}$. By splitting Equation 8 into a symmetric and an antisymmetric part and performing contractions and manipulations, it can be shown that [10]

$$\Delta_{\lambda}{}^{[\mu\nu]} = 0 \Rightarrow S_{\mu\nu}{}^{\lambda} = 0. \tag{10}$$

This straightforwardly implies two things: a) Any torsion is introduced by matter fields for which $\Delta_{\lambda}{}^{[\mu\nu]}$ is non-vanishing; b) torsion is not propagating, since it is given algebraically in terms of the matter fields through $\Delta_{\lambda}{}^{[\mu\nu]}$. It can, therefore, only be detected in the presence of such matter fields. In the absence of the latter, space-time will have no torsion.

Obviously, there are certain types of matter fields for which $\Delta_{\lambda}{}^{\mu\nu} = 0$. Characteristic example is: a scalar field, since in this case, the covariant derivative can be replaced with a partial derivative. Therefore, the connection does not enter the matter action. On the contrary, particles with spin, such as Dirac fields, generically have a non-vanishing hyper-momentum and will, therefore, introduce torsion. A more complicated case is that of a perfect fluid with vanishing vorticity. If we set torsion aside, or if we consider a fluid describing particles that would initially not introduce any torsion then, as for a usual perfect fluid in GR, the matter action can be written in terms of three scalars: the energy density, the pressure, and the velocity potential [16]. Therefore, such a fluid will lead to a vanishing $\Delta_{\lambda}{}^{\mu\nu}$. However, complications arise when torsion is taken into account, even though it can be argued that the spins of the individual particles composing the fluids will be randomly oriented, and therefore the expectation value for the spin should add up to zero, fluctuations around this value will affect space-time [10]. Of course, such effects will be largely suppressed, especially in situations in which the energy density is small, such as late-time cosmology.

Modified expression of Mach's principle

Curvature in the Einstein general relativity is one of the main concepts to be considered, and all calculations, related to the field equations, return to curvature. Curvature can explain experimental observations such as the motion of the planets in the solar system, time delay, bending of light near the stars, and the convergence of light in lensing effect. Therefore, Mach's principle in general relativity, based on the concept of curvature, expressed as follows: *"Matter tells space-time how to curve."*

On the other hand, in the previous section, it is shown that metric-affine $f(R)$ gravity allows the presence of torsion. Torsion is merely introduced by specific forms of matter; those for which the matter action has a dependence on the connections. Therefore, the form of Mach's principle is corrected as follows: as "matter tells space-time how to curve", *"matter will also tell space-time how to twirl"* [10]. But we also do not accept this new sentence, because we believe that torsion has the more comprehensive concept than the twirl and curvature.

For a more clear issue, we consider how to move a bolt while being wound on a wooden surface. If a bit of pressure put on it just the tip of the bolt rotates (two-dimensional motion) on a wooden surface, and in this case, it just means a twirl. But if we put more pressure on it, along with the rotation of bolt tip, it can also penetrate into the wooden surface (third dimension of the transition can move perpendicular to the surface or not). Obviously, with increasing the pressure on the screw bolt, the bolt goes into the wood; and in this case, we see only a hole on the wooden surface. In this example, the rotary motion to add a transitional move can represent that torsion is more accurate. Note that the concepts of this example can be generalized to the super surface and space-time with the higher dimensions.

According to the above physical interpretation for the torsion, it seems that the high energy area, near the black holes and at the Planck energy limit, torsion of space-time, is more realistic than the twirl and curvature. For example, the intense of gravitational attraction around the black hole horizon, first, to divert the direction of objects and light (bending in the path). Then due to the increased gravitational force for objects closer to the horizon, they are rotating around the center of the black hole. Finally, when objects are passed through the horizon, they fall into the black hole and are swallowed. Consequently, due to the appearance of torsion of space-time in the high energy range, we can also modify the expression of Mach's principle in reference [10] as follows: *"Matter will also tell space-time how to twist".* We also recommend that the metric-affine formalism is more likely to be introduced in the high energy physics regions.

Conclusions

In this paper, we first overview the formalism of $f(R)$ gravity. Then, we take a closer look in the metric-affine formalism and the concept of torsion of space-time. If one accepts the interpretation presented in this paper for torsion of space-time, then as it was discussed, torsion will play a major role in the formation of the geometry of space-time near the black holes and the early universe cosmology. Thus, the metric-affine formalism is more likely to be introduced in these regions; and when considering the lower energy limit, we can make use of other formalisms, i.e., the metric and Palatiny formalisms. It seems that the study of metric-affine formalism and the torsion of space-time in the high energy physics require a lot of research in the future.

Competing interests
The authors declare that they have no competing interests.

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Authors' contributions
MM designed and performed all the steps of proof in this research and also wrote the paper. EY participated in the design of the study. All authors read and approved the final manuscript.

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