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Holographic uncomplexity in the hyperscaling violating backgrounds

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Abstract
In this paper, using complexity equals action proposal, we investigate holographic complexity for hyperscaling violating theories on different subregions of space-time enclosed by the null boundaries. By recalling the computation of on-shell action for certain subregions of the intersection between the Wheeler DeWitt patch, as well as, the future interior of a two-sided black brane, we are interested in compute uncomplexity in the hyperscaling violating theories [1]. We show that the dynamical exponent plays a crucial role in computing the rate of complexification. However, at the late time,the rate of the complexity growth is independent of the hyperscaling parameters. Moreover, we compute holographic uncomplexity in hyperscaling violating backgrounds and show when a black hole is formed, uncomplexity is the total space-time volume which is accessible to an observer who decides to pass over the horizon. Therefore, uncomplexity could be considered as a resource which indicates the computational power of a system. In fact, it can extract and analyze the useful information from the system.

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1. Introduction

Investigating new concepts in different aspects of physics to explain what happens around us has been always interesting. In fact, everything began with human's curiosity and which led to a scientific revolution. Obtaining information about unknown subjects and also finding a way to understand them, as much as possible, is fascinating for everyone especially physicists. For example in first place, a black hole was known as a mathematical object. However, several works were done afterwards to detect this attractive creature and explore the world behind the event horizon which is still a mystery. Scientist are curious to know what is happening behind the horizon. This question and the same ones lead to many developments in different fields of physics and the other branches of science. Recently, entanglement entropy and holographic complexity are the most important computational tools in black hole physics, condensed matter physics, quantum information and etc. Entanglement entropy is used to extract information from entangled systems [2–6]. In other words, entanglement entropy of a system is the amount of information that the correspondence observer, who has no access to this region, receives [7]. Furthermore, complexity as well as entanglement entropy is used as a useful tool to understand condensed physical systems especially black hole physics [8–14]. As a matter of fact, holographic complexity explains how difficult it is to implement the computation of unitary operations of the observers. In the language of quantum circuits, complexity is defined as the minimal elementary gates required for computation of the unitary operations [2], [4]. It means complexity is the amount of the difficulty that maps an unentangled and pure state (reference state) to the corresponding orthogonal bases. Therefore, the complexity of a target quantum state can be defined as: The state complexity is the minimal operator complexity of any unitary operator that maps the system from a reference state (typically the complete unentangled state) to the target state [15, 16].

Complexity is a quantity which grows linearly by time for a temporal exponential function $K : C(t) = Kt$. At the time $t : e^k$, complexity reaches its possible maximal value $C_{max}$and for a long time remains in this state. This is the period of complexity equilibrium in which the complexity moves to the maximum limit $C_{max} : e^K$ [16–18].

These conjectures have been used as powerful computational tools to help us to study condensed systems or systems with many degrees of freedom [8, 9]. It might be interesting to study complexity from thermodynamic point of view [18]. In fact, the definitions of holographic complexity, explained above, show the evolution of classical entropy. At first, consider a classical system with the minimal entropy . In this system, the gas reaches equilibrium after that the evolution happens for the entropy, not complexity. However, for the classical case, the linear growth of entropy, in the number of degrees of freedom, insists only for the time polynomial. The maximum entropy is in the order of the number of degrees of freedom, and the recurrence time is simply exponential and not doubly exponential [16–18].

Simply, the quantum complexity for a k-qubit-system behaves like the entropy of a classical system with $2k$ degrees of freedom. In the definition of complexity from thermodynamic point of view, at any moment, the ensemble average of the computational complexity of the quantum system **Q** is proportional to the classical positional entropy of the auxiliary system **A** [16]. This means that the entropy grows with time and eventually reaches its maximum value, but the system never reaches thermal equilibrium if the initial velocities are not Maxwell-Boltzmann distributed. This is almost like the gas of absolutely free particles located on a very large surface with negative Riemann's curvature. Despite the kinetic energy of every particle is conserved, the positions spread out and finally fill the space. Matching these two concepts -average complexity and axillary entropy- one can provide an approximate similarity between the growth and the evolution of computational complexity. At first, a large number of particles (almost 2^k) are situated close to the origin of a large pack of the volume $\exp 2^k$. As we know, the velocities are Maxwell-Boltzmann distributed. Then, the gas begins to expand and the entropy of the gas grows. Finally, the gas fills the box and reaches equilibrium. Afterwards, the system stays in equilibrium for a large time but on timescales $\exp 2^k$ recurrences happen [16]. One can make the computations much more simple by using another quantity. Uncomplexity is a new concept introduced to identify how much computational power the system has, in some special cases.

Uncomplexity of a pure state is defined by the difference between maximum complexity C_{max} and the correspondence complexity of the pure state $C(t)$

$$\Delta C(t) = C_{max} - C(t) \tag{1}$$

For a holographic system with strongly interactions in dual black hole geometry, it has been conjectured that the complexity of the state with entropy S is increased linearly with time until it is saturated in its maximum value e^s [19]. Before the system reaches maximum complexity, the desired state has some computational power given by its uncomplexity. In other words, from the perspective of black hole physics, one can say uncomplexity of a black hole is related to its interior space that is accessible for the falling observer who wants to jump in [16–19].

This article has been organized as follows: In section 2, according to our computational framework and by using holography principle, we briefly review holography principle and AdS/CFT correspondence. In section 3, using complexity equal action proposal, we discuss on-shell action and also recall the process to find on-shell action for WDW patch and its subregions, future interior and past interior in hyperscaling violating theory. Uncomplexity is computed in hyperscaling violating background and is discussed from two points of view, thermodynamic and black hole physics, in section 4. At the end, in section 5, we present a discussion of our result.

2. Holography principle

Recently, the research on the black hole indicates that there might be a relationship between black hole physics and information quantum theory. Moreover, inspired by the studies in the black hole physics, it has been claimed according to the compatibility between gravity and quantum mechanics laws, the universe might behave like a hologram [2]. Holography, as a powerful and useful tool, helps us to study quantum field theories strongly coupled. More precisely, one can do some conceptual and useful computation on some quantities which belong to the space-time with one dimension less. On the other hand, according to AdS/CFT correspondence, everything in quantum field theory maps to its quantity in gravitational field theory with one more dimension [7–9]. This duality illustrates that quantum gravity has a meaningful deep relation with quantum information theory (surface and volume) [7–9], [20–23]. From AdS/CFT perspective it has been proposed the interior growth of a black brain is the dual of complexity growth of quantum information theory [20, 24]. If this conjecture works and leads to conceptual result, one can expect quantum information might play an important role in understanding the nature of space-time [2–6]. On the other hand, in many condensed matter systems, the theory is conformally invariant in critical points [1]. In such theories, when the temporal and spatial coordinates are rescaled with a constant, the system remains invariant. However, there are some systems which do not scale similarly at their critical points. From holographic perspective, these systems are dual to Lifshitz and hyperscaling violating geometries [25, 26]. The purpose of this paper, is to find an explicit form for uncomplexity by following the complexity computation given in Ref. [1] and expanding the results more precisely in hyperscaling violating backgrounds. We should say that such geometries have anisotropic and hyperscaling violating components in their nature [12, 25].

3. The total action in the hyperscaling violating background

It was proposed the complexity related to boundary state at a specific time t, is proportional to the value of the on-shell gravitational action $A(t)$of a certain bulk region [27–35]. This bulk region is the dependence domain of a Cauchy slice anchored on the boundary at time t. This conjecture is well-known as complexity equals action (CA) conjecture and the bulk region is named the Wheeler DeWitt (WDW) patch. The CAconjecture is defined by [4, 8, 9]

$$C = \frac{A_{WDW}}{\pi} \tag{2}$$

On the other side, the rate of the computation done by the system is limited by the energy of the system. This limitation is a famous universal bound known as Lloyd's bound [36] and is given by

$$\frac{dC}{d\tau} \leq \frac{2E}{\pi} \tag{3}$$

where E is the average energy of the system at time t.
In addition, that there is another holographic proposal for finding holographic complexity of the boundary state known as complexity equals volume (CV) [21–23], which has been applied in several works [37–39]. The proposal expresses that the complexity is dual to the codimension-1 volume of the maximal space-like slice anchored at the two given boundary times.
According to AdS/CFT correspondence, AdS geometries are dual to the conformally symmetric field theories. Furthermore, field theories which are scale invariant but not conformal invariant are very important. For instance, in addressing the Landau-Fermi liquids, one needs Lifshitz metrics in dual gravity theory where the spatial and time coordinates of the theory have been scaled differently. Hence studying the holographic dual models for such systems seems to be necessary. In fact, to make such a homogeneity for physical systems which do not respect conformal invariance and, in their critical points, represent a rather different scaling in space and time, one should couple Einstein gravity to a massive vector. In the theory with the Lifshitz fixed point, space and time scale differently as below [25]

$$t \to \zeta^z t, x_i \to \zeta x_i, r \to \zeta r \tag{4}$$

where z is dynamical critical exponent and in conformal field theory sets to 1. The Lifshitz invariant theory is spatially isotropic and homogeneous and admits the non-relativistic scaling symmetry 4. In addition, a full class of scaling metrics which means hyperscaling violating geometries, can be obtained by considering both dilaton scalar field and an Abelian gauge field. Then, the corresponding action is written as below

$$A = \frac{1}{8\pi G_N}\int d^{d+2}x\sqrt{-g}[R - \frac{1}{2}(\partial\varphi)^2 + V(\varphi) - \frac{1}{4}e^{\eta\varphi}(F_{\mu\nu})^2] \tag{5}$$

where G_N is Newton constant and the potential of the scalar field and the vector field are defined by [11]

$$V(\varphi) = V_0 e^{\xi\varphi}, A^t = \frac{L}{r_f^{\frac{\theta}{d}}}\sqrt{\frac{2(z-1)}{d+z-\theta}}\frac{1}{r^{d+z-\theta}},$$

$$e^{-\varphi} = r^{q=\sqrt{2(d-\theta)(z-1-\theta_e)}} \tag{6}$$

where L is the geometry's radius and r_f is a dynamical scale, where the metric may not be a good description for UV complete theory above it. In the above equation, η, ξ and V_0 are free parameters of the model given by [11]

$$V_0 = L_e^2(d+z-\theta-1)(d+z-\theta), \xi = \frac{2\theta}{d\sqrt{2(d-\theta)(z-1-\theta)}}$$

$$\eta = \frac{2\theta(d-1)-2d^2}{\sqrt{2(d-\theta)(z-1-\theta)}} \tag{7}$$

Obviously, one can find that the vector field produces an anisotropy of the theory while non-trivial scalar potential leads to hyperscaling violating factor. For simplicity, it is useful to characterize an effective hyperscaling violating exponent $\theta_e = \frac{\theta}{d}$, an effective dimension $d_e = d - \theta$ and also an effective scale $L_e = \frac{L}{r_f^{\theta_e}}$.
For simplification in the rest of this paper, we set $L_e = 1$ [1]. The solutions are given by [11,25]

$$ds_{d+2}^2 = \frac{1}{r^{2(1-\theta_e)}}(-\frac{f(r)}{r^{2(z-1)}}dt^2 + \frac{dr^2}{f(r)} + \Sigma_{i=1}^d dx_i^2) \tag{8}$$

The function $f(r)$ is defined as follows [11]

$$f(r) = 1 - (\frac{r}{r_h})^{d+z-\theta} \tag{9}$$

where r_h is the radius of horizon. The metric is not scale invariant and under the scale transformation Eq. (4), transforms like blow

$$ds \to \zeta^{\theta_e} ds \tag{10}$$

It should be mentioned that according to null energy condition one has [10,11]

$$(d-\theta)(d(z-1)-\theta) \geq 0, (z-1)(d+z-\theta) \geq 0 \tag{11}$$

In this equation, one can assume that $d > \theta$ which leads to $z \geq 1$. After some straightforward calculations in action and by using hyperscaling violating geometry, it is shown that action density in this background is obtained as follows [1]

$$\sqrt{-g}(R - \frac{1}{2}(\partial\varphi)^2 + V_0 e^{\zeta\varphi} - \frac{1}{4}e^{\eta\varphi}F^2)$$

$$= -2(1-\theta_e)(d_e+z)\frac{1}{r^{d_e+z+1}} \tag{12}$$

Also Hawking temperature and entropy in hyperscaling violating theory are defined by

$$T = \frac{d_e+z}{4\pi r_h}, S_{th} = \frac{V_d}{4G_N r_h^{d_e}} \tag{13}$$

In above equations, V_d is the volume of the spatial coordinate parametrized by x_i, $i = 1, \ldots d$.
According to "Complexity = Action" proposal, one needs to evaluate the on-shell action for three regions. The first one defines and computes on-shell action inside the WDW patch as shown in Fig.1. The second one is the future interior of the black hole and the third is the past interior. For all of these regions, it is known that the complete action should have certain Gibbons-Hawking terms defined at those boundaries. Moreover, the null boundaries as well as the joint points (points of intersection of these null boundaries with any other boundary) have their own stories and it is crucial to add the corresponding Gibbons-Hawking terms as well as certain joint

actions. According to the well-defined variational principle, one can write the following action [1, 20]

$$A^{(0)} = \frac{1}{16\pi G_N}\int d^{d+2}x\sqrt{-g}[R-\frac{1}{2}(\partial\varphi)^2+V_0e^{\zeta\varphi}-\frac{1}{4}e^{\eta\varphi}F^2]$$

$$\pm\frac{1}{8\pi G_N}\int_{\Sigma_s^{d+1}}K_s d\Sigma_s \pm \frac{1}{8\pi G_N}\int_{\Sigma_n^{d+1}}K_n dSd\lambda$$

$$+\frac{1}{8\pi G_N}\int_{\Sigma_t^{d+1}}K_t d\Sigma_t \pm \frac{1}{8\pi G_N}\int_{J_d} adS \quad (14)$$

Here, the first term is the contribution of bulk action and λ is the null coordinate, which is defined on the null segments; space-like, null boundaries and the time-like and also joint points are denoted by $\Sigma^{d+1}s, \Sigma^{d+1}n, \Sigma^{d+1}t$ and J_d, respectively. The extrinsic curvatures of the corresponding boundaries are given by K_s, K_n and K_t. On the other hand, at the intersection of the boundaries, the function is defined by the logarithm of the inner product of the corresponding normal vectors. It is worth to mention that the relative position of the boundaries and the bulk region of interest identify the sign of different terms in the above action [20].
In order to preserve the invariance under a reparametrization of the null generators, an extra term is needed, which should be added to the action to remove the ambiguity. In Ref. [20] it has been shown that such term might be given by

$$A^{amb} = \frac{1}{8\pi G}\int_{\Sigma_n^{d+1}} d^dxd\lambda\sqrt{\gamma}\Theta\log\frac{|\Theta|}{d_e} \quad (15)$$

in which γ is the determinant of the induced metric on the joint point where by definition, two null segments intersect. Θ is defined by [20]

$$\Theta = \frac{1}{\sqrt{\gamma}}\frac{\partial\sqrt{\gamma}}{\partial\lambda} \quad (16)$$

Therefore, up to this level, the on-shell action is $A = A^{(0)} + A^{amb}$[17].
The symmetry of the Penrose diagram in Fig.1, demands that a symmetric configuration with times $\tau_R = \tau_L = \frac{\tau}{2}$ should be considered. According to Fig. 1, it is obvious that there are four null boundaries of the corresponding WDW patch, which are given by [1]

$$N_1: t = t_R - r^*(\varepsilon) + r^*(r), N_2: t = -t_L + r^*(\varepsilon) - r^*(r),$$

$$N_3: t = t_R + r^*(\varepsilon) - r^*(r), N_4: t = -t_L - r^*(\varepsilon) + r^*(r) \quad (17)$$

and also the location of the joint point **m** is given by (note that in our notation, we have used $r^*(r) \leq 0$)

$$\tau \equiv t_r + t_L = 2(r^*(\varepsilon) - r^*(r_m)) \quad (18)$$

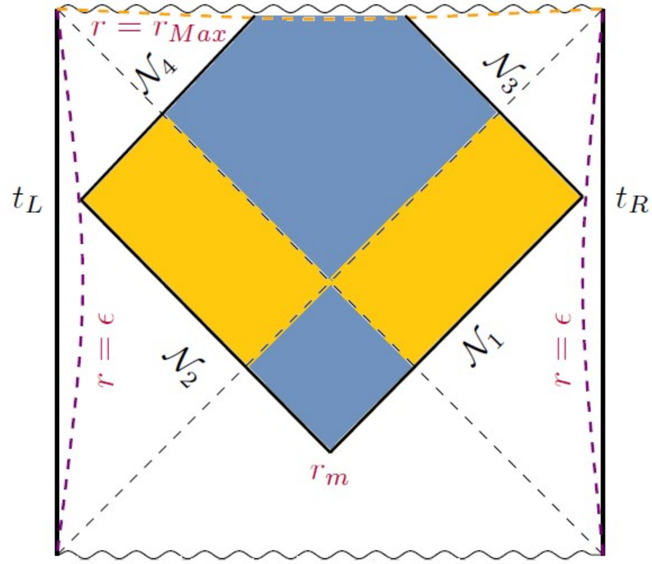

Figure 1. Penrose diagram of the WDW patch of an eternal AdS black hole. N_i are null boundaries and it is supposed that $t_R = t_L$. To find the complexity, the on-shell action should be computed on this patch.

In this paper, we are interested in computing uncomplexity in hyperscaling violating backgrounds. To do this, we follow the computation of complexity for the on-shell action of the interior region of an eternal static neutral black brane in the generic dimensions, presented in Ref. [1], in hyperscaling violating metrics, which are dual to a thermal state on the boundary.
Now, by computing the on-shell action over the corresponding WDW patch, the overall action becomes as below

$$\tilde{A}_{WDW} = A^{bulk}_{WDW} + A^{surf}_{WDW} + A^{joint}_{WDW} + A^{amb}_{WDW}$$

$$= \frac{V_d}{8\pi G_N}(\frac{(d_e+z-2)(\tau+\tau_c)}{2r_h^{d_e+z}} - \frac{\log|f(r_m)|}{r_m^{d_e}} +$$

$$\frac{2(z-\theta_e+1)}{d_e}(\frac{2}{\varepsilon^{d_e}}+\frac{1}{r_m^{d_e}}) + \frac{2\log\varepsilon^{-2\theta_e}}{\varepsilon^{d_e}} - \frac{\log r_m^{2\theta_e}}{r_m^{d_e}}) \quad (19)$$

As one can see the resultant on-shell action is UV-divergent and here, we follow Ref. [40] to introduce a proper counter term to the action, which is given by [1]

$$A^{ct} = \frac{1}{8\pi G_N}\int d\lambda d^dx\sqrt{\gamma}\Theta(\frac{1}{2}\xi\phi + \frac{z-1}{d_e}) \quad (20)$$

$$= -\frac{V_d}{8\pi G_N}(\frac{2\log\varepsilon^{-2\theta_e}}{\varepsilon^{d_e}} - \frac{\log r_m^{-2\theta_e}}{r_m^{d_e}} + \frac{2(z-\theta_e-1)}{d_e}(\frac{2}{d_e} - \frac{1}{r_m^{d_e}}))$$

Now, by adding Eq. (19)and Eq. (20) together, one may find

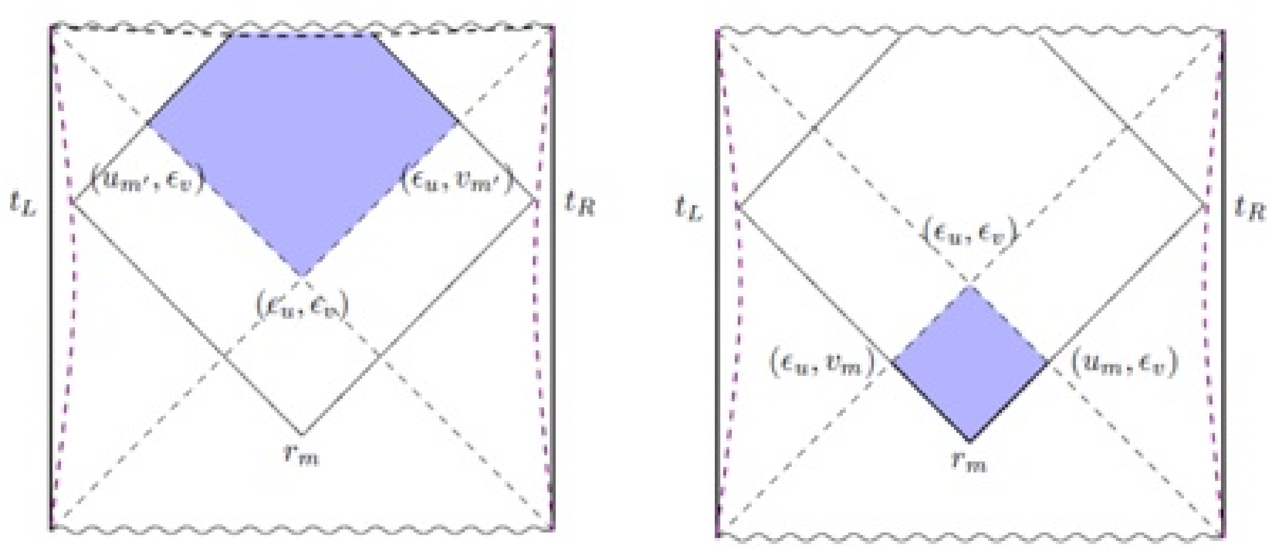

Figure 2. Intersection of WDW patch with the future (left panel) and past (right panel) interior of black brane.

$$A_{WDW} = \tilde{A}_{WDW} + A^{ct} = \frac{V_d}{8\pi G_N}\Big(\frac{d_e+z-2}{2r_h^{d_e+z}}(\tau+\tau_c) - \frac{\log|f(r_m)|}{r_m^{d_e}}\Big) \tag{21}$$

On the other hand, the growth rate of the complexity is given by

$$\frac{dC_{WDW}}{d\tau} = \frac{1}{\pi}\frac{dA_{WDW}}{d\tau} = \frac{2M}{\pi}\Big(\frac{d_e+z-1}{d_e} + \frac{1}{2}\tilde{f}(r_m)\log|f(r_m)|\Big) \tag{22}$$

in which

$$\tilde{f}(r) = \Big(\frac{r_h}{r}\Big)^{d_e+z} - 1, M = \frac{V_d}{16\pi G_N}\frac{d_e}{r_h^{d_e+z}} \tag{23}$$

M stands for the mass of a black hole, which is proportional to the energy of the black brane:

$$M = \frac{d_e}{d_e+z-1}E \tag{24}$$

According to the computation provided for WDW patch and implementing the whole calculation for future interior of a black hole (FI), shown in left panel of Fig.2, the total action for this subregion is given by

$$A_{FI} = \frac{V_d}{8\pi G_N}\Big(\frac{(d_e+z-2)}{2r_h^{d_e+z}}(\tau+\tau_c) + \frac{\log|u_{\acute{m}}v_{\acute{m}}| + c_0}{r_h^{d_e}}\Big)$$

$$= -\frac{V_d}{8\pi G_N}\Big(\frac{(d_e+z-1)\tau}{r_h^{d_e+z}} + \frac{(d_e+z-2)\tau_c}{2r_h^{d_e+z}} + \frac{c_0}{r_h^{d_e}}\Big) \tag{25}$$

Then, the rate of complexification is

$$\frac{dC_{FI}}{d\tau} = \frac{1}{\pi}\frac{dA_{FI}}{d\tau} = \frac{2M}{\pi}\frac{d_e+z-1}{d_e} = \frac{2E}{\pi} \tag{26}$$

Using the calculation, for the action related to past interior of a black hole, (right panel of Fig.2, one can see

$$A_{PI} = \frac{V_d}{8\pi G_N}\Big(\frac{\log|u_m v_m| + c_0}{r_h^{d_e}} - \frac{\log|f(r_m)|}{r_m^{d_e}}\Big)$$

$$= \frac{V_d}{8\pi G_N}\Big(\frac{c_0}{r_h^{d_e}} - \frac{(d_e+z)}{2r_h^{d_e+z}}\tau - \frac{\log|f(r_m)|}{r_m^{d_e}}\Big) \tag{27}$$

And the growth rate of complexity is as follows

$$\frac{dC_{PI}}{d\tau} = \frac{1}{\pi}\frac{dA_{PI}}{d\tau} = \frac{M}{\pi}\tilde{f}(r_m)\log|f(r_m)| \tag{28}$$

As a result, one can see the growth rate of complexity of whole WDW patch, Eq. (22), is equal to the sum of growth rate of future interior Eq. (26) and past interior Eq. (28). So, we have

$$\frac{dC_{WDW}}{d\tau} = \frac{dCFI}{d\tau} + \frac{dC_{PI}}{d\tau}$$

$$= \frac{2M}{\pi}\Big(\frac{d_e+z-1}{d_e} + \frac{1}{2}\tilde{f}(r_m)\log|f(r_m)|\Big) \tag{29}$$

On the other hand, adding the equations (25) and (27) together, one reaches the equation of action of (21), and in fact this is an important result. So,

$$A_{FI} + A_{PI} = \frac{V_d}{8\pi G_N}\Big[\Big(\frac{(d_e+z-2)\tau}{2r_h^{d_e+z}} - \frac{\log|f(r_m)|}{r_m^{d_e}}\Big)$$

$$+\Big(\frac{(d_e+z-2)\tau_c}{2r_h^{d_e+z}} + \frac{2c_0}{r_h^{d_e}}\Big)\Big] \tag{30}$$

where τ_c is critical time and, as it is obvious the second term in the above equation, is time-independent and does not have any contribution in the growth rate of complexity. In fact, this result shows that the action terms which belong to the region outside of the horizon of the black hole identified by yellow regions in Fig.1, are time-independent. Therefore, they have no contributions in computing the complexification.

Another outcome is that according to the growth rate of WDW patch, one can see that the Lloyd's bound is violated and this comes from the contribution of the joint points located in the past interior of the black brain. Therefore, one can say joint points have a necessary role in computing and understanding complexity.

Moreover, although Lloyd's bound was violated in hyperscaling violating geometry, the black hole is still Schwarzschild black brain. However, it should be mentioned at late time $\tau \to \tau_c$, the rate of complexity approaches to $2M$.

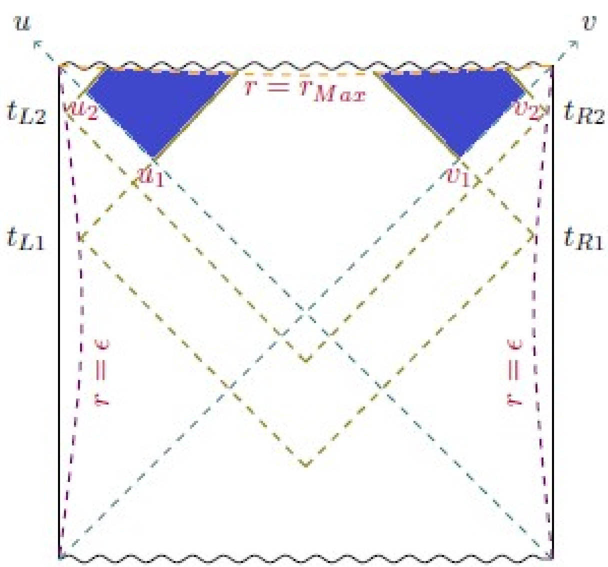

Figure 3. space-time region corresponds to holographic uncomplexity

4. Uncomplexity

In essence, the interest in studying complexity theory, in different fields of physics like computer science, quantum computer, quantum circles, quantum information and etc. began with interesting questions. For example, how can we explain the interior growth of a black hole based on holography principle? Or how much information or useful work can one do after a system reaches its maximum complexity? In other words, if a system reaches its maximum complexity, does it become unusable and out of reach or does it still have some computation power?

In this section, we are going to study the newest quantity in mathematical physics known as uncomplexity. In fact, quantum complexity is a new unknown mathematical subject to most physicists [18]. It is a difficult issue with few quantitative results and, at least for the moment, no experimental guidance. As a matter of fact, Lloyd proved that every computer had main computational power and the rate of the computation in any computer was bounded by the energy of that system [36]. By simulating a black hole, as a physical system, one can explore and investigate this ambiguous concept from black hole physics point of view. Also, we compute and discuss uncomplexity as a resource in hyperscaling violating backgrounds. In fact, the relation between black hole physics and complexity provides a new way to consider uncomplexity as a "space-time" resource based on classical general relativity (GR) [18]. In particular, classical general relativity has introduced a new method about the rejuvenating power of a clean qubit. To understand uncomplexity as a resource from GR perspective, let's assume Alice is a black hole researcher situated just outside the AdS one-sided black hole at boundary time t. She decides to jump from AdS boundary into the black hole. The volume of the space-time is the resource that Alice cares about-without which she will disappear at the horizon [18,19]. We recall that the quantum state of the black hole interior for $t > 0$ has a growing complexity which is dual to the growing space-time volume behind the horizon. As mentioned before, complexity is given by WDW patch. The part of the WDW patch outside the horizon has a time-independent divergence, which can be regulated by considering only the portion of the space behind the horizon. However, we do not care about this part because in general, it has no role in finding complexity. As it is shown in Fig.3, uncomplexity is proportional to the volume of the triangular wedges which stops increasing in $t_{max} : e^s$. This volume is finite and in the limit $t \to e^s$ approaches zero [18].

We can deduce an interesting point from Fig.3. The blue region may be identified with the union of all interior locations behind the horizon where Alice can visit if she jumps into the black hole at any time after t. So, we can define that uncomplexity denotes the space-time source which is available to an observer who intends to pass the horizon. More clearly, consider that Alice attempts to jump into the black hole which has reached maximum complexity in advance but surprisingly at the horizon she faces an obstacle. This is exactly the same as trying to work with a computer which has reached its maximum complexity and it is not able to work anymore or it could not extract more information. In this case, the very interesting and important question that arises is when the resource is ended, can Alice do something to rejuvenate the resource? All that Alice can do is to throw in one thermal photon and wait for a scrambling time. This action helps that the horizon gets clear for an additional exponential time. In computer science, we can do the same action by adding one new clean qubit to restore the computational power of a maximally complex system. So, one can say the obstacle, created at the horizon, is due to the maximum complexity and injecting one clean qubit to a quantum computer i.e. black hole will disappear it for a while. Surely, in this situation, the qubit is a thermal quanta

i.e. photon. Then, one can say, uncomplexity for a black hole is related to the information of its interior growth and is accessible for the observer who attempts to jump in [19].
In thermodynamics and statistical mechanics, there is an important question: Can these fields of physics be useful for analyzing the growth rate and evolution of complexity in general quantum systems? Basically, from thermodynamics perspective, complexity behaves like entropy in statistical systems. So, the free parameter in free energy equation is uncomplexity. Free energy in fact is a resource which represents the amount of the energy with which one can do some useful work.
Now, we compute uncomplexity in hyperscaling violating backgrounds by using complexity equals action proposal. According to the definition provided for uncomplexity, it is shown that this quantity is proportional to subregions duality [17]. To make the concept clearer, consider one-sided black hole which develop from empty AdS space. At $t=0$, some matter in form of spherical shell is injected from the boundary, then a black hole is created. After that, the interior growth starts to increase. This growth is proportional to the increase of complexity. Equivalently, one can say that the growth is fueled by the computational power which in fact is the uncomplexity of the state. At a very late time cutoff t, when there is no more uncomplexity to exploit, the growth will stop. At any time t, the leftover uncomplexity corresponds to the potential of the growth of the space-time, i.e. the leftover interior space-time, in which one can safely enter into the black hole, decreases linearly with time. We can quantify the space-time by different quantities which all of them decrease linearly with time, i.e. action [8,9], volume of maximal surface [21,41], space-time volume, and so on.
The aim of this paper is to study uncomplexity by considering a time slice and the action corresponds to WDW patch in a wider family of states supporting both anisotropic and also hyperscaling violating exponents, means Lifshitz geometry. As pointed out, the gap between possible maximum complexity and a state complexity is called uncomplexity [42]. In other words, uncomplexity is a space to increase complexity and a state complexity is called uncomplexity. According to Fig.3, the on-shell action of the blue triangular wedges is given by the on-shell action which belongs to future interior of the black hole

$$A_{UC}=A_{FI2}-A_{FI1}=\frac{V_d}{8\pi G_N}\frac{(d_e+z-1)}{r_h^{d_e+z}}(\tau_2-\tau_1)$$

$$=2M\frac{(d_e+z-1)}{d_e}(\tau_2-\tau_1)=2E(\tau_2-\tau_1) \tag{31}$$

where τ is the real boundary time. It is worth noting that τ_2 should be thought of a time cutoff. Moreover, one can investigate the $\tau_2\to\infty$ limit for some fixed time . On the other side, one can set the time cutoff in terms of the entropy of the system, means $\tau_2 : \frac{e^{S_{th}}}{2\pi T}$ in which the system reaches the maximum complexity [17].

As mentioned above, the uncomplexity is the difference between maximum complexity and a sate complexity at a given time. So, it is obvious that Eq. (31) cannot satisfy the definition of uncomplexity, simply it cannot fill the gap. An important point is that complexity has two elements, one comes from the boundary term and the other from joint points. So, it is clear the computed term for uncomplexity, Eq. (31), does not include the joint points contributions completely. It has been proved that the joint points play a crucial role in studying holographic complexity. Therefore, the contribution of joint points should be considered in computing uncomplexity. Now, by using Eq. (21), one reaches

$$\Delta A_{UC}^{WDW}=A_2^{WDW}-A_1^{WDW}$$

$$=\frac{V_d}{8\pi G_N}(\frac{d_e+z-2}{2r_h^{d_e+z}}(\tau_2-\tau_1)-\frac{\log|f(r_{m2})|}{r_{m2}^{d_e}}+\frac{\log|f(r_{m1})|}{r_{m1}^{d_e}}) \tag{32}$$

It is evident that in Eq. (32)a joint point term has appeared which cannot be seen in Fig.3. Therefore, this equation is not equal to A_{UC}. It is important to note that when both r_{m1} and r_{m2} approach the horizon r_h then A_{UC} approaches zero. By considering τ_2 as a time cutoff which is very large $r_{m2}\to r_h$, the Eq. (32) leads to

$$\Delta A_{UC}^{WDW}=A_2^{WDW}-A_1^{WDW}$$

$$=\frac{V_d}{8\pi G_N}\frac{d_e+z-1}{r_h^{d_e+z}}(\tau_2-\tau_1)$$

$$-\frac{V_d}{8\pi G_N}(\frac{c_0}{2r_h^{d_e+z}}-\frac{d_e+z}{2r_h^{d_e+z}}\tau_1-\frac{\log|f(r_{m1})|}{r_{m1}^{d_e}})$$

$$\approx 2E(\tau_2-\tau_1)-\frac{V_d}{8\pi G_N}(\frac{c_0}{2r_h^{d_e+z}}-\frac{(d_e+z)}{2r_h^{d_e+z}}\tau_1-\frac{\log|f(r_{m1})|}{r_{m1}^{d_e}}) \tag{33}$$

The second term of this expression is related to the past interior of the black hole, i.e. the action of Eq. (27).

5. Conclusion

In this paper, we have used ”complexity equals action” proposal, and studied the holographic complexity on certain subregions enclosed by null boundaries, including the WDW patch for geometries with hyperscaling violating factor. We have generalized the results of Ref. [17] for future and past interior of the black hole, to hyperscaling violating metric.
Although the complexity and the rate of complexification

depend on the dynamical exponent z and the hyperscaling violation component θ, qualitatively, the rate of complexity growth behaves the same as that of Schwarzschild black hole. Also, we have found that, for this geometries, the value of E which appears on the Lloyd's inequality is always greater than (or equal to) the mass of the black brane. Thus, these kinds of geometries do not respect Lloyd's bound.

One of the important results which plays a critical role in computing complexity and even uncomplexity is the contribution of the joint point action. Our result is fully affected by this term and no one can ignore its role. One can say the violation of Lloyd's bound comes from the joint point part. In hyperscaling violating background, we have used an action term to remove the ambiguity but during the calculation, we have found out that the computed action for WDW patch and its subregions are still UV divergent. To remove this divergence, one needs to add another term. In fact, this divergence might come from Lifshitz geometry. Then, we have used the main counter term action and computed this action in hyperscaling violating metric to get rid of infinity and make our computation finite.

Moreover, by computing uncomplexity it has been shown when a black hole is formed, the resource, uncomplexity is the total space-time volume which is accessible to an observer who decides to pass over the horizon. In other words, uncomplexity should be the amount of how complicated the remaining operations can be while limiting one to acting just on one system. It is also interesting to investigate the concept of uncomplexity from the thermodynamics perspective. In thermodynamics, the free energy $F = E - TS$ actually is a resource that represents the amount of the energy by which one can do useful work. The entropy of axillary system and average complexity are equal so:

$$F_a = E_a - T_a S_a = E_a - T_a \Delta C$$

Since energy and temperature, E_a and T_a respectively, are constants due to the constant number of particles, so the equivalent of the free parameter in the above equation is uncomplexity. Finally, one can say that uncomplexity is a resource and indicates computational power of a system which can extract some useful information from the system.

Conflict of interest statement:
The authors declare no competing interests.

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