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The generalized representation of Dirac equation in two dimensions

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Abstract
Since the discovery of the Dirac equation, much research has been done on the construction of various sets consisting of Dirac matrices that all of which follow the Cliford Algebra. But there is never notice to the relationship between the internal elements of these matrices. In this work, the general form of 2×2 Dirac matrices for 2+1 dimension is found. In order to find this general representation, all relations among the elements of the matrices are found, and the generalized Lorentz transform matrix is also found under the effect of the general representation of Dirac matrices. As we know, the well known equation of Dirac, $\left(i\gamma^{\mu}\partial_{\mu}-m\right)\Psi=0$, is consist of matrices of even dimension known as the Dirac matrices. Our motivation for this study was lack of the general representation of these matrices despite the fact that more than nine decades have been passed since the discovery of this well known equation. Everyone has used a specific representation of this equation according to their need; such as the standard representation known as Dirac-Pauli Representation, Weyl Representation or Majorana representation. In this work, the general form which these matrices can have is found once for all.

Keywords
Dirac equation, General Dirac representation, Generalized Lorentz transform matrix.

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1. Introduction

Quantum Mechanics and Special Relativity are considered two major revolutions in twentieth-century physics. The Dirac equation is undoubtedly one of the best examples of the link between these two fields. Although this equation was first proposed for the electrons, but it has performed great in describing muons, quarks, and tauons and its application in the fundamental and theoretical physics fields is obvious to everyone, including Elementary particle, Quantum Chromodynamics, phenomenological models of hadrons, in standard models, and even in cosmology [1–4]. In [5] and [6] , it was demonstrated that the Dirac equation not only can be applied to Fermionic states but also to Bosonic states.

In the last few decades, it has been demonstrated that electrons in the shells closest to the heavy nucleus influence the bonds of their included molecules in which the relativistic effects are observable and some models such as Dirac-Hartree-Fock (DHF) are suggested describing the characterization of heavy elements. The use of relativistic methods is essential for theoretical models in the quantum chemistry fields, including actinides, lanthanides, and transition metals [7]. The Dirac equation has many applications in the fields of electronic transfers (ET) [8], Photonic Structures, and Supercooled Materials in optic networks [9–11].

Understanding the Dirac fermions is essential for the modern condensed matter physics (CMP). There is a fundamental similarity among a wide range of materials such as d-wave superconductors, graphene, and topological insulators so that their low energy fermionic excitations act similar to massless Dirac fermions. The Dirac fermions specific behaviors are integrated into a unified framework for a class of materials named ”Dirac materials” in the condensed matter systems since many of the special properties of the mentioned materials are induced by the Dirac spectrum of their particles. In other words, the materials include low energy excitations with Dirac points involving linear dispersion relation. In the absence of a mass term, all of the materials are gap-free. Dirac materials create a different class of material against, for example, metals and semiconductors. Different types of Dirac materials have been discovered so far, from normal state crystal materials to asymmetric quantum fluids such as Si, Na3Bi, Cd3As2 [12] , and topological insulators such as Bi2Se3 [13]. Thus far, the applications of the Dirac equation and relativistic quantum have been addressed in different fields of physics, including in CMP. However, some of the CMP such as graphene can be considered as cost-effective laboratories to test some of the relativistic quantum phenomena, including testing the Klein paradox.

After an investigation of the importance of the Dirac equation, it is time to address the aim of the present article. In 1926, Erwin Schrodinger introduced his famous wave equation named ”Schrodinger's wave equation” that was associated with many achievements in the quantum field; however, there was an important point, that is, this equation was not invariant under Lorentz transformations. Hence, the first quantum relativistic equation was proposed by Klein- Gordon to apply to spin-zero particles, but, in 1928, Paul Dirac, one of the founders of quantum mechanics and quantum electrodynamics, proposed his relativistic wave equation for the electron to solve the negative probability problem in the Klein-Gordon equation [14].

The Dirac equation for a free particle or particles with constant potential is in the form of $\left(i\gamma^{\mu}\partial_{\mu} \mp m\right)\Psi_{\mathbf{k}}(\mathbf{r},t) = 0$ in the natural units ($\hbar = c = 1$) (as usual, the minus (-) sign is used). As in the most relativistic quantum books, including [14], the γ^{μ}s matrices that are known as the Dirac matrix or Dirac representation, is a matrix with even dimension $(2\times 2\ or\ 4\times 4\ or\ \cdots)$ that is not unique, but it should follow some given conditions. Dirac matrices as well as the dominated algebraic relations have been extensively studied in detail, but none of them have focused on the relationship between the internal elements of those matrices. Researchers have always used a specific representation to introduce matrices based on their needs. For instance, in 4×4 Dirac matrices, 16 independent matrises can be applied to the 15+1 D space. These 16 states comprise a complete set based which any arbitrary matrices can be expanded as a linear combination of those 16 states. Those different sets can be transformed in 8 steps [15]. Pauli had proven that for the given any two Dirac matrix sets, there would be a similarity conversion by which those matrices can be transformed. Different methods of making those 16-element matrices with specific groups like $O(4,1), O(3,1), O(1,1)\otimes O(3), \ldots$ were classified by E. De Vries and A. J. Vanzanten [16]. Moreover, Cliford algebra can be applied to study different characteristics of Dirac matrices [17–22]. In other words Cliford algebra was originated from Pauli-Dirac matrices [17]. Since, the targeted particles are fermions with 1/2 spins and a unique representation can be selected to highlight the nature of one type of these fermions. For instance, the Majorana representation can lead to the real wave functions. However, these spinors can be the wave functions of particles that are also the anti-particles of themselves.

Nevertheless, ever since the discovery of the Dirac equation, only a few specific and useful answer were considered, such as "Standard or Dirac-Pauli Representation" (that we show this with S-index),"Super Symmetric Representation", "Weyl or Spinor representation" and "Majorana representation" here. Up to now, no one has tried to determine all the acceptable answers for γ^{μ}s matrix. In this work, the γ^{μ} matrices (2×2) are "general representation" (we emphasize this with G-index) is determined with all conditions governing its elements. It is obvious that of answers corresponding to the Dirac equation, i.e, from $\Psi_{\mathbf{k}}(\mathbf{r},t)$. to the problem for operators of spin, helicity, Lorentz transformation, and parity undergo some changes. In sum, all the concepts in which the general representation is generalized will change with new Dirac matrices of γ^{μ}s, although their original definition does not change, including Dirac Lagrangian, Dirac Hamiltonian, and continuity equation of probability in Dirac, etc.

Accordingly, the application range of the Dirac equation will be broader than before, and in the case of deviation between the theoretical results and experimental data, answers that are more consistent may be obtained using the freedom in the elements of γ^{μ} matrices. It means one can build unique matrices special for such problems. As Weyl proposed some other matrices to describe massless relativistic particles with spin-1/2 known as Weyl fermions, required matrices, and Majorana fermions that are themselves antiparticles [23], now one can determine the correct matrices corresponding to some other problems that follow the general principles of Dirac equation. This can be associated with changes in the relationship between the theoretical physics and the experimental world.

The rest of this paper is structured as follows. Section 2 briefly addresses the history of the Dirac equation, then the general Dirac matrices are determined for the case of 2×2. Section 3 evaluates the γ^{μ}s matrices and Dirac spinors under the influence of Lorentz transformation and parity. Finally, the last section corresponds to the results and future work prospects.

2. Generalized representation in two dimensions

(All the calculations were done in the Natural Units; $\hbar = c = 1$)

As we know, the Klein-Gordon equation for the free particle is $(p^{\mu}p_{\mu} - m^2)\Phi = 0$, where $\mu = 0,1,2,\ldots,D$ and D is the dimension of the space. $p^{\mu} = (i\partial/\partial t, -i\partial/\partial x^i) \Rightarrow p_{\mu} = (i\partial/\partial t, i\partial/\partial x^i) = i\partial_{\mu}$ The essential problem of the Klein-Gordon equation is the possibility of negative probability which is due to the second degree time derivative in it. Dirac believed that it was possible to take the square root of the Klein-Gordon equation so that the second order time derivative becomes the first order time derivative. But clearly, the Klein-Gordon equation is not a perfect square. Hence, he assumed that $p^{\mu}p_{\mu} = (\gamma^{\mu}p_{\mu})^2 = \gamma^{\mu}p_{\mu}\gamma^{\nu}p_{\nu}$ can be written using unspecified coefficients called the gamma coefficients. Assuming $[p_{\mu},\gamma^{\nu}] = 0$, then $\gamma^{\mu}\gamma^{\nu}p_{\mu}p_{\nu} = p^{\mu}p_{\mu}$ has to hold. As we know, it's possible to write $\gamma^{\mu}\gamma^{\nu}p_{\mu}p_{\nu} = g^{\mu\nu}p_{\mu}p_{\nu}$ using Minkowski metric tensor ($diagg^{\mu\nu} = (1,-1,-1,\ldots)$). It might seem that by removing the term $p_{\mu}p_{\nu}$ from this equation, one can write $\gamma^{\mu}\gamma^{\nu} = g^{\mu\nu}$. But this cannot be done because of the presence of the indices of ν and μ in other terms of this equation. Hence, the symmetric property in Minkowski metric tensor should be considered. Since the $g^{\mu\nu}$ matrix is symmetric, $\gamma^{\mu}\gamma^{\nu}$ has to be written symmetrically as well. After doing some calculations and removing the antisymmetric term, the following fundamental equation is obtained:

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}I \tag{1}$$

This anti-commute relation confirms that the gamma coefficients cannot be numbers. Hence, Dirac searched for their next from; the gamma matrices. According to the Eq. 1, other useful results will be obtained, which is the focus of this manuscript and helps us to know the elements of these matrices better. Here we describe 13 properties of Dirac matrices and prove them in Appendix A. (In the whole article,

the Greek letters start from zero and the English letters from one)
I. The gamma matrices are matrices of even dimension.

II. $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}I = 2g^{\mu\nu}I$.

III. If there is an invertible matrix that can transform the gamma matrices to their similar ones, then the similar matrices follow this fundamental relation: $\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}I$.

IV. The gamma matrices can be diagonalized.

V. $(\gamma_D^0)^2 = I$.

VI. The eigenvalues of the matrix γ^0 is ± 1.

VII. $(\gamma_D^i)^2 = -I$.

VIII. The eigenvalues of the matrix γ^i is $\pm i$.
IX. The gamma matrices are traceless.
X. $\gamma^\mu \gamma^\nu \neq 0$.

XI. The gamma matrices are normal which means: $[\gamma^\mu, (\gamma^\mu)^\dagger] = 0$.

XII. $(\gamma^0)^\dagger = \gamma^0$.

XIII. $(\gamma^i)^\dagger = -\gamma^i$.

As we know, the Dirac equation is a D+1 dimensional equation; meaning that time is always considered as one dimension in this equation. On the other hand, each of the gamma matrices corresponds to a dimension of space-time. Hence, in which there is only one number, the number is assigned to the dimension of time; and since j=1,2,...,D; D is assigned to the dimension of space. Now, let's consider the smallest matrix; 2×2 gamma matrix.

$$\gamma^0 = \begin{pmatrix} c_0 & a_0 - ib_0 \\ a_0 + ib_0 & -c_0 \end{pmatrix},$$
$$\gamma^1 = i\begin{pmatrix} c_1 & a_1 - ib_1 \\ a_1 + ib_1 & -c_1 \end{pmatrix},$$
$$\gamma^2 = i\begin{pmatrix} c_2 & a_2 - ib_2 \\ a_2 + ib_2 & -c_2 \end{pmatrix}$$

where

$$\begin{aligned} a_0 &= c_1 b_2 - b_1 c_2 \ , \ b_0 = a_1 c_2 - c_1 a_2 \ , \ c_0 = b_1 a_2 - a_1 b_2 \\ a_1 &= b_0 c_2 - c_0 b_2 \ , \ b_1 = c_0 a_2 - a_0 c_2 \ , \ c_1 = a_0 b_2 - b_0 a_2 \\ a_2 &= c_0 b_1 - b_0 c_1 \ , \ b_2 = a_0 c_1 - c_0 a_1 \ , \ c_2 = b_0 a_1 - a_0 b_1 \end{aligned} \tag{2}$$

According to the mentioned properties of I to XIII, the following relations among the elements of the gamma matrices are obtained:

$$a_0^2 + b_0^2 + c_0^2 = a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2 = 1$$
$$a_0^2 + a_1^2 + a_2^2 = b_0^2 + b_1^2 + b_2^2 = c_0^2 + c_1^2 + c_2^2 = 1$$
$$a_0 a_1 + b_0 b_1 + c_0 c_1 = a_0 a_2 + b_0 b_2 + c_0 c_2$$
$$= a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$
$$a_0 b_0 + a_1 b_1 + a_2 b_2 = a_0 c_0 + a_1 c_1 + a_2 c_2$$
$$= b_0 c_0 + b_1 c_1 + b_2 c_2 = 0$$

Which can be summarized using the Einstein summation convention as the following:

$$\begin{gathered} a_\mu = -b_\nu c_\theta \in_{\mu\nu\theta} \ , \ b_\mu = -c_\nu a_\theta \in_{\mu\nu\theta} \ , \ c_\mu = -a_\nu b_\theta \in_{\mu\nu\theta} \\ a_\mu a_\mu = b_\mu b_\mu = c_\mu c_\mu = a_\mu b_\mu = a_\mu c_\mu = b_\mu c_\mu = 1, \\ a_\mu a_\nu + b_\mu b_\nu + c_\mu c_\nu = \delta_{\mu\nu}, \end{gathered} \tag{3}$$

Furthermore, the following relations hold among the gamma matrices themselves:

$$\gamma^0\gamma^1 = -i\gamma^2, \gamma^1\gamma^2 = +i\gamma^0, \gamma^2\gamma^0 = -i\gamma^1, -i\gamma^0\gamma^1\gamma^2 = I \tag{4}$$

All properties of the elements of the gamma matrices can be summarized in the following normal matrix:

$$A \equiv \begin{pmatrix} c_0 & c_1 & c_2 \\ b_0 & b_1 & b_2 \\ a_0 & a_1 & a_2 \end{pmatrix}; AA^T = A^T A = I, |A| = 1. \tag{5}$$

According to the matrices algebra, it can be proved that no other 2×2 gamma matrix could be found that is linearly independent of the other three matrices. In other words, no fourth matrix could be found that held all the mentioned 13 properties along with other three matrices simultaneously. Hence, it can be concluded that this 2×2 matrix is at best useful for the 2+1 dimension space. It worth mentioning that if we assume $c_0 = b_1 = a_2 = 1, Etc = 0$, then the normal or standard representation of Dirac equation is obtained;

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \gamma^2 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

2.1 The general form of Dirac equation and its spinors

In this subsection, the general form of Dirac equation and its solutions are discussed in addition to comparing them to the standard representation of Dirac equation. Based on what was stated, the non-differential form of Dirac equation for the 2+1 dimension space has to be as Eq. 7

$$(\gamma^0 E - \gamma^1 k_1 - \gamma^2 k_2 - mI) u(E, \vec{k}) e^{i(k_1 x + k_2 y - Et)} = 0$$

$$\Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \tag{7}$$

where

$$A = c_0 E - ic_1 k_1 - ic_2 k_2 - m,$$

$$B = (a_0 - ib_0)E - i(a_1 - ib_1)k_1 - i(a_2 - ib_2)k_2,$$

$$C = (a_0 + ib_0)E - i(a_1 + ib_1)k_1 - i(a_2 + ib_2)k_2,$$

$$D = -c_0E + ic_1k_1 + ic_2k_2 - m$$

Which is written as follows in the Standard Representation of Dirac equation:

$$\begin{pmatrix} E-m & -(k_1+ik_2) \\ k_1-ik_2 & -(E+m) \end{pmatrix} \begin{pmatrix} u_{1S} \\ u_{2S} \end{pmatrix} = 0 \tag{8}$$

The determinant of this matrix has to be zero so that the Eq. (9) could have non-trivial solutions. Hence, the important and obvious result $E = \pm\sqrt{k_1^2 + k_2^2 + m^2}$ is obtained. The solutions of the Equation (9) for two given eigenvalues of $\pm E$ are respectively:

$$u_+ = N \begin{pmatrix} (c_0E+m) - i(c_1k_1+c_2k_2) \\ i(b_0E - a_1k_1 - a_2k_2) + (a_0E + b_1k_1 + b_2k_2) \end{pmatrix}$$
$$u_- = N \begin{pmatrix} i(b_0E - a_1k_1 - a_2k_2) - (a_0E + b_1k_1 + b_2k_2) \\ (c_0E+m) + i(c_1k_1+c_2k_2) \end{pmatrix}$$
$$N = \frac{1}{\sqrt{2E(E + c_0m + c_1k_2 - c_2k_1)}} \tag{9}$$

Which in the standard representation are reduced to the following solutions:

$$u_{S+} = \frac{1}{\sqrt{2E(E+m)}} \begin{pmatrix} E+m \\ k_1 - ik_2 \end{pmatrix}$$
$$u_{S-} = \frac{1}{\sqrt{2E(E+m)}} \begin{pmatrix} -(k_1+ik_2) \\ E+m \end{pmatrix} \tag{10}$$

3. The gamma matrices and the Dirac spinors under the Lorentz transformation

If Λ is a Lorentz transformation and $(x' = \Lambda x)$ then,

$$\begin{cases} \psi'(x') = S(\Lambda)\psi(x) \Rightarrow \psi(x) = S^{-1}(\Lambda)\psi'(x') \\ \bar{\psi} = \psi^\dagger\gamma^0 \Rightarrow \bar{\psi}'(x') = \bar{\psi}(x)S^{-1}(\Lambda). \end{cases} \tag{11}$$

The prime means an inertial system O' that moves with velocity v/c relative to the system O, $(m' = m, (\gamma^\mu)' = \gamma^\mu)$.

$$O \to \left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = 0,$$
$$O' \to \left(i\gamma^\mu\partial'_\mu - m\right)\psi'(x') = 0$$
$$\Rightarrow i\gamma^\mu\partial'_\mu\psi'(x') - m\psi'(x') = 0. \tag{12}$$

∂_μ is a vector because it converts like a vector under the Lorentz transformation, ie:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu_\mu\partial'_\nu,$$
$$(11) \Rightarrow \left(i\gamma^\mu\Lambda^\nu_\mu\partial'_\nu - m\right)S^{-1}(\Lambda)\psi'(x') = 0$$
$$\Rightarrow i\gamma^\mu\Lambda^\nu_\mu\partial'_\nu S^{-1}\psi'(x') - mS^{-1}\psi'(x') = 0 \ . \tag{13}$$

We multiply $S(\Lambda)$ at the left on (13) until the second term of this relation equals to the second term of relation (14):

$$iS\gamma^\mu\Lambda^\nu_\mu\partial'_\nu S^{-1}\psi'(x') - m\psi'(x') = 0 \tag{14}$$

$\partial'_\nu S^{-1} = S^{-1}\partial'_\nu$ because the elements S and S^{-1} are constants and the differential operator ∂' or ∂ passes through them. By using (12), (14):

$$iS\gamma^\mu\Lambda^\nu_\mu S^{-1}\partial'_\nu\psi'(x') = i\gamma^\nu\partial'_\nu\psi'(x'), \tag{15}$$

then

$$\gamma^\nu = S\gamma^\mu\Lambda^\nu_\mu S^{-1} \longleftrightarrow S^{-1}\gamma^\nu S = \gamma^\mu\Lambda^\nu_\mu. \tag{16}$$

In the 1+1-dimensional space we have just one "boost" ($v = tanh\theta$)

$$\begin{cases} x'^0 = (cosh\theta)x^0 - (sinh\theta)x^1, \\ x'^1 = (cosh\theta)x^1 - (sinh\theta)x^0 \end{cases} \tag{17}$$

$$\Rightarrow \begin{cases} \Lambda^0_0 = \Lambda^1_1 = cosh\theta \ , \ \Lambda^1_0 = \Lambda^0_1 = -sinh\theta \\ S = exp(-\frac{\theta}{2}\gamma^0\gamma^1) = exp(-\frac{\theta}{2}\bar{\gamma}^2) = Icosh\frac{\theta}{2} - \bar{\gamma}^2 sinh\frac{\theta}{2}. \end{cases} \tag{18}$$

Thus for GR

$$S = \begin{pmatrix} cosh\theta/2 \ - c_2 sinh\theta/2 & -(a_2 - ib_2)sinh\theta/2 \\ -(a_2 + ib_2)sinh\theta/2 & cosh\theta/2 \ + c_2 sinh\theta/2 \end{pmatrix}. \tag{19}$$

Similarly, it can prove that the following relation is also held:

$$S^{-1}\gamma^1 S = \gamma^0\Lambda^1_0 + \gamma^1\Lambda^1_1 = -\gamma^0 sinh\theta + \gamma^1 cosh\theta. \tag{20}$$

Also, we know the Dirac spinors under the Lorentz transformations will be as follows:

$$\psi'(x') = S(\Lambda)\psi(x) \ \ or \ \ u'_\pm = Su_\pm. \tag{21}$$

By changing the internal elements of gammas, the definitions that we have for scalar (such as $\bar{\psi}\psi = \psi\dagger\gamma^0\psi$) and pseudo-scalar and vector and pseudo-vector don't change because, in spite of using the gammas in these definitions, their reaction to the Lorentz transformation that shows the type of being scalar or vector of them, does not related to the internal elements of gammas.

Altogether, concepts in which the Dirac gamma matrices are used will be generalized because of generalizing these matrices; however, the fundamental definition of them does not change such as the Dirac Lagrangian, the Dirac Hamiltonian, the probability continuity equation in Dirac, etc.

4. Conclusion

It was shown in Sec. 2 that how the fundamental Eq. (1) could result in the general representation of Dirac equation (9) as well as other useful relations like the properties of the gamma matrices (I-XIII) in addition to the relation among the elements of the gamma matrices (7). After finding the general form of the Lorentz transform operator, $S(\Lambda)$, in Sec. 3, its effects on the general Dirac spinors were studied. And finally, the general form of the parity operator and its effects on Dirac spinors were studied. The main novelty of this manuscript which makes it important is that despite of all works done in the past nine decades for proving the multiplicity of the representation of Dirac matrices, no efforts have been done in the books and papers to introduce the general form of Dirac matrices. Instead, one specific representation of Dirac matrices is used according to the requirement of the study;such as the standard representation known as Dirac-Pauli Representation, or some especial forms of Supersymmetric Representation or Weyl Representation or Majorana Representation. Sometimes even different forms of Weyl Representation or Majorana Representation are introduced in the papers without mentioning their common origin. Finding the general form of this equation in the present manuscript (for 2+1 dimension) has opened a new way to quick develop of this well known and useful equation in all fields of Physics. It might also lead to the opening of new windows on the influence of Dirac equation; such as the discovery of new particles correspond to its new representation or the discovery of new symmetries in theoretical Physics. It worth mentioning that there are many works to be done for developing the general form of Dirac matrices;such as finding the general form of Dirac matrices for 3+1 dimension of space-time,as well as greater dimensions. In addition to the discovery of different applications of this form of representation; such as calculating the reflection and transmission coefficients of different particles. Especially, studying Klein tunneling using this representation that is been doing by the authors of the present manuscript right now.

A. Proving the properties of Dirac matrices

I. Proof:

$$\{\gamma^0,\gamma^i\}=0\Rightarrow\gamma^0\gamma^i=-\gamma^i\gamma^0$$

$$\Rightarrow|\gamma^0\gamma^i|=|-\gamma^i\gamma^0|=(-1)^N|\gamma^i\gamma^0|$$

$$\Rightarrow|\gamma^0||\gamma^i|=(-1)^N|\gamma^i||\gamma^0|\Rightarrow N=Odd.$$

II. Proof: We know that $[g_{\mu\nu},\gamma^\lambda]=[g_{\mu\nu},g_{\theta\varphi}]=0$

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=g_{\mu\theta}\gamma^\theta g_{\nu\varphi}\gamma^\varphi+g_{\nu\varphi}\gamma^\varphi g_{\mu\theta}\gamma^\theta$$

$$=g_{\mu\theta}g_{\nu\varphi}(\gamma^\theta\gamma^\varphi+\gamma^\varphi\gamma^\theta)=2g_{\mu\theta}g^{\theta\varphi}g_{\nu\varphi}I=2g_{\mu\nu}I.$$

III. Proof: $\gamma'^\mu=A\gamma^\mu A^{-1},\gamma'^\nu=A\gamma^\nu A^{-1}$ then

$$\{\gamma'^\mu,\gamma'^\nu\}=A\gamma^\mu A^{-1}A\gamma^\nu A^{-1}+A\gamma^\nu A^{-1}A\gamma^\mu A^{-1}$$

$$=A(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)A^{-1}=2g^{\mu\nu}I.$$

IV. Proof: if $P^{-1}\gamma^\mu P=\gamma^\mu_D$, that index-D means diagonal matrix. And we saw $(\gamma^\mu)^2=\pm I$ so

$$(\gamma^\mu_D)^2=P^{-1}\gamma^\mu PP^{-1}\gamma^\mu P=P^{-1}(\gamma^\mu)^2P=\pm I.$$

V. Proof:If $P^{-1}\gamma^0P=\gamma^0_D$ and $I=P^{-1}IP=P^{-1}(\gamma^0)^2P=P^{-1}P\gamma^0_DP^{-1}P\gamma^0_DP^{-1}P=(\gamma^0_D)^2$.

VI. Proof: $I=(\gamma^0_D)^2=\begin{pmatrix}\lambda_1^2 & 0 & \cdots\\ 0 & \lambda_1^2 & \cdots\\ \vdots & \vdots & \ddots\end{pmatrix}\Rightarrow\lambda_i=\pm1.$

VII. Proof: $P^{-1}\gamma^iP=\gamma^i_D$ and $-I=-P^{-1}IP=P^{-1}(\gamma^i)^2P=P^{-1}P\gamma^i_DP^{-1}P\gamma^i_DP^{-1}P=(\gamma^i_D)^2$.

VIII. Proof: $-I=(\gamma^i_D)^2=-\begin{pmatrix}\lambda_1^2 & 0 & \cdots\\ 0 & \lambda_1^2 & \cdots\\ \vdots & \vdots & \ddots\end{pmatrix}\Rightarrow\lambda_i=\pm i.$

IX. Proof: We know that $tr(abc)=tr(cab)=tr(bca)$

$$if\ \ \mu\neq\nu\Rightarrow\gamma^\mu\{\gamma^\mu,\gamma^\nu\}=0$$

$$\Rightarrow\pm\gamma^\nu=-\gamma^\mu\gamma^\nu\gamma^\mu\Rightarrow tr(\pm\gamma^\nu)$$

$$=-tr(\gamma^\mu\gamma^\nu\gamma^\mu)=-tr(\pm\gamma^\nu)\Rightarrow tr(\gamma^\nu)=0.$$

X. Proof:

$$|\gamma^\mu\gamma^\nu|=|A\gamma^\mu_DA^{-1}B\gamma^\mu_DB^{-1}|=|\gamma^\mu_D\gamma^\mu_D|\neq0\Rightarrow\gamma^\mu\gamma^\nu\neq0.$$

XI. Proof: Because $\gamma^\mu=A\gamma^\mu_DA^{-1}$ and A is orthonormal.
XII. Proof: We know that $(\gamma^0_D)\dagger=\gamma^0_D$ then $(\gamma^0)\dagger=A^\dagger\gamma^0_DA=\gamma^0$.
XIII. Proof: We know that $(\gamma^i_D)\dagger=-\gamma^i_D$ then $(\gamma^i)\dagger=A^\dagger(\gamma^i_D)^\dagger A=-A^\dagger\gamma^i_DA=-\gamma^i$.

Conflict of interest statement:
The authors declare that they have no conflict of interest.

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