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Dynamics of quantum entanglement in three-spin system with cluster interaction

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Abstract
The present paper investigates the spin-1/2 XX model with three-spin interaction (TSI) from a view point of bipartite entanglement. The three-spin system is initially chosen by well-known W entangled state. By analyzing the time-dependency of the concurrence between the nearest and the next-nearest neighbor pair of spins, we show that where the quantum phase transition (QPT) of the infinite-size system might happen and can be observed, even from the short-time dynamics of such a finite spin system.

Keywords
Quantum entanglement, Cluster interaction, Concurrence, Three-spin interaction, Critical point.

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1. Introduction

Over the past few decades, many efforts have been performed to characterize the entanglement in solid state systems [1–6]. Low-dimensional spin systems are suitable candidates to understand the entanglement between particles. Specially, systems consist of two or three spins known as the two-qubit or three-qubit systems has attracted much attention. Two-spin systems are the simplest quantum systems that can be entangled so far [7–12]. Also, two- spin systems are appropriate systems for quantum information processing [2]. The three-spin systems are known as a special and practical category in studying the quantum correlations [13–23]. Specially, it should be noted that the entanglement behavior in three-qubit system is significantly different from two-qubit systems. Moving from two-spin system to the three-spin system ones seemed a must, since it is the simplest system if we are concerned about the concept of multi-partite entanglement, which has great value in quantum information processing. Equivalence classes of multipartite states are investigated and it is found that there are two inequivalent types of genuine tripartite entanglement, represented by the GHZ and W states [13]. The W-state has the maximal entanglement for three-spin systems and defined as

$$|W>=\frac{1}{\sqrt{3}}[|\uparrow\uparrow\downarrow>+|\downarrow\uparrow>+|\downarrow\uparrow\uparrow>] \quad (1)$$

And the GHZ-state is non-biseparable and introduce as

$$|GHZ>=\frac{1}{\sqrt{3}}(|\uparrow\uparrow\uparrow>+|\downarrow\downarrow\downarrow>) \quad (2)$$

In experiment, the observation of three-photon polarization-entangled W state is reported [14]. By the time, it is known that in addition to photons, it is possible to create a three-spin system experimentally and also there are some measure to determine their amount of entanglement [15,24–29]. Creation of maximally entangled three-spin GHZ-state and W-state with a trapped-ion quantum computer is also reported [15]. From recent research, it is known that a variety a novel spin-1/2 Hamiltonians as three-spin interaction (TSI) can be generated in different configurations of an optical lattice [30–32]. Theoretically, the effect of the TSI on the ground state phase diagram of spin-1/2 XX chain model is studied [33]. In following, we briefly summarize the results obtained within the analytical fermionization approach. The Hamiltonian of the spin-1/2 XX model with three spin interaction is defined as

$$H = J\sum_{n=1}^{N}[S_n^x S_{(n+1)}^x + S_n^y S_{(n+1)}^y] - J'\sum_{n=1}^{N}[S_n^x S_{(n+1)}^z S_{(n+2)}^x + S_n^y S_{(n+1)}^z S_{(n+2)}^y] \quad (3)$$

where S_n is the spin-1/2 operator on the n-the site and the model is considered in the thermodynamic limit $N \longrightarrow \infty$. $J > 0$ denotes the antiferromagnetic exchange coupling and $J' > 0$ is the coupling constant of the three-spin interaction. Using the Jordan-Winger transformations

$$S_n^+ = a_n^+ (e^{i\pi \sum_{l<n} a_l^+ a_l})$$
$$S_n^z = a_n^+ a_n - \frac{1}{2} \quad (4)$$

And applying the Fourier transformation

$$a_n = \frac{1}{\sqrt{N}}\sum_k e^{-ikn} a_k,$$

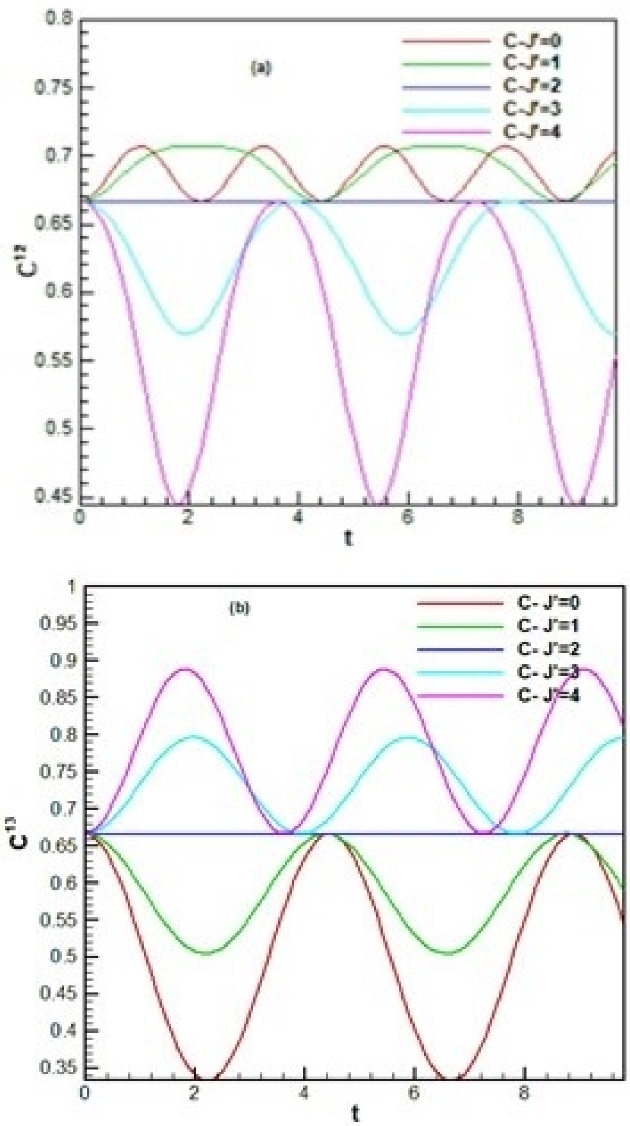

Figure 1. The time behavior of the concurrence between (a) NN and (b) NNN pair spins for different values of TSI interaction.

the Hamiltonian will be diagonalized as

$$H = \sum_k \varepsilon(k) a_k^+ a_k \tag{5}$$

Where the energy spectrum is $\varepsilon(k) = J\cos(k) - 1/2\acute{J}\cos(2k)$. Using the equation $\frac{d\varepsilon(k)}{dk}|_{k_0} = 0$, the energy spectrum maximizes at $k_0 = 0$. It is known that the system is at its critically when $\varepsilon(k=0)$ vanishes. For an infinite model, the quantum phase transition happens at the critical value of the three-spin interaction as $\acute{J}_c = 2J$. Recently, we considered a three-spin system with the same Hamiltonian and with open boundary condition [22]. At zero temperature, by focusing on the entanglement, between each pair of spins, we showed the quantum critical point $\acute{J}_c = 2J$, cac be observed even foe a such finite spin system. Exactly at $\acute{J}_c = 2J$, The ground state of the mentioned three-spin system will become the W-state. By increasing temperature from zero, a shift toward the greater values of $\acute{J}$ is observed because of increasing thermal fluctuations. Motivated by these results, the present paper focuses on the entanglement dynamics of the mentioned three-spin system. We have calculated the time dependent entanglement between different pair spins. Significant differences have been observed in dynamics of entanglement between nearest-neighbor pair spins with respect to the next-to-nearest neighbor pair of spins.

This paper is organized as follows: in the next section, the three-spin system has been introduced in details. In section III the results of entanglement between different pair of spins are presented. Finally, we summarize our results in section IV.

2. The model

Our model is a system consist of three spins interacting with each other via Heisenberg and cluster interactions. The Hamiltonian of the model is written as

$$H = J(S_1^x S_2^x + S_2^x S_3^x + S_1 y S_2^y + S_2^y S_3^y) + \acute{J}(S_1^x S_2^z S_3^x + S_1^y S_2^z S_3^y) \tag{6}$$

The eigenstates and eigenvalues of the Hamiltonian are determined as:

$$E_1 = 0, |\psi_1\rangle = |000\rangle$$

$$E_2 = 0, |\psi_2\rangle = |111\rangle$$

$$E_3 = -\frac{\acute{J}}{4}, |\psi_3\rangle = \frac{1}{\sqrt{2}}(|100\rangle - |001\rangle)$$

$$E_4 = \frac{\acute{J}}{4}, |\psi_4\rangle = \frac{1}{\sqrt{2}}(|011\rangle - |110\rangle)$$

$$E_5 = \frac{1}{8}(-\acute{J} - \sqrt{32J^2 + \acute{J}^2}),$$

$$|\psi_5\rangle = \frac{1}{\sqrt{2+\eta_+^2}}(|110\rangle - \eta_+|101\rangle + |011\rangle)$$

$$E_6 = \frac{1}{8}(\acute{J} - \sqrt{32J^2 + \acute{J}^2}),$$

$$|\psi_6\rangle = \frac{1}{\sqrt{2+\eta_-^2}}(|001\rangle + \eta_-|010\rangle + |100\rangle)$$

$$E_7=\frac{1}{8}(-\acute{J}+\sqrt{32J^2+\acute{J}^2}),$$

$$|\psi_7\rangle=\frac{1}{\sqrt{2+\eta_-^2}}(|110\rangle-\eta_-|101\rangle+|011\rangle)$$

$$E_8=\frac{1}{8}(\acute{J}+\sqrt{32J^2+\acute{J}^2}),$$

$$|\psi_8\rangle=\frac{1}{\sqrt{2+\eta_+^2}}(|001\rangle+\eta_+|010\rangle+|100\rangle) \tag{7}$$

where

$$\eta_+=\frac{2(3J\acute{J}+J\sqrt{32+\acute{J}^2})}{\acute{J}^2+8J^2+\acute{J}\sqrt{32J^2+\acute{J}^2}}$$

$$\eta_-=\frac{2(3J\acute{J}-J\sqrt{32+\acute{J}^2})}{\acute{J}^2+8J^2-\acute{J}\sqrt{32J^2+\acute{J}^2}} \tag{8}$$

The W state is considered as the initial state of the system. In order to study the dynamics of the entanglement, we use the time evolved state, in order to study the dynamics of the entanglement, we used the time evolved state

$$|\psi(t)\rangle=e^{-iHt}|\psi(0)\rangle \tag{9}$$

Where H is the Hamiltonian of the three-spin system. We consider $\hbar=1$. Then, the entangled state of the system at time t is obtained as

$$|\psi(t)\rangle=T_1|001\rangle+T_2|010\rangle+T_1|100\rangle \tag{10}$$

where

$$T_1=\frac{1}{\sqrt{3}}(\frac{(2+\eta_-)e^{-iE_6t}}{\sqrt{3}(2+\eta_-^2)}+\frac{(2+\eta_+)e^{-iE_8t}}{\sqrt{3}(2+\eta_+^2)})$$

$$T_2=\frac{1}{\sqrt{3}}(\frac{\eta_-(2+\eta_-)e^{-iE_6t}}{\sqrt{3}(2+\eta_-^2)}+\frac{\eta_+(2+\eta_+)e^{-iE_8t}}{\sqrt{3}(2+\eta_+^2)}) \tag{11}$$

The Concurrence is a measure for determining the entanglement of a bipartite system. Wootters firstly suggested the concurrence between two spin-1/2 particle as [34]

$$C=max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4,0\} \tag{12}$$

where, λ_i are the square roots of the eigenvalues of $R=\rho\tilde{\rho}$ in descending order, with $\tilde{\rho}=(\sigma^y\bigotimes\sigma^y)\rho^*(\sigma^y\bigotimes\sigma^y)$. ρ^* is the complex conjugate of ρ.The concurrence goes from 0 to 1 (zero for untangled states and 1 for completely entangled states.). The reduced density matrix of two particle system is written as

$$\rho=\begin{pmatrix}\rho_{11}&0&0&0\\0&\rho_{22}&\rho_{23}&0\\0&\rho_{23}^*&\rho_{33}&0\\0&0&0&\rho_{44}\end{pmatrix} \tag{13}$$

The concurrence can be easily obtained by relation

$$C=2max\{|\rho_{23}|-\sqrt{\rho_{11}\rho_{44}},0\} \tag{14}$$

We have calculated the elements of the reduced density matrix for nearest-neighbor pair of spins as

$$\rho_{11}=0 \tag{15}$$

$$\rho_{22}=\rho_{44}=\frac{(2+\eta_-)^2}{3(2+\eta_-^2)^2}+\frac{(2+\eta_+)(2+\eta_-)e^{-iE_6t}e^{iE_8t}}{3(2+\eta_+^2)(2+\eta_-^2)}$$
$$+\frac{(2+\eta_+)^2}{3(2+\eta_+^2)^2}+\frac{(2+\eta_+)(2+\eta_-)e^{iE_6t}e^{-iE_8t}}{3(2+\eta_+^2)(2+\eta_-^2)} \tag{16}$$

$$\rho_{23}=\frac{\eta_-(2+\eta_-)^2}{3(2+\eta_-^2)^2}+\frac{\eta_+(2+\eta_+)(2+\eta_-)e^{iE_6t}e^{-iE_8t}}{3(2+\eta_+^2)(2+\eta_-^2)}$$
$$+\frac{\eta_+(2+\eta_+)^2}{3(2+\eta_+^2)^2}+\frac{\eta_-(2+\eta_+)(2+\eta_-)e^{-iE_6t}e^{iE_8t}}{3(2+\eta_+^2)(2+\eta_-^2)} \tag{17}$$

$$\rho_{33}=\frac{\eta_-^2(2+\eta_-)^2}{3(2+\eta_-^2)^2}+\frac{\eta_-\eta_+(2+\eta_+)(2+\eta_-)e^{-iE_6t}e^{iE_8t}}{3(2+\eta_+^2)(2+\eta_-^2)}$$
$$+\frac{\eta_+^2(2+\eta_+)^2}{3(2+\eta_+^2)^2}+\frac{\eta_-\eta_+(2+\eta_+)(2+\eta_-)e^{iE_6t}e^{-iE_8t}}{3(2+\eta_+^2)(2+\eta_-^2)} \tag{18}$$

And for next-nearest-neighbor pair of spins which in the standard basis $\{|00\rangle,|10\rangle,|01\rangle,|11\rangle\}$ are given by

$$\rho_{22}^{red}=\frac{(2+\eta_-)^2}{3(2+\eta_-^2)^2}+\frac{(2+\eta_+)(2+\eta_-)e^{-iE_6t}e^{iE_8t}}{3(2+\eta_+^2)(2\eta_-^2)}$$
$$+\frac{(2+\eta_+)^2}{3(2+\eta_+^2)^2}+\frac{(2+\eta_+)(2+\eta_-)e^{iE_6t}e^{-iE_8t}}{3(2+\eta_+^2)(2+\eta_-^2)} \tag{19}$$

$$\rho_{44}^{red}=\frac{\eta_-^2(2+\eta_-)^2}{3(2+\eta_-^2)^2}+\frac{\eta_-\eta_+(2+\eta_+)(2+\eta_-)e^{-iE_6t}e^{iE_8t}}{3(2+\eta_+^2)(2+\eta_-^2)}$$
$$+\frac{\eta_+^2(2+\eta_+)^2}{3(2+\eta_+^2)^2}+\frac{\eta_-\eta_+(2+\eta_+)(2+\eta_-)e^{iE_6t}e^{-iE_8t}}{3(2+\eta_+^2)(2+\eta_-^2)} \tag{20}$$

We are interested in estimating the entanglement between the nearest neighbor spins (spins S_1 and S_2) and between the next-to nearest neighbor spins (spins S_1 and S_3) as a function of time. The former is denoted by C^{12} and the latter by C^{13}.

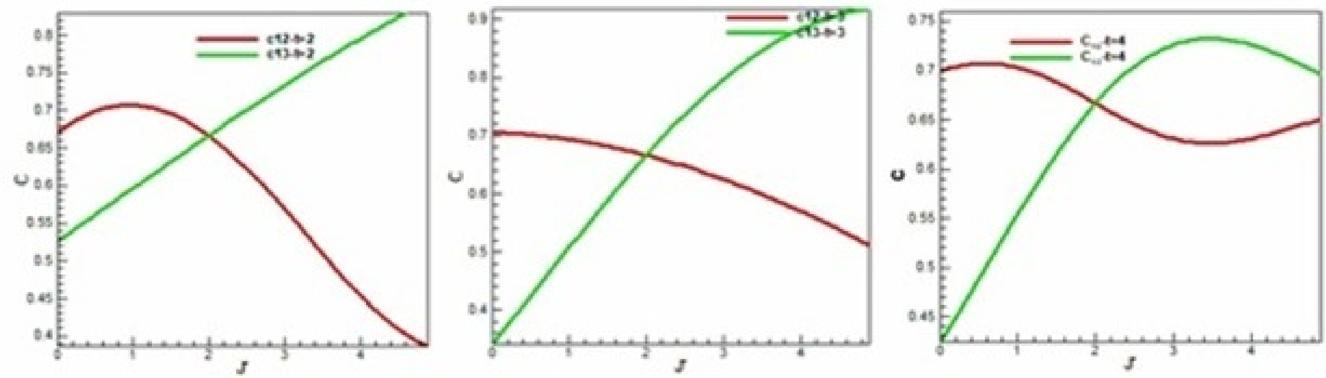

Figure 2. Concurrence between (NN) pair spins and (NNN) pair spins versus, $\acute{J}$ the two curves crossing each other at the critical point $\acute{J} = 2$ for limited time.

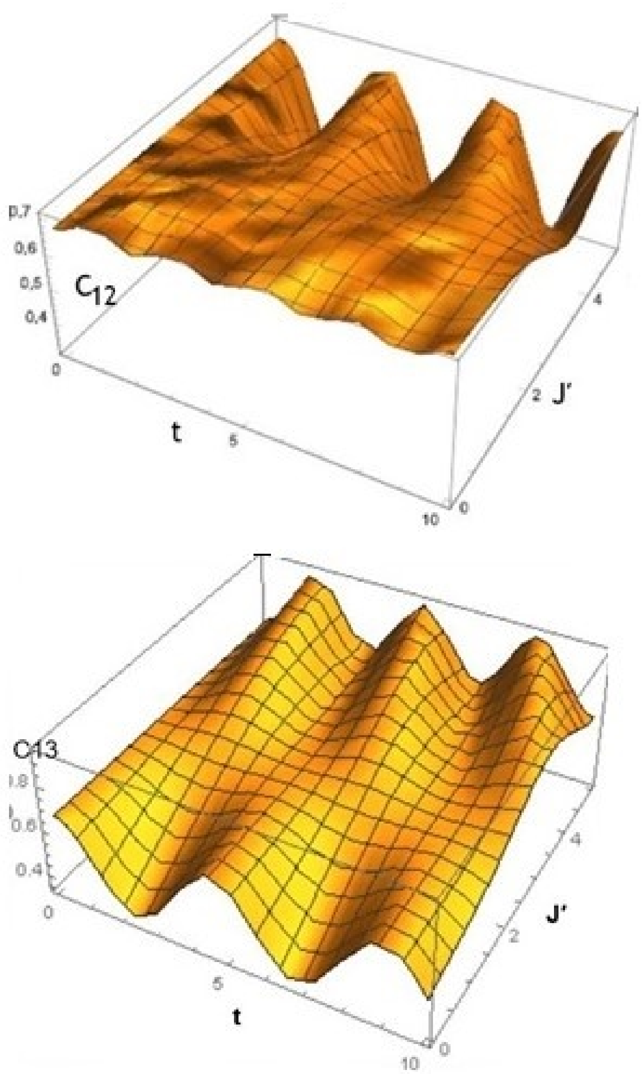

Figure 3. The concurrence as a function of both the time t and the three-spin coupling for (a) C^{12} and (b) C^{13} parameters with choosing W state as initial state for three-qubit system.

3. Result and discussion

Now, we study the time dependence of the entanglement between the nearest neighbor spins and next-to nearest neighbor spins in different strengths of the three- spin coupling. We found out that the concurrence dynamics is periodic over the time. Such a periodic condition is not met for $\acute{J} = 2$ since for this value of $\acute{J}$, the system is in its ground state. namely W-state. The ground state of the system is one of the eigenstates of the system Hamiltonian, therefore, there isn't any dynamic behavior for the system. As it can be seen in Fig. (1), at the start moment, the C^{12} and C^{13} have the same value of 0.66. As observed in figures, in the absence of the three-spin interaction ($\acute{J} = 0$), since the Heisenberg interaction between the nearest neighbor spins happens faster than the next nearest neighbor spins, it is obvious that the frequency producing W-states are different. In this situation, oscillation frequency of the $C^{12}(t)$ is twice of that of the $C^{13}(t)$. Also, as figures indicate the dynamic behavior of the C^{12} and C^{13} have a 180° phase difference for $\acute{J} < 2$ and $\acute{J} > 2$. This sudden change indicates a critical behavior of the system. As the three-qubit interaction is applied to the system, the initial concurrence increases slightly. But the remarkable point is that in the presence of the three-spin interaction, the frequency of the C^{12} and C^{13} becomes the same. The maximum points of the C^{12} are located at the minimum points of C^{13} and vice versa. For $\acute{J} = 2$, as seen in Fig. (1), behavior of both C^{12} and C^{13} are exactly the same. There isn't any dynamical behavior and the C^{12} and C^{13} are constant over the time. This is because for $\acute{J} = 2$ the system is at the ground state. In such situation, the entanglement between the nearest neighbor spin pairs the entanglement between next-to-nearest neighbor spin pairs, are the same.

For $\acute{J} = 3$, the period of oscillation of the entanglement is smaller compared to former cases. Moreover, in fig. (1) an important event can be observed; a phase difference of 180° in oscillation of the C^{12} and C^{13}, which indicates a phase transition in the system.

For $\acute{J} = 4$ as Fig. (1) shows, by increasing of $\acute{J}$, the period of oscillations are reduced. Generally, it can be concluded that by raising the three-spin interaction strength, the frequency of oscillations increase. Thus, the point of $\acute{J}=2$, which depends on different parameters such as initial impurity state of the system, and coupling strengths between particles, has a unique behavior over the time. For $\acute{J}=2$, system doesn't have any dynamic behavior and therefore the entanglement of the system will have the same initial value of 0.66

Fig. (2) presents the concurrence in terms of the three-qubit interaction coupling. There are some noticeable findings in this figure. Initially the system is in an entangled state. Although, over time, system goes through different states, the entanglement behavior of the system for critical point, $\acute{J}=2$, is constant.

In figure (2), variations of the nearest-neighbor entanglement and next-nearest-neighbor entanglement are suddenly changed at a certain point of $\acute{J}=2$. If the entanglement diagram is ascending up to this point, by passing this point, the entanglement behavior becomes descending.

This behavior is surprising. This critical point has been observed in a short time, too. The concurrence value indicates that the system is in its ground state where C^{12} and C^{13} are equal and for the first time, the W-state is reproduced. Moreover, the obtained results are in accordance with the results of reference [17]. In that reference, it has been shown that the critical point corresponds to the same model in non-dynamic conditions for the three-qubit system. It is worthwhile to mention where W state can be generated the first time in this point. Because of the symmetry, W-state is maximally robust under tracing out on each of three spins. In other word, the remaining reduced operator will have the greatest amount of entanglement, which is why it is called robust. As we know, in W-state all pair of spins are equally entangled and the amount of concurrence is 0.66. Meanwhile, what is reported in Ref. [22] as an indication of an occurrence of quantum phase transition (QPT) in the infinite-size system that might happen can be recognized even for a finite-size spin system. Fig. (3) displays the 3D diagrams of C^{12} and C^{12} in terms of the three-spin coupling $\acute{J}$ and the time t. We can see obviously that for C^{12}, for low $\acute{J}$, the amplitude of entanglement fluctuations is very small. By raising the $\acute{J}$, the periodic behavior of the entanglement increases. Unlike the C^{12}, dynamics of entanglement between pair spins S_1 and S_3, C^{13}, presents periodic behavior with significant amplitude for low $\acute{J}$. Although in $\acute{J}=2$, C^{13} doesn't show any dynamical behavior, in higher $\acute{J}$, the fluctuations of the entanglement with high amplitude appear.

4. Conclusion

In this paper, we have considered the W-state as the initial state for studying the dynamical behavior of the entanglement in a three-spin system with the XX Hamiltonian and quantum cluster interaction. Our results have shown some interesting features in the dynamics of quantum correlations between the nearest- neighbor pair of spins and the next-nearest-neighbor spin pairs in the mentioned system with (TSI) interaction. We found out that the time evolution of the concurrence is periodic. The results show that the concurrence fluctuations are same for the nearest- neighbor spins and next- nearest-neighbor pair of spins and increase by raising $\acute{J}$. Thus, the higher the strength, the less time is required to reach W-state. The W-state is revived. Another observation in the figures was that in the absence of the cluster interaction, the frequency of oscillation of the $C^{12}(t)$ is twice that of the $C^{13}(t)$. An important result found in this study is that at $\acute{J}=2$ the system is in the ground state of the Hamiltonian, so no dynamical behavior was seen in Figures C^{12} and C^{13}, the entanglement of the W-state keeps constant in time. Because of the more useful of W states in the entangled systems.

Conflict of interest statement:

The authors declare that they have no conflict of interest.

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