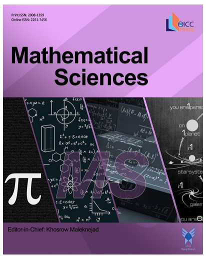

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Research Article

Proportional Hazard Bivariate Chen Model and Its Mathematical Features with Applications on Environmental, Climatic and Medical Data

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Abstract:
This paper introduces a new bivariate model, the Bivariate Chen Proportional Hazard Rate (BCPH) model, designed to effectively model diverse real-life datasets. This model is formulated within the proportional hazard framework, employing Chen marginal distributions. Its main statistical and reliability characteristics such as product moments, Pearson's correlation, survival, and hazard functions are thoroughly derived and analyzed. Parameter estimation is accomplished through the maximum likelihood method and inference functions for margins, with simulation studies confirming the consistency and efficiency of the estimators. The BCPH model is further applied to various environmental, climatic, and medical datasets, including mercury fish length, precipitation temperature, and kidney infection recurrence times. Comparative evaluations using AIC, BIC, CAIC, and HQIC criteria demonstrate that the BCPH distribution provides a superior fit compared to existing bivariate models, highlighting its robustness, flexibility, and suitability for real world applications.

Keywords: Bivariate Chen distribution; Proportional hazard rate model; Simulation; Maximum likelihood estimation; Inference functions for margins

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1. Introduction

The construction of bivariate distributions has long been a crucial area of research in statistics, as these models combine the individual behaviors of two variables (marginals) with their underlying dependence structure. Various techniques have been proposed for developing bivariate models, as exemplified by the work of [1]. In this paper, we employ the proportional hazard rate (PHR) framework to introduce a new bivariate model.

In survival analysis and reliability theory, the proportional hazard rate (PHR) and proportional reversed hazard rate (PRHR) models are frequently employed to study failure time distributions (see [2]). These models provide frameworks for analyzing the effect of covariates on the time until an event occurs, such as the failure of a machine or death in medical studies. The PHR model is widely used in the statistical literature for generating bivariate distributions. In the univariate case, Lehmann (1953) introduced the proportional hazard rate model for hypothesis testing, which later became known as the Lehmann alternatives. [3] developed the proportional reversed hazard rate (PRHR) model, allowing for flexible parametric distributions. Both frameworks have been extended by various researchers to enhance their applicability across diverse types of survival and reliability data. For instance, [4] proposed a bivariate Kumaraswamy-type exponential distribution, while

[5] introduced the bivariate Rayleigh proportional hazard rate model and applied it to COVID-19 data. Moreover, the proportional hazard bivariate Kumaraswamy distribution was discussed by [6]. [7] proposed a bivariate power Lomax Sarmanov distribution and its statistical properties.

The Chen distribution, introduced by Chen (2000), is a flexible probability distribution that serves as an alternative to the Weibull distribution, particularly in modeling lifetime data in reliability analysis and survival studies. It is characterized by two positive shape parameters, $\alpha > 0$ and $\beta > 0$, which govern the distribution's form and enable it to accommodate a wide variety of data patterns. For a random variable X following the Chen distribution with parameters α and β, the probability density function (PDF) is given by:

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1} e^{x^{\beta}} e^{-\alpha(e^{x^{\beta}}-1)}, \quad x > 0, \alpha, \beta > 0. \tag{1}$$

The cumulative distribution function (CDF) is:

$$F(x;\alpha,\beta) = 1 - e^{-\alpha(e^{x^{\beta}}-1)}, \ x > 0, \alpha, \beta > 0. \tag{2}$$

Where, α is a scale parameter and β is a shape parameter.

The Proportional hazard rate (PHR) model is defined as

$$H(x) = 1 - (1 - F(x))^{\lambda}, \qquad x \geq 0, \lambda > 0, \tag{3}$$

where, $F(x)$ represents the baseline cumulative distribution (CDF) and $H(x)$ represents the CDF generated by the PHR model based on dependence parameter λ.

A bivariate PHR model was proposed as follows

$$H(x,y) = 1 - (1 - F_1(x)F_2(y))^{\lambda}, \quad x, y \in \mathbb{R} \text{ and } 0 < \lambda \leq 1, \tag{4}$$

where $F_1(x)$ and $F_2(y)$ are the univariate marginal CDFs.

The joint probability density for PHR model is defined using

$$h(x,y) = \sum_{j=1}^{\infty} \binom{\lambda}{j} (-1)^{j+1} j^2 \times f_1(x)(F_1(x))^{j-1} f_2(y)(F_2(y))^{j-1}, \tag{5}$$

For Bivariate Chen Proportional hazard rate, the Univariate Chen distributions are taken as the marginal distributions. The univariate PHR transformation for the marginals is given by

$$H_i(x) = 1 - (1 - F_i(x))^{\lambda}, \qquad i = 1, 2, \tag{6}$$

where, λ is the dependence parameter. Many useful expansions will be utilized throughout this paper such as

$$\left\{1 - e^{a(1-e^{x^b})}\right\}^{j-1} = \sum_{k=0}^{j-1} \binom{j-1}{k} (-1)^k e^{ka(1-e^{x^b})}, \tag{7}$$

$$e^{tx} = \sum_{m=0}^{\infty} \frac{(tx)^m}{m!}, \tag{8}$$

$$e^{-kae^{x^b}} = \sum_{i=0}^{\infty} \frac{(-1)^i (k)^i a^i e^{ix^b}}{i!}. \tag{9}$$

This paper expands upon several concepts related to the proportional hazard model. The Bivariate Chen Proportional Hazard Rate (BCPH) distribution is derived in Section 2. Section 3 presents its statistical characteristics, including product moments, Pearson's correlation coefficient, the moment generating function, and the r^{th} conditional moments. Reliability characteristics are provided in Section 4. Section 5 discusses two techniques for parameter estimation, while Section 6 introduces a simulation study. Finally, multiple applications of the BCPH model are illustrated in Section 7.

2. Bivariate Chen Proportional Hazard Rate (BCPH) Distribution

The cumulative distribution function (CDF) of the univariate Chen distribution is given by

$$F_i(x) = 1 - e^{\alpha_i(1-e^{x^{\beta_i}})}, \ i = 1, 2, \tag{10}$$

where $\alpha_i > 0$ and $\beta_i > 0$ are shape parameters.

Using Eqs. (4) and (10), the CDF of the Bivariate Chen Proportional Hazard Rate(BCPH) distribution is defined as

$$H(x,y) = 1 - \left\{e^{\alpha_1(1-e^{x^{\beta_1}})} + e^{\alpha_2(1-e^{y^{\beta_2}})} - e^{\alpha_1(1-e^{x^{\beta_1}})+\alpha_2(1-e^{y^{\beta_2}})}\right\}^{\lambda} \tag{11}$$

for $x, y > 0, \alpha_1, \alpha_2, \beta_1, \beta_2, \lambda > 0$. From Eq. (4), the joint probability density function (PDF) of the BCPH model can be expressed as

$$h(x,y) = \alpha_1\alpha_2\beta_1\beta_2 \sum_{j=1}^{\infty} \binom{\lambda}{j} (-1)^{j+1} j^2 x^{\beta_1-1} y^{\beta_2-1} \times e^{x^{\beta_1}} \times e^{y^{\beta_2}} \times e^{\alpha_1(1-e^{x^{\beta_1}})+\alpha_2(1-e^{y^{\beta_2}})} \times \left\{1 - e^{\alpha_1(1-e^{x^{\beta_1}})} - e^{\alpha_2(1-e^{y^{\beta_2}})} + e^{\alpha_1(1-e^{x^{\beta_1}})+\alpha_2(1-e^{y^{\beta_2}})}\right\}^{j-1}. \tag{12}$$

for $x, y > 0, \alpha_1, \alpha_2, \beta_1, \beta_2, \lambda > 0$.

Figure 1 and Figure 2 show the cdf of BCPH model plot and contours with different parameter values.

3. Statistical Characteristics of BCPH Distribution

Product moments, pearson's correlation coefficient, moment generating function, and r^{th} conditional moments for BCPH model are derived as follows:

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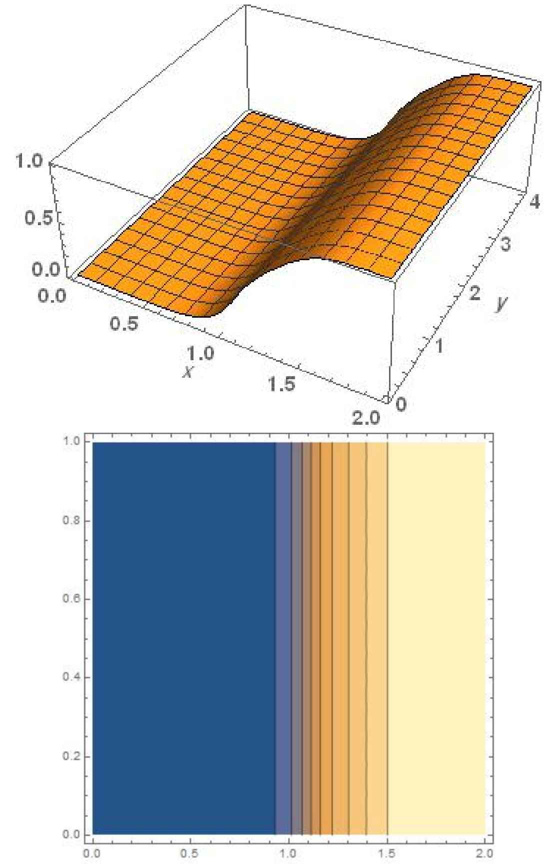

Figure 1. Cumulative Plots and Contours of BCPH model for parameter values $\alpha_1 = 0.3$, $\alpha_2 = 0.4, \beta_1 = 2, \beta_2 = 5, \lambda = 0.9$

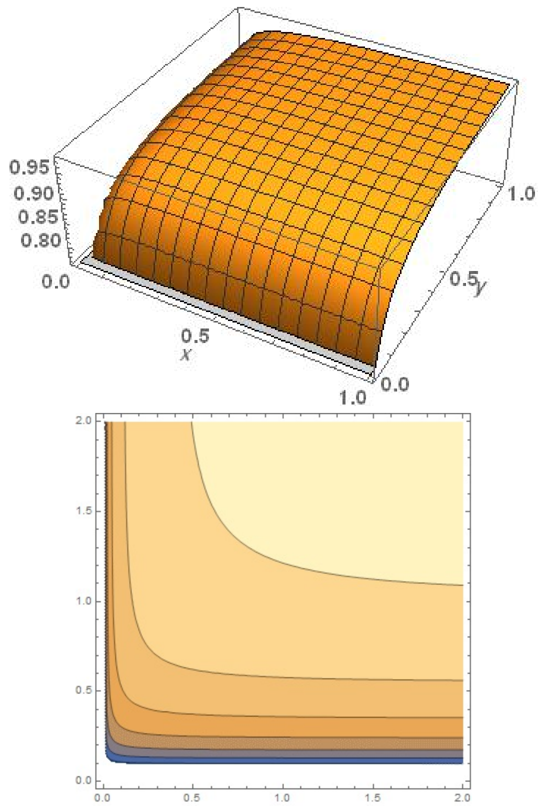

Figure 2. Cumulative Plots and Contours of BCPH model for parameter values $\alpha_1 = 5.2$, $\alpha_2 = 4.5, \beta_1 = 0.1, \beta_2 = 0.2, \lambda = 0.5$

3.1 Product Moments

Product moments are generalizations of moments that play a crucial role in characterizing the relationships between multiple random variables, including measures such as correlation and covariance. The product moment for the BCPH model is given by:

$$E\left(X^{r_1}Y^{r_2}\right) = \int_0^\infty\int_0^\infty x^{r_1}y^{r_2}h(x,y)\,dx\,dy. \tag{13}$$

Theorem 3.1 If $(X,Y) \sim$ BCPH (x,y), for r_1,$r_2 \geq 1$, then the r^{th} product moment of X and Y, denoted by $E\left(X^{r_1}Y^{r_2}\right)$, is given by

$$\begin{aligned} E\left(X^{r_1}Y^{r_2}\right) = \sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j-1}j^2 \\ \times\prod_{n=1}^{2}\Bigg[\alpha_n\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{(k+1)\alpha_n} \\ \times\sum_{i=0}^{\infty}\frac{(-1)^i(k+1)^i\alpha_n^i}{i!} \\ \times\left\{(-1-i)^{\left(-1-\frac{r_n}{\beta_n}\right)}\Gamma\left(1+\frac{r_n}{\beta_n}\right)\right\}\Bigg], \end{aligned} \tag{14}$$

In particulare, the product moment $E(XY)$ is

$$\begin{aligned} E\left(XY\right) = \sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j-1}j^2 \\ \times\prod_{n=1}^{2}\Bigg[\alpha_n\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{(k+1)\alpha_n} \\ \times\sum_{i=0}^{\infty}\frac{(-1)^i(k+1)^i\alpha_n^i}{i!} \\ \times\left\{(-1-i)^{\left(-1-\frac{1}{\beta_n}\right)}\Gamma\left(1+\frac{1}{\beta_n}\right)\right\}\Bigg], \end{aligned} \tag{15}$$

Proof. By substituting Eq. (12) into Eq. (13), we obtain

$$\begin{aligned} E\left(X^{r_1}Y^{r_2}\right) &= \sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j-1}j^2\int_0^\infty\int_0^\infty \alpha_1\alpha_2\beta_1\beta_2 \\ &\quad x^{r_1+\beta_1-1}y^{r_2+\beta_2-1}e^{x^{\beta_1}}e^{y^{\beta_2}}e^{\alpha_1(1-e^{x^{\beta_1}})+\alpha_2(1-e^{y^{\beta_2}})} \\ &\quad\times\left[(1-e^{\alpha_1(1-e^{x^{\beta_1}})})(1-e^{\alpha_2(1-e^{y^{\beta_2}})})\right]^{j-1}dx\,dy, \\ &= \sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j-1}j^2\, I_1(r_1,j)\, I_2(r_2,j), \end{aligned} \tag{16}$$

where,

$$\begin{aligned} I_1 = \int_0^\infty \alpha_1\beta_1\, x^{r_1+\beta_1-1}\, e^{x^{\beta_1}}\, e^{\alpha_1(1-e^{x^{\beta_1}})} \\ \times\left(1-e^{\alpha_1(1-e^{x^{\beta_1}})}\right)^{j-1}dx, \end{aligned} \tag{17}$$

$$\begin{aligned} I_2 = \int_0^\infty \alpha_2\beta_2\, y^{r_2+\beta_2-1}\, e^{y^{\beta_2}}\, e^{\alpha_2(1-e^{y^{\beta_2}})} \\ \times\left(1-e^{\alpha_2(1-e^{y^{\beta_2}})}\right)^{j-1}dy. \end{aligned} \tag{18}$$

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By using the expansions Eqs. (7) and (9), we have

$$I_1 = \alpha_1 \sum_{k=0}^{j-1} \binom{j-1}{k} (-1)^k e^{(k+1)\alpha_1} \times \sum_{i=0}^{\infty} \frac{(-1)^i (k+1)^i \alpha_1^i}{i!} \left\{ (-1-i)^{\left(-1-\frac{r_1}{\beta_1}\right)} \Gamma\left(1+\frac{r_1}{\beta_1}\right) \right\}. \quad (19)$$

similarity,

$$I_2 = \alpha_2 \sum_{k=0}^{j-1} \binom{j-1}{k} (-1)^k e^{(k+1)\alpha_2} \times \sum_{i=0}^{\infty} \frac{(-1)^i (k+1)^i \alpha_2^i}{i!} \left\{ (-1-i)^{\left(-1-\frac{r_2}{\beta_2}\right)} \Gamma\left(1+\frac{r_2}{\beta_2}\right) \right\}. \quad (20)$$

By substituting Eqs. (19) and (20) into Eq. (16), the proof is complete.

3.2 Pearson's Correlation Coefficient

Pearson's correlation coefficient is a statistical measure used to assess the strength and direction of the linear relationship between two continuous variables. For the random vector (X, Y), the Pearson's correlation coefficient is calculated using the following equation:

$$corr(x, y) = \frac{E(xy) - E(x)E(y)}{\sigma_x\ \sigma_y}. \quad (21)$$

To calculate $corr(x, y)$, we utilized Eq. (15) and conclude $E(x)$, $E(y)$, σ_x and σ_y as follows, The present study provides a comprehensive derivation for (E(X)), thereby filling a gap in the literature, as previous studies (e.g., [8]; [9]) did not report this result.

$$\begin{aligned} E(x) &= \int_0^{\infty} x f_1(x) dx \quad (22) \\ &= \alpha_1 e^{\alpha_1} \int_0^{\infty} \beta_1 x^{\beta_1} e^{x^{\beta_1} - \alpha_1 e^{x^{\beta_1}}} dx, \\ &= \alpha_1 e^{\alpha_1} \int_1^{\infty} u^{\frac{1}{\beta_1}} e^{u(1-\alpha_1)} du, \\ &= \alpha_1 e^{\alpha_1} (\alpha_1 - 1)^{\left(\frac{-1}{\beta_1}+1\right)} \Gamma\left(\frac{1}{\beta_1}+1\right). \end{aligned}$$

Similarly, $E(y)$ can be written as follow

$$E(y) = \alpha_2 e^{\alpha_2} (\alpha_2 - 1)^{\left(\frac{-1}{\beta_2}+1\right)} \Gamma\left(\frac{1}{\beta_2}+1\right). \quad (23)$$

Also,

$$\begin{aligned} E(x^2) &= \int_0^{\infty} \alpha_1 \beta_1 x^{\beta+1} e^{x^{\beta_1}} e^{\alpha_1 (1-e^{x^{\beta_1}})} dx \quad (24) \\ &= \alpha_1 e^{\alpha_1} \frac{\Gamma\left(\frac{1}{\beta_1}+1\right)}{(1-\alpha_1)^{\frac{1}{\beta_1}+1}}. \end{aligned}$$

Similarly,

$$E(y^2) = \alpha_2 e^{\alpha_2} \frac{\Gamma\left(\frac{1}{\beta_2}+1\right)}{(1-\alpha_2)^{\frac{1}{\beta_2}+1}}. \quad (25)$$

Thus, σ_x can be concluded from Eqs. (22) and (24) as follows,

$$\sigma_x = \left(\frac{\alpha_1 e^{\alpha_1} \Gamma\left(1+\frac{1}{\beta_1}\right)}{(1-\alpha_1)^{1+\frac{1}{\beta_1}}} - \alpha_1^2 e^{2\alpha_1} (\alpha_1 - 1)^{\left(\frac{-2}{\beta_1}+2\right)} \Gamma\left(\frac{1}{\beta_1}+1\right)^2 \right)^{\frac{1}{2}}. \quad (26)$$

Similarly,from Eqs. (23) and (25), we get

$$\sigma_y = \left(\frac{\alpha_2 e^{\alpha_2} \Gamma\left(1+\frac{1}{\beta_2}\right)}{(1-\alpha_2)^{1+\frac{1}{\beta_2}}} - \alpha_2^2 e^{2\alpha_2} (\alpha_2 - 1)^{\left(\frac{-2}{\beta_2}+2\right)} \Gamma\left(\frac{1}{\beta_2}+1\right)^2 \right)^{\frac{1}{2}}. \quad (27)$$

By substituting Eqs. (15), (22), (23), (26) and (27) into Eq. (21) the Pearson's correlation coefficient can be concluded.

3.3 Moment Generating Function

The moment generating function for bivariate distribution is a useful tool that captures essential information about its probability distribution. The moment generating function can be calculated from

$$M_{(x,y)}(t_1, t_2) = \sum_{j=1}^{\infty} \binom{\lambda}{j} (-1)^{j+1} j^2 M_x(t_1, \alpha_1, \beta_1) M_y(t_2, \alpha_2, \beta_2), \quad (28)$$

Theorem 3.2 The joint moment generating function of (X, Y) for BCPH model is given by

$$M_{(X,Y)}(t_1, t_2) = \sum_{j=1}^{\infty} \binom{\lambda}{j} (-1)^{j-1} j^2 \times \prod_{n=1}^{2} \left[\sum_{r=0}^{\infty} \frac{t_n^r}{r!} \left\{ \sum_{k=0}^{j-1} \binom{j-1}{k} \times (-1)^k e^{(k+1)\alpha_n} \sum_{i=0}^{\infty} \frac{(-1)^i (k+1)^i \alpha_n^i}{i!} \times (-1-i)^{\left(-1-\frac{r}{\beta_n}\right)} \Gamma\left(1+\frac{r}{\beta_n}\right) \right\} \right]. \quad (29)$$

Proof. To derive the moment generating function for BCPH model, we begin by calculating the function $M_x(t_1, \alpha_1, \beta_1)$ as follows:

$$\begin{aligned} M_x(t_1, \alpha_1, \beta_1) &= \int_0^{\infty} e^{t_1 x} h_1(x) (H_1(x))^j dx \\ &= \int_0^{\infty} \alpha_1 \beta_1 e^{t_1 x} x^{\beta_1 - 1} e^{x^{\beta_1}} e^{\alpha_1 (1-e^{x^{\beta_1}})} \quad (30) \\ &\quad \times \left\{ 1 - e^{\alpha_1 (1-e^{x^{\beta_1}})} \right\}^{j-1} dx. \end{aligned}$$

By substituting Eqs. (7), (8) and (9) into Eq. (30), we obtain

$$M_x(t_1,\alpha_1,\beta_1) = \sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{(k+1)\alpha_1} \tag{31}$$
$$\times \sum_{r=0}^{\infty}\frac{(t_1 x)^r}{r!}\sum_{i=0}^{\infty}\frac{(-1)^i (k+1)^i \alpha_1^i}{i!}$$
$$\times \int_0^\infty \alpha_1\beta_1 x^{\beta_1+r-1} e^{-(i+1)x^{\beta_1}}\,dx,$$
$$= \sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{(k+1)\alpha_1}$$
$$\times \sum_{r=0}^{\infty}\frac{t_1^r}{r!}\sum_{i=0}^{\infty}\frac{(-1)^i (k+1)^i \alpha_1^{(i+1)}}{i!}$$
$$\times (-1-i)^{-1-\frac{r}{\beta_1}}\Gamma\left(1+\frac{r}{\beta_1}\right).$$

Furthermore, $M_y(t_2,\alpha_2,\beta_2)$ can be derived as $M_x(t_1,\alpha_1,\beta_1)$ and shown in the following form,

$$M_y(t_2,\alpha_2,\beta_2) = \sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{(k+1)\alpha_2} \tag{32}$$
$$\times \sum_{r=0}^{\infty}\frac{t_2^r}{r!}(-1-\mathrm{i})^{-1-\frac{r}{\beta_2}}\Gamma\left(1+\frac{r}{\beta_2}\right).$$

By substituting Eqs. (31) and (32) into Eq. (28), the joint moment generating function $M_{(X,Y)}(t_1,t_2)$ is obtained.

3.4 Conditional Moments

The conditional distribution is the probability distribution of a random variable, calculated based on the knowledge that another random variable has already taken a specific value. In the following theorem, the conditional moment for BCPH model is concluded.

Theorem 3.3 If $(X,Y)\sim$ BCPH(x,y), then the r^{th} conditional moment is calculated as follows

$$E(y^r|x=t) = \frac{e^{\alpha_1(1-e^{t^{\beta_1}})(1-\lambda)}}{\lambda} \tag{33}$$
$$\times\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\left(1-e^{\alpha_1(1-e^{t^{\beta_1}})}\right)^{j-1}$$
$$\times\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{k\alpha_2}\alpha_2$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(k+1)^i\alpha_2^i}{i!}$$
$$\times\frac{r(-1-i)^{\frac{-r+\beta_2}{\beta_2}}\Gamma\left[\frac{r}{\beta_2}\right]}{\beta_2}.$$

Thus,

$$E(y\mid x=t) \tag{34}$$
$$= \frac{e^{\alpha_1(1-e^{t^{\beta_1}})(1-\lambda)}}{\lambda}\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2$$
$$\times\left(1-e^{\alpha_1(1-e^{t^{\beta_1}})}\right)^{j-1}\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{k\alpha_2}\alpha_2$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(k+1)^i\alpha_2^i}{i!}\frac{(-1-\mathrm{i})^{\frac{1+\beta_2}{\beta_2}}}{\beta_2}\Gamma\left(\frac{1}{\beta_2}\right).$$

We have that,

$$E(y|x=t) = \int_0^\infty y h(y/t)\,dy \tag{35}$$
$$= \int_0^\infty \frac{\alpha_2\beta_2}{\lambda}y^{\beta_2}e^{y^{\beta_2}}e^{\alpha_1(1-e^{t^{\beta_1}})(1-\lambda)}e^{\alpha_2(1-e^{y^{\beta_2}})}$$
$$\times\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\left\{\left(1-e^{\alpha_1(1-e^{t^{\beta_1}})}\right)\right.$$
$$\left.\times\left(1-e^{\alpha_2(1-e^{y^{\beta_2}})}\right)\right\}^{j-1}dy,$$

By utilizing Eqs. (7) and (9), we get

$$E(y|x=t) = \frac{e^{\alpha_1(1-e^{t^{\beta_1}})(1-\lambda)}}{\lambda}$$
$$\times\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\left(1-e^{\alpha_1(1-e^{t^{\beta_1}})}\right)^{j-1}$$
$$\times\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{k\alpha_2}\sum_{i=0}^{\infty}\frac{(-1)^i(k+1)^i\alpha_2^i}{i!}$$
$$\times\int_0^\infty \alpha_2\beta_2 y^{\beta_2}e^{(i+1)y^{\beta_2}}dy,$$
$$= \frac{e^{\alpha_1(1-e^{t^{\beta_1}})(1-\lambda)}}{\lambda}$$
$$\times\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\left(1-e^{\alpha_1(1-e^{t^{\beta_1}})}\right)^{j-1}$$
$$\times\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{k\alpha_2}\sum_{i=0}^{\infty}\frac{(-1)^i(k+1)^i\alpha_2^i}{i!}$$
$$\times\frac{\alpha_2(-1-i)^{\frac{-1+\beta_2}{\beta_2}}\Gamma\left(\frac{1}{\beta_2}\right)}{\beta_2}.$$

as similar

$$E(y^2|x=t) = \frac{e^{\alpha_1(1-e^{t^{\beta_1}})(1-\lambda)}}{\lambda} \tag{36}$$
$$\times\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\left(1-e^{\alpha_1(1-e^{t^{\beta_1}})}\right)^{j-1}$$
$$\times\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{k\alpha_2}\sum_{i=0}^{\infty}\frac{(-1)^i(k+1)^i\alpha_2^i}{i!}$$
$$\times\int_0^\infty \alpha_2\beta_2 y^{\beta_2+1}e^{(i+1)y^{\beta_2}}dy,$$

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Hence,

$$E(y^2|x=t) = \frac{e^{\alpha_1(1-e^{t^{\beta_1}})(1-\lambda)}}{\lambda} \tag{37}$$
$$\times \sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\left(1-e^{\alpha_1(1-e^{t^{\beta_1}})}\right)^{j-1}$$
$$\times \sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{k\alpha_2}\sum_{i=0}^{\infty}\frac{(-1)^i(k+1)^i\alpha_2^i}{i!}$$
$$\times \frac{\alpha_2(-1-i)^{\frac{-2+\beta_2}{\beta_2}}\Gamma\left(\frac{2}{\beta_2}\right)}{\beta_2},$$

Therefore, the desired result of the conditional moment of order r is obtained as follows,

$$E(y^r|x=t) = \frac{e^{\alpha_1(1-e^{t^{\beta_1}})(1-\lambda)}}{\lambda}$$
$$\times \sum_{j=1}^{\infty}\left(\begin{array}{c}\lambda\\ j\end{array}\right)(-1)^{j+1}j^2\left(1-e^{\alpha_1(1-e^{t^{\beta_1}})}\right)^{j-1}$$
$$\times \sum_{k=0}^{j-1}\left(\begin{array}{c}j-1\\ k\end{array}\right)(-1)^k$$
$$\times e^{k\alpha_2}\alpha_2\sum_{i=0}^{\infty}\frac{(-1)^i(k+1)^i\alpha_2^i}{i!}$$
$$\times \frac{r(-1-i)^{\frac{-r+\beta_2}{\beta_2}}\Gamma\left[\frac{r}{\beta_2}\right]}{\beta_2}.$$

4. Reliability Characteristics of BCPH Distribution

The reliability characteristics of the BCPH distribution are displayed in this section specifically the stress-strength parameter, bivariate survival function, and bivariate hazard rate function.

4.1 Stress-Strength Parameter

The stress-strength parameter, often denoted as R, is a measure of the probability that a system will remain reliable when subjected to stress. A higher value of R indicates a more reliable system, as there's a greater chance the system's strength can handle the stress.

Theorem 4.1 Let X and Y be independent random variables representing stress and strength, respectively, with the BCPH probability density function (PDF). Then, the stress-strength parameter $R = P(X < Y)$ is given by

$$R = \sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j \tag{38}$$
$$\times \sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{\alpha_2(k+1)}\sum_{m=0}^{j}\binom{j}{m}(-1)^m e^{m\alpha_1}$$
$$\times \sum_{i=0}^{\infty}\frac{(-1)^{2i}m^i(k+1)^i\alpha_1^i\alpha_2^i}{(i!)^2}\left(\frac{\alpha_2\beta_2 e^{iy^{\beta_1}}}{\beta_2+i\beta_2}\right).$$

Proof.

$$R = P(X<Y) = \int_0^{\infty}\int_0^{y} h(x,y)\,dxdy,$$

From Eq. (12), we have

$$R = \left(\begin{array}{c}\lambda\\ j\end{array}\right)(-1)^{j+1}j^2\int_0^{\infty}\int_0^{y}\alpha_1\alpha_2\beta_1\beta_2 x^{\beta_1-1}y^{\beta_2-1}$$
$$\times e^{x^{\beta_1}}\times e^{y^{\beta_2}}\times e^{\alpha_1(1-e^{x^{\beta_1}})+\alpha_2(1-e^{y^{\beta_2}})}$$
$$\left\{1-e^{\alpha_1(1-e^{x^{\beta_1}})}-e^{\alpha_2(1-e^{y^{\beta_2}})}\right.$$
$$\left.+e^{\alpha_1(1-e^{x^{\beta_1}})+\alpha_2(1-e^{y^{\beta_2}})}\right\}^{j-1} dxdy,$$

Let begin with this integral

$$\int_0^{y}\alpha_1\beta_1 x^{\beta_1-1}\times e^{x^{\beta_1}}\times e^{\alpha_1\left(1-e^{x^{\beta_1}}\right)}\left\{1-e^{\alpha_1\left(1-e^{x^{\beta_1}}\right)}\right\}^{j-1}dx$$
$$= \int_0^{1-e^{\alpha_1\left(1-e^{y^{\beta_1}}\right)}} u^{j-1}du = \frac{1}{j}\left(1-e^{\alpha_1\left(1-e^{y^{\beta_1}}\right)}\right)^{j}, \tag{39}$$

To complete the rest of the proof, it must be substituted the Eq. (40) into R. Then R becomes as follows

$$R = \frac{1}{j}\int_0^{\infty}\alpha_2\beta_2 y^{\beta_2-1}\times e^{y^{\beta 2}}\times e^{\alpha_2\left(1-e^{y^{\beta_1}}\right)}$$
$$\times\left\{1-e^{\alpha_2\left(1-e^{y^{\beta_2}}\right)}\right\}^{j-1}\left(1-e^{\alpha_1\left(1-e^{y^{\beta_1}}\right)}\right)^{j},$$

By using Eq. (9), we have

$$R = \frac{1}{j}\sum_{k=0}^{j-1}\left(\begin{array}{c}j-1\\ k\end{array}\right)(-1)^k\times e^{(k+1)\alpha_2}$$
$$\times\sum_{k=0}^{j}\left(\begin{array}{c}j\\ k\end{array}\right)(-1)^k\left(e^{\alpha_2(k+1)}e^{k\alpha_1}\right)$$
$$\times\int_0^{\infty}\alpha_2\beta_2 y^{\beta_2-1}\times e^{-(k+1)\alpha_2 e^{y^{\beta_2}}}\times e^{-k\alpha_1 e^{y^{\beta_1}}}\times e^{y^{\beta_2}}dy,$$

By using expansion from Eq. (7), we have

$$R = \frac{1}{j}\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{(k+1)\alpha_2} \tag{40}$$
$$\times\sum_{m=0}^{j}\binom{j}{m}(-1)^m e^{m\alpha_1}$$
$$\times\sum_{n=0}^{\infty}\frac{k^n(k+1)^n\alpha_1^n\alpha_2^n}{(n!)^2}$$
$$\times\int_0^{\infty}\alpha_2\beta_2\, y^{\beta_2-1}e^{y^{\beta_2}(1+i)}\, e^{iy^{\beta_1}}dy$$

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$$= \binom{\lambda}{j}(-1)^{j+1} j \times \sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{\alpha_2(k+1)}$$
$$\times \sum_{m=0}^{j}\binom{j}{m}(-1)^m e^{m\alpha_1}$$
$$\times \sum_{i=0}^{\infty}\frac{(-1)^{2i}k^i(k+1)^i\alpha_1^i\alpha_2^i}{(i!)^2}\left(\frac{\alpha_2\beta_2 e^{iy^{\beta_1}}}{\beta_2+i\beta_2}\right).$$

4.2 Bivariate Survival Function

The survival function, also known as the reliability function, plays a vital role in analyzing time-to-event data, which is prevalent across many fields. It provides important insights into the probability of survival over time, the distribution of survival times, and the factors that influence survival outcomes. The bivariate survival function of the BCPH distribution is given by:

$$S(x,y) = 1 - H_1(x) - H_2(y) + H(x,y),$$

By using Eq. (11), the bivariate survival function is derived as

$$\begin{aligned} S(x,y,\lambda) = {} & e^{\alpha_1(1-e^{x^{\beta_1}})} + e^{\alpha_2(1-e^{y^{\beta_2}})} \\ & -\{e^{\alpha_1(1-e^{x^{\beta_1}})} + e^{\alpha_2(1-e^{y^{\beta_2}})} \\ & -e^{\alpha_1(1-e^{x^{\beta_1}})+\alpha_2(1-e^{y^{\beta_2}})}\}^{\lambda}. \end{aligned} \tag{41}$$

Figure 3 show the survival function for BCPH model plot and contours.

4.3 The Bivariate Hazard Rate Function

Understanding the interconnecting of events is crucial so the bivariate hazard rate function is a good tool to do this. The bivariate hazard function describes the instantaneous conditional probability of one event happening at a specific time given that both events have survived up to that time. It captures the dependence between the events. The form of the bivariate hazard rate function is given by [10]

$$r(x,y) = \frac{h(x,y)}{S(x,y)},$$

From Eqs. (12) and (41) then the hazard rate function for BCPH distribution is calculated as follows

$$\begin{aligned} r(x,y) = {} & \left\{x^{1-\beta_1}y^{1-\beta_2}e^{\alpha_1\left(e^{x^{\beta_1}}-1\right)-x^{\beta_1}-y^{\beta_2}}\left(e^{\alpha_1\left(1+e^{x^{\beta_1}}\right)}\right.\right. \\ & \left.\left.-z^{\lambda}+e^{\alpha_2\left(1+e^{y^{\beta_2}}\right)}\right)\right\}\Big/\left\{\alpha_1\alpha_2\beta_1\beta_2\lambda e^{\alpha_2\left(1+e^{y^{\beta_2}}\right)}\right. \\ & \left.\times z^{\lambda-2}\left(1-\lambda\left(e^{\alpha_1\left(1+e^{x^{\beta_1}}\right)}-1\right)\left(e^{\alpha_2\left(1+e^{y^{\beta_2}}\right)}-1\right)\right)\right\}. \end{aligned} \tag{42}$$

where, $z = e^{\alpha_1\left(1+e^{x^{\beta_1}}\right)} - e^{\alpha_1+\alpha_2-\alpha_1 e^{x^{\beta_1}}-\alpha_2 e^{y^{\beta_2}}} + e^{\alpha_2\left(1+e^{y^{\beta_2}}\right)}$ [11] suggested another way to calculate hazard gradient of a bivariate random vector as follows:

$$r(x,y) = \left(\frac{-\partial \ln S(x,y)}{\partial x}, \frac{-\partial \ln S(x,y)}{\partial y}\right),$$

Hence,

$$r(x,y) = \left(\alpha_1\beta_1 e^{x^{\beta_1}}x^{\beta_1-1}, \alpha_2\beta_2 e^{y^{\beta_2}}y^{\beta_2-1}\right). \tag{43}$$

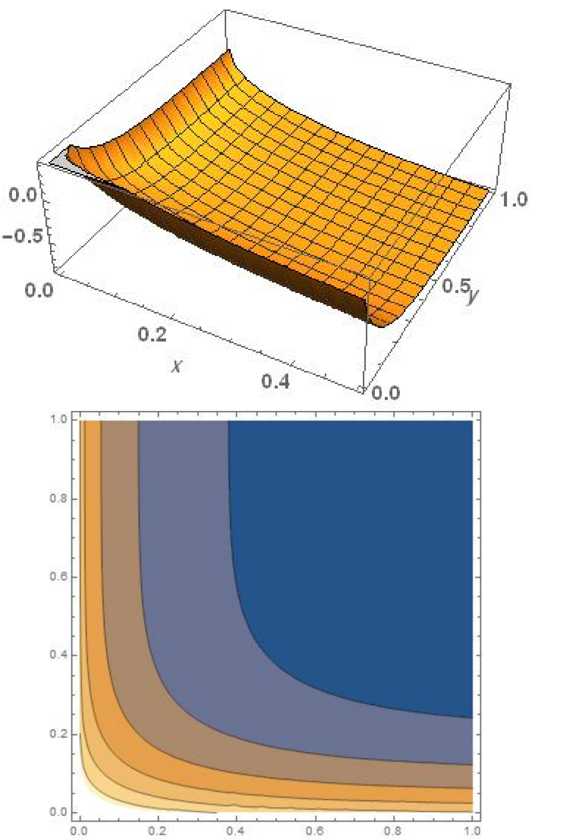

Figure 3. Bivariate Survival Plots and Contours of BCPH distribution for parameter values $\alpha_1 = 2, \alpha_2 = 5, \beta_1 = 0.5, \beta_2 = 2.8, \lambda = 0.01$

5. Parameter Estimation

Parameter estimation allows to make predictions, understand the underlying process better, and test different hypotheses. In this section maximum likelihood estimation and inference function for margins estimation are concluded for BCPH model in order to reflect real world system.

5.1 Maximum Likelihood Estimation

Maximum likelihood estimation method (MLE) aims to find the parameter values that make the observed data appear most probable. In this section, the MLE is applied to estimate the parameters of BCPH distribution. The log-likelihood function for BCPH distribution is shown as follows

$$\begin{aligned} l(x,y) = {} & n\ln[\lambda\alpha_1\alpha_2\beta_1\beta_2] + (\beta_1-1)\sum_{i=1}^{n}\ln[x_i] \\ & + (\beta_2-1)\sum_{i=1}^{n}\ln[y_i] + \sum_{i=1}^{n}x_i^{\beta_1} + \sum_{i=1}^{n}y_i^{\beta_2} \\ & +(\lambda-2)\sum_{i=1}^{n}\ln\left[e^{\alpha_1\left(1-x_i^{\beta_1}\right)} - e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}\right. \\ & \left.+e^{\alpha_2\left(1-y_i^{\beta_2}\right)}\right] + \sum_{i=1}^{n}\alpha_1\left(1-x_i^{\beta_1}\right) \end{aligned} \tag{44}$$

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$$+\sum_{i=1}^{n}\alpha_2\left(1-y_i^{\beta_2}\right)+\sum_{i=1}^{n}\ln\left[1-\lambda\left\{-e^{\alpha_1\left(1-x_i^{\beta_1}\right)}+e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-e^{\alpha_2\left(1-y_i^{\beta_2}\right)}+1\right\}\right],$$

When the first derivative is applied to Eq. (44), the maximum likelihood equations for parameters $\lambda, \alpha_1, \alpha_2, \beta_1$ and β_2 of BCPH model is given as follows

$$\frac{\partial l}{\partial\alpha_1}=\frac{n}{\alpha_1}+\sum_{i=1}^{n}\left(1-x_i^{\beta_1}\right)+(\lambda-2)\sum_{i=1}^{n}\frac{\left(1-x_i^{\beta_1}\right)e^{\alpha_1\left(1-x_i^{\beta_1}\right)}-\left(1-x_i^{\beta_1}\right)e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}}{e^{\alpha_1\left(1-x_i^{\beta_1}\right)}-e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}+e^{\alpha_2\left(1-y_i^{\beta_2}\right)}}$$
$$-\sum_{i=1}^{n}\frac{\lambda\left(\left(1-x_i^{\beta_1}\right)e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-\left(1-x_i^{\beta_1}\right)e^{\alpha_1\left(1-x_i^{\beta_1}\right)}\right)}{1-\lambda\left(-e^{\alpha_1\left(1-x_i^{\beta_1}\right)}+e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-e^{\alpha_2\left(1-y_i^{\beta_2}\right)}+1\right)}=0, \tag{45}$$

$$\frac{\partial l}{\partial\alpha_2}=\frac{n}{\alpha_2}+(\lambda-2)\sum_{i=1}^{n}\frac{\left(1-y_i^{\beta_2}\right)e^{\alpha_2\left(1-y_i^{\beta_2}\right)}-\left(1-y_i^{\beta_2}\right)e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}}{e^{\alpha_1\left(1-x_i^{\beta_1}\right)}-e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}+e^{\alpha_2\left(1-y_i^{\beta_2}\right)}}$$
$$-\sum_{i=1}^{n}\frac{\lambda\left(\left(1-y_i^{\beta_2}\right)e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-\left(1-y_i^{\beta_2}\right)e^{\alpha_2\left(1-y_i^{\beta_2}\right)}\right)}{1-\lambda\left(-e^{\alpha_1\left(1-x_i^{\beta_1}\right)}+e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-e^{\alpha_2\left(1-y_i^{\beta_2}\right)}+1\right)}+\sum_{i=1}^{n}\left(1-y_i^{\beta_2}\right)=0, \tag{46}$$

$$\frac{\partial l}{\partial\beta_1}=\frac{n}{\beta_1}+\sum_{i=1}^{n}\alpha_1x_i^{\beta_1}\left(-\ln[x_i]\right)+\sum_{i=1}^{n}x_i^{\beta_1}\ln[x_i]+\sum_{i=1}^{n}\ln[x_i]$$
$$+(\lambda-2)\sum_{i=1}^{n}\frac{\alpha_1x_i^{\beta_1}\ln[x_i]e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-\alpha_1x_i^{\beta_1}\ln[x_i]e^{\alpha_1\left(1-x_i^{\beta_1}\right)}}{e^{\alpha_1\left(1-x_i^{\beta_1}\right)}-e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}+e^{\alpha_2\left(1-y_i^{\beta_2}\right)}}$$
$$-\sum_{i=1}^{n}\frac{\lambda\left(\alpha_1x_i^{\beta_1}\ln[x_i]e^{\alpha_1\left(1-x_i^{\beta_1}\right)}-\alpha_1x_i^{\beta_1}\ln[x_i]e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}\right)}{1-\lambda\left(-e^{\alpha_1\left(1-x_i^{\beta_1}\right)}+e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-e^{\alpha_2\left(1-y_i^{\beta_2}\right)}+1\right)}=0, \tag{47}$$

$$\frac{\partial l}{\partial\beta_2}=\frac{n}{\beta_2}+(\lambda-2)\sum_{i=1}^{n}\frac{\alpha_2y_i^{\beta_2}\ln[y_i]e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-\alpha_2y_i^{\beta_2}\ln[y_i]e^{\alpha_2\left(1-y_i^{\beta_2}\right)}}{e^{\alpha_1\left(1-x_i^{\beta_1}\right)}-e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}+e^{\alpha_2\left(1-y_i^{\beta_2}\right)}}$$
$$-\sum_{i=1}^{n}\frac{\lambda\left(\alpha_2y_i^{\beta_2}\ln[y_i]e^{\alpha_2\left(1-y_i^{\beta_2}\right)}-\alpha_2y_i^{\beta_2}\ln[y_i]e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}\right)}{1-\lambda\left(-e^{\alpha_1\left(1-x_i^{\beta_1}\right)}+e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-e^{\alpha_2\left(1-y_i^{\beta_2}\right)}+1\right)}$$
$$+\sum_{i=1}^{n}\alpha_2y_i^{\beta_2}\left(-\ln[y_i]\right)+\sum_{i=1}^{n}y_i^{\beta_2}\ln[y_i]+\sum_{i=1}^{n}\ln[y_i]=0, \tag{48}$$

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\frac{e^{\alpha_1\left(1-x_i^{\beta_1}\right)}-e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}+e^{\alpha_2\left(1-y_i^{\beta_2}\right)}-1}{1-\lambda\left(-e^{\alpha_1\left(1-x_i^{\beta_1}\right)}+e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}-e^{\alpha_2\left(1-y_i^{\beta_2}\right)}+1\right)}$$
$$+\sum_{i=1}^{n}\ln\left[e^{\alpha_1\left(1-x_i^{\beta_1}\right)}-e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)}+e^{\alpha_2\left(1-y_i^{\beta_2}\right)}\right]=0, \tag{49}$$

5.2 Inference Function for Margins (IFM)

While estimating parameters for a full multivariate model is usually preferred, using marginal distributions can be an alternative in certain situations. This method can offer computational advantages when the specific dependence structure between variables is unknown.[12] uses this method for estimating distribution parameters by maximizing the marginal likelihood functions. In order to have the estimates $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2$ and $\hat{\lambda}$, the log-likelihood function equation of these parameters must be calculated.

The log-likelihood function equation of parameters α_1, β_1 and λ can be written as given

$$l_1 = \ln[\alpha_1] + n\ln[\beta_1] + n\alpha_1 + (\beta_1 - 1)\sum_{i=1}^{n}\ln[x_i] + \sum_{i=1}^{n} x_i^{\beta_1} - \sum_{i=1}^{n}\alpha_1 x_i^{\beta_1},$$

By taking the first derivative for α_1 and β_1, we have

$$\frac{\partial l_1}{\partial \alpha_1} = n + \frac{n}{\alpha_1} - \sum_{i=1}^{n} e^{x_i^{\beta_1}} = 0.$$

$$\frac{\partial l_1}{\partial \beta_1} = \frac{n}{\beta_1} + \sum_{i=1}^{n}\ln[x_i] + \sum_{i=1}^{n}\ln[x_i]\, x_i^{\beta_1} - \sum_{i=1}^{n}\alpha_1 e^{x_i^{\beta_1}}\ln[x_i]\, x_i^{\beta_1} = 0.$$

The log-likelihood function function of parameters α_2, β_2 and λ can be defined as given

$$l_2 = \ln[\alpha_2] + n\ln[\beta_2] + n\alpha_2 + (\beta_2 - 1)\sum_{i=1}^{n}\ln[y_i] + \sum_{i=1}^{n} y_i^{\beta_2} - \sum_{i=1}^{n}\alpha_2 y_i^{\beta_2},$$

By taking the first derivative for α_2 and β_2, we have

$$\frac{\partial l_2}{\partial \alpha_2} = n + \frac{n}{\alpha_2} - \sum_{i=1}^{n} e^{y_i^{\beta_2}} = 0.$$

$$\frac{\partial l_2}{\partial \beta_2} = \frac{n}{\beta_2} + \sum_{i=1}^{n}\ln[y_i] + \sum_{i=1}^{n}\ln[y_i] y_i^{\beta_2} - \sum_{i=1}^{n}\alpha_2 e^{y_i^{\beta_2}}\ln[y_i] y_i^{\beta_2} = 0.$$

By taking the first derivative for λ and equating it to zero, the likelihood function can be get as following equation

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \frac{e^{\alpha_1\left(1-x_i^{\beta_1}\right)} - e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)} + e^{\alpha_2\left(1-y_i^{\beta_2}\right)} - 1}{1 - \lambda\left(-e^{\alpha_1\left(1-x_i^{\beta_1}\right)} + e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)} - e^{\alpha_2\left(1-y_i^{\beta_2}\right)} + 1\right)} + \sum_{i=1}^{n}\ln\left[e^{\alpha_1\left(1-x_i^{\beta_1}\right)} - e^{\alpha_1\left(1-x_i^{\beta_1}\right)+\alpha_2\left(1-y_i^{\beta_2}\right)} + e^{\alpha_2\left(1-y_i^{\beta_2}\right)}\right] = 0.$$

6. Simulation

This study investigates the maximum likelihood estimation (MLE) performance for the parameters of a bivariate distribution constructed using a proportional hazard rate (PHR) with specified marginal distributions. The true parameter values used in the simulation were $\alpha_1 = 1.5$, $\alpha_2 = 3$, $\beta_1 = 2$, $\beta_2 = 4$, and $\lambda = 3$. We conducted simulation experiments with increasing sample sizes $n = 50, 100, 150, 200$ and performed 1000 replications for each scenario. The results were evaluated in terms of the average estimates, bias, and mean squared error (MSE).

Across all sample sizes, the MLEs demonstrated desirable convergence properties. As expected, increasing the sample size led to reduced bias and MSE for all parameters. For instance, at $n = 50$, the largest observed bias was for the dependence parameter λ, with a value of -0.075 and an MSE of 0.0545. However, as the sample size increased to $n = 200$, the bias for λ reduced substantially to -0.015 and the MSE decreased to 0.0221, indicating improved estimation accuracy with more data.

Similarly, the marginal parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ showed decreasing trends in bias and MSE with increasing sample size. For example, the bias for α_1 reduced from -0.015 at $n = 50$ to -0.007 at $n = 200$, while the corresponding MSE dropped from 0.0047 to 0.0012. The parameter β_2, which showed the largest positive bias among the marginals across all settings, improved from a bias of 0.012 at $n = 50$ to 0.021 at $n = 200$, although the MSE also decreased, suggesting better concentration of estimates around the true value.

Overall, the results confirm the consistency of the MLE approach under the specified bivariate PHR model. The bias and MSE metrics collectively suggest that moderate to large sample sizes (e.g., $n \geq 150$) are sufficient for reliable parameter estimation. Notably, the dependence parameter λ is more sensitive to sample size compared to the marginal parameters, which aligns with typical findings in modelling where dependence parameters require more information to estimate precisely. These findings support the use of the proposed MLE procedure for joint modeling in reliability and survival analysis contexts, especially when sample sizes are adequate.

7. Real Data Applications

This research presents the analysis of three datasets from diverse fields, including environmental, climatic, and medical domains. Each dataset involves examining the survival of two related variables. The BCPH model is fitted to these data, and its parameter estimates are compared with those of other established models based on the specific characteristics of each dataset.

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Table 1. Bias and MSE for estimation of parameters α, β, and λ

n	Estimation	Bias	MSE
50	$\hat{\alpha}_1 = 1.485$	-0.015	0.0047
	$\hat{\alpha}_2 = 2.970$	-0.030	0.0078
	$\hat{\beta}_1 = 1.989$	-0.011	0.0043
	$\hat{\beta}_2 = 4.012$	0.012	0.0070
	$\hat{\lambda} = 2.925$	-0.075	0.0545
100	$\hat{\alpha}_1 = 1.482$	-0.018	0.0031
	$\hat{\alpha}_2 = 2.981$	-0.019	0.0042
	$\hat{\beta}_1 = 1.997$	-0.003	0.0025
	$\hat{\beta}_2 = 4.021$	0.021	0.0050
	$\hat{\lambda} = 2.954$	-0.046	0.0378
150	$\hat{\alpha}_1 = 1.490$	-0.010	0.0018
	$\hat{\alpha}_2 = 2.985$	-0.015	0.0026
	$\hat{\beta}_1 = 2.003$	0.003	0.0017
	$\hat{\beta}_2 = 4.018$	0.018	0.0035
	$\hat{\lambda} = 2.970$	-0.030	0.0292
200	$\hat{\alpha}_1 = 1.493$	-0.007	0.0012
	$\hat{\alpha}_2 = 2.989$	-0.011	0.0018
	$\hat{\beta}_1 = 2.006$	0.006	0.0012
	$\hat{\beta}_2 = 4.021$	0.021	0.0024
	$\hat{\lambda} = 2.985$	-0.015	0.0221

7.1 Mercury Concentrations Data

This section introduces the relationship between total mercury concentration and fish size, a common and important topic in environmental and food safety research. This study focuses on two types of fish: the Mediterranean Horse and the Mackerel Horse (Trachurus trachurus). [13] examined mercury concentrations in these two species and investigated how these concentrations vary with fish size, highlighting implications for consumer mercury exposure.

A comparison is conducted between the BCPH distribution and other models, including the Bivariate Farlie-Gumbel-Morgenstern Kumaraswamy (BFGMK) model [14] and the Bivariate Cubic Kumaraswamy (BCK) model, to demonstrate the superior performance of the BCPH model. Goodness-of-fit criteria such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), and Consistent AIC (CAIC) are employed to support this conclusion. Table 2 presents the lengths of the first fish species (Mediterranean Horse) alongside the mean total mercury concentrations.

For extracting meaningful information from this data, statistical analysis must be done. Table 3 involves statistical analysis for Mediterranean Horse fish and mean total of mercury concentration.

In Table 4, several goodness-of-fit test statistics for the marginal Chen distribution are presented, including the Kolmogorov–Smirnov (K–S) test, Cramér–von Mises (W_n^2) test, Anderson–Darling (A_n^2) test, and Watson (U_n^2) test. These tests assess the adequacy of the model fit to the data. It is observed that The K–S, Anderson–Darling, and Watson statistics were all small, indicating that the Chen model provides a good fit for this dataset.

Table 2. Length of Mediterranean Horse Fish and Mean Total of Mercury Concentrations

Group No.	Fish Length (m)	Mean Total Mercury ($mg\,kg^{-1}$)
1	0.185	0.00009
2	0.199	0.00009
3	0.299	0.00021
4	0.234	0.00016
5	0.243	0.00021
6	0.246	0.00026
7	0.265	0.00028
8	0.291	0.00041
9	0.296	0.00054
10	0.341	0.00070
11	0.369	0.00155
12	0.397	0.00162

Table 3. Descriptive statistics (mean, median, quartiles, minimum, and maximum) for Mediterranean Horse data

Statistics	Fish Length (m)	Mean Total of Mercury ($mg\,kg^{-1}$)
Minimum	0.185	0.00009
1st Quartile	0.23625	0.0001725
Median	0.27800	0.0002700
Mean	0.28042	0.0005100
3rd Quartile	0.33050	0.000600
Maximum	0.397	0.00162
Std. Error of Mean	0.018773	0.00015402

In Table 5, BCPH model is compared with other two models, Bivariate Cubic Kumaraswamy (BCK) model and Bivariate Farlie-Gumbel Morgenstern Kumaraswamy (BFGMK) model.

Table 5 compares the BCPH model with other bivariate models. The BCPH model demonstrates superior performance, having the lowest values for AIC, BIC, AICC, CAIC, and HQIC criteria.

The length of the second species of fish (Mackerel Horse) and the mean total of mercury concentrations are presented in Table 6.

Table 7 contains the statistical analysis for length of Mackerel Horse fish and mean total of mercury concentration.

In Table 8, it is observed that the BCPH model provides a good fit to the length of the Mackerel Horse fish and the mean total mercury concentrations. This conclusion is supported by goodness-of-fit test statistics applied to the marginal Chen distribution, including the Kolmogorov–Smirnov (K–S), Cramér–von Mises (W_n^2),

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Table 4. Goodness-of-fit statistics (K–S, W_n^2, A_n^2, and U_n^2 tests) for Mediterranean Horse data

Variable	MLEs	P-value	K–S	W_n^2	A_n^2	U_n^2
Fish length (m)	$\hat{\lambda} = 12.88$, $\hat{\alpha} = 19.33$, $\hat{\beta} = 4.59$	0.8770	0.1703	0.0622	0.3696	0.0573
Mean total mercury (mg kg^{-1})	$\hat{\lambda} = 32.97$, $\hat{\alpha} = 49.46$, $\hat{\beta} = 1.02$	0.7207	0.2004	0.0621	0.4803	0.0583

Table 5. The measures of AIC, BIC, CAIC, and HQIC for the Mediterranean Horse Data

Distribution	MLEs	Log	AIC	BIC	CAIC	HQIC
BCPH	$\hat{\lambda} = 2.3388$ $\hat{\alpha}_1 = 12.4098$, $\hat{\alpha}_2 = 195.8805$ $\hat{\beta}_1 = 2.8735$, $\hat{\beta}_2 = 0.8408$	85.8493	−161.699	−159.274	−151.699	−162.596
BCK	$\hat{\theta} = 0.62665$ $\hat{\alpha}_1 = 1.6562$, $\hat{\alpha}_2 = 0.3169$ $\hat{\beta}_1 = 5.1597$, $\hat{\beta}_2 = 7.7509$	78.7900	−147.590	−145.170	−137.590	−148.490
BFGMK	$\hat{\theta} = 0.76561$ $\hat{\lambda}_1 = 5.0296$, $\hat{\lambda}_2 = 0.32009$ $\hat{\beta}_1 = 15.5721$, $\hat{\beta}_2 = 11.3926$	54.9400	−99.870	−97.450	−89.870	−100.770

Table 6. Length of Mackerel Horse fish and mean total of mercury concentrations

Group No. No.	Fish (m) (m)	Mean total of mercury (mg kg^{-1})
1	0.195	0.00017
2	0.214	0.00020
3	0.244	0.00016
4	0.250	0.00026
5	0.290	0.00043
6	0.231	0.00029
7	0.271	0.00042
8	0.285	0.00044
9	0.321	0.00059
10	0.326	0.00139
11	0.344	0.00142
12	0.372	0.00241

Table 7. Descriptive statistics (mean, median, quartiles, minimum, and maximum) for Mackerel Horse data

Statistic	Fish length (m)	Mean total mercury (mg kg^{-1})
Minimum	0.1950	0.00016
1st Quartile	0.23425	0.0002150
Median	0.2780	0.0004250
Mean	0.27858	0.0006817
3rd Quartile	0.32475	0.0011900
Maximum	0.3720	0.00241
Std. Error of Mean	0.15743	0.00020094

Anderson–Darling (A_n^2), and Watson (U_n^2) tests, all indicating the suitability of the BCPH model for this dataset.

Table 9 presents a comparison of the BCPH model with other models, including the Bivariate Nelsen-Ten Kumaraswamy (BNTK) [14], Bivariate Farlie-Gumbel-Morgenstern Kumaraswamy (BFGMK), and Bivariate Cubic Kumaraswamy (BCK) models. Using criteria such as AIC, BIC, CAIC, and HQIC, it is evident that the BCPH model achieves the lowest values, indicating it is the best-fitting model among those considered.

7.2 Climatic Data

[15] displays bivariate data set which refers to the Environmental Climatic bivariate data set. From the 51 largest cities on USA, these data are collected by National Climatic Data Center (NCDC) of the USA and provided on the website, https://www.ncei.noaa.gov/access. These bivariate date represent the relation between the average precipitation in millimeters and the average maximum temperature. Let X refers to average precipitation (mm) and Y refers to Average maximum temperature (c^0). Table 10 illustrate the relationship between the average precipitation in millimeters and the average maximum temperature.

Statistical analysis is an essential tool in various fields such as climatic field for exploring and presenting data. Table 11 contains the statistical analysis for average precipitation and average maximum temperature.

These data are fit to BCPH model Table 12 introduces goodness of fit tests such as K-S, W_n^2 , A_n^2 and U_n^2. They are applied to the marginal chen distribution , the result is that the value of The K–S, Anderson–Darling, and Watson statistics were all small, indicating good overall agreement of chen distribution.

This research provide a comparison between BCPH model and some bivariate models such as Bivariate Kumaraswamy Type Exponential (BKE) model [4], Bi-

Table 8. Goodness-of-fit statistics (K–S, W_n^2, A_n^2, and U_n^2 tests) for Mackerel Horse data

Data Type	MLEs	P-value	K–S	W_n^2	A_n^2	U_n^2
Length of Mackerel Horse fish (m)	$\hat{\lambda} = 28.59$, $\hat{\alpha} = 42.89$, $\hat{\beta} = 5.92$	0.9865	0.1307	0.0293	0.1900	0.0276
Mean total mercury concentration (mg kg^{-1})	$\hat{\lambda} = 26.22$, $\hat{\alpha} = 39.34$, $\hat{\beta} = 0.96$	0.6412	0.2141	0.6307	0.1047	0.1047

Table 9. The measures of AIC, BIC, CAIC, and HQIC for Mackerel Horse Data

Distribution	MLEs	Log	AIC	BIC	CAIC	HQIC
BCPH	$\hat{\lambda} = 21.4382$ $\hat{\alpha}_1 = 7.5698$, $\hat{\alpha}_2 = 16.0416$ $\hat{\beta}_1 = 3.6785$, $\hat{\beta}_2 = 0.6907$	84.000	−158.572	−156.147	−148.572	−159.469
BNTK	$\hat{\theta} = 0.8570$ $\hat{\alpha}_1 = 1.4632$, $\hat{\alpha}_2 = 0.3200$ $\hat{\beta}_1 = 1.6897$, $\hat{\beta}_2 = 2.9168$	83.396	−156.793	−154.386	−146.793	−157.690
BFGMK	$\hat{\theta} = 0.8809$ $\hat{\lambda}_1 = 1.8580$, $\hat{\lambda}_2 = 0.3606$ $\hat{\beta}_1 = 6.6354$, $\hat{\beta}_2 = 9.9984$	77.430	−144.866	−142.441	−134.866	−145.763
BCK	$\hat{\theta} = 0.9266$ $\hat{\lambda}_1 = 1.9851$, $\hat{\lambda}_2 = 0.2643$ $\hat{\beta}_1 = 1.4010$, $\hat{\beta}_2 = 2.2750$	55.450	−100.896	−98.470	−90.895	−101.793

Table 10. Climatic Data Set

No.	X	Y	No.	X	Y
1	99	12	18	71	19
2	61	17	19	44	5
3	86	7	20	13	14
4	113	13	21	52	20
5	96	5	22	97	8
6	99	2	23	146	11
7	83	12	24	52	26
8	57	2	25	52	1
9	80	6	26	29	3
10	79	4	27	108	10
11	75	5	28	135	18
12	70	14	29	102	6
13	15	6	30	48	10
14	62	2	31	66	23
15	87	4	32	90	7
16	95	18	33	22	20
17	81	4	34	72	4
18	35	176	35	83	11
19	36	107	36	51	77

Table 11. Descriptive statistics (mean, median, quartiles, minimum, and maximum) for Climatic Data

Statistics	X	Y
Minimum	13	1
1st Quartile	52.00	5.00
Median	77.00	10.00
Mean	76.76	10.39
3rd Quartile	96.00	14.00
Maximum	176	26
Std. Error of Mean	4.645	0.907

variate Farlie-Gumbel Morgenstern Weibull (FGMBW) model [16] , and Bivariate cubic Rayleigh (BCR) [5] model by employing AIC, BIC, CAIC and HQIC criteria. These compared models are mentioned as follows

From Table 13, the result of comparison is that BCPH distribution is more better than other bivariate distributions.

7.3 Infection Kidney Data

This Dataset is taken from *McGilchrist* and *Aisbett* [17], which represents the recurrence time of infection for 30 kidney patients who are using a portable dialysis machine. The catheter must be withdrawn, the infection must be treated, and then the catheter must be reinserted if an infection develops at the site of catheter insertion. Recurrence times are times from insertion until the next infection. Assume that X refers to the first recurrence time (days) and Y refers to the second recurrence

Table 12. Goodness-of-fit statistics (K–S, W_n^2, A_n^2, and U_n^2 tests) for Climatic Data

Variable	MLEs	P-value	K-S	W_n^2	A_n^2	U_n^2
X	$\hat{\lambda} = 0.0336,\ \hat{\alpha} = 0.06713,\ \hat{\beta} = 0.40261$	0.59022	0.10812	0.12733	0.82813	0.12559
Y	$\hat{\lambda} = 0.14661,\ \hat{\alpha} = 0.21991,\ \hat{\beta} = 0.49244$	0.65137	0.10301	0.08693	0.59012	0.08403

Table 13. The Measures AIC, BIC, CAIC and HQIC of Climatic Data

Distribution	MLEs	Log	AIC	BIC	CAIC	HQIC
BCPH	$\hat{\lambda} = 1.45109$ $\hat{\alpha}_1 = 0.00214,\ \hat{\alpha}_2 = 0.02336$ $\hat{\beta}_1 = 0.39504,\ \hat{\beta}_2 = 0.48389$	−413.401	836.801	846.460	838.135	840.492
BCR	$\hat{\alpha}_1 = 60.7392,\ \hat{\alpha}_2 = 8.65753$ $\hat{\theta} = 0.45$	−416.110	838.219	844.015	838.730	840.434
BKE	$\hat{\theta} = 0.6242$ $\hat{\lambda}_1 = 0.0199,\ \hat{\lambda}_2 = 0.1453$	−441.988	889.977	895.487	890.487	892.191
FGMBW	$\hat{\theta} = -0.5595$ $\hat{\lambda}_1 = 0.3268,\ \hat{\lambda}_2 = 1.7595$ $\hat{\beta}_1 = 6.1648,\ \hat{\beta}_2 = 15.9845$	−461.348	932.697	942.356	934.030	936.388

Table 14. The first and second recurrence time

No.	X	Y	No.	X	Y
1	8	16	16	17	4
2	23	13	17	185	117
3	22	28	18	292	114
4	447	318	19	22	159
5	30	12	20	15	108
6	24	245	21	152	362
7	7	9	22	402	24
8	511	30	23	13	66
9	53	196	24	39	46
10	15	154	25	12	40
11	7	333	26	113	201
12	141	8	27	132	156
13	96	38	28	34	30
14	149	70	29	2	25
15	536	25	30	130	26

time(days). Table 14 shows this data.

Table 15 contains the statistical analysis for recurrence time data. Statistical analysis is essential tool to form meaningful conclusions and identify trends that might not be readily apparent.

Goodness of fit tests such as K-S, W_n^2 , A_n^2 and U_n^2 tests are given in Table 16 for BCPH model. The K–S, Anderson–Darling, and Watson statistics were all small, indicating good overall agreement between observed and fitted for chen model.

The BCPH model is fitted to the recurrence time data and compared with other models to show the superiority of our model. These models are Bivariate Marshall-Olkin Weibull (BMOW) model [18], Bivariate Rayleigh

Table 15. Statistical Analysis of Infection for Kidney Patients' Data

Statistics	X	Y
Minimum	2	4
1^{st} Quartile	15.00	24.75
Median	36.50	43.00
Mean	120.97	99.10
3^{rd} Quartile	149.75	156.75
Maximum	536	362
Std. Error of mean	28.798	19.154

Proportional Hazard (BRPH) model, and Bivariate Morgenstern Rayleight (BMR) model [19].

From Table 17, the result of comparison is that BCPH distribution is more better than other bivariate distributions.

8. Conclusion

This paper introduces a new probability distribution called the BCPH distribution specifically designed for analyzing data from different fields such as (medical, climactic and environment).The paper explores the statistical properties of the BCPH distribution, including its probability density function and cumulative distribution function, along with reliability characteristics. Pearson's correlation coefficient form is concluded. Different methods for estimating the BCPH distribution's parameters are discussed . The Maximum Likelihood Estimation (MLE) approach shows the best performance. The BCPH model is compared to other existing bivariate distributions for three application. The study suggests BCPH provides a more flexible and better-fitting model.Future research may focus on extending the BCPH model to higher dimensions, exploring Bayesian

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Table 16. Goodness-of-fit statistics (K–S, W_n^2, A_n^2, and U_n^2 tests) of Infection for Kidney Patients' Data

	MLEs	P-value	K-S	W_n^2	A_n^2	U_n^2
X	$\hat{\lambda} = 0.18313,\ \hat{\alpha} = 0.27469,\ \hat{\beta} = 0.23354$	0.48390	0.15298	0.15237	0.99624	0.14830
Y	$\hat{\lambda} = 1.04833,\ \hat{\alpha} = 0.03833,\ \hat{\beta} = 0.265469$	0.4913	0.152108	0.123529	0.78771	0.127435

Table 17. The measures of AIC, BIC, CAIC, and HQIC for Infection in Kidney Patients' Data

Distribution	MLEs	Log	AIC	BIC	CAIC	HQIC
BCPH	$\hat{\lambda} = 0.93963$ $\hat{\alpha}_1 = 0.05393,\ \hat{\alpha}_2 = 0.03416$ $\hat{\beta}_1 = 0.23348,\ \hat{\beta}_2 = 0.26572$	−343.534	697.067	704.073	699.567	700.198
BMOW	$\hat{\lambda}_1 = 0.7363,\ \hat{\lambda}_2 = 89.8322$ $\hat{\beta}_1 = 104.228,\ \hat{\beta}_2 = 93.587$	−349.068	706.137	711.742	710.639	710.377
BRPH	$\hat{\lambda} = 0.95979$ $\hat{\alpha} = 133.816,\ \hat{\beta} = 96.8469$	−396.821	799.642	803.845	800.565	800.987
BMR	$\hat{\alpha}_1 = 140.098,\ \hat{\alpha}_2 = 105.728$	−397.124	800.248	804.451	801.171	801.592

estimation methods, and applying the model to new fields such as finance and environmental risk assessment to further evaluate its versatility and predictive performance.

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Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data that support the findings of this study are available within the article.
Conflict of interests
The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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