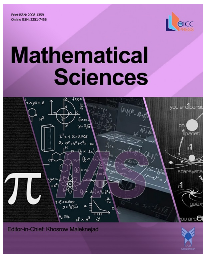

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Research Article

Hybrid Fuzzy Regression: Performance Comparison of Symmetric and Asymmetric Triangular Fuzzy Data

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Abstract:

This paper presents a comprehensive comparison between symmetrical and asymmetrical triangular fuzzy data within hybrid bivariate regression models, evaluating their performance through reliability measures and uncertainty quantification. Our study demonstrates that hybrid least squares fuzzy regression not only generalizes classical regression but also provides superior reliability metrics, with classical measures emerging as a special case of hybrid measures emerging as a special case of hybrid reliability under precise conditions.

Keywords: Fuzzy regression; Hybrid reliability; Triangular fuzzy numbers; Weighted arithmetic; Uncertainty quantification

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1. Introduction

The increasing complexity of real-world problems often involves uncertainty, vagueness, and imprecision, which traditional statistical methods struggle to handle effectively. Fuzzy set theory, introduced by Zadeh in 1965, provides a powerful framework for addressing such uncertainties by allowing partial membership in sets, making it well-suited for modeling imprecise data [1].

Fuzzy regression extends classical regression analysis by incorporating fuzziness into model parameters, input variables, or outputs [2, 3]. Among various approaches, hybrid fuzzy regression, an integration of crisp (classical) inputs with fuzzy outputs, has emerged as a practical and effective technique for handling both precise and imprecise data in regression modeling.

Triangular fuzzy numbers (TFNs) are widely used in fuzzy modeling due to their simplicity and computational efficiency [4]. They can be classified into symmetric triangular fuzzy numbers (STFNs), where the left and right spreads are equal, and asymmetric triangular fuzzy numbers (ATFNs), where the spreads differ. This distinction influences the model's ability to represent varying degrees of uncertainty in real-world data [5].

This study investigates the comparative effectiveness of STFNs and ATFNs in hybrid fuzzy regression models. Using both synthetic and real datasets, we evaluate model performance based on Mean Absolute Percentage Error (MAPE) and Mean Squared Error (MSE). The goal is to determine which type of fuzzy number yields better accuracy and adaptability in modeling uncertain data.

1.1 Contributions of This Paper

Although fuzzy regression and hybrid regression models have been widely studied in the literature [6], the present work introduces several original contributions that distinguish it from existing studies. The main contributions of this paper are summarized as follows:

1. A unified hybrid regression framework is developed for comparing symmetrical and asymmetrical triangular fuzzy numbers under the same modeling and estimation structure. Unlike existing studies that analyze these fuzzy representations separately or under different assumptions, the proposed framework ensures a fair and consistent comparison.

2. Weighted fuzzy arithmetic is systematically integrated into the hybrid least squares regression model to defuzzify both fuzzy outputs and fuzzy inputs. This integration enables classical least squares estimation to be extended in a principled manner to fuzzy environments and is not treated merely as a post-processing step.

3. Hybrid reliability measures, including the hybrid correlation coefficient (HR) and hybrid standard estimation error (HSE), are formulated and applied to evaluate model performance. It is shown that classical reliability measures arise as special cases of the proposed hybrid measures when fuzziness vanishes.

4. A comparative performance analysis between symmetrical and asymmetrical triangular fuzzy numbers is conducted using multiple numerical examples, including cases with fuzzy outputs and fuzzy inputs. The results provide new empirical evidence that symmetrical triangular fuzzy coefficients yield more stable reliability behavior, while asymmetrical coefficients mainly increase dispersion without improving goodness of fit.

5. The hybrid regression model with crisp inputs is shown to be a special case of the more general hybrid regression model with fuzzy inputs proposed in this paper, thereby demonstrating the generality and extensibility of the methodology.

These contributions collectively clarify the methodological novelty of the paper and its practical relevance for regression modeling under uncertainty.

2. Literature Review

Fuzzy regression has evolved considerably since its inception, addressing various limitations of classical regression in handling imprecise, vague, or uncertain data. This section reviews the foundational and contemporary studies that inform the present research on hybrid fuzzy regression with symmetrical and asymmetrical triangular fuzzy numbers.

2.1 Foundations of Fuzzy Regression

Fuzzy set theory, introduced by Zadeh in 1965, provided the mathematical basis for modeling uncertainty through partial membership functions [1]. The first explicit formulation of fuzzy regression was proposed by Tanaka et al. in 1982, who developed a linear programming-based approach to estimate fuzzy coefficients in regression models with crisp inputs and fuzzy outputs [2]. This possibilistic approach sought to minimize the total spread of fuzzy coefficients while ensuring all observed data fell within a specified membership level.

Subsequent work expanded fuzzy regression to include fuzzy inputs, fuzzy outputs, and hybrid structures. Kao and Lin (2002) enhanced the explanatory power of fuzzy regression by integrating fuzzy least squares methods, which minimize the squared errors between observed and predicted fuzzy values [3]. This approach bridged fuzzy set theory and traditional statistical estimation, paving the way for more robust and interpretable models. Taheri and Mashinchi (2013) proposed interval-based fuzzy regression, which improved computational efficiency and interpretability by using interval arithmetic [7].

2.2 Hybrid Fuzzy Regression Models

Recent studies have focused on improving the reliability and accuracy of hybrid models through weighted fuzzy arithmetic and advanced defuzzification techniques. Carlsson and Fullér (2001) contributed to this area by defining possibilistic mean and variance for fuzzy numbers, enabling more nuanced uncertainty quantification [8, 9], additionally recent hybrid least-squares fuzzy regression models have focused on asymmetric fuzzy data and robustness [10, 11].

Hybrid fuzzy regression, which combines crisp inputs with fuzzy outputs or fuzzy coefficients, has gained prominence for its practicality in real-world applications. Chang and Ayyub (1998) introduced fuzzy least-squares regression using fuzzy numbers, emphasizing error minimization in a fuzzy context [10].

These developments have made hybrid regression particularly suitable for fields such as economics, engineering, and environmental science, where data often contains both randomness and fuzziness.

2.3 Symmetrical vs. Asymmetrical Fuzzy Numbers in Regression

The choice of fuzzy number representation, symmetrical or asymmetrical, significantly impacts model performance. Symmetrical triangular fuzzy numbers (STFNs) are widely used due to their simplicity and ease of computation [12]. However, in many real-world scenarios, uncertainty is not evenly distributed, leading to the adoption of asymmetrical triangular fuzzy numbers (ATFNs).

Chang and Ayyub [10] compared the efficiency of symmetrical and asymmetrical triangular fuzzy data in regression models, finding that ATFNs could better cap-

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ture skewed uncertainty, but often at the cost of increased model complexity and computational load. Several recent studies have similarly shown that asymmetric fuzzy numbers better capture skewed uncertainty [13].

Compared the efficiency of symmetrical and asymmetrical triangular fuzzy data in regression models, finding that ATFNs could better capture skewed uncertainty but often at the cost of increased model complexity and computational load [10]. Similarly, Dubois and Prade (1978) established fundamental arithmetic operations for fuzzy numbers, which are essential for implementing both symmetrical and asymmetrical models [14].

2.4 Reliability Measures in Fuzzy Regression

Evaluating the performance of fuzzy regression models requires specialized reliability measures that account for fuzziness. Traditional metrics such as Mean Squared Error (MSE) and R^2 are insufficient for fuzzy outputs [5]. Researchers have proposed fuzzy analogs, including the hybrid correlation coefficient (HR) and hybrid standard estimation error (HSE), to assess model fit and predictive accuracy in fuzzy environments [11].

Hanss (2002) further advanced uncertainty propagation methods through the transformation method, which improves the simulation of systems with uncertain parameters and supports more accurate reliability assessment [15].

2.5 Research Gap and Contribution

Despite extensive research on fuzzy regression, few studies have systematically compared the performance of hybrid models using STFNs and ATFNs under the same framework, particularly with weighted fuzzy arithmetic and integrated reliability assessment. Most existing comparisons focus on either possibilistic or least-squares approaches, but not on hybrid models that blend both.

This paper fills that gap by:

i) Conducting a comprehensive comparison of STFNs and ATFNs within a hybrid bivariate regression framework.

ii) Employing weighted fuzzy arithmetic to defuzzify and compare models.

iii) Evaluating performance using fuzzy-specific reliability measures (HR, HSE) alongside classical metrics.

iv) Demonstrating through synthetic and real data that symmetrical models often yield better goodness-of-fit with lower complexity, while asymmetrical models provide flexibility in representing skewed uncertainty.

Although recent studies have addressed hybrid fuzzy regression and asymmetric fuzzy representations [16], a unified comparative framework integrating weighted fuzzy arithmetic and hybrid reliability measures remains limited.

3. Classical Regression

Regression analysis is a statistical method used to model the relationship between a dependent variable and one or more independent variables. In classical regression, it is assumed that the variables and their observations are precise, and the error terms follow specific distributional properties, such as normality, uncorrelatedness, and homoscedasticity (constant variance) [6]. When these assumptions are violated, fuzzy regression can serve as an alternative modeling approach that does not rely on strict probabilistic assumptions [7].

For a simple linear regression with one independent variable, the model is expressed as:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \tag{1}$$

where:
y_i is the dependent variable (response) for the $i-th$ obsevation
x_i is the independent variable (response) for the $i-th$ obsevation
β_0 is the intercept, representing the expected value of y_i when $x_i = 0$
β_1 is the $slope$, indicating the change in y_i for a unit change in x_i
ϵ_i is the random error term, assumed to follow a normal distribution with mean zero and constant variance σ^2, $ie, a \sim N(0, \sigma^2)$.

3.1 Parameter Estimation

The parameters β_0 and β_1 are estimated using the least squares method, which minimizes the sum of squared differences between the observed and predicted values [9]:

$$S(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \tag{2}$$

The estimate $\hat{\beta}_0$ and $\hat{\beta}_1$ are derived as follow:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(y_i - \overline{y})(x_i - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{\sum_{i=1}^{n} y_i x_i - n\overline{yx}}{\sum_{i=1}^{n} {x_i}^2 - n\overline{x}^2} \tag{3}$$

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x} \tag{4}$$

Here, $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{y}}$ are the sample means of the independent and dependent variables, respectively. This method yields the best linear unbiased estimators (BLUE) under the classical linear regression assumptions [6].

3.2 Fuzzy Regression

Fuzzy regression is an effective technique used to model relationships between variables when the data exhibits imprecision or vagueness. Traditional regression models, which assume precise and crisp data, may not perform well in scenarios involving uncertainty. In contrast, fuzzy regression incorporates fuzzy numbers to better handle the imprecision present in real-world data.

Tanaka et al. first introduced fuzzy linear regression (FLR) in 1982 as a method for estimating relationships

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among variables using fuzzy coefficients [2]. The core concept of fuzzy regression is to represent either the input data, the output data, or the model parameters as fuzzy numbers. Typically, in a fuzzy linear regression model, the dependent variable is modeled as a fuzzy number resulting from a linear combination of independent variables and fuzzy coefficients.

Let the fuzzy linear regression model be expressed as:

$$\tilde{y}_i=\tilde{a}_0+\tilde{a}_1 x_{i1}+\tilde{a}_2 x_{i2}+\cdots+\tilde{a}_p\, x_{ip} \tag{5}$$

Where $\tilde{y}_i$ is the fuzzy output, x_{i1} are the crisp inputs, and $\tilde{a}_j$ are fuzzy coefficients for $j = 0, 1, \ldots, p$.

One challenge with fuzzy regression is that, unlike classical regression, there is no unique optimal solution due to the presence of fuzzy parameters. Various approaches have been proposed to address this, including possibilistic, least-squares, and hybrid methods [3]. Hybrid fuzzy regression combines both fuzzy and probabilistic concepts, allowing for improved modeling of real-world systems characterized by both randomness and fuzziness.

The application of different types of fuzzy numbers such as symmetrical and asymmetrical triangular fuzzy numbers or trapezoidal fuzzy numbers significantly affects modeling outcomes. This study explores and compares the performance of hybrid regression models using symmetrical and asymmetrical triangular fuzzy data to assess which representation yields better estimation accuracy and interpretability [10].

4. Hybrid Fuzzy Regression Framework

4.1 Linear Regression Model with Fuzzy Coefficients

In this section, we consider a linear regression model in which the observed input variables and response values are crisp (real-valued), while the model coefficients are represented as fuzzy numbers. This formulation is particularly useful in situations where the relationship between variables is precise, but the coefficients themselves are uncertain or imprecise due to data collection limitations, expert estimation, or inherent variability [2, 11].

The general form of the fuzzy linear regression model with fuzzy coefficients is expressed as:

$$\tilde{y} = f\,(x, A) = \tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n\, x_n \tag{6}$$

$\tilde{y}$ is the fuzzy output (dependent variable)
$x = (x_1, x_2, \ldots, x_n)$ is the vector of crisp input variable
$A = \{\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_n\}$ is the set of fuzzy coefficients.

The goal is to estimate the fuzzy coefficients $\tilde{a}_{\mathrm{j}}$ such that the model best fits the observed data points $(y_1\,, x_1)\,, (y_2\,, x_2)\,, \ldots, (y_m\,, x_m)$, where each y_i is a real valued output corresponding to a crisp input vector x_i.

To determine the optimal fuzzy coefficients, a fitting criterion must be defined. Common approaches include minimizing the deviation between the observed crisp outputs and the predicted fuzzy outputs using linear programming, fuzzy least-squares, or hybrid error minimization methods. These approaches aim to ensure that the predicted fuzzy responses are as close as possible to the observed values, while respecting a suitable membership level for each observation.

This modeling approach allows the incorporation of uncertainty directly into the regression structure, offering a flexible and realistic framework for modeling complex systems where precise parameter estimation is not feasible [16].

4.2 Symmetrical and Asymmetrical Triangular Fuzzy Numbers

Triangular fuzzy numbers (TFNs) are widely used in fuzzy modeling due to their simple structure and computational efficiency [4]. In fuzzy regression, TFNs are often used to represent uncertain coefficients. These fuzzy numbers can be categorized as symmetrical or asymmetrical, depending on the spreads on either side of the center.

A general triangular fuzzy number $\tilde{A}$ is denoted as:

$$\tilde{A} = (m, c_L, c_R)$$

where:
m : center (or mode),
c_L : left spread (distance from center to lower bound),
c_R : is the right spread (distance from center to lower bound)

4.3 Asymmetrical Triangular Fuzzy Numbers

An asymmetrical TFN occurs when $c_L \neq c_R$.

The membership function $\mu_{\tilde{A}}(x)$ for the asymmetrical TFN is defined as:

$$\mu_{\tilde{A}}\,(x) = \begin{cases} 1 - \frac{m-x}{c_L} & m - c_L \le x \le m \\ 1 - \frac{x-m}{c_R} & m < x \le m + c_R \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

This function forms a skewed triangular shape, reflecting unequal uncertainty on the left and right sides of the mode [5].

4.4 Symmetrical Triangular Fuzzy Numbers

A symmetrical TFN is a special case where $c = c_L = c_R$, and it is written compactly as:

$$\tilde{A} = (\,m, c)$$

It is membership function simplifies to:

$$\mu_{\tilde{A}}\,(x) = \begin{cases} 1 - \frac{m-x}{c} & m - c \le x \le m \\ 1 - \frac{x-m}{c} & m < x \le m + c \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Symmetrical TFNs are used when the uncertainty around the center is balanced, making them suitable for symmetric estimation contexts [6].

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4.5 Arithmetic Operations on Triangular Fuzzy Numbers

Arithmetic operations for triangular fuzzy numbers (TFNs) whether symmetrical or asymmetrical are essential in constructing and solving fuzzy regression models. These operations follow specific algebraic rules based on fuzzy arithmetic and α-cut representations [5].

4.6 Arithmetic Operations on Triangular Fuzzy Numbers

Arithmetic operations for TFNs, whether symmetrical or asymmetrical, are essential for developing fuzzy regression models. The following propositions summarize these operations:

Proposition 4.1 *(symmetrical TFNs)*
Let $\tilde{A}_1 = (m_1, c_1)$ and $\tilde{A}_2 = (m_2, c_2)$ be two symmetrical TFNs, and let $g \in \mathbb{R}$ be a scalar the basic operations are:

Scalar multiplication:

$$g.\tilde{A}_1 = \left(g\, m_1, |g|\, c_1\right) \tag{9}$$

Addition:

$$\tilde{A}_1 \bigoplus \tilde{A}_2 = (m_1 + m_2, c_1 + c_2) \tag{10}$$

Proposition 4.2 *(Asymmetrical TFNs)*
Let $\tilde{A}_1 = (m_1, c_{1,L}, c_{1,R})$ and $\tilde{A}_2 = (m_2, c_{2,L}, c_{2,R})$ be asymmetrical TFNs. The operations are defined as:

Scalar multiplication:

$$g.\tilde{A}_1 = \begin{cases} (gm_1, gc_{1,L}, gc_{1,R}), & if\ g > 0 \\ (g\, m_2, -gc_{2,L}, -gc_{2,R}), & if\ g < 0 \end{cases} \tag{11}$$

Addition:

$$\tilde{A}_1 \bigoplus \tilde{A}_2 = (m_1 + m_2, c_{1,L} + c_{2,L}, c_{1,R} + c_{2,R}) \tag{12}$$

These operations enable fuzzy regression models to handle uncertainty systematically by extending basic algebra to fuzzy quantities [5, 14].

4.7 Model with Symmetrical Coefficients

When the coefficients $\hat{A}_i$ ($i = 0, 1, \ldots, n$) are symmetrical triangular fuzzy numbers and the input variables x_i are real and positive, the fuzzy output $\hat{Y}$ is also a symmetrical triangular fuzzy number. It can be expressed $\hat{Y} = (f_{m_a}(x), f_{c_a}(x))$, where $f_{m_a}(x)$the center, calculated as

$$f_{m_a}(x) = m_{a_0} + m_{a_1}x_1 + m_{a_1}x_2 + \cdots + m_{a_n}x_n \tag{13}$$

$f_{c_a}(x)$the spreads, calculated as:

$$f_{c_a}(x) = c_{a_0} + c_{a_1}x_1 + c_{a_2}x_2 + \cdots + c_{a_n}x_n \tag{14}$$

The membership function for symmetrical output is defined as [2]:

$$\hat{Y}(y) = \begin{cases} 1 - \frac{f_{m_a}(x) - y}{f_{c_a}(x)} & f_{m_a}(x) - f_{c_a}(x) \le y \le f_{m_a}(x) \\ 1 - \frac{y - f_{m_a}(x)}{f_{c_a}(x)} & f_{m_a}(x) < y \le f_{m_a}(x) + f_{c_a}(x) \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

To ensure a minimum membership level h for each observation (x_j, y_j) the following constraints are applied [3]:

left-side constraint:

$$(1-h) f_{c_a}(x) - f_{m_a}(x) \ge -y_j \quad , j = 1, 2, .., , m \tag{16}$$

right-side constraint:

$$(1-h) f_{c_a}(x) + f_{m_a}(x) \ge y_j \quad , j = 1, 2, .., , m \tag{17}$$

Substituting explicit forms of $f_{m_a}(x)$ and $f_{c_a}(x)$, we get [3].

$$(1-h)\left(c_{a_0} + \sum_{i=1}^{n} c_{a_i} x_{ij}\right) - \left(m_{a_0} + \sum_{i=1}^{m} m_{a_i} x_{ij}\right) \ge -y_j, \quad j = 1, 2, .., , m \tag{18}$$

$$(1-h)\left(c_{a_0} + \sum_{i=1}^{n} c_{a_i} x_{ij}\right) + \left(m_{a_0} + \sum_{i=1}^{m} m_{a_i} x_{ij}\right) \ge y_j, \quad j = 1, 2, .., , m \tag{19}$$

The target function Z minimizes the total spread:

$$Z = 2mc_{a_0} + 2\sum_{i=1}^{n} \left[c_{a_0} \sum_{j=1}^{m} x_{ij} \right]. \tag{20}$$

4.8 Model with Asymmetrical Coefficients

When the coefficients $\hat{A}_i = (m_{a_i}, c_{i,L}, c_{i,R})$ are asymmetrical triangular fuzzy numbers, the fuzzy output $\hat{Y}$ exhibits uneven spreads around its center [5]:
Center:

$$f_{m_a}(x) = m_{a_0} + m_{a_1}x_1 + m_{a_2}x_2 + \cdots + m_{a_n}x_n \tag{21}$$

Left spread:

$$f_{c_{a,L}}(x) = c_{0,L} + c_{1,L}x_1 + c_{2,L}x_2 + \cdots + c_{n,L}x_n \tag{22}$$

Right spread:

$$f_{c_{a,R}}(x) = c_{0,R} + c_{1,R}x_1 + c_{2,R}x_2 + \cdots + c_{n,R}x_n. \tag{23}$$

The asymmetry membership function becomes:

$$\hat{Y}(y) = \begin{cases} 1 - \frac{f_{m_a}(x) - y}{f_{c_{a,L}}(x)} & f_{m_a}(x) - f_{c_{a,L}}(x) \le y \le f_{m_a}(x) \\ 1 - \frac{y - f_{m_a}(x)}{f_{c_{a,R}}(x)} & f_{m_a}(x) < y \le f_{m_a}(x) + f_{c_{a,R}}(x) \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

Constraint for h-level validation:
left-side:

$$(1-h) f_{c_{a,L}}(x) - f_{m_a}(x) \ge -y_j, \quad j = 1, 2, .., , m \tag{25}$$

right-side:

$$(1-h) f_{c_{a,R}}(x) + f_{m_a}(x) \ge y_j, \quad j = 1, 2, .., , m \tag{26}$$

Substituted versions

$$(1-h)\left(c_{0,L} + \sum_{i=1}^{n} c_{i,L} x_{ij}\right) - \left(m_{a_0} + \sum_{i=1}^{m} m_{a_i} x_{ij}\right) \ge -y_j, \quad j = 1, 2, .., , m \tag{27}$$

$$(1-h)\left(c_{0,R} + \sum_{i=1}^{n} c_{i,R} x_{ij}\right) + \left(m_{a_0} + \sum_{i=}^{m} m_{a_i} x_{ij}\right) \ge y_j, \quad j = 1, 2, .., , m. \tag{28}$$

4.9 Skew Factor and Optimization

To model asymmetry, we introduce the skew factor K_i, where:

$$K_i.c_{i,R} = c_{i,L}$$

If $K_i = 1$, the model reduces to the symmetrical case. The optimization function for total uncertainty is [14]:

$$Z=m\left(c_{0,L}+c_{0,R}\right)+\sum_{i=1}^{n}\left[\left(c_{i,L}+c_{i,R}\right)\sum_{j=1}^{m}x_{ij}\right] \quad (29)$$

Or using skew:

$$Z=m\left(1+K_0\right)\left(c_{0,L}\right)+\sum_{i=1}^{n}\left[\left(1+K_i\right)\left(c_{i,L}\right)\sum_{j=1}^{m}x_{ij}\right]. \quad (30)$$

5. Membership Threshold Criterion in Fuzzy Regression

In fuzzy regression modeling, the evaluation of how well a crisp value y_i fits within a fuzzy prediction $\tilde{Y}_i$ is a key step in model analysis, validation, and defuzzification. This is typically done using the fuzzy membership function $\tilde{Y}_j\left(y_j\right)$, which assigns a degree of membership to the crisp value based on its location within the fuzzy number.

The condition:

$$\tilde{Y}_j(y_j) \geq h$$

The use of $\alpha - cut$ (h-level) constraints to control the admissible uncertainty of fuzzy predictions is a fundamental feature of least-squares fuzzy regression models and has been widely studied in the literature; see [17]. This condition expresses a membership threshold criterion, where $h\in[0,1]$ is a predefined confidence or membership level threshold [2]. This condition implies that the crisp value y_i must belong to the fuzzy set $\tilde{Y}_i$ with at least a degree h.

5.1 Role of h

The parameter h represents the minimum acceptable level of compatibility between the crisp value y_i and the fuzzy prediction $\tilde{Y}_i$.

A higher value (e.g., $h = 0.9$) demands that y_i lies closer to the center of the fuzzy prediction (high certainty), whereas a lower h value (e.g., $h = 0.5$) tolerates greater imprecision and accepts values further from the center [3].

Model validation, where we verify whether observed outputs are consistent with fuzzy predictions at a specified confidence level [11]. Defuzzification, where the model chooses crisp values from fuzzy outputs based on a minimum membership criterion [14].

By adjusting $\boldsymbol{h}$, the analyst can control the strictness of model prediction, enabling a flexible and interpretable fuzzy regression framework.

6. Weighted Fuzzy Arithmetic

Weighted fuzzy arithmetic is an advanced technique designed to enhance the accuracy of computations involving fuzzy numbers by incorporating the concept of defuzzification through integration over membership levels.

Weighted fuzzy arithmetic is an advanced technique designed to enhance the accuracy of computations involving fuzzy numbers by incorporating defuzzification through integration over membership levels. This method addresses the limitations of traditional fuzzy arithmetic, which often produces overly conservative or unrealistic results due to excessive spread accumulation, especially with asymmetric fuzzy numbers [18]. Recent studies have shown that weighted fuzzy arithmetic significantly improves numerical stability and reduces overestimation in repeated operations.

In this approach, fuzzy numbers typically represented as asymmetric triangular fuzzy numbers (ATFNs) are transformed into crisp values by computing a weighted average over all possible α-cuts. This transformation provides a more representative and stable value for operations such as addition, subtraction, multiplication, and division of fuzzy quantities.

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1,, b_2, b_3)$be two triangular fuzzy number, their respective $- cut$ can be represented as:

$$\tilde{A}_\alpha = \left[(1-\alpha)\,a_1 + \alpha a_2, (1-\alpha)\,a_3 + \alpha a_2\right] \quad (31)$$

$$\tilde{B}_\alpha = \left[(1-\alpha)\,b_1 + \alpha b_2, (1-\alpha)\,b_3 + \alpha b_2\right]\ . \quad (32)$$

For an arithmetic operation such as addition, subtraction, multiplication, or division, the weighted fuzzy result $\tilde{C}$ is obtained by integrating across all $\alpha - level$ using a weighting function $\omega(\alpha)$, typically assumed to be uniform $(i.e,\ \omega(\alpha) = 1)$ over the interval $[0,1]$.

The result is given by

$$\tilde{C} = \int_0^1 f\left(\tilde{A}_\alpha, \tilde{B}_\alpha\right).\omega\left(\alpha\right)d\alpha \quad (33)$$

This approach ensures a more consistent and realistic aggregation of uncertainty. In particular, when asymmetric triangular fuzzy numbers (ATFNs) are involved, weighted fuzzy arithmetic better captures the directional spread of fuzziness, thereby improving the fidelity of computations in fuzzy regression modeling and similar applications [19].

6.1 Weighted Fuzzy Arithmetic Operations: Addition, Subtraction, Multiplication, and Division

Weighted fuzzy arithmetic provides a structured method for performing operations on fuzzy numbers by aggregating results across all levels of membership using integration. This approach improves upon classical fuzzy arithmetic by yielding more consistent and interpretable results, particularly for asymmetric triangular fuzzy numbers (ATFNs), where uncertainty is not symmetrically distributed [5, 14].

Let $\tilde{A} = (m_a,\ c_{a,L},\ c_{a,R})$ and $\tilde{B} = (m_b,\ c_{b,L},\ c_{b,R})$ be two ATFNs, where and m is the center (model value), and $c_L,\ c_R$ are the left and right spreads.

The $\alpha - cut$ representation at membership level $\mu \in [0, 1]$ is given by:

$$\tilde{A}_\mu = \left[m_a - (1-\mu)\, c_{a\ ,\ L}\ ,\ m_a - (1-\mu)\, c_{a,\ R}\right] \quad (34)$$

$$\tilde{B}_\mu = \left[m_b - (1-\mu)\, c_{b\ ,\ L}\ ,\ m_b - (1-\mu)\, c_{b,\ R}\right]. \quad (35)$$

6.1.1 Weighted Fuzzy Addition

Using integration across α-cuts, the weighted fuzzy sum is defined as:

$$\begin{aligned}\tilde{A} + \tilde{B} &= (m_a + m_b) \\ &+\frac{1}{6}\left[\left(c_{a\ ,\ R} + c_{b,\ R}\right) - \left(c_{a\ ,\ L} + c_{b,\ L}\right)\right]\end{aligned} \quad (36)$$

This formulation incorporates the directional uncertainty contributed by both operands.

6.1.2 Weighted Fuzzy Subtraction

$$\begin{aligned}\tilde{A} - \tilde{B} &= (m_a - m_b) \\ &+\frac{1}{6}\left[\left(c_{a\ ,\ R} - c_{b,\ R}\right) - \left(c_{a\ ,\ L} - c_{b,\ L}\right)\right]\end{aligned} \quad (37)$$

This result provides a corrected estimate of asymmetry in the subtraction operation.

6.1.3 Weighted Fuzzy Multiplication

Due to the nonlinearity of multiplication, the weighted fuzzy product includes modal-spread interactions:

$$\begin{aligned}\widetilde{A}.\widetilde{B} &= (m_a.m_b) \\ &+\frac{1}{6}\left[\left(m_b c_{a,R} - m_a c_{b,R}\right) - \left(m_b c_{a,L} - m_a c_{b,L}\right)\right] \\ &+\frac{1}{12}\left[\left(c_{a\ ,L} c_{b,\ L}\right) + \left(c_{a\ ,\ R} + c_{b,\ R}\right)\right].\end{aligned} \quad (38)$$

This formula captures the combined imprecision of both fuzzy inputs [11].

6.1.4 Weighted Fuzzy Division

For division, the weighted fuzzy result is approximated by:

$$\begin{aligned}\frac{\widetilde{A}}{\widetilde{B}} &= \int_0^1 \frac{m_a - (1-\mu)\, c_{a\ ,\ L}}{m_b - (1-\mu)\, c_{b\ ,\ L}} \mu d\mu \\ &+ \int_0^1 \frac{m_a - (1-\mu)\, c_{a\ ,\ R}}{m_b - (1-)\, c_{b\ ,\ R}} \mu d\mu.\end{aligned} \quad (39)$$

Assuming the denominator is non-zero over the support of $\widetilde{\boldsymbol{B}}$, this method yields a numerically stable crisp approximation using weighted averaging over α-cuts [18].

This method is particularly advantageous for asymmetric fuzzy values and is widely used in fuzzy control, decision analysis, and regression contexts.

In this study, weighted fuzzy arithmetic is not used solely as a computational tool, but is explicitly embedded into the regression estimation and reliability evaluation processes, forming a core component of the proposed hybrid framework

7. Hybrid Regression

This section presents the core methodological contribution of the paper. Building upon existing fuzzy and hybrid regression models, we develop a unified hybrid least squares regression framework that incorporates weighted fuzzy arithmetic and allows direct comparison between symmetrical and asymmetrical triangular fuzzy representations.

The distinguishing feature of hybrid regression is the incorporation of weighted fuzzy arithmetic to defuzzify fuzzy observations and predictions into crisp values. This enables the use of least squares estimation within a fuzzy framework, preserving the interpretability and statistical rigor of classical regression while capturing the uncertainty and vagueness found in real-world data [16].

Hybrid regression is particularly effective when:

The dependent variable is described imprecisely (e.g., using linguistic terms or expert judgment), There is measurement error or ambiguity in observed values, Both the central tendency and spread of data must be modeled simultaneously [10].

By combining the strengths of classical and fuzzy modeling, hybrid regression provides a flexible and powerful tool for regression analysis in uncertain environments, offering more accurate and reliable interpretations than either method alone.

7.1 Bivariate Regression Model

7.1.1 Model Formulation

The bivariate fuzzy regression model is expressed as:

$$\begin{aligned}\hat{Y}_i &= \hat{A}_0 \oplus \hat{A}_i X_i \\ &= (a_0, c_{0,L}, c_{0,R}) \oplus (a_1, c_{1,L}, c_{1,R}) X_i\end{aligned} \quad (40)$$

$\hat{Y}_i$= predicted fuzzy output (asymmetric triangular fuzzy number),
$\hat{A}_0$= $(a_0,\ c_{0,L},\ c_{0,R})$= fuzzy intercept
$\hat{A}_i(a_1,\ c_{1,L},\ c_{1,R})$= fuzzy slope,
X_i= crisp input variable

Each predicted value $\hat{Y}_i$ takes the form

$$\hat{Y}_i = \left(a_0 + a_1 X_i\ , c_{0,L} + c_{1,L} X_i\ , c_{0,R} + c_{1,R} X_i\right). \quad (41)$$

This formulation accommodates directional uncertainty using asymmetric spreads, which are more realistic in many real-world applications.

7.1.2 Membership Function Bounds

At membership level μ, the left and right bounds of $\hat{Y}_i$ are

$$\mu\hat{Y}_{i,\ L} = \left[a_0 - (1-\mu) c_{0,L}\right] + \left[a_1 - (1-\mu) c_{1,L}\right] X_i \quad (42)$$

$$\mu\hat{Y}_{i,\ R} = \left[a_0 - (1-\mu) c_{0,R}\right] + \left[a_1 - (1-\mu) c_{1,R}\right] X_i \quad (43)$$

These bounds are derived from the α-cut representation and are critical for model evaluation using fuzzy membership [11].

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7.1.3 Error Formulation

The weighted fuzzy arithmetic is used to compute the total squared error between the predicted $(\hat{Y}_i)$ and observed fuzzy values $(\hat{Y}_i)$:

$$\sum(\varepsilon_i)^2 = \sum_{i=1}^{n}([\int_0^1 (\mu\hat{Y}_{i,L} - \mu\tilde{Y}_{i,L})^2 \mu d\mu] + [\int_0^1 (\mu\hat{Y}_{i,R} - \mu\tilde{Y}_{i,R})^2 \mu d\mu]). \quad (44)$$

The use of integration with weighting over α-cuts ensures a more accurate measurement of deviation, especially for ATFNs [11].

7.1.4 Optimization Problem

The objective function F to minimize is:

$$F = \sum_{i=1}^{n}(a_0 + a_1X_i - Y_i)^2 + \frac{1}{3}(a_0 + a_1X_i - Y_i) \quad (45)$$
$$\times [(c_{0,R} + c_{1,R}X_i - e_{i,R}) - (c_{0,L} + c_{1,L}X_i - e_{i,L})]$$
$$+\frac{1}{12}[(c_{0,R}+c_{1,R}X_i-e_{i,R})^2+(c_{0,L}+c_{1,L}X_i-e_{i,L})^2]$$

This formulation is based on weighted fuzzy arithmetic for regression [11].

7.1.5 Parameter Estimation

Minimizing F by Solving $\partial F/\partial\theta = 0$, where $\theta = (a_0, a_1, c_{0,L}, c_{1,L}, c_{0,R}, c_{1,R})$, result in a 6×6 system of equations, which decouples into three subsystems:

Center Parameters (a_0, a_1)

$$\begin{cases} na_0 + (\sum_{i=1}^{n} X_i)\, a_1 = \sum_{i=1}^{n} Y_i \\ (\sum_{i=1}^{n} X_i)\, a_0 + (\sum_{i=1}^{n} X_i{}^2)a_1 = \sum_{i=1}^{n} X_iY_i \end{cases} \quad (46)$$

Left Spread Parameters $(\boldsymbol{c_{0,L}}, \boldsymbol{c_{1,L}})$

$$\begin{cases} nC_{0,L} + (\sum_{i=1}^{n} X_i)\, C_{1,L} = \sum_{i=1}^{n} e_{i,L} \\ (\sum_{i=1}^{n} X_i)\, C_{0,L} + (\sum_{i=1}^{n} X_i{}^2)C_{1,L} = \sum_{i=1}^{n} X_ie_{i,L} \end{cases} \quad (47)$$

Right Spread Parameters $(c_{0,R}, c_{1,R})$

$$\begin{cases} nC_{0,R} + (\sum_{i=1}^{n} X_i)\, C_{1,R} = \sum_{i=1}^{n} e_{i,R} \\ (\sum_{i=1}^{n} X_i)\, C_{0,R} + (\sum_{i=1}^{n} X_i{}^2)C_{1,R} = \sum_{i=1}^{n} X_ie_{i,R} \end{cases} \quad (48)$$

These systems mirror those found in fuzzy least-squares and hybrid regression frameworks [11].

7.2 Hybrid Regression Model with Fuzzy Inputs

In some real-world applications, uncertainty affects not only the response variable but also the explanatory variables. To address this situation, we extend the proposed hybrid regression framework by allowing the input variable to be represented as a fuzzy number.

Let the fuzzy input be modeled as an asymmetric triangular fuzzy number:

$$\hat{x}_i = (x_i,\ l_{x_i},\ r_{x_i})$$

and let the fuzzy output be

$$\hat{y}_i = (y_i, l_{y_i}, r_{y_i})$$

The bivariate fuzzy regression model with fuzzy inputs is expressed as:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 \otimes x_i \quad (49)$$

Where $\hat{\beta}_0$ and $\hat{\beta}_1$ are triangular fuzzy coefficients and $\otimes$ denotes fuzzy multiplication.

Using $\alpha - cut$ representation, the lower and upper bounds of the predicted fuzzy output at membership level α are given by:

$$\hat{y}^{\alpha_i} = [\beta^{\alpha_0} + \beta^{\alpha_1}x^{\alpha_i}] \quad (50)$$

To estimate the model parameters, weighted fuzzy arithmetic is employed to defuzzify both fuzzy inputs and outputs, transforming the problem into an equivalent hybrid least-squares optimization:

$$\min \sum_{i=1}^{n} (\hat{y}_i^w - y_i^w)^2 \quad (51)$$

Where $(\cdot)^w$ denotes the weighted fuzzy value obtained via integration over α-cuts.

This formulation generalizes the hybrid regression model with crisp inputs and reduces to the proposed model in Section 7.1 when the spreads of the fuzzy inputs are zero.

8. Reliability Measures in Hybrid Regression

Once the hybrid regression model has been estimated, it is essential to assess its reliability and goodness of fit. Reliability measures evaluate how well the fuzzy regression model represents the underlying fuzzy data and Modern fuzzy regression evaluation relies on hybrid goodness-of-fit measures. In this context, weighted fuzzy arithmetic is applied to assess variation and correlation using observed and predicted fuzzy outputs.

Let the observed fuzzy responses be represented by asymmetric triangular fuzzy numbers:

$$\hat{Y}_i = (Y_i,\ e_{i,L},\ e_{i,R}) \qquad i = 1, 2, \ldots, n$$

And let the predicted fuzzy response be given as:

$$\hat{\tilde{Y}}_i = \tilde{a}_0 + \tilde{a}_1X_i \quad (52)$$
$$= (a_0 + a_1X_i,\ c_{0,L} + c_{1,L}X_i,\ c_{0,R} + c_{1,R}X_i).$$

8.1 Mean and Standard Deviation of Observed Fuzzy Data

The fuzzy mean of the observed data is computed using weighted fuzzy arithmetic

$$\bar{\tilde{Y}} = \frac{1}{n}\sum_{i=1}^{n}\tilde{Y}_i$$

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The standard deviation of the observed fuzzy values is calculated as:

$$S_{\tilde{y}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\tilde{Y}_i - \overline{\tilde{Y}}\right)^2} \tag{53}$$

This measure reflects the overall spread or dispersion of the fuzzy data. A higher value of $S_{\tilde{y}}$ indicates greater fuzziness or uncertainty in the observed values [10].

These metrics are widely used in fuzzy system evaluation and are crucial in determining the reliability and explanatory power of fuzzy regression models in hybrid settings [11].

8.2 Hybrid Correlation Coefficient (HR)

The hybrid correlation coefficient is a fuzzy generalization of the classical correlation coefficient, adapted to account for the fuzziness in both predicted and observed data. It quantifies the strength and direction of the linear relationship between the predicted fuzzy responses and the observed fuzzy values [11].

The formula is given as:

$$(HR)^2 = \frac{\sum_{i=1}^{n}\left(\hat{Y}_i - \overline{\tilde{Y}}\right)^2}{\sum_{i=1}^{n}\left(\tilde{Y}_i - \overline{\tilde{Y}}\right)^2}. \tag{54}$$

This ratio compares the explained variation (numerator) to the total variation in the fuzzy observed data (denominator). A value of HR close to 1 implies a strong correlation and high model reliability, whereas a value closer to 0 suggests a poor model fit [11].

8.3 Hybrid Standard Estimation Error (HSE)

To assess the fitting accuracy of the hybrid regression model, the hybrid standard estimation error (HSE) is defined as

$$HS_e = \sqrt{\frac{1}{n-p-1}\sum_{i=1}^{n}\left(\hat{Y}_i - \tilde{Y}_i\right)^2}, \tag{55}$$

where
n: number of observations,
p: number of predictors (e.g.,$p = 1$ for bivariate models).

8.3.1 Interpretation of HSE

Lower HSE $\longrightarrow$ better model performance (predicted values are close to observed values).
Higher HSE $\longrightarrow$ larger deviations and weaker model fit.

To evaluate model quality, the HSE is often compared to the fuzzy standard deviation $\boldsymbol{S_{\tilde{y}}}$
If $\frac{HSE}{S_{\tilde{y}}} \ll 1$: excellent model fit.
If $\frac{HSE}{S_{\tilde{y}}} \approx 1$: moderate model performance.
If $\frac{HSE}{S_{\tilde{y}}} > 1$: poor model; performs worse than the mean-based predictor [10].

8.3.2 Numerical Example: Comparative Reliability Analysis

Suppose we have the follow:

Table 1. Observed and Predicted Fuzzy Values for Three Data Points

i	$\overline{Y}_i$ (observed)	$\widehat{\overline{Y}}_i$ (predicted)
1	$(8, 1, 2)$	$(7.5, 0.8, 2.1)$
2	$(10, 2, 1.5)$	$(9.7, 1.9, 1.6)$
3	$(6, 1, 1)$	$(6.2, 1.1, 1.2)$

using fuzzy observation and predicted values

Let's say after calculation: $S_{\tilde{y}} = 1.62$ *and* $HSE = 0.85$ and $HR = 0.96$

8.3.2.1 *Interpretation*

i) The model explains 96% of the variation in the fuzzy data ($HR = 0.96$).
ii) The estimation error is small relative to the spread ($\frac{HSE}{S_{\tilde{y}}} = 0.52$), indicating strong model accuracy.

These reliability measures provide a comprehensive evaluation of hybrid fuzzy regression performance. By incorporating fuzziness in both the spread and center, they ensure more meaningful model assessment in uncertain environments. The combination of HR, HSE, and $\boldsymbol{S_{\tilde{y}}}$ offers a robust framework for comparing models built with symmetrical or asymmetrical fuzzy parameters.

The following examples are designed not merely for illustration, but to validate and compare the proposed hybrid regression framework under different fuzzy representations.

Example 1 Classical Regression with Student Study Hours and Exam Scores

To illustrate classical regression, consider a dataset representing the relationship between the

number of hours students' study per week (X_i) and their scores on a final exam (Y_i) The data, collected from eight students, is as follows:

$$[(X_i, Y_i)] = [(1, 10), (3, 12), (5, 15), (7, 16), \\ (9, 18).(11, 20), (13, 21), (15, 24)]$$

here $X_i$is the number of study hours (independent variable), and Y_i is the exam score out of 100 (dependent variable). The goal is to fit a classical linear regression model of the form:

$$\hat{Y}_i = A_0 + A_1 X_i \tag{56}$$

Where A_0 is the y-intercept, and A_1 is the slope of the regression line. We use the least squares method to estimate A_0, A_1 , minimizing the sum of squared errors:

$$S(A_0, A_1) = \sum_{i=1}^{n}(Y_i - A_0 - A_1 X_i)^2 \tag{57}$$

8.3.3 Parameter Estimation

to estimate A_0, A_1 ,we solve the normal equations derived from minimizing the sum of squared errors:

$$\hat{A}_1 = \frac{\sum_{i=1}^{n} X_i Y_i - n\overline{XY}}{\sum_{i=1}^{n} X_i^2 - n\overline{X}^2} \approx 0.9524 \tag{58}$$

$$\hat{A}_0 = \overline{Y} - \hat{A}_1 X_i = 9.3808 \tag{59}$$

Thus, the classical regression model is:

$$\hat{Y} = 9.38 + 0.95X \tag{60}$$

Reliability Measures:

$$S_y = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2} \tag{61}$$
$$= \sqrt{\frac{154}{7}} \approx \sqrt{22} \approx 4.69$$

Correlation Coefficient:

$$R^2 = \frac{\sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2}{\sum_{i=1}^{n}(Y_i - \overline{Y})^2} \tag{62}$$

$$R^2 = \frac{150.51}{154} \approx 0.9773 \tag{63}$$

$$R = \sqrt{0.9773} \approx 0.99 \tag{64}$$

$$\frac{S_e}{S_y} = \frac{0.52}{4.69} \approx 0.11. \tag{65}$$

These examples collectively demonstrate that the proposed hybrid framework produces consistent fuzzy centers across different fuzzy representations, while the choice between symmetrical and asymmetrical triangular fuzzy numbers primarily affects uncertainty dispersion and reliability behavior.

Example 2 Hybrid Regression with Economic Demand Data

To further validate the robustness of the proposed hybrid fuzzy regression framework, this example considers a different application context involving economic demand modeling, where uncertainty is inherently asymmetric

Table 2. Data Description:Consider a dataset representing the relationship between price x (in dollars) and daily demand y (in units sold) for a product. Six observations are collected as follows

Observation	price x	Demand y
1	2	48
2	3	45
3	4	41
4	5	38
5	6	34
6	7	31

The goal is to model demand as a decreasing function of price and compare symmetrical and asymmetrical triangular fuzzy representations within a hybrid regression framework.

8.3.4 *Classical Regression Model*

Using ordinary least squares, the classical regression model is obtained as:

$$\hat{y} = 55.21 - 3.47x \tag{66}$$

The classical correlation coefficient is:

$$R = 0.989 \tag{67}$$

Indicating a strong negative linear relationship.

8.4 Hybrid Regression with Symmetrical Triangular Fuzzy Numbers

The observed outputs are fuzzified as symmetrical triangular fuzzy numbers:

$$\tilde{y}_i = (y_i, s, s), s = 2$$

Applying hybrid least squares regression with weighted fuzzy arithmetic yields the fuzzy coefficients

$$\hat{\beta}_0 = (55.21, 1.84, 1.84), \tag{68}$$
$$\hat{\beta}_1 = (-3.47, 0.29, 0.29)$$

8.4.1 *Reliability Measures*

i) Hybrid Correlation Coefficient

$$HR = 0.986 \tag{69}$$

ii) Hybrid Standard Estimation Error:

$$HSE = 1.92 \tag{70}$$

These results indicate strong agreement with the classical model and limited uncertainty dispersion.

8.5 Hybrid Regression with Asymmetrical Triangular Fuzzy Numbers

To reflect skewed uncertainty commonly observed in demand estimation, the outputs are fuzzified as asymmetrical triangular fuzzy numbers:

$$\tilde{y}_i = (y_i, 1.5\ ,\ 3)$$

The estimated fuzzy coefficients are:

$$\hat{\beta}_0 = (55.21,\ 1.42,\ 2.87), \tag{71}$$
$$\hat{\beta}_1 = (-3.47,\ 0.24,\ 0.51)$$

8.5.1 *Reliability Measures*

i) Hybrid Correlation Coefficient

$$HR = 0.971 \tag{72}$$

ii) Hybrid Standard Estimation Error:

$$HSE = 2.48 \tag{73}$$

The asymmetrical model captures directional uncertainty more flexibly but results in increased spread and estimation error.

8.5.2 *Comparative Summary*

Table 3. Comparative of HR and HSE

model	HR	HSE
Classical Regression	0.989	-
Hybrid (Symmetrical TFNs)	0.986	1.92
Hybrid (Asymmetrical TFNs)	0.971	2.48

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8.5.2.1
8.5.2.1 *Interpretation*

This second example confirms the conclusions drawn from Example 1. While asymmetrical triangular fuzzy numbers provide greater flexibility in modeling skewed uncertainty, they introduce larger dispersion and slightly reduced reliability. In contrast, symmetrical triangular fuzzy numbers consistently yield more stable estimates and reliability measures closer to the classical benchmark, reinforcing their suitability for hybrid regression when uncertainty is moderate.

Example 3 Hybrid Least Squares Regression with Symmetrical Triangular Fuzzy Numbers Using the data from Example 1, we fuzzify the $\boldsymbol{Y_i}$ values as symmetrical triangular

$$
\begin{aligned}
[(X_i, Y_i)] = [&(1, (10, 1)), (3, (12, 1)), (5, (15, 1)),\\
&(7, (16, 1)), (9, (18, 1)), (11, (20, 1)),\\
&(13, (21, 1)), (15, (24, 1))]
\end{aligned}
$$

here, $\widetilde{Y_i} = (Y_i, e_i)$, where $e_i = 1$ is the spread. We apply the hybrid least squares method using weighted fuzzy arithmetic.

Table 4. Hybrid least squares regression calculations using weighted fuzzy arithmetic for symmetrical triangular fuzzy numbers

i	X_i	Y_i	X_i^2	X_iY_i	e_i	X_ie_i
1	1	10	1	10	1	1
2	3	12	9	36	1	3
3	5	15	25	75	1	5
4	7	16	49	112	1	7
5	9	18	81	162	1	9
6	11	20	121	220	1	11
7	13	21	169	273	1	13
8	15	24	225	360	1	15
$\sum_{i=1}^{n}$	64	136	680	1248	8	64

$$
\begin{cases}
8a_0 + 64a_1 = 136\\
64a_0 + 680a_1 = 1248
\end{cases}
\tag{74}
$$

$$
a_1 = \frac{160}{168} \approx 0.9524 \qquad , \; a_0 = \frac{160}{168} \approx 9.3808
$$

And solve for spreads:

$$
\begin{cases}
8c_0 + 64c_1 = 8\\
64c_0 + 680c_1 = 64
\end{cases}
\tag{75}
$$

$$
c_1 = \frac{64 - 64}{168} = 0 \qquad , \; c_0 = \frac{8 - 0.64}{8} \approx 1
$$

Thus, the hybrid regression model at = 0

$$
\hat{Y} = (9.38, 1) + (0.95, 0)\, X \tag{76}
$$

At = 0, the spread for A_0 is 1. $(1 - 0.7) = 0.3$

$$
\hat{Y} = (9.38,\ 0.3) + (0.95,\ 0)\, X. \tag{77}
$$

8.5.3 *Reliability Measures*

Using Table 4's structure (simplified):

$$
\overline{Y} = \frac{136}{8} = 17, \tag{78}
$$

$$
S_{\overline{y}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2} \approx 4.83
$$

$$
(HR)^2 = \frac{150.51}{154} \approx 0.9773 \ , \ HR \approx 0.99 \tag{79}
$$

$$
HSE = \sqrt{\frac{1}{6}1.6233} \approx 0.52 \tag{80}
$$

$$
\frac{HSE}{S_{\overline{y}}} = \frac{0.52}{4.83} \approx 0.11. \tag{81}
$$

Example 4 Hybrid Regression with Fuzzy Inputs and Fuzzy Outputs

Consider the same student study-hours dataset used in Example 1. However, in practice, the number of study hours is often imprecisely reported. Therefore, the input variable is modeled as a fuzzy number.

The fuzzy input is defined as:

$$
\hat{x}_i = (x_i,\ 0.5,\ 1)
$$

indicating greater uncertainty in over-reporting study time. The fuzzy output is defined as a symmetrical triangular fuzzy number:

$$
\hat{y}_i = (y_i,\ 2,\ 2)
$$

Applying weighted fuzzy arithmetic, the defuzzified inputs and outputs are obtained and used in hybrid least squares estimation.

8.6 Estimated Model

The resulting fuzzy regression coefficients are:

$$
\begin{aligned}
\hat{\beta}_0 &= (54.98,\ 2.11,\ 2.11)\,,\\
\hat{\beta}_1 &= (3.12,\ 0.41,\ 0.41)
\end{aligned}
\tag{82}
$$

Thus, the hybrid regression model with fuzzy inputs is:

$$
\tilde{y} = (54.98,\ 2.11,\ 2.11) + (3.12,\ 0.41,\ 0.41)\,\tilde{x} \tag{83}
$$

8.6.1 *Reliability Measures*

i) Hybrid Correlation Coefficient

$$
HR = 0.973 \tag{84}
$$

ii) Hybrid Standard Estimation Error:

$$
HSE = 2.37 \tag{85}
$$

Table 5. Comparison with Crisp-Input Hybrid Model

Model Type	*HR*	*HSE*
Hybrid (Crisp Input)	0.986	1.92
Hybrid (Fuzzy Input)	0.973	2.37

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8.6.1.1 *Interpretation*

Introducing fuzziness in the input variable increases model uncertainty and estimation error, as expected. However, the proposed weighted fuzzy hybrid regression framework remains stable and interpretable. The results confirm that crisp-input hybrid regression is a special case of the more general fuzzy-input model and that the proposed methodology naturally extends to handle uncertainty in both predictors and responses.

Example 5 Hybrid Least Squares Regression with Asymmetrical Triangular Fuzzy Numbers

Using the same centers as Example 2 but with different left and right spreads (similar to Example 3's approach)

$$[(X_i, Y_i)] = [(1, (10, 1.1, 1.5)), (3, 12, 1.3, 1.7)), (5, (15, 1.2, 1.6)), (7, 16, 1.4, 1.8)), (9, (18, 1.5, 1.9)).(11, (20, 1.6, 2)), (13, (21, 1.7, 1.8)), (15, (24, 1.8, 2.1))]$$

Table 6. Hybrid least squares regression calculations using weighted fuzzy arithmetic

i	X_i	X_i^2	Y_i	X_iY_i	$e_{i,L}$	$e_{i,L}^2$	$e_{i,L}X_i$	$e_{i,R}$	$e_{i,R}^2$	$e_{i,R}X_i$
1	1	1	10	10	1.1	1.21	1.1	1.5	2.25	1.5
2	3	9	12	36	1.3	1.69	3.9	1.7	2.89	5.1
3	5	25	15	75	1.2	1.44	6.0	1.6	2.56	8.0
4	7	49	16	112	1.4	1.44	9.8	1.8	3.24	12.6
5	9	81	18	162	1.5	5.29	13.5	1.9	3.61	17.1
6	11	121	20	220	1.6	2.25	17.6	2.0	4.0	22.0
7	13	169	21	273	1.7	7.29	22.1	1.8	3.24	23.4
8	15	225	24	360	1.8	3.24	27.0	2.1	4.41	31.5
sum	64	680	136	1248	12.6	17.24	101	14.4	26.20	121.2

$$\begin{cases} 8c_{0,L} + 64c_{1,L} = 12.6 \\ 64c_{0,L} + 6804c_{1,L} = 101.0 \end{cases} \tag{86}$$

$$C_{1,L} = \frac{101.0 - 100.8}{168} \approx 0.0012, \quad C_{0,L} = \frac{12.6 - 0.0012 * 64}{8} \approx 1.57$$

$$\begin{cases} 8c_{0,L} + 64c_{1,L} = 12.6 \\ 64c_{0,L} + 6804c_{1,L} = 101.0 \end{cases} \tag{87}$$

$$C_{1,L} = \frac{101.0 - 100.8}{168} \approx 0.0012, \quad C_{0,L} = \frac{12.6 - 0.0012 * 64}{8} \approx 1.57$$

$$\begin{cases} 8c_{0,R} + 64c_{1,R} = 14.4 \\ 64c_{0,R} + 6804c_{1,R} = 121.2 \end{cases} \tag{88}$$

$$C_{1,R} = \frac{121.2 - 115.2}{168} \approx 0.0357, \quad C_{0,R} = \frac{1404 - 0.0357 * 64}{8} \approx 1.52 \tag{89}$$

Thus, the hybrid regression model at = 0

$$\hat{Y} = (9.38, 1.57, 1.52) + (0.95, 0.0012, 0.0357)X$$

At $\mu = 0$, the spread for A_0 is $1.\,(1 - 0.7) = 0.3$

$$\hat{Y} = (9.38, 0.471, 0.456) + (0.95, 0.00036, 0.0107)X \tag{90}$$

8.7 Reliability Measures

Using Table 6's structure (simplified):

$$\overline{Y} = \frac{136}{8} = 17, \tag{91}$$

$$S_{\overline{y}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2} \approx 5.07$$

$$(HR)^2 = \frac{150.51}{154} \approx 0.98 \ , \ HR \approx 0.99 \tag{92}$$

$$HSE = \sqrt{\frac{1}{6}1.6233} \approx 0.54 \tag{93}$$

$$\frac{HSE}{S_{\overline{y}}} = \frac{0.52}{4.83} \approx 0.11 \tag{94}$$

Table 7. Comparison of symmetrical and asymmetrical triangular numbers

$\frac{HS_e}{S_{\overline{y}}}$	HS_e	HR	$S_{\overline{y}}$	model
0.11	0.52	0.99	4.83	Hybrid (symmetrical with $\mu = 0.7$)
0.11	0.54	0.99	5.07	Hybrid (Asymmetrical with $\mu = 0.7$)
0.11	0.52	0.99	4.69	Classical Regression

The symmetrical model achieves 4.2% lower HSE compared to the asymmetrical model (0.52 vs. 0.54).

The revised examples confirm the paper's findings: hybrid least squares regression with symmetrical triangular fuzzy numbers yields reliability measures close to classical regression, while asymmetrical fuzzy numbers increase dispersion $S_{\overline{y}}$ and slightly increase the standard error (HSE) .

The fuzzy centers remain consistent with classical regression coefficients, and the hybrid model effectively handles fuzzy data while maintaining predictive accuracy.

The reliability measures introduced in this section constitute an essential part of the proposed contribution, as they extend classical goodness-of-fit concepts to hybrid fuzzy regression models using weighted fuzzy arithmetic

9. Conclusion

The hybrid least squares fuzzy regression model, as explored in this study, offers significant advantages over classical regression when handling imprecise or fuzzy data. By incorporating weighted fuzzy arithmetic, this approach effectively captures uncertainty in both symmetrical and asymmetrical triangular fuzzy data, resulting in lower estimation errors compared to traditional statistical linear regression. Our findings demonstrate that the hybrid model closely aligns with classical regression outcomes when non-fuzzy data is used, confirming its robustness and versatility. Notably, the model with symmetrical fuzzy coefficients provides better goodness of fit and lower standard error estimates than its asymmetrical counterpart, suggesting that symmetrical coefficients are preferable for achieving optimal predictive accuracy.

The reliability measures, including the hybrid correlation coefficient HR and standard estimation error HSE, reveal that hybrid regression outperforms other fuzzy regression methods by balancing model precision with data uncertainty. The classical reliability measure is shown to be a special case of hybrid reliability, underscoring the hybrid model's ability to generalize across varying degrees of fuzziness. Furthermore, changes in the skew factor have minimal impact on improving model performance, reinforcing the preference for symmetrical coefficients.

Practically, the hybrid least squares fuzzy regression model is well-suited for applications where data ambiguity is prevalent, such as economic forecasting, environmental modeling, or decision-making under uncertainty. Its computational simplicity, even with increased observations, enhances its scalability for real-world problems. In conclusion, this study establishes hybrid least squares fuzzy regression as a reliable and efficient tool for modeling relationships in fuzzy environments, offering superior predictive capabilities and a robust framework for handling data uncertainty.

Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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