To appear in: **Mathematical Sciences**

Online ISSN: 2251-7456 Print ISSN: 2008-1359

Received: 01-September-2025

Revised: 27-November-2025

Accepted: 27-January-2026

DOI: https://doi.org/10.57467/mathsci.jdwx.q7r2.0034

ORIGINAL RESEARCH

A survey: Kannan fixed point theorem in hyperconvex spaces

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Abstract. We prove that every closed ball in a non-reflexive Banach space ℓ^∞ is a proximinal set. We also define a concept of $\hbar$-proximal quasi-normal structure on a pair of nonempty and admissible subsets of a metric space and prove that every nonempty and admissible pair of subsets of a hyperconvex metric space has the $\hbar$-proximal quasi-normal structure. We then discuss best proximity theory by considering two family of cyclic maps entitled cyclic Kannan contractions and cyclic relatively Kannan nonexpansive mappings and show that every cyclic Kannan contraction defined on a union of two nonempty and admissible subsets of a hyperconvex metric space has a best proximity point and by using the geometric notion of $\hbar$-proximal quasi-normal structure, we conclude a similar result for cyclic relatively Kannan nonexpansive maps which preserves distance. In a special case, we obtain Baillon's fixed point theorem for Kannan nonexpansive self-mapping.

Key words: Best proximity point; Hyperconvex space; Relatively Kannan nonexpansive; Proximinal pair.

2010 Mathematics Subject Classification: 47H09, 41A65, 46B20

1 Introduction

The proofs of the existence of fixed point theorems for various classes of maps are based on two main ideas. The first one may be concluded from metric assumptions and by using the iterative sequences whereas the second technique of proofs of fixed point theorems can be done by considering the topological and geometric properties.

The fundamental fixed point theorem (**FPT** for convenience) w.r.t. metric assumptions is a *Banach contraction principle* (**BCP** for brief) which states that if $(\mathcal{M}, d)$ is a complete metric space and $\eta : \mathcal{M} \to \mathcal{M}$ is a contraction, that is, there exists a number $\alpha \in (0, 1)$ such that

$$d(\eta x, \eta y) \leq \alpha d(x, y), \quad \forall x, y \in \mathcal{M}, \tag{1}$$

then η has a unique fixed point and for any $x_0 \in \mathcal{M}$ the iterative sequence $\{\eta^n(x_0)\}$ converges to the fixed point of η. We recall that if in inequality (1) we have $\alpha = 1$, then the mapping η is said to be *nonexpansive*. The study of fixed points for nonexpansive self-mappings needs to use the technique of topological and geometric properties and most of them cannot be concluded using the iterative sequences approaches. For instance, the function $\eta : [1, +\infty) \to [1, +\infty)$ which is

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defined by $\eta x = x + \frac{1}{x}$ is a nonexpansive self-mapping which is fixed point free. Besides, if we use the notion of compactness as a topological property and the notion of convexity as a geometric property, then by using the well-known Schauder's **FPT** we deduce that every nonexpansive self-map defined on a compact and convex subset of a Banach space X has a fixed point. On the other hand, if we consider the Banach space c_0 consisting of all real sequences which are convergent to zero equipped with the supremum norm and define a map η from the closed unit ball $\mathfrak{B}_{c_0}$ into itself with

$$\eta(x_1, x_2, \dots) = (1, x_1, x_2, \dots), \quad \forall (x_1, x_2, \dots) \in \mathfrak{B}_{c_0},$$

then η is an isometry map and so is a nonexpansive mapping which is fixed point free. As we know, the closed unit ball is noncompact because of the fact that c_0 is an infinite dimensional space. Hence, for nonexpansive self-mappings which are defined on a nonempty, bounded, closed and convex subset of a Banach space X, we must consider another geometric concept. We mention that a convex subset $\mathcal{G}$ of a Banach space X has a *normal structure* in the sense of Brodskii and Milman ([6]) if for every nonempty, bounded, closed and convex subset $\mathcal{U}$ of $\mathcal{G}$ with $\operatorname{diam}(\mathcal{U}) > 0$ it is the case that there is a point $p \in \mathcal{U}$ such that $\sup\{\|p - x\|; x \in \mathcal{U}\} < \operatorname{diam}(\mathcal{U})$. For instance, every compact and convex subset of a Banach space X has the normal structure (see [15] for more details). A Banach space X is said to have normal structure provided that each nonempty and convex subset of X has the normal structure. Hilbert spaces and more generally, uniformly convex Banach spaces are samples of Banach spaces having the normal structure.

In 1965, W.A. Kirk ([16]) established the following topological **FPT** for nonexpansive self-mappings by employing the notion of normal structure.

Theorem 1.1. (Kirk's **FPT**) *Let $\mathcal{G}$ be a nonempty, weakly compact and convex subset of a Banach space X and let $\eta : \mathcal{G} \to \mathcal{G}$ be a nonexpansive mapping. If X has the normal structure, then $\mathfrak{Fix}(\eta) \neq \emptyset$, where $\mathfrak{Fix}(\eta)$ stands for the set of all fixed points of the map η.*

There are many extensions of the aforesaid metric and topological **FPT**s but most of them are based on the continuity of the considered self-map. In 1968, Kannan introduced the following category of mappings.

Definition 1.2. ([13]) *Let $(\mathcal{M}, d)$ be a metric space. A self-mapping $\eta : \mathcal{M} \to \mathcal{M}$ is called*

(i) *a Kannan contraction if there exists $\alpha \in [0, \frac{1}{2})$ such that*

$$d(\eta x, \eta y) \leq \alpha\Big(d(x, \eta x) + d(y, \eta y)\Big), \quad \forall x, y \in \mathcal{M};$$

(ii) *Kannan nonexpansive whenever*

$$d(\eta x, \eta y) \leq \frac{1}{2}\Big(d(x, \eta x) + d(y, \eta y)\Big), \quad \forall x, y \in \mathcal{M};$$

A similar result of **BCP** is valid for Kannan contraction maps, but the importance of Kannan's **FPT** w.r.t. **BCP** is that Kannan contractions may not be continuous, whereas the continuity of the contraction map η in **BCP** has a key role in the proof of the existence of a fixed point.

Another observation about the comparison between **BCP** and Kannan's **FPT** is that Kannan's theorem characterizes the metric completeness, that is, a metric space $\mathcal{M}$ is complete if and only if every Kannan contraction self-map defined on $\mathcal{M}$ has a fixed point (see [18]), whereas **BCP** cannot characterize the completeness of metric spaces (see [8] for more details).

From that side, the existence of a fixed point for a Kannan nonexpansive self-mapping needs a stronger geometric notion of normal structure, named *quasi-normal structure*.

Definition 1.3. *A subset $\mathcal{G}$ of Banach space X is said to have quasi-normal structure if for every nonempty, bounded, closed and convex subset $\mathcal{U}$ of $\mathcal{G}$ with $\operatorname{diam}(\mathcal{U}) > 0$, then there exists a point $x^\star \in \mathcal{U}$ such that*

$$\|x - x^\star\| < \operatorname{diam}(\mathcal{U}), \quad \forall x \in \mathcal{U}.$$

The point $x^\star \in \mathcal{U}$ is said to be a non-diametral point. Also, a Banach space X is said to have quasi-normal structure provided that every nonempty and convex subset of X has the quasi-normal structure.

From the above definition, we find out that

$$\text{normal structure} \Rightarrow \text{quasi-normal structure}.$$

The next is a Kannan version of Kirk's **FPT** which was established by C. Wong in [21].

Theorem 1.4. (Kannan's **FPT**) *Let $\mathcal{G}$ be a nonempty, weakly compact and convex subset of a Banach space X and let $\eta : \mathcal{G} \to \mathcal{G}$ be a Kannan nonexpansive mapping. If X has the quasi-normal structure, then $\mathfrak{Fix}(\eta) \neq \emptyset$.*

It is worth noticing that in both Theorem 1.1 and Theorem 1.4, the weak compactness assumption of the set $\mathcal{G}$ can be replaced by the condition that $\mathcal{G}$ is bounded, closed and convex subset of a Banach space X which is reflexive. This means that these theorems are true in Hilbert or in more general, uniformly convex Banach spaces such as ℓ^p for $1 < p < \infty$ which have the normal structure. A natural question arises as to what happens in a nonreflexive Banach space such as ℓ^∞. To answer to this question which is well-known as Baillon's **FPT** ([5]), we recall some essential concepts related to *hyperconvex metric spaces*.

Let $(\mathcal{M}, d)$ be a metric space. A closed ball with center at $x_0 \in \mathcal{M}$ and radius $r > 0$ will be displayed by $B[x_0, r]$. We will say that a set $\mathcal{G} \subseteq \mathcal{M}$ is *admissible* if $\mathcal{G}$ is a nonempty intersection of a family of closed balls.

Also, for a nonempty subset $\mathcal{G}$ of $\mathcal{M}$ the cover of $\mathcal{G}$ is denoted by $\operatorname{cov}(\mathcal{G})$ and defined by

$$\operatorname{cov}(\mathcal{G}) = \bigcap \Big\{ \mathcal{C};\ \mathcal{C} \ \text{ is a closed ball containing } \mathcal{G} \Big\}.$$

In this way, $\mathcal{G} \subseteq \mathcal{M}$ is admissible if and only if $\operatorname{cov}(\mathcal{G}) = \mathcal{G}$. The set of all nonempty and admissible subsets of a metric space $(\mathcal{M}, d)$ is denoted by $\mathfrak{A}(\mathcal{M})$.

The notion of Hyperconvexity on metric spaces which is a suitable property for closed balls in metric spaces, was introduced by Aronszajn and Panitchpakdi in 1965 as below.

Definition 1.5. ([4]) *Let $(\mathcal{M}, d)$ be a metric space. Then $\mathcal{G} \subseteq \mathcal{M}$ is said to be hyperconvex if for every family $\{x_i\}_{i \in I}$ in the set $\mathcal{G}$ and each family $\{r_i\}_{i \in I}$ of positive numbers if $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$, then*

$$\Big(\bigcap_{i \in I} B[x_i; r_i] \Big) \cap \mathcal{G} \neq \emptyset.$$

Particularly, if $\mathcal{G} = \mathcal{M}$, then we say that $\mathcal{M}$ is a hyperconvex metric space.

For instance the nonreflexive Banach spaces L^∞, l^∞ are hyperconvex (see [15] for more details). Or if we consider the Euclidean plan $\mathbb{R}^2$ with the river metric d_{riv} defined by

$$d_{\text{riv}}(\mathbf{x}, \mathbf{y}) = \begin{cases} |x_2 - y_2|, & \text{if } x_1 = y_1, \\ |x_2| + |y_2| + |x_1 - y_1|, & \text{if } x_1 \neq y_1, \end{cases}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, is a hyperconvex metric space (see Lemma 2 of [7]). For additional information regarding extensions of hyperconvex spaces, the reader is referred to [2].

Note that all hyperconvex metric spaces are complete and each admissible subset of a hyperconvex space is hyperconvex too ([15]). The next theorem is a counterpart result of Kirk's **FPT** in the framework of hyperconvex spaces (We refer the reader to [9, 20] for alternative approaches to extending this problem).

Theorem 1.6. (Baillon's **FPT** [5]) *Let $(\mathcal{M}, d)$ be a bounded hyperconvex metric space and $\eta : \mathcal{M} \to \mathcal{M}$ be a nonexpansive mapping. Then $\mathfrak{Fix}(\eta) \neq \emptyset$.*

As a corollary of Baillon's **FPT** we conclude that every nonexpansive self-mapping defined on a closed ball $B[0;1]$ in a Banach space ℓ^∞ has a fixed point. Note that $B[0;1]$ in ℓ^∞ is not weakly compact and so this result cannot be concluded from Kirk's **FPT**. On the other hand, it follows from Lemma 4.1 of [15] that if $\mathcal{G}$ is a bounded subset of a hyperconvex metric space $(\mathcal{M}, d)$ then $\mathcal{G}$ has a non-diametral point and so, the Banach space ℓ^∞ has the normal structure, automatically.

In this paper, we obtain a similar result of Baillon's **FPT** for Kannan nonexpansive maps. To this end, we consider cyclic maps defined on the union of two nonempty subsets of a hyperconvex metric space that satisfy the Kannan-type contraction and Kannan-type nonexpansive conditions, and we investigate the existence of best proximity points under suitable sufficient conditions.

2 Preliminaries

The notation and terminology used in this paper are standard. For convenience of the reader we recall some basic facts.

Let $(\mathcal{G}, \mathcal{H})$ be a nonempty pair of subsets of a metric space $(\mathcal{M}, d)$. A mapping $\eta : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is said to be *cyclic* if $\eta(\mathcal{G}) \subseteq \mathcal{H}$ and $\eta(\mathcal{H}) \subseteq \mathcal{G}$.

We fix the following notations:

$$\delta_u(\mathcal{G}) = \sup\Big\{d(u, a) : a \in \mathcal{G}\Big\}, \quad \forall u \in \mathcal{M};$$
$$\delta(\mathcal{G}, \mathcal{H}) = \sup\Big\{d(a, b) : (a, b) \in \mathcal{G} \times \mathcal{H}\Big\}.$$

From now on, we say that a pair $(\mathcal{G}, \mathcal{H})$ of nonempty subsets of a metric space $\mathcal{M}$ has a property if both the sets $\mathcal{G}$ and $\mathcal{H}$ have that property; For example our meaning about the closedness of the pair $(\mathcal{G}, \mathcal{H})$ is that both $\mathcal{G}$ and $\mathcal{H}$ are closed in $\mathcal{M}$.

Definition 2.1. *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty pair of subsets of a metric space $(\mathcal{M}, d)$ and η be a cyclic mapping on $\mathcal{G} \cup \mathcal{H}$. A point $u^\star \in \mathcal{G} \cup \mathcal{H}$ is called a best proximity point of the mapping η whenever*

$$d(u^\star, \eta u^\star) = \operatorname{dist}(\mathcal{G}, \mathcal{H}) = \inf\Big\{d(a, b);\ (a, b) \in \mathcal{G} \times \mathcal{H}\Big\}.$$

The set of all best proximity points of the cyclic mapping η is displayed by $\mathfrak{Best}(\eta)$, that is,

$$\mathfrak{Best}(\eta) = \Big\{u^\star \in \mathcal{G} \cup \mathcal{H} \text{ s.t. } d(u^\star, \eta u^\star) = \operatorname{dist}(\mathcal{G}, \mathcal{H})\Big\}.$$

The *proximal pair* of $(\mathcal{G}, \mathcal{H})$ is denoted by $(\mathcal{G}_0, \mathcal{H}_0)$ and defined as

$$\mathcal{G}_0 = \Big\{u \in G\colon d(u, v') = \operatorname{dist}(\mathcal{G}, \mathcal{H}) \text{ for some } v' \in \mathcal{H}\Big\},$$

$$\mathcal{H}_0 = \Big\{v \in \mathcal{H}\colon d(u', v) = \operatorname{dist}(\mathcal{G}, \mathcal{H}) \text{ for some } u' \in \mathcal{G}\Big\}.$$

In general, the proximal pair $(\mathcal{G}_0, \mathcal{H}_0)$ can be empty. But if $(\mathcal{G}, \mathcal{H})$ is a nonempty pair of bounded, closed and convex subsets of a reflexive Banach spaces X, then its proximal pair is nonempty. Moreover, just recently, the current authors proved the following auxiliary lemma.

Lemma 2.2. (Lemma 2.5 of [12]) *If $(\mathcal{G}, \mathcal{H})$ is a pair of nonempty and admissible subsets of a hyperconvex metric space $(\mathcal{M}, d)$, then the proximal pair $(\mathcal{G}_0, \mathcal{H}_0)$ is also nonempty and admissible.*

As an important application of Lemma 2.2 we show that every closed ball in the Banach space ℓ^∞ is a proximinal set. Let $\mathcal{G}$ be a nonempty subset of a normed linear space X. Define a set-valued map $\mathbb{P}_\mathcal{G} : X \to 2^\mathcal{G}$ with

$$\mathbb{P}_\mathcal{G}(x) = \big\{u \in \mathcal{G};\ \|x - u\| = \operatorname{dist}(\{x\}, \mathcal{G})\big\}, \quad \forall x \in X.$$

The elements of $\mathbb{P}_\mathcal{G}(x)$ are known as *best approximations of x in $\mathcal{G}$*. The set $\mathcal{G}$ is said to be *proximinal* provided that $\mathbb{P}_\mathcal{G}(x) \neq \emptyset$ for any $x \in X$. Moreover, $\mathcal{G}$ is said to be a *Chebyshev set* if $\mathbb{P}_\mathcal{G}(x)$ is singleton for any $x \in X$. It is well-known that if $\mathcal{G}$ is a nonempty, bounded, closed and convex subset of a reflexive Banach space X, then $\mathcal{G}$ is a proximinal set, and furthermore, if the norm of the reflexive Banach space X satisfies the strict convexity, then the set $\mathcal{G}$ will be a Chebyshev set (see [11] for more details). Therefore, any closed ball in the Banach space ℓ^p $(1 < p < \infty)$ is a Chebyshev set. We now have the following observation on the non-reflexive Banach space ℓ^∞ as a famous hyperconvex space.

Theorem 2.3. *Every closed ball in ℓ^∞ is a proximinal set.*

Proof. For convenience, we consider the closed ball $B[0; 1]$ and let $p = (p_1, p_2, \dots) \in \ell^\infty$ be such that $\|p\|_\infty > 1$. Set $\mathcal{G} = B[0; 1]$ and $\mathcal{H} = \{p\}$. Since $\mathcal{H} = \bigcap_{n\in\mathbb{N}} B[p; \frac{1}{n}]$, we have $\mathcal{H} \in \mathfrak{A}(\mathcal{M})$. Since ℓ^∞ is hyperconvex, by using Lemma 2.2, the pair $(\mathcal{G}, \mathcal{H})$ is proximinal, that is, there exists a point $u \in \mathcal{G}$ for which $\|p - u\| = \operatorname{dist}(\{p\}, \mathcal{G})$. Thus $u \in \mathbb{P}_\mathcal{G}(p)$ for any $p \in \ell^\infty$ (note that in the case that $p \in \mathcal{G}$, then $\mathbb{P}_\mathcal{G}(p) = p$) and so, the set $\mathcal{G}$ is a proximinal set. It is worth noticing that for any $x \in \mathcal{H}$ we have

$$\|p - x\|_\infty \geq \|p\|_\infty - \|x\|_\infty = \|p\|_\infty - 1,$$

which implies that $\operatorname{dist}(\mathcal{G}, \mathcal{H}) \geq \|p\|_\infty - 1$. Besides, if we consider $u = \big(\frac{p_1}{\|p\|_\infty}, \frac{p_2}{\|p\|_\infty}, \dots\big)$, then $u \in \mathcal{G}$ and $\|p - u\|_\infty = \|p\|_\infty - 1$. Thus, $\operatorname{dist}(\mathcal{G}, \mathcal{H}) = \|p\|_\infty - 1$. It is interesting to note that the set $\mathcal{G}$ is not Chebyshev because of the fact that ℓ^∞ is not strictly convex. For instance, if we consider $q := (2, 0, 0, \dots) \in \ell^\infty$, then any point $y = (1, y_1, y_2, \dots) \in \mathcal{G}$ with $|y_n| \leq 1$ for all $n \in \mathbb{N}$ is a best approximation of the point q. $\square$

Remark 2.4. *It is worth mentioning that every closed ball in L^∞ is also proximinal (see [4]). More generally, any admissible subset in a hyperconvex metric space (M, d) is proximinal. Indeed, if*

$$\mathcal{G} := \bigcap_{i\in I} B[x_i; r_i] \in \mathfrak{A}(\mathcal{M}),$$

and $x \in M$, then by setting $s := \operatorname{dist}(\{x\}, \mathcal{G})$ and considering the family of closed balls

$$\mathcal{F} \ := \ \Big\{ B[x_i; r_i],\ B[x; s] \Big\}_{i\in I},$$

we have

$$d(x_i, x) \leq d(x_i, u) + d(u, x) \leq r_i + d(u, x), \quad \forall u \in \mathcal{G}.$$

Now, by taking infimum over $u \in \mathcal{G}$, we obtain $d(x_i, x) \leq r_i + s$. In view of the fact that M is hyperconvex, $\bigcap \mathcal{F} \neq \emptyset$. Let $v \in \mathcal{F}$. Thus $v \in \mathcal{G}$ and that $d(v, x) \leq s$ which concludes that $d(x, v) = \operatorname{dist}(\{x\}, \mathcal{G})$.

Definition 2.5. *A nonempty pair $(\mathcal{G},\mathcal{H})$ in a metric space $(\mathcal{M},d)$ is said to be proximinal provided that for any $(x,y)\in\mathcal{G}\times\mathcal{H}$, there is an element $(x',y')\in\mathcal{G}\times\mathcal{H}$ such that*

$$d(x,y')=\operatorname{dist}(\mathcal{G},\mathcal{H})=d(x',y),$$

or equivalently, $\mathcal{G}_0=\mathcal{G}$ and $\mathcal{H}_0=\mathcal{H}$.

In 2013, Abkar and Gabeleh introduced a concept of *proximal quasi-normal structure* in Banach spaces as below.

Definition 2.6. ([1]) *A convex pair $(\mathcal{G},\mathcal{H})$ in a Banach space X is said to have proximal quasi-normal structure* (**PQNS** for brief) *if for any bounded, closed, convex and proximinal pair $(\mathcal{K}_1,\mathcal{K}_2)\subseteq(\mathcal{G},\mathcal{H})$ for which $\operatorname{dist}(\mathcal{K}_1,\mathcal{K}_2)=\operatorname{dist}(\mathcal{G},\mathcal{H})$ and $\delta(\mathcal{K}_1,\mathcal{K}_2)>\operatorname{dist}(\mathcal{K}_1,\mathcal{K}_2)$, there exits a point $(p,q)\in\mathcal{K}_1\times\mathcal{K}_2$ such that*

$$\max\Big\{d(p,y),d(x,q)\Big\}<\delta(\mathcal{K}_1,\mathcal{K}_2),$$

for all $(x,y)\in\mathcal{K}_1\times\mathcal{K}_2$.

By a proof of Proposition 2.1 of [10] we conclude that every nonempty and convex pair $(\mathcal{G},\mathcal{H})$ in a uniformly convex Banach space X has **PQNS**.

We now present a counterpart concept of **PQNS** in the setting of metric spaces. For a nonempty pair $(\mathcal{G},\mathcal{H})$ in a metric space $(\mathcal{M},d)$ set

$$\nabla(\mathcal{G},\mathcal{H}):=\Big\{(\mathcal{K}_1,\mathcal{K}_2)\subseteq(\mathcal{G},\mathcal{H});\ \mathcal{K}_1,\mathcal{K}_2\in\mathfrak{A}(\mathcal{M}),\ \operatorname{dist}(\mathcal{G},\mathcal{H})=\operatorname{dist}(\mathcal{K}_1,K_2)<\delta(\mathcal{K}_1,\mathcal{K}_2)\Big\}.$$

Definition 2.7. *An admissible pair $(\mathcal{G},\mathcal{H})$ in a metric space $(\mathcal{M},d)$ is said to have $\hbar$-proximal quasi-normal structure* ($\hbar$-**PQNS** briefly) *if for any $(\mathcal{K}_1,\mathcal{K}_2)\in\nabla(\mathcal{G},\mathcal{H})$, there exists a point $(p,q)\in\mathcal{K}_1\times\mathcal{K}_2$ so that*

$$\max\Big\{d(p,y),d(x,q)\Big\}<\delta(\mathcal{K}_1,\mathcal{K}_2),$$

for all $(x,y)\in\mathcal{K}_1\times\mathcal{K}_2$.

The next lemma shows that $\hbar$-**PQNS** is an inherent property of admissible pairs in hyperconvex spaces.

Lemma 2.8. *If $(\mathcal{G},\mathcal{H})$ is a nonempty and admissible pair in a hyperconvex metric space $(\mathcal{M},d)$, then $(\mathcal{G},\mathcal{H})$ has an $\hbar$-***PQNS**.

Proof. Let $(\mathcal{K}_1,\mathcal{K}_2)\in\nabla(\mathcal{G},\mathcal{H})$ and for $r:=\delta(\mathcal{K}_1,\mathcal{K}_2)+\operatorname{dist}(\mathcal{G},\mathcal{H})$ define

$$\mathcal{U}:=\Big(\bigcap_{y\in\mathcal{K}_2}B[y;\frac{r}{2}]\Big)\cap\mathcal{K}_1,$$
$$\mathcal{V}:=\Big(\bigcap_{x\in\mathcal{K}_1}B[x;\frac{r}{2}]\Big)\cap\mathcal{K}_2.$$

Suppose $v_1,v_2\in\mathcal{K}_2$. The proximinality of $(\mathcal{K}_1,\mathcal{K}_2)$ implies that there exists an element $u_1\in\mathcal{K}_1$ such that $d(u_1,v_1)=\operatorname{dist}(\mathcal{G},\mathcal{H})$. Thus

$$d(v_1,v_2)\le d(v_1,u_1)+d(u_1,v_2)\le r.$$

By the hyperconvexity of $\mathcal{M}$,

$$\bigcap_{v\in\mathcal{K}_2}B[v;\frac{r}{2}]\neq\emptyset.$$

Since $\mathcal{K}_1$ is admissible, we may assume that $\mathcal{K}_1 = \bigcap_{i\in I} B[u_i; r_i]$, where $u_i \in \mathcal{M}$ and $r_i > 0$ for all $i \in I$. Consider the family of closed balls $\Big\{B[u_i; r_i],\ B[v; \frac{r}{2}];\ i \in I \text{ and } v \in \mathcal{K}_2\Big\}$. Let $i \in I$ and $v \in \mathcal{K}_2$. Then there is a point $u \in \mathcal{K}_1$ with $d(u, v) = \operatorname{dist}(\mathcal{G}, \mathcal{H})$ and so,

$$d(u_i, v) \le d(u_i, u) + d(u, v) \le r_i + \operatorname{dist}(\mathcal{G}, \mathcal{H}) \le r_i + \frac{r}{2}.$$

Again using the hyperconvexity of $\mathcal{M}$ we obtain $\mathcal{U} \neq \emptyset$. By an equivalent argument, $\mathcal{V} \neq \emptyset$. Suppose $(p, q) \in \mathcal{U} \times \mathcal{V}$. Then $(p, q) \in \mathcal{K}_1 \times \mathcal{K}_2$ and for any $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ we have

$$\max\Big\{d(p, y), d(x, q)\Big\} \le \frac{r}{2} = \frac{1}{2}\Big(\delta(\mathcal{K}_1, \mathcal{K}_2) + \operatorname{dist}(\mathcal{G}, \mathcal{H})\Big) < \delta(\mathcal{K}_1, \mathcal{K}_2),$$

and hence the lemma. $\square$

In a particular case of Lemma 2.8, we conclude that *every nonempty and admissible subset of a hyperconvex metric space has the quasi-normal structure in the sense of Definition* 1.3.

Motivated by the notion of Kannan contractions and Kannan nonexpansive maps the following classes of cyclic mappings was considered in [1].

Definition 2.9. *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty pair in a metric space $(\mathcal{M}, d)$. Then $\eta : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ is said to be*

(1) *cyclic Kannan contraction if η is a cyclic mapping and there exists $\alpha \in [0, \frac{1}{2})$ such that*

$$d(\eta x, \eta y) \le \alpha\Big(d(x, \eta x) + d(y, \eta y)\Big) + (1 - 2\alpha)\operatorname{dist}(\mathcal{G}, \mathcal{H}), \quad \forall (x, y) \in \mathcal{G} \times \mathcal{H},$$

(2) *cyclic relatively Kannan nonexpansive if η is a cyclic mapping such that*

$$d(\eta x, \eta y) \le \frac{1}{2}\Big(d(x, \eta x) + d(y, \eta y)\Big), \quad \forall (x, y) \in \mathcal{G} \times \mathcal{H},$$

(3) *a mapping which preserves distance whenever $d(\eta x, \eta y) = \operatorname{dist}(\mathcal{G}, \mathcal{H})$ if $d(x, y) = \operatorname{dist}(\mathcal{G}, \mathcal{H})$ for an element $(x, y) \in \mathcal{G} \times \mathcal{H}$.*

We finish this section by stating the following best proximity theorems for cyclic contractions and relatively Kannan nonexpansive maps in Banach spaces.

Theorem 2.10. (see Theorem 2.3 of [1]) *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty weakly compact and convex pair in a Banach space X. If $\eta : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ is a cyclic Kannan contraction, then $\mathfrak{Best}(\eta) \neq \emptyset$.*

Theorem 2.11. (see Theorem 3.8 of [1]) *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty weakly compact and convex pair in a Banach space X and assume that $(\mathcal{G}, \mathcal{H})$ has a* **PQNS**. *If $\eta : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ is a cyclic relatively Kannan nonexpansive mapping which preserves distance, then $\mathfrak{Best}(\eta) \neq \emptyset$.*

Note in Theorem 2.10 and Theorem 2.11, the weak compactness of the considered pair $(\mathcal{G}, \mathcal{H})$ plays an important role in the proof of them (for additional approaches to related problems, see also [3, 17, 19]). As we will see, we obtain the same conclusions of these theorems in hyperconvex setting and drop the weak compactness assumption in some particular cases.

3 Main Results

We begin the results of this section by studying the existence of best proximity points for cyclic contractions in hyperconvex spaces.

Theorem 3.1. *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty pair of admissible pair in a hyperconvex space $(\mathcal{M}, d)$. If $\eta : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is a cyclic Kannan contraction, then $\mathfrak{Best}(\eta) \neq \emptyset$.*

Proof. Let $\mathfrak{F}$ denote the set of all nonempty admissible pairs $(\mathcal{U}, \mathcal{V})$ such that η is cyclic on $\mathcal{U} \cup \mathcal{V}$. Then $\mathfrak{F}$ is nonempty since $(\mathcal{G}, \mathcal{H}) \in \mathfrak{F}$. Suppose $\{(\mathcal{U}_j, \mathcal{V}_j)\}_{j \in \digamma}$ is a descending chain in the set $\mathfrak{F}$ and let $\mathcal{A} = \bigcap_{j \in \digamma} \mathcal{U}_j$ and $\mathcal{B} = \bigcap_{j \in \digamma} \mathcal{V}_j$. We assert that $(\mathcal{A}, \mathcal{B}) \in \mathfrak{F}$. Since $\mathcal{U}_j \in \mathfrak{A}(\mathcal{M})$, we may assume that $\mathcal{U}_j = \bigcap_i B[x_{j_i}, r_{j_i}]$ for all $j \in \digamma$. Let $j, j' \in \digamma$ and $\mathcal{U}_j \subseteq \mathcal{U}_{j'}$. Then for any $u \in \mathcal{U}_j$ we have

$$d(x_{j_i}, x_{j'_i}) \leq d(x_{j_i}, u) + d(u, x_{j'_i}) \leq r_{j_i} + r_{j'_i}, \quad \forall i,$$

and using the hyperconvexity we obtain $\bigcap_{j \in \digamma} \bigcap_i B[x_{j_i}, r_{j_i}] \neq \emptyset$ which means that $\mathcal{A} \in \mathfrak{A}(\mathcal{M})$. By a similar manner, $\mathcal{B} \in \mathfrak{A}(\mathcal{M})$. Applying Zorn's lemma to find a minimal element for the set $\mathfrak{F}$, say $(\mathcal{K}_1, \mathcal{K}_2)$. Note that $\Big(\mathrm{cov}\big(\eta(\mathcal{K}_2)\big), \mathrm{cov}\big(\eta(\mathcal{K}_1)\big)\Big) \subseteq \big(\mathcal{K}_1, \mathcal{K}_2\big)$. Moreover, $\eta\big(\mathrm{cov}(\eta(\mathcal{K}_2))\big) \subseteq \mathrm{cov}\big(\eta(\mathcal{K}_1)\big)$ and also $\eta\big(\mathrm{cov}(\eta(\mathcal{K}_1))\big) \subseteq \mathrm{cov}\big(\eta(\mathcal{K}_2)\big)$. By the minimality of $(\mathcal{K}_1, \mathcal{K}_2) \in \mathfrak{F}$,

$$\mathrm{cov}\big(\eta(\mathcal{K}_2)\big) = \mathcal{K}_1, \quad \mathrm{cov}\big(\eta(\mathcal{K}_1)\big) = \mathcal{K}_2.$$

Assume $p \in \mathcal{K}_1$, then $\mathcal{K}_2 \subseteq B[p; \delta_p(\mathcal{K}_2)]$. If $y \in \mathcal{K}_2$, then by the fact that η is a cyclic Kannan contraction,

$$\begin{aligned} d(\eta p, \eta y) &\leq \alpha\Big(d(p, \eta p) + d(y, \eta y)\Big) + (1 - 2\alpha)\mathrm{dist}(\mathcal{G}, \mathcal{H}) \\ &\leq 2\alpha\delta(\mathcal{K}_1, \mathcal{K}_2) + (1 - 2\alpha)\mathrm{dist}(\mathcal{G}, \mathcal{H}). \end{aligned}$$

Therefore, for all $y \in \mathcal{K}_2$,

$$\eta y \in \eta(\mathcal{K}_2) \subseteq B[\eta p; 2\alpha\delta(\mathcal{K}_1, \mathcal{K}_2) + (1 - 2\alpha)\mathrm{dist}(\mathcal{G}, \mathcal{H})].$$

Hence,

$$\mathcal{K}_1 = \mathrm{cov}\big(\eta(\mathcal{K}_2)\big) \subseteq B[\eta p; 2\alpha\delta(\mathcal{K}_1, \mathcal{K}_2) + (1 - 2\alpha)\mathrm{dist}(\mathcal{G}, \mathcal{H})].$$

This implies

$$d(x, \eta p) \leq 2\alpha\delta(\mathcal{K}_1, \mathcal{K}_2) + (1 - 2\alpha)\mathrm{dist}(\mathcal{G}, \mathcal{H}), \quad \forall x \in \mathcal{K}_1,$$

so that

$$\delta_{\eta p}(\mathcal{K}_1) \leq 2\alpha\delta(\mathcal{K}_1, \mathcal{K}_2) + (1 - 2\alpha)\mathrm{dist}(\mathcal{G}, \mathcal{H}). \tag{2}$$

Similarly, if $q \in \mathcal{K}_2$, then

$$\delta_{\eta q}(\mathcal{K}_2) \leq 2\alpha\delta(\mathcal{K}_1, \mathcal{K}_2) + (1 - 2\alpha)\mathrm{dist}(\mathcal{G}, \mathcal{H}). \tag{3}$$

Now, we set

$$\begin{aligned} \mathcal{E}_1 &:= \Big\{x \in \mathcal{K}_1 : \delta_x(\mathcal{K}_2) \leq 2\alpha\delta(\mathcal{K}_1, \mathcal{K}_2) + (1 - 2\alpha)\mathrm{dist}(\mathcal{G}, \mathcal{H})\Big\}, \\ \mathcal{E}_2 &:= \Big\{y \in \mathcal{K}_2 : \delta_y(\mathcal{K}_1) \leq 2\alpha\delta(\mathcal{K}_1, \mathcal{K}_2) + (1 - 2\alpha)\mathrm{dist}(\mathcal{G}, \mathcal{H})\Big\}. \end{aligned}$$

Then, $(\mathcal{E}_1, \mathcal{E}_2)$ is nonempty and it is easy to see that

$$\mathcal{E}_1 = \bigcap_{y\in\mathcal{K}_2} B\Big[y;\ 2\alpha\delta(\mathcal{K}_1,\mathcal{K}_2) + (1-2\alpha)\mathrm{dist}(\mathcal{G},\mathcal{H})\Big] \cap \mathcal{K}_1,$$
$$\mathcal{E}_2 = \bigcap_{x\in\mathcal{K}_1} B\Big[x;\ 2\alpha\delta(\mathcal{K}_1,\mathcal{K}_2) + (1-2\alpha)\mathrm{dist}(\mathcal{G},\mathcal{H})\Big] \cap \mathcal{K}_2.$$

Moreover, by (2) and (3), we see that η is cyclic on $\mathcal{E}_1 \cup \mathcal{E}_2$. By the minimality of $(\mathcal{K}_1, \mathcal{K}_2)$, we must have $\mathcal{E}_1 = \mathcal{K}_1$ and $\mathcal{E}_2 = \mathcal{K}_2$. Thus, we obtain

$$\delta_x(\mathcal{K}_2) \leq 2\alpha\delta(\mathcal{K}_1,\mathcal{K}_2) + (1-2\alpha)\mathrm{dist}(\mathcal{G},\mathcal{H}), \quad \forall x \in \mathcal{K}_1,$$

which implies that

$$\delta(\mathcal{K}_1,\mathcal{K}_2) = \sup_{x\in\mathcal{K}_1} \delta_x(\mathcal{K}_2) \leq 2\alpha\delta(\mathcal{K}_1,\mathcal{K}_2) + (1-2\alpha)\mathrm{dist}(\mathcal{G},\mathcal{H}).$$

Therefore, $\delta(\mathcal{K}_1,\mathcal{K}_2) = \mathrm{dist}(\mathcal{G},\mathcal{H})$, that is

$$d(x,\eta x) \leq \delta(\mathcal{K}_1,\mathcal{K}_2) = \mathrm{dist}(\mathcal{G},\mathcal{H}), \quad \forall x \in \mathcal{K}_1.$$

So, $\mathcal{K}_1 \subseteq \mathfrak{Best}(\eta) \cap \mathcal{G}$. Similarly, we have $\mathcal{K}_2 \subseteq \mathfrak{Best}(\eta) \cap \mathcal{H}$ and the result follows.

□

Let us illustrate Theorem 3.1 with the next example.

Example 3.1. Consider the hyperconvex metric space $(\mathbb{R}^2, d_{\mathrm{riv}})$ and let

$$\mathcal{G} = \{(0,x) : 0 \leq x \leq \frac{1}{2}\}, \quad \mathcal{H} = \{(1,y) : 0 \leq y \leq 1\}.$$

Then it is easy to check that $\mathcal{G} = B\big[(0,\frac{1}{4}),\frac{1}{4}\big]$ and $\mathcal{H} = B\big[(1,\frac{1}{2}),\frac{1}{2}\big]$ which ensures that $\mathcal{G},\mathcal{H} \in \mathfrak{A}(\mathcal{M})$. Also, $\mathrm{dist}(\mathcal{G},\mathcal{H}) = 1$ Define $\eta : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ with

$$\eta(0,x) = (1,x^2), \quad \eta(1,y) = (0,\frac{y}{2}), \quad \forall (x,y) \in [0,\frac{1}{2}] \times [0,1].$$

Thus

$$\begin{aligned} d_{\mathrm{riv}}\Big(\eta(0,x),\eta(1,y)\Big) &= d_{\mathrm{riv}}\Big((1,x^2),(0,\frac{y}{2})\Big) = 1 + x^2 + \frac{y}{2} \\ &\leq \frac{1}{3}\Big(d_{\mathrm{riv}}((0,x),\eta(0,x)) + d_{\mathrm{riv}}((1,y),\eta(1,y))\Big) + (1-(2)\frac{1}{3})\mathrm{dist}(\mathcal{G},\mathcal{H}), \end{aligned}$$

and so η is a cyclic Kannan contraction mapping with $\alpha = \frac{1}{3}$. Using Theorem 3.1 to obtain best proximity points for η. We note that $\mathfrak{Best}(\eta) = \Big\{(0,0),(1,0)\Big\}$.

In what follows, we use the concept of $\hbar$-**PQNS** to establish the following best proximity point theorem for relatively Kannan nonexpansive maps.

Theorem 3.2. *Let $(\mathcal{G},\mathcal{H})$ be a nonempty pair of admissible pair in a hyperconvex space $(\mathcal{M},d)$. Suppose $\eta : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ is a cyclic relatively Kannan nonexpansive which preserves distance. Then $\mathfrak{Best}(\eta) \neq \emptyset$.*

Proof. By Lemma 2.2, $\mathcal{G}_0, \mathcal{H}_0 \in \mathfrak{A}(\mathcal{M})$. Also, since η preserves distance, we conclude that η is cyclic on $\mathcal{G}_0 \cup \mathcal{H}_0$. Let $\prod$ denote the set of all nonempty, admissible and proximinal pairs $(\mathcal{E}, \mathcal{F})$ that are subsets of $(\mathcal{G}, \mathcal{H})$ and $\operatorname{dist}(\mathcal{E}, \mathcal{F}) = \operatorname{dist}(\mathcal{G}, \mathcal{H})$ and that η is cyclic on $\mathcal{E} \cup \mathcal{F}$. Note that $(\mathcal{G}_0, \mathcal{H}_0) \in \prod \neq \emptyset$. Assume $\{(\mathcal{E}_j, \mathcal{F}_j)\}_{j \in \digamma}$ is a descending chain in $\prod$ and put $\mathcal{U} = \bigcap_{j \in \digamma} \mathcal{E}_j$ and $\mathcal{V} = \bigcap_{j \in \digamma} \mathcal{F}_j$. By a similar proof of Theorem 3.1, $\mathcal{U}, \mathcal{V} \in \mathfrak{A}(\mathcal{M})$ and it is easy to see that η is cyclic on the set $\mathcal{U} \cup \mathcal{V}$. Now, let $x \in \mathcal{U}$. Then $x \in \mathcal{E}_j$ for all $j \in \digamma$. By the fact that each pair $(\mathcal{E}_j, \mathcal{F}_j)$ is proximinal, we must have

$$\mathcal{L}_j := B[x, \operatorname{dist}(\mathcal{G}, \mathcal{H})] \bigcap \mathcal{F}_j \in \mathfrak{A}(\mathcal{M}), \quad \forall j \in \digamma.$$

Since the family $\{\mathcal{F}_j\}_{j \in \digamma}$ is decreasing, we conclude that $\{\mathcal{L}_j\}_{j \in \digamma}$ is also a nested descending net of admissible sets which ensures that $\mathcal{L} := \bigcap_{j \in \digamma} \mathcal{L}_j$ is a nonempty subset of $\mathcal{V}$ and so, $B[x; \operatorname{dist}(\mathcal{G}, \mathcal{H})] \bigcap \mathcal{V}$ is nonempty too. Thus there is a point $y \in \mathcal{V}$ for which $d(x, y) = \operatorname{dist}(\mathcal{G}, \mathcal{H})$, that is, the pair $(\mathcal{U}, \mathcal{V})$ is proximinal with $\operatorname{dist}(\mathcal{U}, \mathcal{V}) = \operatorname{dist}(\mathcal{G}, \mathcal{H})$.

Now applying Zorn's lemma to obtain a minimal element of $\prod$, say $(\mathcal{Y}_1, \mathcal{Y}_2)$. Choose a number $\rho > 0$ such that $\rho \geq \operatorname{dist}(\mathcal{G}, \mathcal{H})$ and consider a point $(p, q) \in \mathcal{Y}_1 \times \mathcal{Y}_2$ for which $d(p, q) = \operatorname{dist}(\mathcal{G}, \mathcal{H})$ and

$$\max \Big\{ d(p, \eta p), d(q, \eta q) \Big\} \leq \rho.$$

Set

$$\mathcal{Y}_1^\rho = \Big\{ x \in \mathcal{Y}_1 : d(x, \eta x) \leq \rho \Big\},$$
$$\mathcal{Y}_2^\rho = \Big\{ y \in \mathcal{Y}_2 : d(y, \eta y) \leq \rho \Big\},$$

and define

$$\mathcal{R}_1^\rho = \operatorname{cov}\Big(\eta(\mathcal{Y}_1^\rho)\Big) \subseteq \mathcal{Y}_2, \quad \mathcal{R}_2^\rho = \operatorname{cov}\Big(\eta(\mathcal{Y}_2^\rho)\Big) \subseteq \mathcal{Y}_1.$$

Then $(q, p) \in \mathcal{R}_1^\rho \times \mathcal{R}_2^\rho$ and so, $\operatorname{dist}(\mathcal{R}_1^\rho, \mathcal{R}_2^\rho) = \operatorname{dist}(\mathcal{G}, \mathcal{H})$. Let $x \in \mathcal{R}_1^\rho$. If $d(x, \eta x) = \operatorname{dist}(\mathcal{G}, \mathcal{H})$, then $x \in \mathcal{Y}_2^\rho$. Suppose $d(x, \eta x) > \operatorname{dist}(\mathcal{G}, \mathcal{H})$ and define $h := \sup\{d(\eta v, \eta x) : v \in \mathcal{Y}_1^\rho\}$. Then we have

$$\eta(\mathcal{Y}_1^\rho) \subseteq B[\eta x; h],$$

and so $\mathcal{R}_1^\rho \subseteq B[\eta x; h]$. In view of the fact that $x \in \mathcal{R}_1^\rho$, we have $d(x, \eta x) \leq h$. Besides, by the definition of the number h, for any $\varepsilon > 0$ there is a point $v' \in \mathcal{Y}_1^\rho$ so that $h - \varepsilon < d(\eta x, \eta v')$. Thereby,

$$\begin{aligned} d(x, \eta x) - \varepsilon &\leq h - \varepsilon \\ &\leq d(\eta x, \eta v') \leq \frac{1}{2}\Big(d(x, \eta x) + d(v', \eta v')\Big) \\ &\leq \frac{1}{2} d(x, \eta x) + \frac{\rho}{2}. \end{aligned}$$

Hence, $d(x, \eta x) \leq \rho + 2\varepsilon$ which concludes that $x \in \mathcal{Y}_2^\rho$. This shows that $\mathcal{R}_1^\rho \subseteq \mathcal{Y}_2^\rho$ and so,

$$\eta(\mathcal{R}_1^\rho) \subseteq \eta(\mathcal{Y}_2^\rho) \subseteq \operatorname{cov}\big(\eta(\mathcal{Y}_2^\rho)\big) = \mathcal{R}_2^\rho.$$

Equivalently, we can see that $\eta(\mathcal{R}_2^\rho) \subseteq \mathcal{R}_1^\rho$, means that η is cyclic on $\mathcal{R}_2^\rho \cup \mathcal{R}_1^\rho$. We assert that

$\delta(\mathcal{R}_2^\rho, \mathcal{R}_1^\rho) \le \rho$. By a result of [14] we have

$$\begin{aligned}
\delta(\mathcal{R}_2^\rho, \mathcal{R}_1^\rho) &= \delta\Big(\mathrm{cov}\big(\eta(\mathcal{Y}_2^\rho)\big), \mathrm{cov}\big(\eta(\mathcal{Y}_1^\rho)\big)\Big)\\
&= \delta\Big(\eta(\mathcal{Y}_2^\rho), \eta(\mathcal{Y}_1^\rho)\Big)\\
&= \sup\Big\{d(\eta z, \eta w) : (z,w) \in \mathcal{Y}_1^\rho \times \mathcal{Y}_2^\rho\Big\}\\
&\le \sup\Big\{\frac{1}{2}\big(d(z,\eta z) + d(w,\eta w)\big) : (z,w) \in \mathcal{Y}_1^\rho \times \mathcal{Y}_2^\rho\Big\}\\
&\le \frac{1}{2}(\rho+\rho) = \rho.
\end{aligned}$$

Suppose that $\rho_0 := \inf\big\{d(u,\eta u) : u \in \mathcal{Y}_1 \cup \mathcal{Y}_2\big\}$. Then $\rho_0 \ge \mathrm{dist}(\mathcal{G},\mathcal{H})$. Let $\{\rho_n\}$ be a strictly decreasing sequence such that $\rho_n \to \rho_0$. Therefore, $\{\mathcal{R}_1^{\rho_n}\}$ and $\{\mathcal{R}_2^{\rho_n}\}$ are decreasing sequences in $\mathfrak{A}(\mathcal{M})$ and by the countably compactness of $\mathfrak{A}(\mathcal{M})$ if we set

$$\mathcal{R}_1^{\rho_0} := \bigcap_n \mathcal{R}_1^{\rho_n}, \quad \mathcal{R}_2^{\rho_0} := \bigcap_n \mathcal{R}_2^{\rho_n},$$

then $\mathcal{R}_1^{\rho_0}, \mathcal{R}_2^{\rho_0} \in \mathfrak{A}(\mathcal{M})$. Furthermore, by the above discussion, η is cyclic on $\mathcal{R}_1^{\rho_0} \cup \mathcal{R}_2^{\rho_0}$ and since $\mathrm{dist}(\mathcal{R}_2^{\rho_n}, \mathcal{R}_1^{\rho_n}) = \mathrm{dist}(\mathcal{G},\mathcal{H})$ for all $n \in \mathbb{N}$, we obtain $\mathrm{dist}(\mathcal{R}_2^{\rho_0}, \mathcal{R}_1^{\rho_0}) = \mathrm{dist}(\mathcal{G},\mathcal{H})$. Hence $(\mathcal{R}_2^{\rho_0}, \mathcal{R}_1^{\rho_0}) \in \prod$ and by the minimality of $(\mathcal{Y}_1,\mathcal{Y}_2)$ we deduce that

$$\mathcal{R}_2^{\rho_0} = \mathcal{Y}_1, \quad \mathcal{R}_1^{\rho_0} = \mathcal{Y}_2.$$

So,

$$d(u,\eta u) \le \rho_0, \quad \forall u \in \mathcal{Y}_1 \cup \mathcal{Y}_2.$$

On the other hand, if $\rho_0 > \mathrm{dist}(\mathcal{G},\mathcal{H})$ by using Lemma 2.8, since the pair $(\mathcal{G},\mathcal{H})$ has $\hbar$-**PQNS**, there exists a point $(u^*,v^*) \in \mathcal{Y}_1 \times \mathcal{Y}_2$ such that

$$\max\Big\{d(u^*,y), d(x,v^*)\Big\} < \delta(\mathcal{Y}_1,\mathcal{Y}_2) \le \rho_0, \quad \forall (x,y) \in \mathcal{Y}_1 \times \mathcal{Y}_2.$$

This implies that

$$\max\Big\{d(u^*,\eta u^*), d(v^*,\eta v^*)\Big\} < \rho_0,$$

which is impossible. Thus $\rho_0 = \mathrm{dist}(\mathcal{G},\mathcal{H})$, that is,

$$(\mathcal{Y}_1,\mathcal{Y}_2) \subseteq \big(\mathfrak{Best}(\eta|_{\mathcal{G}_0}), \mathfrak{Best}(\eta|_{\mathcal{H}_0})\big),$$

and we are finished.

$\square$

Remark 3.3. *Although any hyperconvex metric space can be isometrically embedded into the Banach space L^∞, Theorem 3.1 and Theorem 3.2 cannot be drived from Theorem 2.10 and Theorem 2.11, respectively, because admissible sets in the Banach space L^∞ may not be weakly compact. For example, the unit closed ball in L^∞ is an admissible set that is not weakly compact.*

The next corollary which is a straightforward consequence of Theorem 3.2, cannot be obtained from Theorem 2.11.

Corollary 3.4. *Consider the closed balls $\mathcal{G} = B[x_0; r]$ and $\mathcal{H} = B[y_0; s]$ in the Banach space ℓ^∞. If η is a cyclic relatively nonexpansive mapping which preserves distance on $\mathcal{G} \cup \mathcal{H}$, then $\mathfrak{Best}(\eta) \neq \emptyset$.*

Baillon's **FPT** for Kannan nonexpansive self-maps can be concluded from Theorem 3.2 by taking $\mathcal{G} = \mathcal{H}$.

Corollary 3.5. *Let $\mathcal{G}$ be a nonempty and admissible subset of a hyperconvex metric space $\mathcal{M}$. If $\eta : \mathcal{G} \to \mathcal{G}$ is a Kannan nonexpansive map, then $\mathfrak{Fix}(\eta) \neq \emptyset$.*

The next example shows the useability of Corollary 3.5.

Example 3.2. Consider the hyperconvex Banach space ℓ^∞ and let $\mathcal{G} = B[0; 2]$. Then $\mathcal{G}$ is and admissible set which is not weakly compact in ℓ^∞. Indeed, a Banach space is reflexive iff the closed unit ball is weakly compact and since ℓ^∞ is not reflexive, $\mathcal{G}$ is not weakly compact. Define a function $r : \mathbb{R} \to [0, 2]$ with

$$r(t) = \max\Big\{0, \min\{2, t\}\Big\}, \quad \forall t \in \mathbb{R}.$$

Also, define a function $s : [0, 2] \to [0, 2]$ with

$$s(t) = \begin{cases} 2, & \text{if} \quad t = 0; \\ 1, & \text{if} \quad 0 < t < 2; \\ 0, & \text{if} \quad t = 2. \end{cases}$$

Assume that $\eta : \mathcal{G} \to \ell^\infty$ be defined as follows:

$$\eta(x) = \Big(s\big(r(x_1)\big), 0, 0, \dots\Big), \quad \forall x = (x_1, x_2, \dots) \in \mathcal{G}.$$

Clearly, $\|\eta(x)\|_\infty \leq 2$ for all $x \in \mathcal{G}$ and so η maps $\mathcal{G}$ into itself. We claim that η is a Kannan nonexpansive mapping. For elements $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \mathcal{G}$ we consider the following cases:

Case 1. If $r(x_1), r(y_1) \in (0, 2)$, then $\eta(x) = \eta(y) = (1, 0, 0, \dots)$ and the result follows.

Case 2. If $r(x_1) \in (0, 2)$ and $r(y_1) = 2$, then $\eta(x) = (1, 0, 0, \dots)$ and $\eta(y) = (0, 0, \dots)$ which implies that

$$\|\eta(x) - \eta(y)\|_\infty = 1 \leq \frac{1}{2}\Big(|1 - x_1| + \underbrace{|y_1|}_{\geq 2}\Big) \leq \frac{1}{2}\Big(\|x - \eta(x)\|_\infty + \|y - \eta(y)\|_\infty\Big).$$

Case 3. If $r(x_1) = r(y_1) = 2$, then $\eta(x) = \eta(y) = (0, 0, \dots)$ and the result follows.

Case 4. If $r(x_1) = 0, r(y_1) = 2$, then $x_1 \leq 0$, $\eta(x) = (2, 0, 0, \dots)$ and $y_1 \geq 2$, $\eta(y) = (0, 0, \dots)$. We now have

$$\|\eta(x) - \eta(y)\|_\infty = 2 \leq \frac{1}{2}\Big((2 - x_1) + y_1\Big) \leq \frac{1}{2}\Big(\|x - \eta(x)\|_\infty + \|y - \eta(y)\|_\infty\Big).$$

Thus η is a Kannan nonexpansive map. Therefore, by Corollary 3.5, $\mathfrak{Fix}(\eta) \neq \emptyset$. It is worth noticing that if $u = (u_1, u_2, \dots) \in \mathcal{G}$ is a fixed point of the map η, then we must have $u_j = 0$ for all $j \geq 2$ and that $s\big(r(u_1)\big) = u_1$ which ensures that $u_1 = 1$, that is, $u = (1, 0, 0, \dots)$ is a unique fixed point of η. It is remarkable to note that in this example η is not nonexpansive. Indeed, if we set $\mathbf{x} := (0, 0, \dots)$ and $\mathbf{y} = (0.9, 0, 0, \dots)$, then $\eta(\mathbf{x}) = (2, 0, 0, \dots)$ and $\eta(\mathbf{y}) = (1, 0, 0, \dots)$ which concludes that

$$\|\eta(\mathbf{x}) - \eta(\mathbf{y})\|_\infty = 1 > 0.9 = \|\mathbf{x} - \mathbf{y}\|_\infty.$$

Another observation is that the existence of a fixed point for the map η cannot be obtained from Theorem 1.4 because of the fact that $\mathcal{G}$ is not weakly compact.

Acknowledgments. The authors would like to thank the anonymous referees for their valuable comments and suggestions.

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