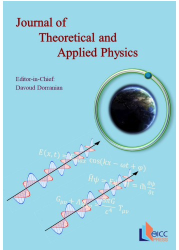

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Holographic entanglement entropy in the fourth-order scale-invariant gravity

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Abstract:
In this paper, we use the holographic method to compute the entanglement entropy for different regions in the fourth-order scale-invariant theory of gravity. In four dimensions, the action of scale-invariant gravity contains a parameter where the moduli space has some distinguished points, precisely, the space of solutions contains the Log-gravity and for this specific solution, we compute the entanglement entropy. We also use the numerical method to investigate mutual information and tripartite information. Moreover, we make a comment on the sign of these quantities.

Keywords: Scale invariance; Conformal gravity; Log-gravity

1. Introduction

In principle, scale-invariance is an important feature of a theory implies that if the variables, such as energy, length or other scales are multiplied by a certain factor, then the content of the theory does not change, and in this sense, it may present a universality of the theory. The underlying transformation of scale-invariance is identified by the dilatation, which dilatations can be considered as a subclass of a larger family of transformations named by conformal symmetry. The scale-invariant theories have many applications in most ares of physics: For example, the theory of phase transitions in statistical mechanics; the strength of particle interactions in elementary particle physics are described by the scale-invariant theory. Moreover, it is argued that a scale-invariant function shall be used to describe the power spectrum of the spatial distribution of the cosmic microwave background. Additionally, this theory has its own application in gravity theory in describing the issue of dark matter [1–3]. Einstein's theory of gravity is known as a standard theory of gravity, however, at very large and small distances. This theory needs to be modified in order to be compatible with observation. The issues of dark matter and dark energy and also the difficulties due to quantum gravity are some problems that should be addressed by a modified gravity theory. In this way, large-scale observation forces us to eliminate any fixed scale of space-time, which means a scale-invariant theory of gravity might be substitute with a modified version of gravity. Even though there is no satisfactory re-normalizable theory of gravity, the scale and conformal techniques have been widely used in general relativity.

In our previous paper [4, 5], we investigated the scale-invariant gravity theory in four dimensions, and showed that the theory has a critical point, in which, the theory admits several non-trivial solutions such as a logarithmic solution. In this paper, in order to investigate the other features of the theory, we use the holographic methods which based on Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence, and consider some non-local probes of entanglement. More precisely, we compute the holographic entanglement entropy (HEE) and mutual information for the logarithmic solution in the scale-invariant theory in four dimensions. First, let us review the theory.

In four dimensions the action of the scale-invariant gravity is given by

$$\mathscr{I} = -\frac{\kappa}{32\pi}\int \sqrt{-g}\, d^4x \times$$

$$\left[(\sigma_0-6)R^2_{\mu\nu\rho\sigma} - 2(\sigma_0-12)R^2_{\mu\nu} + (\frac{\sigma_0}{3}-5)R^2\right] \qquad (1)$$

in this notation, one receives conformal gravity at point $\sigma_0 \to \infty$ on the other hand, at $\sigma_0 = 0$ the R^2 gravity is obtained. We also argued there is a critical point at $\sigma_0 = 6$ where the theory reduces to the log gravity . The corresponding equations of motion are given by

$$\left(\nabla^{\sigma}\nabla^{\rho}-\frac{1}{2}R^{\sigma\rho}\right)C_{\mu\sigma\nu\rho}=\frac{1}{2\sigma_0}\times$$
$$\left(RR_{\mu\nu}-g_{\mu\nu}\frac{R^2}{4}-\nabla_{\mu}\nabla_{\nu}R+g_{\mu\nu}\Box R\right) \tag{2}$$

where $\Box=\nabla^{\mu}\nabla_{\mu}$. The model has some black hole solutions [6]

$$ds^2=\frac{L^2}{r^2}\left(-F(r)\,dt^2+\frac{dr^2}{F(r)}+d\Sigma_{2,k}^2\right),$$
$$F(r)=\lambda+kr^2+c_3r^3 \tag{3}$$

where L stands for the radius of curvature and $k=1,-1$ and 0 corresponds to $\Sigma_{2,k}=S^2,H_2$ and R^2, respectively. It is worth mentioning that

$$R_{\mu\nu}=\frac{3\lambda}{L^2}g_{\mu\nu},\quad \lambda=\pm1,0$$

One can find that an AdS wave solution as follows (see also [7,8])

$$ds^2=\frac{L^2}{r^2}\left(dr^2+dy^2-2dx_-dx_++k(x_+,r)\,dx_+^2\right)$$
$$k(x_+,r)=c_0(x_+)+c_3(x_+)r^3 \tag{4}$$

In order to explore the space of solutions, the AdS wave Ansatz can be utilized as follows

$$ds^2=\frac{L^2}{r^2}\left(dr^2+dy^2-2dx_-dx_++k(x_+,r)\,dx_+^2\right) \tag{5}$$

where this Ansatz together with Eq. (2) leads one to write

$$k(x_+,r)=c_0(x_+)+c_3(x_+)r^3+$$
$$b_1(x_+)r^{\frac{3}{2}-\frac{1}{2}\sqrt{\frac{\sigma_0+48}{\sigma_0}}}+b_2(x_+)r^{\frac{3}{2}+\frac{1}{2}\sqrt{\frac{\sigma_0+48}{\sigma_0}}} \tag{6}$$

Noting that the case $\sigma_0=6$, a logarithmic solution is obtained as follows

$$k(x_+,r)=c_0(x_+)+b_0(x_+)\log r+\left(c_3(x_+)+b_3(x_+)\log r\right)r^3 \tag{7}$$

By making use of the Ansatz

$$ds^2=dr^2+dy^2-2dx_-dx_++k(x_+,r)\,dx_+^2, \tag{8}$$

the equations of motion reduces to

$$\sigma_0\frac{\partial^4k}{\partial^4r}=0, \tag{9}$$

and for $\sigma_0\neq0$ the above equation has a non-trivial solution given by

$$k(x_+,r)=c_0(x_+)+c_1(x_+)r+c_2(x_+)r^2+c_3(x_+)r^3. \tag{10}$$

Moreover, the theory admits the Lifshitz solution which is given by

$$ds^2=\frac{L^2}{r^2}\left(-\frac{dt^2}{r^{2z}}+dr^2+dx_1^2+dx_2^2\right),$$

$$z=\frac{\sigma_0-6+\sqrt{(\sigma_0-6)(4\sigma_0+3)}}{\sigma_0+3} \tag{11}$$

Therefore, at different points of moduli space, the model has some non-trivial solutions. It is worth noting that for $\sigma_0=6$ the action Eq.1 reads

$$\mathscr{I}=-\frac{3\kappa}{32\pi}\int d^4x\sqrt{-g}\left[4R_{\mu\nu}^2-R^2\right]. \tag{12}$$

We note, however, that although the above action has degenerate equations of motion giving rise to a logarithmic solution, modifying the IR limit by adding a linear scalar curvature term that would remove this solution. It is worth to mention that at $\sigma_0=0$, one obtains the R^2 gravity. In order to make the finite action, one might use the Gauss-Bonnet term, where the corresponding equations of motion become

$$\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)R+\left(g_{\mu\nu}\Box-\nabla_{\mu}\nabla_{\nu}\right)R=0 \tag{13}$$

One can show that all the Einstein solutions, previously mentioned, are the solutions to the above equations, as well. However, $R=0$ while $R_{\mu\nu}\neq0$ can be considered as a new class of solutions. Namely, one can say that the theory has a larger class of the solutions, such as the Ricci scalar flat which is given by

$$ds^2=\frac{L^2}{r^2}\left(-F(r)\,dt^2+\frac{dr^2}{F(r)}+d\Omega_{2,k}^2\right),$$
$$F(r)=kr^2+c_3r^3+c_4r^4, \tag{14}$$

where c_3 and c_4 are two constant, noting that the Ricci scalar becomes zero whereas, due to the $c_4\neq0$, the Ricci tensor is non-zero. This can be stated as follows

$$R_{\mu\nu}=-\frac{c_4r^4}{2L^2}\left(\eta_{\mu}^{\rho}g_{\rho\nu}+\eta_{\nu}^{\rho}g_{\rho\mu}\right),$$
$$\text{with }\ \eta_{\mu}^{\nu}=\mathrm{diag}(-1,-1,1,1). \tag{15}$$

It is important to mention that for $k=1,c_4=c^2$ and $c_3=-2c$ where $F(r)=r^2(1-cr)^2$ the solution reduces to an extremal black hole solution and the near horizon geometry is given by $\mathrm{AdS}_2\times S^2$. By making use of $r=\frac{1}{\xi}$ one obtains

$$ds^2=L^2\left[-\left(1-\frac{c}{\xi}\right)^2dt^2+\frac{d\xi^2}{\left(1-\frac{c}{\xi}\right)^2}+\xi^2d\Omega_2^2\right], \tag{16}$$

with the following geometry

$$ds^2=L^2c^2\left(-\rho^2dt^2+\frac{d\rho^2}{\rho^2}+d\Omega_2^2\right), \tag{17}$$

where we have used $\xi-c=c^2\rho$.
By setting $k=0$ one can obtain another class of solutions. Let us set $c_4=1$ and suppose c_3 is a free parameter. Therefore by setting $r=\frac{1}{\rho}$ the corresponding metric is given by

$$ds^2=L^2\rho^2\left(-(1+c_3\rho)\frac{dt^2}{\rho^4}+\frac{d\rho^2}{1+c_3\rho}+dx_1^2+dx_2^2\right). \tag{18}$$

The above solution can be considered as the hyperscaling violating metric with $\theta = 4, z = 3$, noting that a $d+2$ dimensional hyperscaling violating geometry is

$$ds^2 = \frac{L^2}{r^2} r^{2\frac{\theta}{d}} \left(-f(r) \frac{dt^2}{r^{2(z-1)}} + \frac{dr^2}{r} + d\vec{x}_d^2 \right),$$

$$\text{with } f(r) = 1 - m\, r^{d-\theta+z}. \tag{19}$$

The rest of the paper is organized as follows. In the next section, we use holographic methods and study the entanglement entropy for three different entangling regions: Strip, circle and cusp. We also will explore different features of the model, especially at the critical point, we compute the mutual information and also the tripartite information. We use numerical analysis, to explore the sign of these quantities, which become important the monogamy of mutual information.

2. Holographic entanglement entropy

In this section in order to further explore the scale-invariant theory, let us use the holographic method and study the HEE in this theory. As mentioned in the previous section the theory has critical points leading to some non-trivial solutions. It is worth to mention that due to higher order terms, to compute the HEE one should use the generalized Ryu-Takayanagi prescription, [9–11]. First let us review the proposal briefly. For the following action

$$\mathscr{I} = -\frac{\kappa}{32\pi} \int \sqrt{-g}\, d^4x \left(aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right), \tag{20}$$

thus it is argued that in order to obtain the HEE one should minimize the following functional

$$S = \frac{\kappa}{8} \int \sqrt{\gamma} d^2\zeta \left[2aR + b\left(R_{\mu\nu} n_i^\mu n_i^\nu - \frac{1}{2}\mathscr{K}^i \mathscr{K}_i \right) + 2c\left(R_{\mu\nu\rho\sigma} n_i^\mu n_j^\nu n_i^\rho n_j^\sigma - \mathscr{K}^i_{\mu\nu}\mathscr{K}_i^{\mu\nu} \right) \right], \tag{21}$$

where $i = 1, 2$ refers to the two transverse directions to a co-dimension two hypersurfaces in the bulk, n_i^μ are two mutually orthogonal unit vectors to the hypersurface. $\mathscr{K}^{(i)}$ are the traces of two extrinsic curvature tensors defined by

$$\mathscr{K}^{(i)}_{\mu\nu} = \pi^\sigma_{\ \mu} \pi^\rho_{\ \nu} \nabla_\rho (n_i)_\sigma, \quad \text{with} \quad \pi^\sigma_{\ \mu} = \varepsilon^\sigma_{\ \mu} + \xi \sum_{i=1,2} (n_i)^\sigma (n_i)_\mu\,. \tag{22}$$

In the above relation, γ is the induced metric on the hypersurface whose coordinates are denoted by ζ, and, $\xi = 1$ and $\xi = -1$ are used for time-like and space-like vectors, receptively. Now we use this method and study the HEE for three different entangling regions: circle, strip and cone.

2.1 HEE of a strip

In order to find HEE for a strip entangling region which is given by ($t =$ constant,)

$$-\frac{\ell}{2} \le y \le \frac{\ell}{2}, \qquad 0 \le x \le H, \tag{23}$$

it is useful to parametrize the AdS metric as follows

$$ds^2 = \frac{L^2}{r^2}(-dt^2 + dr^2 + dx^2 + dy^2). \tag{24}$$

The co-dimension two hypersurface in the bulk can be parametrized by setting $y = f(r)$. After doing some algebra, one obtains

$$S = \kappa H \left[(\sigma_0 - 6)\left(\frac{1}{\varepsilon} - \frac{2\pi\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} \frac{1}{\ell} \right) + (\sigma_0 - 6)\left(-\frac{1}{\varepsilon} \right) \right]. \tag{25}$$

The above equation can be divided in two terms: The first term comes form the dynamical part of the action, whereas, second term is due to the Gauss-Bonnet term. The second one plays the role of regulator in the theory. Now let us compare the HEE with that of Einstein gravity which is

$$S_{\text{Ein}} = \frac{L^2 H}{2G} \left(\frac{1}{\varepsilon} - \frac{2\pi\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} \frac{1}{\ell} \right), \tag{26}$$

which by setting $\kappa = \frac{L^2}{2(\sigma_0 - 6)G}$, the dynamical part is reproduced.

2.2 HEE of a circle

To compute entanglement entropy for a circle it is useful to parametrize the metric as follows

$$ds^2 = \frac{L^2}{r^2}(-dt^2 + dr^2 + d\rho^2 + \rho^2 d\phi), \tag{27}$$

and the entangling region is given by

$$t = \text{contant}, \qquad \rho = H = \text{constant}, \tag{28}$$

by which, using the symmetry of the system, the corresponding co-dimension two hypersurface in the bulk may be parametrized by $\rho = f(r)$. Therefore two mutually orthogonal unit vectors to the hypersurface are given by

$$n_1 = \frac{L}{r}\Big(1, 0, 0, 0\Big),$$

$$n_2 = \frac{L}{r\sqrt{1 + f'(r)^2 + 1}}\Big(0, -f'(r), 1, 0\Big). \tag{29}$$

Using these vectors and going through the minimization procedure one gets following expression for the holographic entanglement entropy (see [12] for five dimensional case)

$$S = \frac{\pi\kappa H}{2}(3(4\lambda_1 + \lambda_2) + 2\lambda_3) \times \int_\varepsilon^H dr \frac{1}{r^2} = \frac{\pi\kappa}{2}(3(4\lambda_1 + \lambda_2) + 2\lambda_3)\left(\frac{H}{\varepsilon} - 1 \right). \tag{30}$$

Now going back to the scale-invariant case where $\lambda_1 = \frac{\sigma_0}{3} - 5, \lambda_2 = -2(\sigma_0 - 12)$ and $\lambda_3 = \sigma_0 - 6$ one finds that the corresponding entanglement entropy is identically zero. Indeed this may be understood as follows. Let us write the action as follows

$$\mathscr{I} = -\frac{\kappa}{32\pi} \int d^4x \sqrt{-g} \times$$

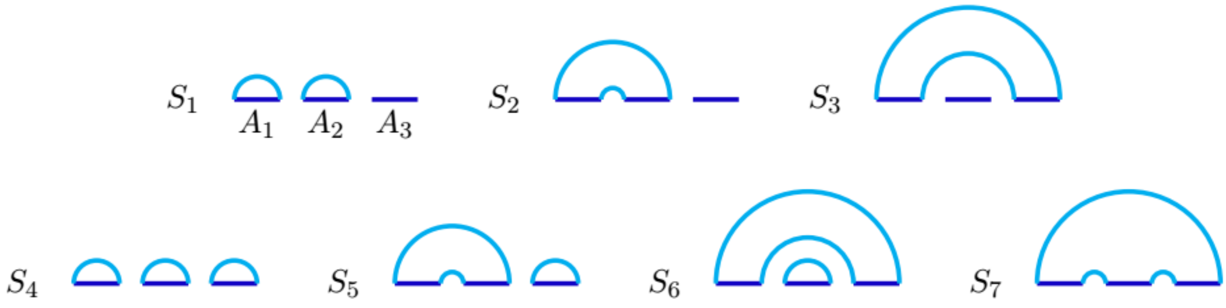


Figure 1. For three subsystem, in reading the holographic entanglement entropy of the union part, one must use the minimal configuration. In this figure, we have plotted the configurations of the two and three regions.

$$\left(2\sigma_0(R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2)+R^2+(\sigma-6)\mathrm{GB}_4\right). \quad (31)$$

Then one can find the contribution of each terms separately. The result is as follows

$$S=-\frac{\pi\kappa}{4}\left[4\sigma_0\left(-\frac{H}{\varepsilon}+1\right)-24\left(-\frac{H}{\varepsilon}+1\right)+4(\sigma_0-6)\left(\frac{H}{\varepsilon}-1\right)\right]=-\pi\kappa\times\left[(\sigma_0-6)\left(-\frac{H}{\varepsilon}+1\right)+(\sigma_0-6)\left(\frac{H}{\varepsilon}-1\right)\right]=0. \quad (32)$$

Therefore it is zero due to the contribution of the Gauss-Bonnet term which plays the role of regulator. Indeed the situation is the same as that in four-dimensional conformal gravity [13]. Actually since the Gauss-Bonnet term does not contribute to the equations of motion the whole dynamics must be encoded in the two first terms in the above action. Therefore if we would like to find the contributions of the dynamical degrees of freedom to the entanglement entropy, we should only consider the dynamical parts of the action in which the corresponding entanglement entropy is found

$$S=\pi\kappa(\sigma_0-6)\left(\frac{H}{\varepsilon}-1\right), \quad (33)$$

which clearly has UV divergent term.
It is also illustrative to compare the contribution of dynamical part to that obtained from just Einstein gravity. Of course for Einstein gravity the holographic entanglement entropy is given by minimizing the area of a co-dimension two hypersurface. The corresponding entanglement entropy is [14, 15]

$$S_{\mathrm{Ein}}=\frac{\pi L^2}{2G}\left(\frac{H}{\varepsilon}-1\right). \quad (34)$$

which is the same as Eq. 33 if one identifies $\kappa=\frac{L^2}{2(\sigma_0-6)G}$. It is obvious that this identification is valid as long as $\sigma_0\neq 6$.

2.3 HEE of a cusp

Finally, it is also straightforward to compute the universal term of the holographic entanglement entropy for the case where the entangling region has a cusp. To proceed we will parametrize the AdS metric as Eq. 27. In this case, the entangling region is given by $t=$ constant, $-\Omega\leq\phi\leq\Omega$. Indeed such computation has already been done for general quadratic action in [16] where it was shown that the Gauss-Bonnet term does not contribute to the universal term and therefore one arrives at

$$S_{\mathrm{univ}}=\kappa(\sigma_0-6)a(\Omega)\log\frac{R}{\varepsilon}, \quad (35)$$

where

$$a(\Omega)=\int_0^\infty dy\left[1-\sqrt{\frac{1+h_0^2(1+y^2)}{2+h_0^2(1+y^2)}}\right], \quad (36)$$

and h_0 can be found from the following equation

$$\Omega=\int_0^{h_0}dh\frac{2h^2\sqrt{1+h_0^2}}{\sqrt{1+h^2}\sqrt{(h_0^2-h^2).(h_0^2+(1+h_0^2)h^2)}} \quad (37)$$

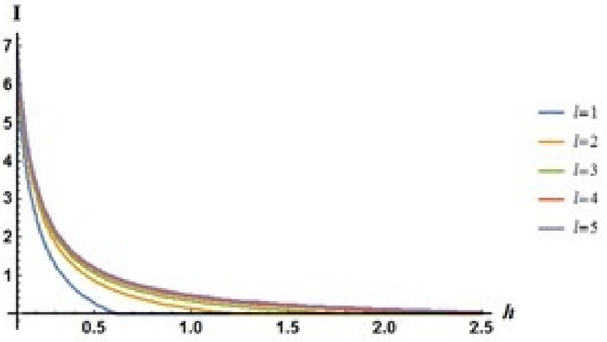
Again the result reduces to that of the Einstein gravity if one identifies $\kappa=\frac{L^2}{2(\sigma_0-6)G}$.
An interesting feature of this new universal term is that in the smooth limit where $\Omega\to\frac{\pi}{2}$ the coefficient of the universal term vanishes, though with certain form. More precisely one has

$$\kappa(\sigma_0-6)a(\Omega)=\frac{\kappa(\sigma_0-6)}{4\pi}\left(\Omega-\frac{\pi}{2}\right)^2\equiv C\left(\Omega-\frac{\pi}{2}\right)^2, \quad \text{at } \Omega\to\frac{\pi}{2}. \quad (38)$$

One can then show that C is related to C_T which is the central charge appearing in the two point function of stress tensor: $C=\frac{\pi^2}{24}C_T$ [16].

3. HEE for the logarithmic solution

The gauge/gravity duality tells us that the log term in the AdS solution might be identified by deformed the dual CFT by an irrelevant operator. This means that such term destroy the conformal symmetry at UV and therefore, applying the AdS/CFT correspondence should be clear. However, following [17], let us assume that the deformation is sufficiently small in a way that this term may be treated perturbatively. Holographic entanglement entropy for logarithmic solution in Log-gravity has been studied in [18]. In what follows we will use the procedure of [18] for the action Eq. (12). To



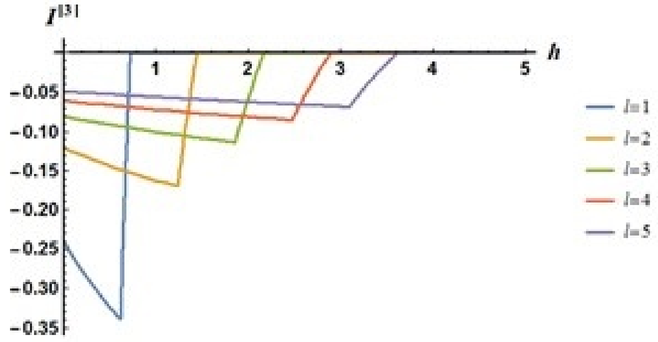


Figure 2. Numerical results for holographic mutual information (left plot) and tripartite information (right plot) as a function of the separation distance: for $\ell = 1, \cdots, 5$. Note in all cases the mutual information is positive while for three regions the tripartite information remains negative.

proceed let us set $c_0 = c_3 = b_3 = 0$ and $b_0 = \beta \neq 0$ in the solution Eq. (7) and therefore one arrives at

$$ds^2 = \frac{L^2}{r^2}\left(\beta \log\left(\frac{r}{L}\right) dx_+^2 - 2dx_- dx_+ + dy^2 + dr^2\right). \quad (39)$$

Although the constant β can be removed by a rescaling, in order to trace the effect of logarithmic term we will keep it in the metric. Obviously setting $\beta = 0$ the solution reduces to an AdS metric. In what follows we will compute holographic entanglement entropy for the above metric at leading order in $\beta \to 0$ limit.

We will consider an entangling region given by $x_+ + x_- = 0$ and $\frac{\ell}{2} \leq y \leq \frac{\ell}{2}$ which is a strip with width ℓ along the y direction. Then the co-dimension two hypersurafce in the bulk may be parametrized by $y = f(r)$. In this case the induced metric reads

$$ds^2 = \frac{L^2}{r^2}\left((1+f'^2)dr^2 + (2+\beta \log\frac{r}{L})dx_+^2\right). \quad (40)$$

On the other hand two unit vectors normal to the co-dimension two hypersurfaces are

$$x_+ + x_- = \text{const.} \qquad n_1 = \frac{L}{r\sqrt{2+\beta\log\frac{r}{L}}}\,(0,1,1,0)\,,$$

$$\rho - f(r) = \text{const.} \qquad n_2 = \frac{L}{r\sqrt{1+f'^2}}\,(-f',0,0,1)\,. \quad (41)$$

It is then straightforward to compute the entropy functional (21) for the case of $\lambda_1 = -3, \lambda_2 = 12$ and $\lambda_3 = 0$ according to the action Eq. (12). Setting $F = 2 + \beta \log\frac{r}{L}$ one finds

$$S_A = -\frac{\kappa}{4}\frac{3H}{\sqrt{2}}\int dr\, \frac{\sqrt{(f'(r)^2+1)F}}{2F^2r^2}\times$$
$$\left(\frac{\left(2rf''(r)F + (\beta - 4F)\left(f'(r)^3 + f'(r)\right)\right)^2}{(f'(r)^2+1)^3} + 12\beta F\right) \quad (42)$$

where H comes from the integration over x_+ and $\sqrt{2}$ is due to our notation of light cone coordinates, $x_\pm = \frac{t\pm x}{\sqrt{2}}$. Now the aim is to minimize the above entropy function and find the corresponding profile of $y = f(r)$ which we will do it for in $\beta = 0$ limit. Indeed at leading order one finds

$$f'(r) = \frac{r^2}{r_t^4 - r^4} + \mathcal{O}(\beta), \quad (43)$$

where r_t is the turning point of the hypersurface. Plugging this expression into the above entropy functional one gets

$$S_A = \frac{9H\kappa\beta}{4}\, r_t^2 \int_\varepsilon^{r_t} dr \frac{1}{r^2\sqrt{r_t^4 - r^4}} + \mathcal{O}(\beta^2)$$
$$= \frac{9H\kappa\beta}{4}\left(\frac{1}{\varepsilon} - \frac{v_0}{r_t}\right) + \mathcal{O}(\beta^2). \quad (44)$$

where $v_0 = \frac{\sqrt{2}\pi^{3/2}}{\Gamma\left(\frac{1}{4}\right)^2}$. On the other hand using the fact that $\ell = 2\int_0^{r_t} dr f' + \mathcal{O}(\beta) = 2r_t v_0 + \mathcal{O}(\beta)$, one arrives at

$$S_A = \frac{9\kappa\beta}{4} H\left(\frac{1}{\varepsilon} - \frac{2\pi\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}\,\frac{1}{\ell}\right) + \mathcal{O}(\beta^2). \quad (45)$$

Interestingly enough if one choose the coupling constant κ properly, the above expression at leading order exactly reduces to that of Einstein gravity, even though the logarithmic solution is not an Einstein solution.

4. Concluding remarks: holographic mutual information

Entanglement entropy as a probe of entanglement has a UV divergence, and due to the UV cut-off, the entanglement entropy can not be considered as an universal quantity. However, for a system that has two disjoint parts, one can define the mutual information. The mutual information quantifies the amount of information which is shared between two systems. This quantity is finite given by

$$I(A_1, A_2) = S(A_1) + S(A_2) - S(A_1 \cup A_2), \quad (46)$$

where for a region A, the $S(A)$ is the entanglement entropy whereas, the entanglement entropy for the union of the two entangling regions is $S(A_1 \cup A_2)$. For three regions, the

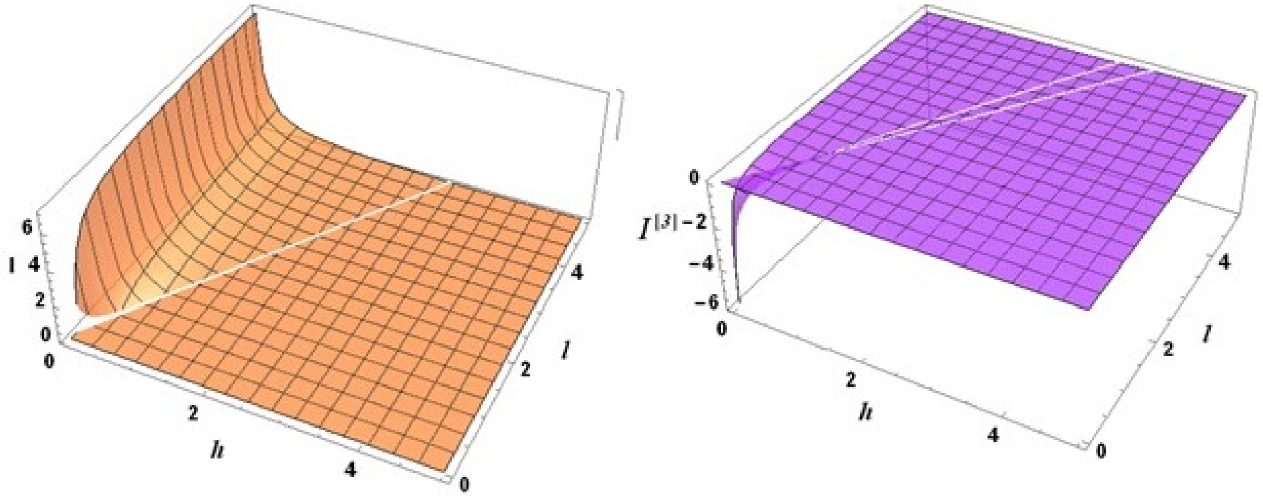


Figure 3. The 3D plot of holographic mutual information and tripartite information as a function of the separation distance and the length of the strips.

corresponding quantity is given by the tripartite information defined as follows

$$I^{[3]}(A_1,A_2,A_3) = S(A_1)+S(A_2)+S(A_3)-S(A_1\cup A_2)-$$
$$S(A_1\cup A_3)-S(A_2\cup A_3)+S(A_1\cup A_2\cup A_3)\,. \quad (47)$$

Based on holographic method, the union parts of the regions play a key role, and in order to investigate this point, it is important to mention that the tripartite information can be rewritten in terms of the mutual information as follows

$$I^{[3]}(A_1,A_2,A_3) = I(A_1,A_2)+I(A_1,A_3)-I(A_1,A_2\cup A_3), \quad (48)$$

and by this modification, the aim is to compute the mutual information.
At the leading order for the logarithmic solution, the holographic entanglement entropy for a strip entanglement region one has

$$S(\ell) = \frac{9\kappa\beta}{4}H\left(\frac{1}{\varepsilon}-\frac{2\pi\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}\,\frac{1}{\ell}\right). \quad (49)$$

In order to compute the mutual information, we should consider two entangling regions; we assume two strips with the same length of ℓ separated by distance h. After doing some calculation, one obtains

$$I = \frac{9\kappa\beta}{4}H\Big(\frac{1}{2\ell+h}+\frac{1}{h}-\frac{2}{\ell}\Big). \quad (50)$$

Similarly, for three strips with the same length ℓ separated by distance h, as mentioned, the key point is to identify the union part of entanglement entropies, noting that the holographic principle forces us to find a minimal configuration in the bulk. For three strips, we have plotted all the union parts in Fig.1. In each case, in computing the $S(A_i\cup A_j)$ and $S(A_1\cup A_2\cup A_3)$ one must find the minimum among the possible diagrams. This needs a numerical computation and the holographic tripartite information is given by

$$I^{[3]}(A,B,C) = 3S(\ell)-\min\{S_1,S_3\}-$$
$$2\min\{S_1,S_2\}+\min\{S_4,S_5,S_6,S_7\}\,, \quad (51)$$

noting that $\min\{S_1,S_2\}$ means that the minimum configuration between S_1 and S_2 must be used. Note that we have used the following simplifications:

$$S(A_i\cup A_j): \begin{cases} 2S(\ell)\equiv S_1 \\ S(2\ell+h)+S(h)\equiv S_2 \\ S(3\ell+2h)+S(\ell+2h)\equiv S_3 \end{cases}$$

and for three entangling regions, one has

$$S(A_1\cup A_2\cup A_3): \begin{cases} 3S(\ell)\equiv S_4 \\ S(3\ell+2h)+S(\ell+2h)+S(\ell)\equiv S_5 \\ S(2\ell+h)+S(\ell)+S(h)\equiv S_6 \\ S(3\ell+2h)+2S(h)\equiv S_7 \end{cases}$$

We have used Mathematica software to compute the numeric analysis of finding the minimal configuration and the results are shown in figures 2 and 3 for mutual information and tripartite information, receptively, for some certain values of ℓ and h. Our numerical analysis shows that the mutual information is always positive, whereas the tripartite information becomes negative, at least this happens for the range of parameters that we have used. Previously. we have observed this feature for some specific gravity theories as well [19–22].

5. Conclusion

In this paper, we have investigated some features of the scale-invariant gravity theory in four dimensions by computing the HEE for the logarithmic solution. Our numerical results indicate that in a such theory, the measure of entanglement for two systems is always positive. On the other hand for three regions the corresponding measure becomes negative. The latter quantity is named tripartite information, and the negativity of tripartite information imposes a condition on the mutual information. This statement can be understood from the relation (48): when $I^{[3]}(A_1,A_2,A_3)$ is negative one receives

$$I(A_1,A_2 \cup A_3) \geq I(A_1,A_2) + I(A_1,A_3).$$

It is worth mentioning that in a generic case in quantum field theory, the above inequality does not hold, and this inequality becomes very important in the context of quantum information theory. This indicates a well-known feature of quantum correlation, namely the monogamy of mutual information. This characteristic of measures of quantum entanglement completely comes from the nature of quantum mechanics. In other words, correlations in the holographic regimes arise from quantum entanglement rather than classical correlations.

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Conflict of interest statement
The authors declare that they have no conflict of interest.

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