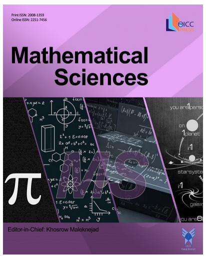

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Research Article

Reliability Analysis of Log-Exponential Transformations under Progressive type-II Censoring with Medical Applications

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Abstract:

In the present paper, we propose a flexible and comprehensive family of distributions derived using the Log-Exponential transformation method within the framework of the progressive type-II censoring scheme. This new family is adaptable to various data types, as its probability density function can assume multiple forms. Beyond presenting this flexible family, we study its statistical properties and invariant estimation techniques, including the maximum likelihood and maximum product spacing estimation. We also conduct a reliability analysis of the new family, with a focus on the modified Weibull distribution as a special case. The Monte Carlo simulations are provided to assess the performance of the proposed estimation methods of the parameters and reliability function under the progressive type-II censoring. Our study emphasizes modeling COVID-19 data using this new family and performing reliability analysis under progressive type-II censoring, accompanied by a thorough discussion on estimation. The applicability of the proposed family is demonstrated using two COVID-19 data sets and comparisons with competitor distributions.

Keywords: COVID-19; Log-Exponential; Modified Weibull; Product spacing; SEM algorithm

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1. Introduction

Statistical probability models are essential for analyzing real-world phenomena and experimental data. However, the complex nature of some real data can limit the effectiveness of some classical probability models. Choosing an appropriate probability model significantly influences statistical inferences, and poor model selection can lead to inaccurate or misleading conclusions. Therefore, researchers have enhanced the distributional flexibility by employing several transformation methods to capture real data characteristics better and facilitate fitting. Also, parameter estimation methods and identifying the most suitable method directly affect the accuracy of statistical inferences, making estimation a critical aspect of lifetime statistical modeling. Concerning all the mentioned topics, some recent flexible lifetime distributions can be found in the research of [1], [2], and [3].

The Moore and Bilikam (MB) family of lifetime distributions [4] is suitable for modeling practical data sets in many disciplines and encompasses several probability distributions, covering fourteen probabilistic models.

The reliability function is one of the challenging issues in complex systems, especially for censored data. Reliability analysis aims to shorten testing duration and reduce costs while maintaining efficient statistical inference. [5] proposed a reliability analysis of the Maxwell-Boltzmann distribution using the generalized progressive type-II hybrid censored design, employing both maximum likelihood (ML) and Bayesian inferential methods. Two real-life data consisting of the failure time points of aircraft windscreens and daily average wind speed in Cairo confirmed the applicability of the Maxwell-Boltzmann distribution. [6] considered the MB family under the progressive type-II censoring strategy, focusing particularly on the Burr XII distribution, and provided the reliability analysis through both frequentist and Bayesian estimation approaches. The Harris extended-exponential distribution under the progressive type-II censoring is proposed by [7], which demonstrated the flexible hazard rate shapes. The reliability research is performed based on both ML and Bayesian estimates. Applications to engineering and veterinary medicine datasets demonstrated the practical advantages of the proposed model.

[8] recommended the Exponentiated Kumaraswamy G-Logarithmic family, which includes four additional parameters but may not adequately fit complex data patterns. [9] proposed a more flexible class, called the reduced Logarithmic-X family, which allows modification of existing distributions by adding parameters. [10, 11] reported a flexible Log-Exponential transformation (LET) method that contains an extra shape parameter.

In reliability analysis and data modeling, some units may be lost or discarded from investigation before failure, resulting in censored samples. This can occur due to shortened testing periods, insufficient resources or cost constraints. Hence, censored modeling based on a specific scheme is an executable solution in such situations. A more flexible censoring scheme is known as progressive type-II censoring due to its flexibility in removing surviving units during the experiment. For further details on the progressive censoring scheme, we cite [12], [13], and [14].

Recently, there has been growing interest among researchers in censored data and the development of flexible lifetime distributions. [15] focused on the progressive type-II censoring competing risks model of the Weibull distribution with binomial removals based on the Bayesian estimation method. [16] provided the statistical inference of the generalized Rayleigh competing risks model under the progressive type-II censoring with different or common parameters of the latent lifetime distributions. [11] presented the Bayesian estimation of the shifted Exponential distribution based on progressive type-II censoring with random removals. [17] estimated parameters of the extended odd Weibull Exponential distribution, concentrating on progressive type-II censoring with random removals and applying maximum product spacing (MPS) and ML estimation methods.

Several studies have focused on progressive and hybrid censoring schemes combined with flexible lifetime distributions and Bayesian inference frameworks. [18] studied Bayesian estimation under the adaptive type-II progressively hybrid censoring of the logistic-exponential distribution, while [19] proposed the Lindley distribution under the unified hybrid censoring with applications in medical survival and biological data sets. These recent developments highlight the growing interest in flexible distributional constructions and motivate the present work.

[20] provided the structural characteristics of progressive hybrid censoring schemes by examining the various data scenarios. The findings demonstrate that the distributions of hybrid censored random variables can be directly derived from the cases of type-I and type-II censored data. Additionally, it is shown that likelihood and Bayesian inference results can also be obtained straightforwardly, which accounts for the similarities observed in the probabilistic and statistical analyses of these censoring methods. In a related study, [21] utilized the joint type-II hybrid censoring plan to collect data from two production lines and applied this plan to analyze three real engineering data sets. The parent populations were assumed to follow the Nadarajah–Haghighi distribution with different shape and scale parameters. Two ML and Bayesian estimation techniques were employed to obtain both point and interval estimates of the model parameters.

[22] introduced a multilateral model for repairable systems that can simultaneously analyze failure modes with right and multiple censored data, subject to both imperfect preventive and corrective maintenance. The applicability of the proposed model is highlighted by the aviation industry case study for the failure data set of one overhaul cycle of three aero engines. [23] considered the partially constant-stress accelerated life test model in the presence of generalized inverted exponential distribution under the progressive type-II censoring. The statistical research for a constant-stress partially accelerated life test of the inverted Kumaraswamy distribution based on progressive hybrid and the inverse Weibull based on the adaptive progressive type-I censoring schemes are conducted by [24] and [18].

Despite the flexibility and adaptability of the LET in modeling different types of data, to the best of our knowledge, no extensions of this transformation have been explored in existing research. Also, censoring designs related to the estimation of parameters and reliability characteristics of the LET have not been well addressed, although they are highly relevant and practically important in many real-world situations. Meanwhile, the MB family, though less frequently cited in the literature, is a common probability model encompassing several key lifetime distributions. To bridge this gap, this paper provides a new combined family based on the flexible LET of the MB family, an extensive class of lifetime distributions. The proposed family in this paper integrates the LET generator with the MB baseline mechanism. The LET-based models focus primarily on transformation mechanisms applied to a baseline distribution, improving tail behavior but offering limited hazard flexibility. Hence, the MB baseline is motivated by its ability to model a wide range of hazard rate shapes, including increasing, decreasing, bathtub, and unimodal characteristics. However, exist-

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ing MB-based families often lack sufficient tail flexibility. In particular, the MB component allows various hazard rates, while the LET includes the tail adaptability, shape of distribution, and model robustness under censoring. As a result, the proposed family nests several existing models as special cases and exhibits superior fitting performance for complex lifetime data. For more in-depth analysis, we explore the ML (Newton-Raphson (NR), expectation-maximization (EM), and stochastic EM (SEM)) and MPS estimation methods of unknown parameters, as well as their reliability characteristics under the progressive type-II censoring scheme.

The remainder of the paper is organized as follows. In section 2, the new LET family is provided along with some statistical properties of the proposed family. In section 3, several estimation approaches of the parameters and reliability function of the LET family are represented. The LET of the modified Weibull submodel is investigated in section 4, with the Monte Carlo simulation discussion of the estimation methods. In section 5, the applicability of the new family using real-world data is demonstrated by exploring two real data sets of new death cases of COVID-19 from Denmark and Ethiopia, showing that the considered model addresses a better fit than other competitors.

2. The new Log-Exponential transformation family

In this section, we introduce a new family called Log-Exponential MB transformation (LEMBT). Regarding the new family, we modulate the LET with the extra shape parameter and MB baseline distribution. Also, some stochastic properties of the model are provided.

Consider the cumulative distribution function (CDF) and the probability density function (PDF) of MB family of baseline distribution as

$$H(x,\beta,\lambda) = 1 - \exp\big(\frac{-g^{\beta}(x)}{\lambda}\big), \qquad \beta,\ \lambda > 0,$$
$$h(x,\beta,\lambda) = \frac{\beta}{\lambda} g'(x) g^{\beta-1}(x) \exp\big(\frac{-g^{\beta}(x)}{\lambda}\big),\ \beta, \lambda > 0,$$

where $g(x)$ is a strictly increasing function of x with $g(0^{+}) = 0$, $g(\infty) = \infty$, and $g'(x)$ denotes the derivative of $g(x)$ with respect to x. We introduce the CDF and PDF of the LEMBT model as below

$$\begin{aligned}
F(x,\beta,\lambda,\theta) &= \frac{\ln(2 - e^{-\theta H(x,\beta,\lambda)})}{\ln(2-e^{-\theta})} \\
&= \frac{\ln\Big(2 - e^{-\theta\big(1-\exp(\frac{-g^{\beta}(x)}{\lambda})\big)}\Big)}{\ln(2-e^{-\theta})}, \qquad \beta,\ \lambda, \theta > 0, \\
f(x,\beta,\lambda,\theta) &= \frac{\theta h(x) e^{-\theta H(x,\beta,\lambda)}}{\ln(2-e^{-\theta})\big(2 - e^{-\theta H(x,\beta,\lambda)}\big)} \\
&= \frac{\theta\beta g'(x) g^{\beta-1}(x) \exp\big(\frac{-g^{\beta}(x)}{\lambda}\big) e^{-\theta\,(1-\exp(\frac{-g^{\beta}(x)}{\lambda}))}}{\lambda \ln(2-e^{-\theta})\big(2 - e^{-\theta\big(1-\exp(\frac{-g^{\beta}(x)}{\lambda})\big)}\big)}.
\end{aligned}$$

The hazard and cumulative hazard rate functions of the LEMBT are represented respectively as

$$h(t,\beta,\lambda,\theta) = \frac{\theta\beta g'(t) g^{\beta-1}(t) \exp\big(\frac{-g^{\beta}(t)}{\lambda}\big) e^{-\theta\,(1-\exp(\frac{-g^{\beta}(t)}{\lambda}))}}{\lambda\Big(\ln(2-e^{-\theta}) - \ln\big(2 - e^{-\theta\big(1-\exp(\frac{-g^{\beta}(t)}{\lambda})\big)}\big)\Big)\big(2 - e^{-\theta\big(1-\exp(\frac{-g^{\beta}(t)}{\lambda})\big)}\big)},$$
$$S(t,\beta,\lambda,\theta) = \ln\big(\ln(2-e^{-\theta})\big) - \ln\Big(\ln\big(\frac{2-e^{-\theta}}{2-e^{-\theta\,\big(1-\exp(\frac{-g^{\beta}(t)}{\lambda})\big)}}\big)\Big).$$

The LEMBT family provides an extra parameter θ to generate an extended version of baseline model, which improves the characteristics of classical models with more flexible kurtosis and skewness. The LET and the MB family represent two distinct approaches with inherent strengths and limitations. The LET method introduces a single additional shape parameter, through a Log-Exponential structure applied to a baseline CDF. While this transformation effectively adds flexibility in modeling tail behavior and can modulate skewness, its capacity to generate diverse hazard rate shapes is inherently constrained by the single extra parameter and the underlying baseline. It primarily scales and distorts the baseline distribution rather than fundamentally expanding its hazard profile. Conversely, the MB family provides a versatile two-parameter baseline generator. By carefully selecting $g(x)$ and tuning β and λ, the MB family can produce several classical lifetime distributions and exhibit various hazard rate shapes. However, its flexibility is ultimately bounded by the form of $g(x)$, and achieving highly complex, non-monotonic hazard patterns may require nesting multiple distributions or choosing intricate, less tractable forms for $g(x)$, potentially leading to over-parameterization. The LEMBT family combines these two approaches, overcoming their individual limitations. We apply the LET not to a simple baseline, but directly to the CDF of the MB family, that yields a three-parameter generator. In essence, while the LET offers a scaling mechanism and the MB family offers a baseline shape mechanism, the LEMBT combines them into a shape-scaling synthesis. This provides a more powerful and adaptable tool for reliability and survival analysis, particularly for contemporary data exhibiting complex underlying patterns. Several submodels of the LEMBT family, based on special cases of parameters, are summarized in Table 1.

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Table 1. Some submodels of the LEMBT family

Parameters	Distribution	CDF
$\theta \to 0$	MB baseline distribution	$F(x,\beta,\lambda) = 1 - \exp\left(\frac{-g^{\beta}(x)}{\lambda}\right)$
$\beta = 1$ and $g(x) = x$	Log-Exponential Exponential	$F(x,\lambda,\theta) = \dfrac{\ln\left(2 - e^{-\theta\left(1-\exp\left(\frac{-x}{\lambda}\right)\right)}\right)}{\ln(2-e^{-\theta})}$
$\beta = 1,\ g(x) = x$ and $\theta \to 0$	Exponential distribution	$F(x,\lambda) = 1 - \exp\left(\frac{-x}{\lambda}\right)$

Notation 2.1 We obtain several Log-Exponential distributions by different choices of the arbitrary function $g(x)$ and β.

2.1 Mixture representation of the PDF of LEMBT family

In this section, we indicate the mixture representation of the PDF of the LEMBT family, which will be used for the subsequent computation of the statistical properties of the LEMBT family. The mixture representation of the PDF of the LEMBT family is represented as

$$f(x,\beta,\lambda,\theta) = \frac{\theta}{\ln(2-e^{-\theta})}\sum_{k=1}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{h}\frac{(k\theta)^h(-1)^{h+i}}{2^k(i+1)!(h-i)!} \times\left(\frac{(i+1)\beta g'(x)g^{\beta-1}(x)e^{-(i+1)\frac{g^{\beta}(x)}{\lambda}}}{\lambda}\right). \quad (1)$$

For more details, see Appendix A. From Equation 1, the PDF of the LEMBT family is expressed as a product of the parameter θ as $\frac{\theta}{\ln(2-e^{-\theta})}$ and the sum of the product of the weighted baseline PDF of MB with parameters $(\beta, \frac{i+1}{\lambda})$ and weight $w = \frac{(-1)^{h+i}(k\theta)^h}{2^k(h-i)!(i+1)!}$. The mixture representation of the PDF of the LEMBT family will be utilized in computing the r-th moments of the family.

2.2 Some statistical properties of the LEMBT family

The moment generating and quantile functions of the LEMBT family of distributions are derived in the following propositions.

Proposition 2.1 The moment generating function (MGF) of the LEMBT family is shown by

$$M_X(t) = \frac{\alpha\theta}{\ln(2-e^{-\theta})} \times\sum_{k=1}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{h}\sum_{l=0}^{\infty}\frac{(-1)^{h+i}(k\theta)^h t^l}{2^k(h-i)!(i+1)!l!}\xi(l,i+1).$$

Proof. By definition of the MGF and Taylor expansion $e^{tx} = \sum_{l=0}^{\infty}\frac{t^l x^l}{l!}$, we have

$$M_X(t) = \frac{\theta}{\ln(2-e^{-\theta})}\sum_{k=1}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{h}\frac{(-1)^{h+i}(k\theta)^h}{2^k(h-i)!(i+1)!} \times\int_0^{\infty}\sum_{l=0}^{\infty}\frac{t^l x^l}{l!}\frac{(i+1)\beta}{\lambda}g'(x)g^{\beta-1}(x)e^{-\frac{(i+1)g^{\beta}(x)}{\lambda}}dx$$
$$= \frac{\theta}{\ln(2-e^{-\theta})}\sum_{k=1}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{h}\sum_{l=0}^{\infty}\frac{(-1)^{h+i}(k\theta)^h t^l}{2^k(h-i)!(i+1)!l!} \times\int_0^{\infty}x^l\frac{(i+1)\beta}{\lambda}g'(x)g^{\beta-1}(x)e^{-\frac{(i+1)g^{\beta}(x)}{\lambda}}dx$$
$$= \frac{\theta}{\ln(2-e^{-\theta})}\sum_{k=1}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{h}\sum_{l=0}^{\infty}\frac{(-1)^{h+i}(k\theta)^h t^l}{2^k(h-i)!(i+1)!l!}\xi(l,i+1).$$

Proposition 2.2 (Quantile function) The quantile function of the LEMBT family is demonstrated as follows $x = g^{-1}\left(\left[\lambda\ln\left[1+\frac{1}{\theta}\ln\left(2-(2-e^{-\theta})^u\right)\right]\right]^{\frac{1}{\beta}}\right)$, where $g^{-1}(.)$ is the inverse of the arbitrary function.

2.3 The reliability analysis

Reliability is the probability that a component or system operates without failure within predefined limits (such as time or space), under specified operating conditions. The reliability function is widely utilized in reliability analysis and lifetime data. Consider the LEMBT family of distributions, the reliability function is represented as

$$R(t,\beta,\lambda,\theta) = \frac{\ln(2-e^{-\theta}) - \ln(2-e^{-\theta H(t,\beta,\lambda)})}{\ln(2-e^{-\theta})} = \frac{\ln(2-e^{-\theta}) - \ln\left(2-e^{-\theta\left(1-\exp\left(\frac{-g^{\beta}(t)}{\lambda}\right)\right)}\right)}{\ln(2-e^{-\theta})}.$$

2.4 Special model of the LEMBT family

As a special case of the LEMBT family, set $g(x) = x^{\eta}e^{\nu x}$ with $\beta = 1$ that leads to the Log-Exponential modified Weibull (ELMW) distribution. The PDF and CDF plots of the ELMW distribution are shown in Figure 1 and Figure 2, for several combinations of the parameters. As can be seen, the ELMW distribution can be applied to every kind of real phenomenon, which covers both monotone (decreasing) and unimodal (inverted bathtub) shapes of PDF. Furthermore, varying the extra parameter θ alters the skewness and kurtosis of the ELMW distribution.

The PDF and CDF of ELMW distribution are represented as follows

$$f(x,\lambda,\theta)=\frac{\theta x^{\eta-1}e^{\nu x}(\eta+\nu x)\exp\left(\frac{-x^{\eta}e^{\nu x}}{\lambda}\right)e^{-\theta\,(1-\exp(\frac{-x^{\eta}e^{\nu x}}{\lambda}))}}{\lambda\ln(2-e^{-\theta})\left(2-e^{-\theta\left(1-\exp(\frac{-x^{\eta}e^{\nu x}}{\lambda})\right)}\right)},$$

$$F(x,\lambda,\theta)=\frac{\ln\left(2-e^{-\theta\,(1-\exp(\frac{-x^{\eta}e^{\nu x}}{\lambda}))}\right)}{\ln(2-e^{-\theta})}.$$

The hazard rate function of the ELMW is demonstrated as follows

$$h(t,\lambda,\theta)=\frac{\theta t^{\eta-1}e^{\nu t}(\eta+\nu t)\exp\left(\frac{-t^{\eta}e^{\nu t}}{\lambda}\right)e^{-\theta\,(1-\exp(\frac{-t^{\eta}e^{\nu t}}{\lambda}))}}{\lambda\Big(\ln(2-e^{-\theta})-\ln\left(2-e^{-\theta\left(1-\exp(\frac{-t^{\eta}e^{\nu t}}{\lambda})\right)}\right)\Big)\left(2-e^{-\theta\left(1-\exp(\frac{-t^{\eta}e^{\nu t}}{\lambda})\right)}\right)}.$$

3. Different estimation approaches of the parameters of LEMBT family

In this section, several estimation approaches of the parameters of the LEMBT family containing the ML (NR, EM, and SEM algorithm) and MPS estimation methods are discussed comprehensively. The EM algorithm treats the progressive censored data as the missing data and the SEM algorithm replaces the E-step with the S-step, which is appropriate for many distributions and the missing data [25]. The performance of these estimation procedures is evaluated through the Monte Carlo simulation scheme.

3.1 The maximum likelihood estimators

Progressive censoring is useful in many scientific fields and life testing studies, as it allows the removal of surviving experimental units before the termination of the test. Here, we concentrate on the progressive type-II censoring scheme.

The progressive censored sample of size n with m failures and $n-m$ progressive censored samples is summarized as follows. Consider the first n units, after each m failure occurred, respectively, $r_1, r_2, \ldots, r_{m-1}$ and r_m, surviving units are randomly withdrawn from the test. At the last failure, the remaining $r_m = n-r_1-r_2-\ldots-r_{m-1}-m$ units are excluded from the experiment.

Consider the sample size n and $t_1 < \ldots < t_m$ be a progressive type-II censored sample from a life test of sample size m, where the ordered lifetimes have a MB family with a specified number of removal, as $(r_1, \ldots, r_m)$, then the likelihood function is given by

$$\begin{aligned}L(\gamma,T)&=C\prod_{i=1}^{m}f(t_i,\gamma)[1-F(t_i,\gamma)]^{r_i}\\&=C\Big(\frac{\theta\beta}{\lambda}\Big)^m\prod_{i=1}^{m}\left(\ln(2-e^{-\theta})\right)^{-(r_i+1)}\\&\times\frac{g'(t_i)g^{\beta-1}(t_i)\exp\left(\frac{-g^{\beta}(t_i)}{\lambda}\right)e^{-\theta\,(1-\exp(\frac{-g^{\beta}(t_i)}{\lambda}))}}{\left(2-e^{-\theta\left(1-\exp(\frac{-g^{\beta}(t_i)}{\lambda})\right)}\right)}\\&\times\Big[\ln(2-e^{-\theta})-\ln\left(2-e^{-\theta\left(1-\exp(\frac{-g^{\beta}(t_i)}{\lambda})\right)}\right)\Big]^{r_i},\end{aligned}$$

where $\gamma=(\beta,\lambda,\theta)$ is an unknown vector of parameters, and $C=n(n-1-r_1)\ldots(n-r_1-\ldots-r_{m-1}-m+1)$ is a constant independent of the parameters. By ignoring C, the log-likelihood function is obtained as

$$\begin{aligned}\mathcal{L}(\beta,\theta,T)&=m\ln\beta+m\ln\theta-m\ln\lambda \qquad (2)\\&-\sum_{i=1}^{m}(r_i+1)\ln\ln\left(2-e^{-\theta}\right)+\sum_{i=1}^{m}\ln g'(t_i)\\&+(\beta-1)\sum_{i=1}^{m}\ln g(t_i)-\sum_{i=1}^{m}\frac{g^{\beta}(t_i)}{\lambda}\\&-\sum_{i=1}^{m}\ln\left(2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}-1\right)\\&+\sum_{i=1}^{m}r_i\ln\ln\left(\frac{2-e^{-\theta}}{2-e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}\right).\end{aligned}$$

The ML estimator of the parameters can be obtained by setting the score of Equation 2 equal to zero, (See Appendix B).

The score equations cannot be solved analytically. Therefore, some numerical methods such as Newton-Raphson must be applied to find the ML estimate of the unknown parameters.

The NR estimates are sensitive to the initial parameter values and converge slowly to real values of parameters. Moreover, when executing the censored scheme, the NR estimators are significantly biased. All these deficiencies persuade us to discuss the different estimation approaches. In the following, the EM and SEM algorithms are provided for more accurate results.

3.1.1 EM algorithm

The EM algorithm was recommended to estimate any missing or incomplete data. The advantages of the EM algorithm are its ease of implementation, computational stability, and satisfactory convergence rate. The progressively type-II censoring can be considered as an incomplete data set, and therefore, the EM algorithm is recommended instead of the NR method to find the ML estimators.

Let $Z=(Z_1, Z_2, \ldots, Z_m)$ with $Z_j=(Z_{j1}, Z_{j2}, \ldots, Z_{jR_j})$, $j=1,\ldots,m$, be the censored data and suppose the censored data as missing. Therefore, the complete data set X, which includes missing and observed data, is constructed from the combination of $(T,Z)=X$. The complete log-likelihood function of

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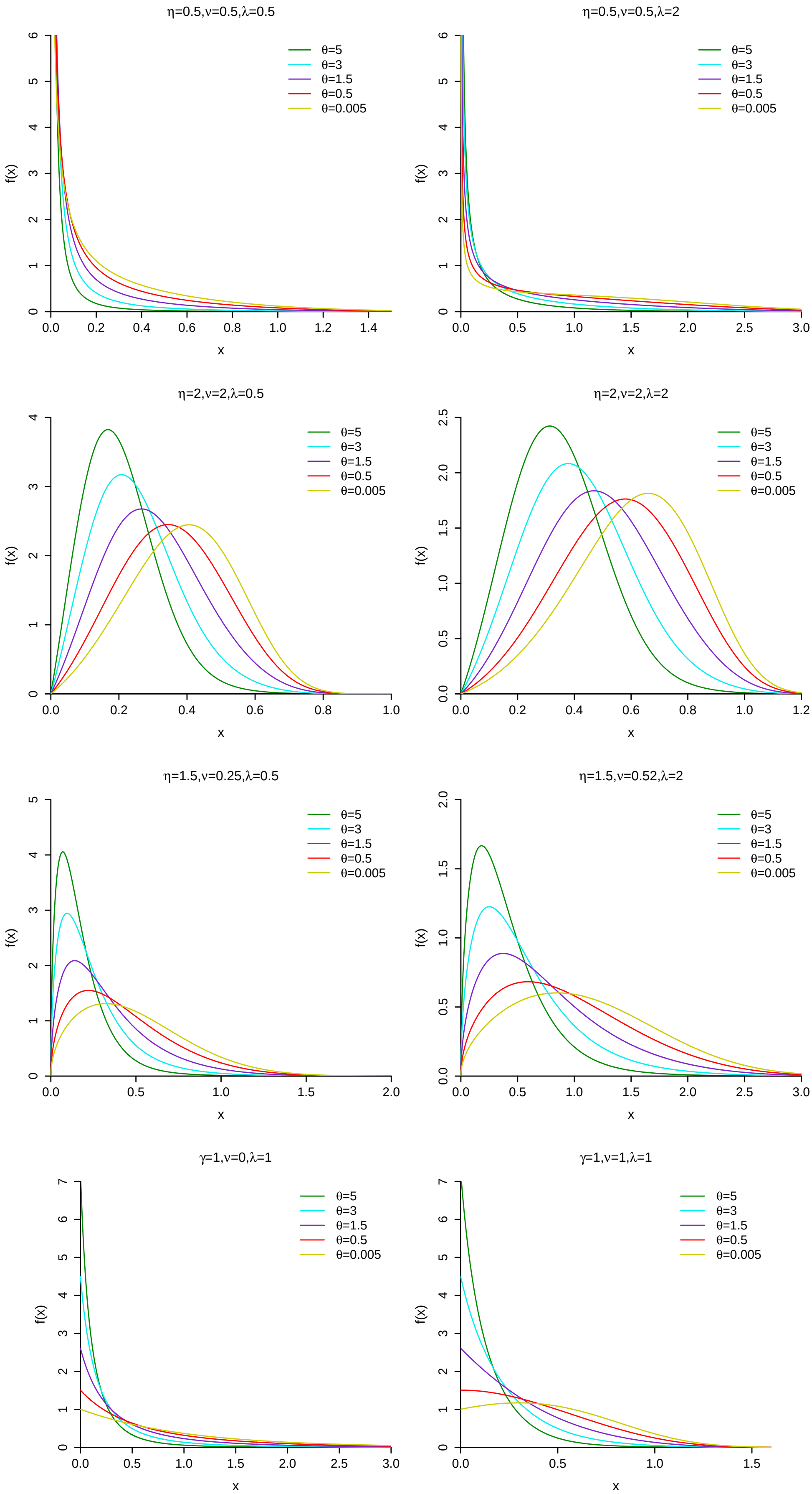

Figure 1. The PDF plots of the ELMW distribution

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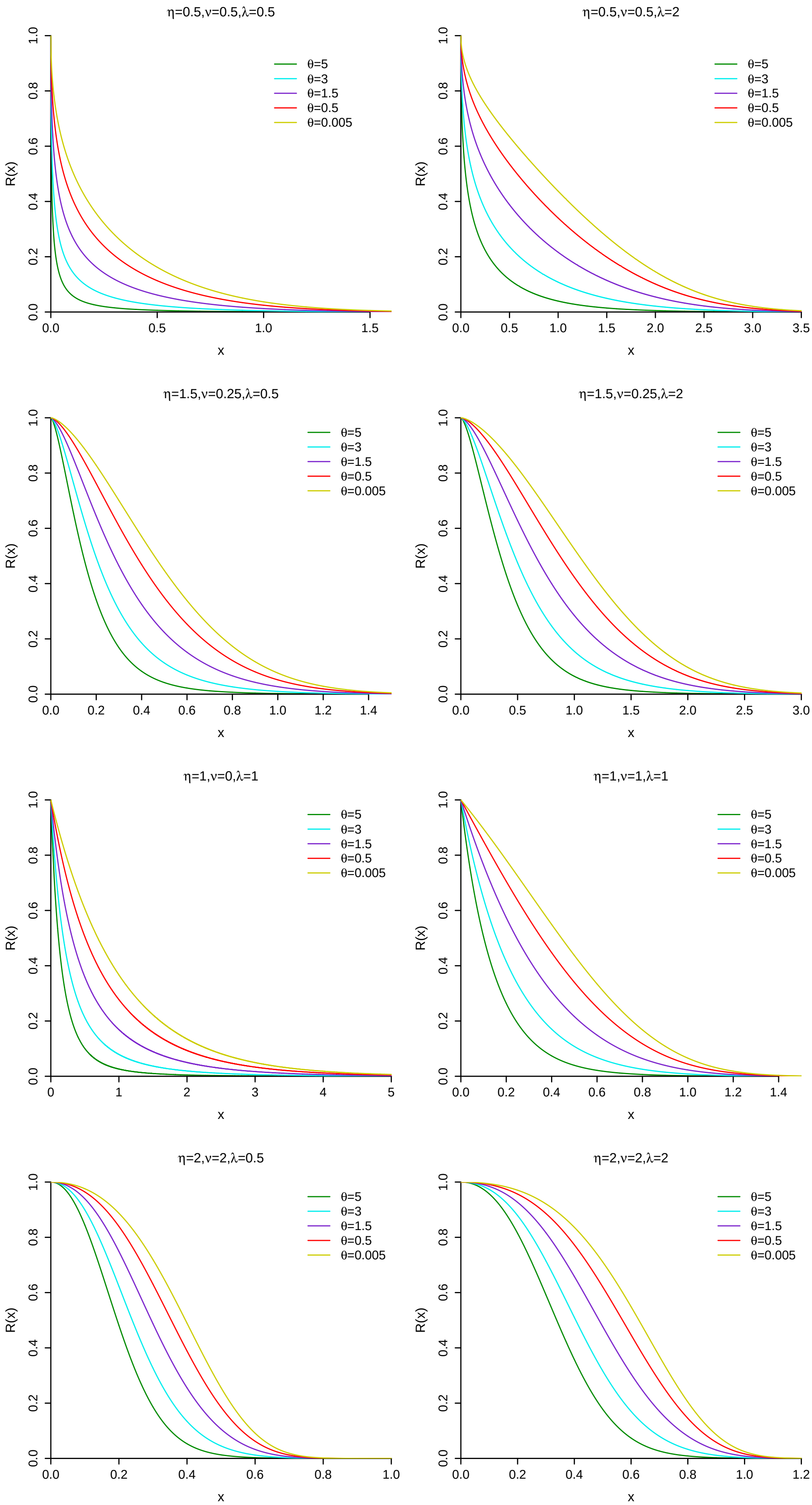

Figure 2. The reliability plots of the ELMW distribution

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LEMBT family based on X is represented by

$$\begin{aligned}\mathcal{L}_c(\gamma, X) &= n\ln\beta + n\ln\theta - n\ln\lambda \\ &-n\ln\ln(2-e^{-\theta}) + \sum_{i=1}^{m}\ln g'(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln g'(z_{ik}) \\ &+(\beta-1)\sum_{i=1}^{m}\ln g(t_i) + (\beta-1)\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln g(z_{ik}) \\ &-\sum_{i=1}^{m}\frac{g^{\beta}(t_i)}{\lambda} - \sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{g^{\beta}(z_{ik})}{\lambda} \\ &-\sum_{i=1}^{m}\ln\left(2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}-1\right) \\ &-\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln\left(2e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}-1\right).\end{aligned} \tag{3}$$

The ML estimators of the vector parameters γ for complete sample X can be achieved by equating the score of Equation 3 to zero, (See Appendix C). The EM algorithm has two steps, E-step and M-step. In the E-step, any function $h(Z_{ik})$ is replaced by its conditional expectation $E(h(Z_{ik})|Z_{ik} > t_i)$. For $z_{ik} > t_i$ the conditional distribution of Z_{ik} given $T_i = t_i$ follows a truncated MB family with left truncation at t_i as

$$\begin{aligned} f_{Z|T}(z_{ik}|t) &= \frac{f(z_{ik})}{1-F(t_i)} = \frac{\theta\beta g'(z_{ik})g^{\beta-1}(z_{ik})}{\lambda(2-e^{-\theta(1-e^{-\frac{g^{\beta}(z_{ik})}{\lambda}})})} \\ &\times\frac{\exp(-\frac{g^{\beta}(z_{ik})}{\lambda})e^{-\frac{g^{\beta}(t_i)}{\lambda}}e^{-\theta(1-e^{-\frac{g^{\beta}(z_{ik})}{\lambda}})}}{\left(\ln(2-e^{-\theta})-\ln(2-e^{-\theta(1-e^{-\frac{g^{\beta}(t_i)}{\lambda}})})\right)}.\end{aligned} \tag{4}$$

The E-step requires computing six conditional expectations that appear in the expected complete-data log-likelihood. These expectations are defined as follows for each $i = 1,\dots,m$,

$$\begin{aligned} E_{1i}(t,\beta,\lambda,\theta) &= E(g^{\beta}(z_{ik})|Z_{ik} > t), \\ E_{2i}(t,\beta,\lambda,\theta) &= E\left(\frac{e^{\frac{-g^{\beta}(z_{ik})}{\lambda}}g^{\beta}(z_{ik})}{2-e^{-\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}\middle|Z_{ik} > t_i\right), \\ E_{3i}(t,\beta,\lambda,\theta) &= E(\ln(g(z_{ik}))|Z_{ik} > t), \\ E_{4i}(t,\beta,\lambda,\theta) &= E(\ln(g(z_{ik}))g^{\beta}(z_{ik})|Z_{ik} > t), \\ E_{5i}(t,\beta,\lambda,\theta) &= E\left(\frac{\ln(g(z_{ik}))g^{\beta}(z_{ik})e^{\frac{-g^{\beta}(z_{ik})}{\lambda}}}{2-e^{-\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}\middle|Z_{ik} > t_i\right), \\ E_{6i}(t,\beta,\lambda,\theta) &= E\left(\frac{1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}}}{2-e^{-\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}\middle|Z_{ik} > t_i\right).\end{aligned} \tag{5}$$

These expectations can be expressed as integrals $E_{ji} = \int_{t_i}^{\infty} h_j(z)f_{Z|T}(z|t_i)dz$, $j = 1,\dots,6$, where the functions $h_j(z)$ correspond to the integrands in Equation 5.

In the M-step, at $(k+1)$-th iteration, the EM estimates of the parameters are obtained by solving the derivate of the complete log-likelihood with respect to each parameter (see Appendix D.).

3.1.2 SEM algorithm

The stochastic EM algorithm is a stochastic version of the EM algorithm that replaces each missing observation with a value randomly generated from the conditional distribution given the current parameter estimates, and the M-step performs a complete-data ML estimation.

For the LEMBT family and the SEM algorithm, r_i number of samples of z_{ik}, such that $z_{ik} > t_i$, $i = 1,2,\dots,m$, $k = 1,2,\dots,r_i$ must be generated from the following conditional CDF

$$\begin{aligned} F_{Z|T}(z_{ik}|t_i) &= \frac{F(z_{ik})-F(t_j)}{1-F(t_j)} \\ &= \frac{\ln\left(2-e^{-\theta\left(1-\exp(\frac{-g^{\beta}(z_{ik})}{\lambda})\right)}\right)-\ln\left(2-e^{-\theta\left(1-\exp(\frac{-g^{\beta}(t_i)}{\lambda})\right)}\right)}{\ln(2-e^{-\theta})-\ln\left(2-e^{-\theta\left(1-\exp(\frac{-g^{\beta}(t_i)}{\lambda})\right)}\right)},\end{aligned}$$

The estimators of γ at the $(k+1)$-step of the algorithm are provided in Appendix E.

3.2 The maximum product spacings estimators

The MPS estimators of the parameters of the LEMBT family can be obtained by maximizing the product of spacings which are defined under progressive type-II censoring sample as follows

$$V(\gamma, T) = C\prod_{i=1}^{m}\Delta_i[1-F(t_i,\gamma)]^{r_i},$$

where C is a constant term that does not depend on γ and

$$\Delta_i = \begin{cases} F(t_1,\gamma), & i = 1 \\ F(t_i,\gamma)-F(t_{i-1},\gamma), & i = 2,\dots,m-1 \\ 1-F(t_m,\gamma), & i = m.\end{cases}$$

The logarithm of the product of spacings is represented below

$$\ln(V(\gamma,T)) = \ln C + \sum_{i=1}^{m}\ln(\Delta_i) + \sum_{i=1}^{m} r_i\ln(1-F(t_i,\gamma)).$$

By differentiating the logarithm of the product of spacings regarding the parameters, the MPS estimators of the parameters under progressive type-II censored samples are obtained as follows

$$\begin{aligned} \frac{\partial\ln(V(\gamma,T))}{\partial\lambda} &= \sum_{i=1}^{m}\frac{D_1(i)}{\Delta_i} - \sum_{i=1}^{m}\frac{r_iC_1(i)}{1-F(t_i,\gamma)} = 0, \\ \frac{\partial\ln(V(\gamma,T))}{\partial\beta} &= \sum_{i=1}^{m}\frac{D_2(i)}{\Delta_i} - \sum_{i=1}^{m}\frac{r_iC_2(i)}{1-F(t_i,\gamma)} = 0, \\ \frac{\partial\ln(V(\gamma,T))}{\partial\theta} &= \sum_{i=1}^{m}\frac{D_3(i)}{\Delta_i} - \sum_{i=1}^{m}\frac{r_iC_3(i)}{1-F(t_i,\gamma)} = 0,\end{aligned} \tag{6}$$

where

$$C_1(i) = \frac{\partial F(t_i,\gamma)}{\partial \lambda} = \frac{-\theta g^{\beta}(t_i) e^{\frac{-g^{\beta}(t_i)}{\lambda}} e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\lambda^2 \ln(2-e^{-\theta})\left(2-e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}\right)},$$

$$C_2(i) = \frac{\partial F(t_i,\gamma)}{\partial \beta} = \frac{\theta \ln(g(t_i)) g^{\beta}(t_i) e^{\frac{-g^{\beta}(t_i)}{\lambda}} e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\lambda \ln(2-e^{-\theta})\left(2-e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}\right)},$$

$$C_3(i) = \frac{\partial F(t_i,\gamma)}{\partial \theta} = \frac{(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}) e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\ln(2-e^{-\theta})\left(2-e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}\right)} - \frac{e^{-\theta}\ln(2-e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})})}{\left(\ln(2-e^{-\theta})\right)^2 (2-e^{-\theta})},$$

and $D_1(i) = \frac{\partial \Delta_i}{\partial \lambda} = C_1(i) - C_1(i-1)$, $D_2(i) = \frac{\partial \Delta_i}{\partial \beta} = C_2(i) - C_2(i-1)$, $D_3(i) = \frac{\partial \Delta_i}{\partial \theta} = C_3(i) - C_3(i-1)$. The Equation 6 cannot be solved analytically, hence the numerical nonlinear optimization algorithms are implemented to solve equations.

Corollary 3.1 Due to the invariant property of the ML and MPS estimators of the model's parameters, the reliability estimators of the LEMBT family based on ML and MPS methods are derived as

$$\hat{R}_{ML} = \frac{\ln(2-e^{-\hat{\theta}_{ML}}) - \ln\left(2-e^{-\hat{\theta}_{ML}\left(1-\exp(\frac{-g^{\hat{\beta}_{ML}}(t)}{\hat{\lambda}_{ML}})\right)}\right)}{\ln(2-e^{-\hat{\theta}_{ML}})},$$

$$\hat{R}_{MPS} = \frac{\ln(2-e^{-\hat{\theta}_{MPS}}) - \ln\left(2-e^{-\hat{\theta}_{MPS}\left(1-\exp(\frac{-g^{\hat{\beta}_{MPS}}(t)}{\hat{\lambda}_{MPS}})\right)}\right)}{\ln(2-e^{-\hat{\theta}_{MPS}})},$$

where $(\hat{\beta}_{ML}, \hat{\lambda}_{ML}, \hat{\theta}_{ML})$ and $(\hat{\beta}_{MPS}, \hat{\lambda}_{MPS}, \hat{\theta}_{MPS})$ are ML and MPS estimators and computed numerically via the statistical packages.

4. Simulation study

In this section, we represent the Monte-Carlo simulation results, which have been conducted to evaluate the performance of the estimates using the ML (via NR, EM, SEM) and MPS estimation methods of the parameters and reliability function of the LEMBT family under progressive type-II censoring for the different combinations of parameters and sample sizes.

All simulations were conducted by statistical software R version 4.4.1 on a desktop computer. The estimation procedures were repeated $h = 1000$ times.

To compare the estimation methods for the ELMW parameters, we consider the values $(\theta, \lambda) = (1.5, 3)$, $(3, 0.5)$. The values (θ, λ) were selected to represent different distributional characteristics, such that $(\theta, \lambda) = (1.5, 3)$ represents a scenario with moderate skewness and a scale parameter that yields a reasonable spread of failure times, whereas $(\theta, \lambda) = (3, 0.5)$ corresponds to a more extreme case with higher skewness and smaller scale, producing data with a heavier right tail and earlier failures. For reliability function estimation, we set with $(\theta, \lambda) = (0.5, 6)$, over the time points $t = 1, 2, 3, 4, 5$.

Inspired of [26], the progressive type-II censored data from the ELMW is generated with the censoring schemes $r_1 = \left(1^3, 0^2, 2^3, 1^3, 0^{m-15}, 2^3, n-m-18\right)$ and $r_2 = \left(2^2, 0^{m-10}, 1^4, 2^3, n-m-14\right)$. Scheme r_1 features alternating periods of removals and no removals, representing an experiment where units are periodically removed for inspection. Scheme r_2 begins with removals, then a long period without removals, followed by a final removal phase, simulating an experiment with initial and final inspections but continuous operation in between.

The sample sizes, $(n, m) = (40, 15)$ with high censoring rate 62.5%, $(n, m) = (40, 20)$ with moderate censoring rate 50%, $(n, m) = (80, 30)$ with high censoring rate 62.5%, and $(n, m) = (80, 50)$ with low censoring rate 37.5%. This design allows us to examine the effects of both total sample size and censoring proportion on estimation accuracy.

4.1 Simulation results of the different estimation methods for the parameters and reliability function of the ELMW distribution

Table 2 and Table 3 represent the mean and RMSE (in brackets) of the ML (NR, EM and SEM algorithms) and MPS estimates of the parameters of the ELMW distribution under the progressive type-II censoring for different combinations of the parameters and censoring scheme.

Results of Table 2 and Table 3:

a) For all estimation methods, the higher values of n lead to better estimates in the sense of affinity to the true value of the parameters and having small RMSE values. The estimates of the parameters of ELMW distribution are convergent to the real values of the parameters.

b) Bias decreases as sample size increases for all methods, confirming consistency. These patterns are observed uniformly across different parameter configurations.

c) The EM algorithm consistently outperforms the NR, SEM and MPS methods in terms of RMSE across all censoring schemes and sample sizes.

d) The average computation times per 1000 replications were as follows

Estimation methods:	NR	EM	SEM	MPS
Times (per second):	45-60	120-180	200-300	90-120

where the min and max computation time belongs to the NR and SEM methods, respectively.

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e) The EM and SEM algorithms, while more computationally intensive, provide more stable estimates than NR, especially for small samples and high censoring rates.
f) All methods converged successfully in all 1000 replications for every parameter and censoring combination. No convergence failures were observed, indicating the robustness of the implemented algorithms. The EM algorithm typically required 15-25 iterations to reach convergence, while the NR method converged in 5-10 iterations.

The RMSE measures the estimation error at a specific point, but it is also worthwhile to study a global risk measure. Therefore, the reliability estimators are compared using integrated mean squared error (IMSE), which is the integration of the total area for t_i and

$$\begin{aligned}\mathrm{IMSE}(\hat{R}(t)) &= \frac{1}{1000}\sum_{i=1}^{1000}\left(\frac{1}{n_t}\sum_{j=1}^{n_t}(\hat{R}_j(t_i) - R(t_i))^2\right)\\ &= \frac{1}{n_t}\sum_{i=1}^{n_1}\mathrm{RMSE}^2(\hat{R}(t_i)),\end{aligned}$$

where n_t is the number of times and $t = 1, 2, 3, 4, 5$.

The reliability function estimator of the ELMW distribution and RMSE values under the progressively type-II censoring for $t = 1, 2, 3, 4, 5$ are represented in Table 4 and Table 5. For ML estimation method, we only consider the SEM algorithm and the results are reported.

As can be seen in Table 4 and Table 5, as the sample size n and the effective sample size m increase, the RMSEs decrease. Hence, by increasing the sample size, the IMSE of the reliability function estimators of the ELMW distribution are decreased. The IMSEs of the SEM estimates of the reliability function are less than MPS, which confirms the superiority of the SEM over the MPS method in reliability function estimation.

5. Real data analysis

In this section, we present the application of the LEMBT family under the progressive type-II censoring, and the model is compared with other related models. Note that for $g(x)$, same as the simulation section, we consider the modified Weibull $\left(g(x) = x^{\gamma}e^{\nu x}\right)$ distribution as a special case of the MB family. The performance of the estimation procedures is evaluated by two real data sets under the progressive type-II censoring scheme, which verify the simulation results.

The first and second data sets represent the daily number of newly reported death cases due to COVID-19 in Denmark, with $n = 69$ sample size (from 19 November 2020 to 26 January 2021) and in Ethiopia, with $n = 45$ sample size (from 11 July to 24 August 2020), respectively. The daily data related to the COVID-19 in the world were sourced from the World Health Organization (https://covid19.who.int), that represent mortality trends from different geographical regions and epidemic phases. Figure 3 shows histogram with the kernel density plot of the COVID-19 data set in Denmark and Ethiopia. It can be seen that the COVID-19 data are right-skewed and platykurtic.

Table 6 gives some statistical measures of COVID-19 data sets, in accordance with Figure 3, indicates that the empirical distribution of both data sets are skewed to the right and platykurtic. Also, the Kolmogorov-Smirnov (K-S) distances between the fitted and the empirical distribution function results are presented in Table 6. Based on the K-S p-values, we conclude that the modified Weibull distribution provides a good fit for both data sets at the 0.05 significance level.

We represent the total time on test (TTT) plot of COVID-19 data in Figure 4 Hence, both data sets have increasing hazard rate functions and the modified Weibull distribution can be suggested for them.

We compare the ELMW distribution of the LEMBT family $\left(g(x) = x^{\gamma}e^{\nu x}\right)$ with some classical distributions. Table 7 shows the ML estimates and goodness of fit statistics (Akaike information criterion (AIC)) for COVID-19 data sets. Based on Table 7, the ELMW distribution has the smallest value of AIC, which confirmed the ELMW distribution works well among the other competitive distributions to modeling the real data set.

The QQ-plot of both real data sets under the ELMW distribution, as the best model, are provided in Figure 5. Based on Figure 5, the applicability of the ELMW in modeling the COVID-19 data set is affirmed.

Although the original COVID-19 data sets from Denmark and Ethiopia are complete, we create artificially progressively censored samples from them. This approach serves a methodological purpose by demonstrating how the proposed estimation techniques for the progressive type-II censoring scheme can be implemented and perform in a realistic data setting, and is a common practice in reliability and survival analysis research (see [27]). It enables the evaluation of estimation performance under censoring using real-world data patterns, thereby bridging the gap between simulation studies and actual applications. Additionally, it provides a realistic, controlled environment to apply and compare the proposed estimation methods on data with a known underlying structure. So, it illustrates how an analyst would use these techniques on data obtained from a planned life-test involving progressive removals.

We generate progressively censored samples of sizes $m_1 = 35$ and $m_2 = 20$ from the COVID-19 data sets. The progressive censoring schemes for the first and second data sets are represented, respectively as $r_1 = \left(3^5, 1^2, 0^8, 2^7, 0^{10}, 1^3\right)$, $r_2 = \left(0^4, 2^5, 0^3, 2^2, 3^3, 0^2, 2^1\right)$.

Based on the censored data, the ML and MPS estimates of the parameters of the ELMW distribution for both COVID-19 data sets are illustrated in Table 8. Finally, the SEM and MPS estimates of the reliability function of the ELMW distribution are represented in Table 9.

Table 2. Simulation results for $(\theta, \lambda) = (1.5, 3)$ of the ELMW distribution with RMSE in brackets

			$r_1 = (1^3, 0^2, 2^3, 1^3, 0^{m-15}, 2^3, n-m-18)$				$r_2 = (2^2, 0^{m-10}, 1^4, 2^3, n-m-14)$			
n	m		NR	EM	SEM	MPS	NR	EM	SEM	MPS
40	15	$\hat{\theta}$	2.12566	1.49636	1.62121	1.69408	2.31497	1.51624	1.63851	1.67674
			(0.71226)	(0.39652)	(0.54576)	(0.83946)	(0.87107)	(0.44091)	(0.56278)	(0.93688)
		$\hat{\lambda}$	2.40877	2.97081	3.14939	3.48384	2.22046	2.90004	3.16642	3.35719
			(0.66902)	(0.51856)	(0.83408)	(0.96132)	(0.84295)	(0.61038)	(0.90641)	(0.99918)
	20	$\hat{\theta}$	1.59973	1.49485	1.62109	1.79766	1.79391	1.50486	1.62542	1.73649
			(0.39142)	(0.31331)	(0.42012)	(0.75852)	(0.44858)	(0.40505)	(0.53142)	(0.82637)
		$\hat{\lambda}$	2.82243	2.97631	3.13814	3.58663	2.64502	2.94918	3.14552	3.46247
			(0.52332)	(0.40706)	(0.70052)	(0.89649)	(0.62506)	(0.54868)	(0.81734)	(0.94281)
80	30	$\hat{\theta}$	2.41108	1.49576	1.62744	1.44061	2.11207	1.48702	1.63214	1.49636
			(0.63497)	(0.27210)	(0.44763)	(0.64741)	(0.72904)	(0.28894)	(0.51578)	(0.66282)
		$\hat{\lambda}$	2.16244	2.96984	3.15595	3.11339	2.05672	3.08787	3.16635	3.19222
			(0.56972)	(0.36409)	(0.52035)	(0.70241)	(0.67244)	(0.40115)	(0.64658)	(0.79423)
	50	$\hat{\theta}$	1.45581	1.49572	1.62048	1.69482	1.59605	1.48247	1.61193	1.60571
			(0.40426)	(0.19832)	(0.36748)	(0.57722)	(0.48249)	(0.22021)	(0.37685)	(0.63484)
		$\hat{\lambda}$	2.87135	2.96646	3.15269	3.18483	2.79306	3.04149	3.12982	3.11689
			(0.43936)	(0.29002)	(0.42971)	(0.54812)	(0.52875)	(0.32994)	(0.55151)	(0.54461)

Table 3. Simulation results for $(\theta, \lambda) = (3, 0.5)$ of the ELMW distribution with RMSE in brackets

			$r_1 = (1^3, 0^2, 2^3, 1^3, 0^{m-15}, 2^3, n-m-18)$				$r_2 = (2^2, 0^{m-10}, 1^4, 2^3, n-m-14)$			
n	m		NR	EM	SEM	MPS	NR	EM	SEM	MPS
40	15	$\hat{\theta}$	2.873764	2.88475	2.89261	2.50748	2.82028	2.87809	2.90571	2.88875
			(0.88688)	(0.78292)	(0.85692)	(0.80755)	(0.91022)	(0.79723)	(0.86705)	(0.80048)
		$\hat{\lambda}$	0.34662	0.45732	0.48719	0.468689	0.29513	0.45843	0.46726	0.60943
			(0.17632)	(0.09583)	(0.11751)	(0.21719)	(0.21998)	(0.10321)	(0.1249)	(0.22633)
	20	$\hat{\theta}$	2.90083	2.871382	2.88234	2.78366	2.90452	2.89855	2.90123	2.63109
			(0.89925)	(0.81325)	(0.84872)	(0.86203)	(0.92577)	(0.88348)	(0.85348)	(0.89574)
		$\hat{\lambda}$	0.444347	0.462948	0.48312	0.57676	0.41865	0.45437	0.489214	0.54438
			(0.16063)	(0.06324)	(0.08751)	(0.23238)	(0.19727)	(0.09226)	(0.11004)	(0.24149)
80	30	$\hat{\theta}$	2.92374	2.86957	2.89866	2.57561	2.84623	2.85176	2.89389	2.54734
			(0.57646)	(0.47274)	(0.53704)	(0.64393)	(0.60392)	(0.48508)	(0.55102)	(0.86699)
		$\hat{\lambda}$	0.419031	0.45996	0.47925	0.55538	0.44429	0.46474	0.48292	0.52839
			(0.13764)	(0.06941)	(0.07061)	(0.17361)	(0.16068)	(0.07653)	(0.08142)	(0.18344)
	50	$\hat{\theta}$	2.900049	2.85789	2.89369	2.97007	2.90036	2.85142	2.88917	2.84549
			(0.59995)	(0.48524)	(0.51732)	(0.64194)	(0.60965)	(0.48449)	(0.532376)	(0.7825)
		$\hat{\lambda}$	0.44969	0.46547	0.483443	0.53306	0.44766	0.46768	0.482427	0.61794
			(0.12046)	(0.05032)	(0.06677)	(0.19643)	(0.12372)	(0.05804)	(0.07089)	(0.20959)

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Table 4. Simulation results of reliability function for $(\theta, \lambda) = (0.5, 6)$ of ELMW with RMSE in brackets

				$r_1 = \left(1^3, 0^2, 2^3, 1^3, 0^{m-15}, 2^3, n-m-18\right)$		$r_2 = \left(2^2, 0^{m-10}, 1^4, 2^3, n-m-14\right)$	
n	m	t	$R(t)$	SEM	MPS	SEM	MPS
40	15	1	0.67681	0.64554	0.61316	0.64745	0.60904
				(9.77022e-04)	(4.04966e-03)	(8.58982e-04)	(4.59012e-03)
		2	0.42388	0.38031	0.35416	0.38364	0.35003
				(1.89823e-03)	(4.86022e-03)	(1.63243e-03)	(5.45302e-03)
		3	0.19923	0.16294	0.15321	0.16045	0.15077
				(1.54383e-04)	(2.11822e-03)	(1.30232e-03)	(2.34887e-03)
		4	0.05762	0.040417	0.04098	0.00334	0.04023
				(4.08303e-04)	(2.76916e-04)	(3.42891e-04)	(3.02528e-04)
		5	0.00702	0.00504	0.00458	0.00482	0.00451
				(1.58153e-06)	(5.95763e-06)	(1.35733e-05)	(6.37986e-06)
IMSE				0.00068	0.00226	0.00083	0.00254
40	20	1	0.67681	0.65925	0.62214	0.65189	0.62325
				(3.07787e-04)	(2.98707e-03)	(6.20198e-04)	(2.86758e-03)
		2	0.42388	0.40011	0.36461	0.38973	0.36446
				(5.65194e-04)	(3.51306e-03)	(1.16595e-03)	(3.53004e-03)
		3	0.19923	0.17826	0.16114	0.16883	0.15943
				(4.39778e-04)	(1.45117e-03)	(9.24669e-04)	(1.58447e-03)
		4	0.05762	0.04676	0.04468	0.04193	0.04296
				(1.17941e-04)	(1.67436e-04)	(2.46287e-04)	(2.14977e-04)
		5	0.00702	0.00474	0.00534	0.00469	0.00482
				(5.18497e-06)	(2.82793e-06)	(1.01825e-05)	(4.85615e-06)
IMSE				0.00028	0.00162	0.00059	0.00164

Table 5. Simulation results of the reliability function for $(\theta, \lambda) = (0.5, 6)$ of ELMW with RMSE in brackets

				$r_1 = \left(1^3, 0^2, 2^3, 1^3, 0^{m-15}, 2^3, n-m-18\right)$		$r_2 = \left(2^2, 0^{m-10}, 1^4, 2^3, n-m-14\right)$	
n	m	t	$R(t)$	SEM	MPS	SEM	MPS
80	30	1	0.67681	0.66893	0.64369	0.67163	0.65601
				(6.19094e-05)	(1.09601e-03)	(2.67213e-05)	(4.31881e-04)
		2	0.42388	0.41202	0.38429	0.41557	0.39835
				(1.40544e-04)	(1.56717e-03)	(6.90966e-05)	(6.51526e-04)
		3	0.19923	0.18745	0.16962	0.19044	0.17944
				(1.38769e-04)	(8.77112e-04)	(7.73817e-05)	(3.917076e-04)
		4	0.05762	0.05079	0.04484	0.05224	0.04869
				(4.66826e-05)	(1.63231e-04)	(2.89045e-05)	(7.97646e-05)
		5	0.00702	0.00545	0.00471	0.00573	0.00532
				(2.47291e-06)	(5.33613e-06)	(1.65711e-06)	(2.88348e-06)
IMSE				7.80758e-05	0.00074	4.07522e-05	0.00031
80	50	1	0.67681	0.66913	0.65601	0.67289	0.66689
				(5.87921e-05)	(4.31881e-04)	(1.52323e-05)	(9.80188e-05)
		2	0.42388	0.41321	0.39835	0.41785	0.41033
				(1.13854e-04)	(6.51526e-04)	(3.63856e-05)	(1.84486e-04)
		3	0.19923	0.18949	0.17944	0.19306	0.18704
				(9.50477e-05)	(3.91707e-04)	(3.80654e-05)	(1.48668e-04)
		4	0.05762	0.05234	0.04869	0.05392	0.05114
				(2.79052e-05)	(7.97646e-05)	(1.36685e-05)	(4.19944e-05)
		5	0.00702	0.00585	0.00532	0.00614	0.00561
				(1.38356e-06)	(2.88348e-06)	(7.84704e-07)	(2.00255e-06)
IMSE				5.93966e-05	0.00031	2.08273e-05	9.50342e-05

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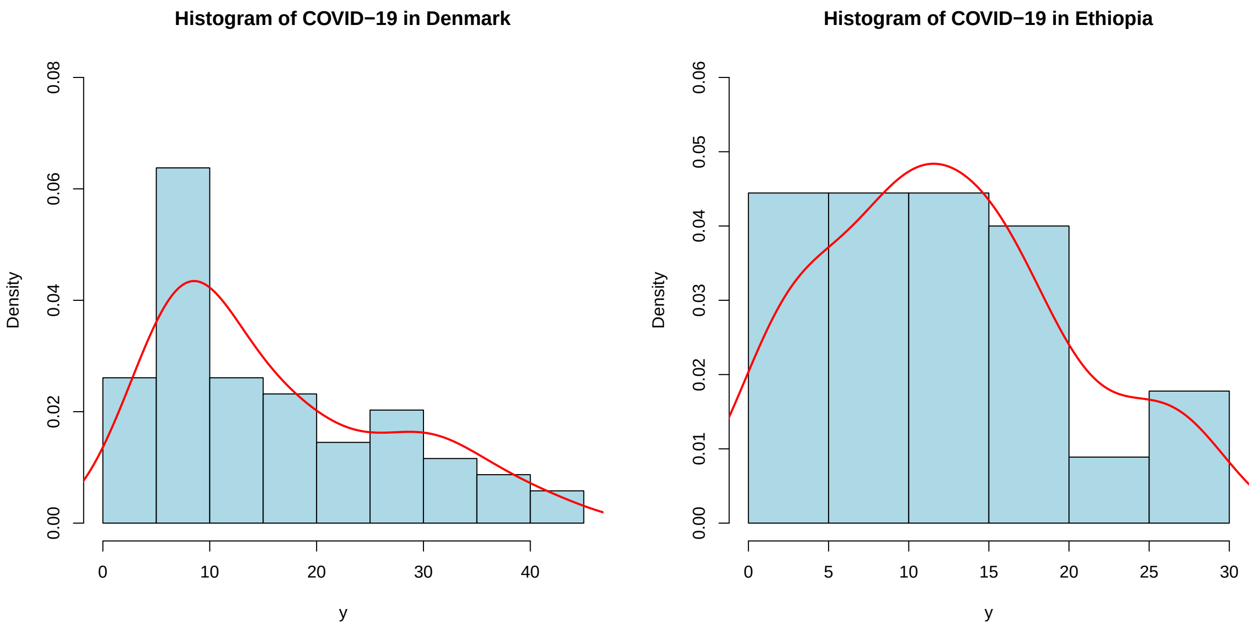

Figure 3. The histogram with red density line of COVID-19 data set

Table 6. Descriptive statistics of the COVID-19 data sets

	Mean	Median	Variance	Skewness	Kurtosis	K-S distance	P-value
First data	18	17	125.73	0.3547	1.9526	0.1191	0.2653
Second data	12.31	11	57.85	0.3883	2.3906	0.1059	0.6689

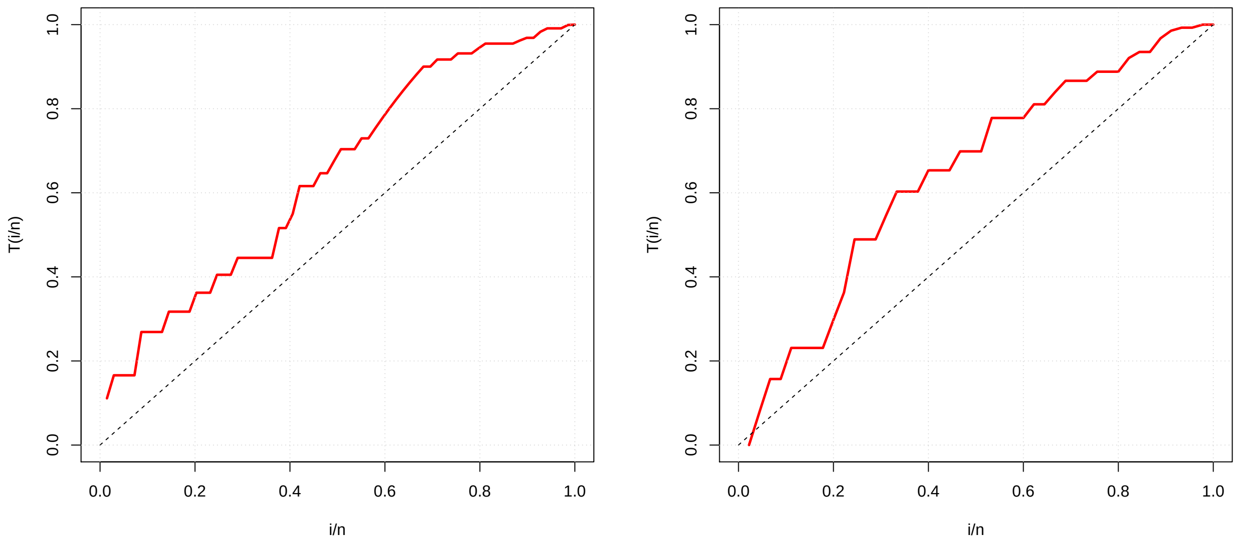

Figure 4. The TTT plot for the data set of Denmark and Ethiopia COVID-19 data (left to right)

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Table 7. MLEs and goodness of fit measures of the COVID-19 data set

	Denmark		Ethiopia	
Models	ML estimations	AIC	ML estimations	AIC
Gamma	$\hat{\alpha}=2.1597, \hat{\beta}=0.1199$	521.69	$\hat{\alpha}=2.0177,\ \hat{\beta}=0.1638$	309.28
Weibull	$\hat{\alpha}=1.6433, \hat{\beta}=20.1396$	519.91	$\hat{\alpha}=1.6275,\ \hat{\beta}=13.7176$	307.08
Gompertz	$\hat{k}=0.0557, \hat{\mu}=0.4292$	518.63	$\hat{k}=0.1341,\ \hat{\mu}=0.1273$	312.42
Par	$\hat{\alpha}=99.5536, \hat{\theta}=89.8233$	986.19	$\hat{\alpha}=65.1849,\ \hat{\theta}=42.0421$	538.18
Burr XII	$\hat{\theta}=0.1057$	521.59		
Quasi Lindley	$\hat{\theta}=0.1229, \hat{p}=0.1762$	519.94	$\hat{\theta}=0.1506, \hat{p}=0.1708$	311.05
Bur	$\hat{c}=14.1973, \hat{k}=0.0266$	640.54	$\hat{c}=152.5606, \hat{k}=0.0029$	369.92
Inverse Weibull	$\hat{\beta}=15.8999, \hat{\theta}=1.2357$	545.98	$\hat{\beta}=8.2503, \hat{\theta}=1.1251$	322.41
Inverse generalized Gamma	$\hat{\beta}=15.6266, \hat{\theta}=0.8711$	539.82	$\hat{\beta}=10.5198, \hat{\theta}=0.8239$	312.39
Generalized inverse Lindley	$\hat{\beta}=16.7477, \hat{\theta}=1.2342$	546.07	$\hat{\beta}=9.0001, \hat{\theta}=1.1192$	322.74
Extended-Weibull	$\hat{a}=0.0134, \hat{b}=1.4861,$ $\hat{c}=1.9199$	521.24	$\hat{a}=0.0141, \hat{b}=1.6269,$ $\hat{c}=0.0152$	309.11
Exponential Weibull	$\hat{\beta}=0.0028, \hat{\lambda}=0.0037$ $\hat{k}=1.8442$	524.72	$\hat{\beta}=0.0001, \hat{\lambda}=0.0406,$ $\hat{k}=2.8916$	307.37
Marshall-Olkin Weibull	$\hat{\alpha}=1.0521, \hat{\beta}=0.0505,$ $\hat{\theta}=1.6271$	521.89	$\hat{\alpha}=2.5209, \hat{\beta}=0.1041,$ $\hat{\theta}=1.3403$	308.55
ELMW	$\hat{\theta}=1.8771, \hat{\lambda}=30.9396$	515.81	$\hat{\theta}=0.3161, \hat{\lambda}=11.3616$	299.21

Table 8. The estimates of ELMW distribution under progressive type-II censoring of COVID-19 data

	parameters	NR	EM	SEM	MPS
Denmark	$\hat{\theta}$	1.2701	1.0267	1.1712	0.92790
	$\hat{\lambda}$	21.9998	28.7335	20.4308	24.5293
Ethiopia	$\hat{\theta}$	0.4497	0.2916	0.4158	0.2828
	$\hat{\lambda}$	12.9823	15.8848	12.7241	16.7622

Table 9. The reliability estimates of ELMW under the progressive type-II censoring of COVID-19 data

	Denmark				Ethiopia			
t	$\hat{R}_{NR}$	$\hat{R}_{EM}$	$\hat{R}_{SEM}$	$\hat{R}_{MPS}$	$\hat{R}_{NR}$	$\hat{R}_{EM}$	$\hat{R}_{SEM}$	$\hat{R}_{MPS}$
1	0.89188	0.92476	0.88929	0.91676	0.88509	0.91392	0.88441	0.90889
2	0.87385	0.91179	0.87086	0.90251	0.85575	0.89126	0.85489	0.88745
3	0.85688	0.89947	0.85352	0.88899	0.82884	0.87022	0.82781	0.86751
4	0.83962	0.88682	0.83589	0.87513	0.80175	0.84881	0.80056	0.85716
5	0.82159	0.87348	0.81748	0.86055	0.77363	0.82628	0.77228	0.83572

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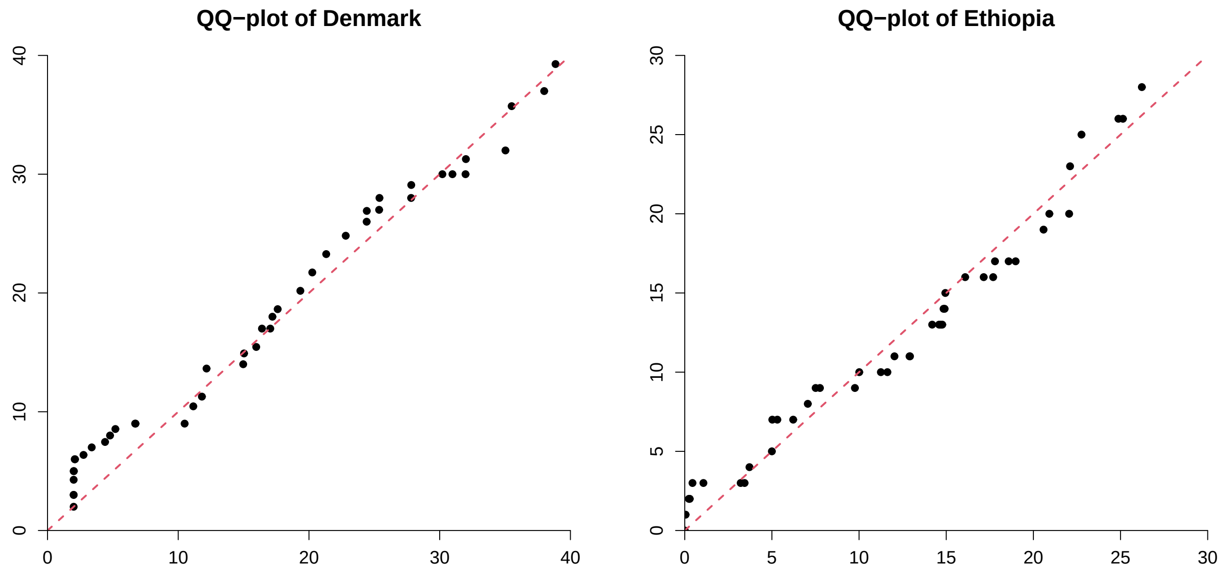

Figure 5. The QQ-plot, for the COVID-19 data set, under ELMW distribution

Conclusion

In this paper, a new flexible family of lifetime distributions, called the Log-Exponential MB Transformation (LEMBT), has been introduced. The proposed family extends existing transformation-based models by combining the Log-Exponential generator with the MB baseline, allowing for greater flexibility in modeling skewness, tail behavior, and hazard rate shapes. Statistical inference for the LEMBT family was developed under progressive type-II censoring using several estimation approaches, including Newton-Raphson, EM, SEM, and maximum product spacing methods. A comprehensive Monte Carlo study demonstrated that all estimators perform satisfactorily as sample size increases, while the EM-based procedures generally provide more stable estimates with smaller RMSE values under moderate and heavy censoring. The applicability of the proposed model was illustrated using COVID-19 mortality data from Denmark and Ethiopia. The Log-Exponential Modified Weibull submodel achieved a better fit than competing models according to information criteria, highlighting its practical relevance for real-life lifetime data. Future work may consider Bayesian inference for the LEMBT family, extensions to hybrid or adaptive censoring schemes, and applications to other biomedical and reliability datasets.

Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Appendix A. The mixture representation of the PDF of the LEMBT family is computed as follows

$$
\begin{aligned}
f(x,\beta,\lambda,\theta) &= \frac{\theta\beta g'(x) g^{\beta-1}(x)\exp\left(\frac{-g^{\beta}(x)}{\lambda}\right) e^{-\theta\left(1-\exp\left(\frac{-g^{\beta}(x)}{\lambda}\right)\right)}}{\lambda\ln(2-e^{-\theta})\left(2-e^{-\theta\left(1-\exp\left(\frac{-g^{\beta}(x)}{\lambda}\right)\right)}\right)}\\
&= \frac{\theta\beta g'(x) g^{\beta-1}(x)\exp\left(\frac{-g^{\beta}(x)}{\lambda}\right)}{\lambda\ln(2-e^{-\theta})}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k} e^{-k\theta\left(1-\exp\left(\frac{-g^{\beta}(x)}{\lambda}\right)\right)}\\
&= \frac{\theta\beta g'(x) g^{\beta-1}(x)\exp\left(\frac{-g^{\beta}(x)}{\lambda}\right)}{\lambda\ln(2-e^{-\theta})}\sum_{k=1}^{\infty}\sum_{h=0}^{\infty}\frac{(-k\theta)^{h}\left(1-\exp\left(\frac{-g^{\beta}(x)}{\lambda}\right)\right)^{h}}{2^{k}h!}\\
&= \frac{\theta\beta g'(x) g^{\beta-1}(x)}{\lambda\ln(2-e^{-\theta})}\sum_{k=1}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{h}\frac{(-k\theta)^{h}(-1)^{i} e^{-(i+1)\frac{g^{\beta}(x)}{\lambda}}}{2^{k}i!(h-i)!}\\
&= \frac{\theta}{\ln(2-e^{-\theta})}\sum_{k=1}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{h}\frac{(k\theta)^{h}(-1)^{h+i}}{2^{k}(i+1)!(h-i)!}\left(\frac{(i+1)\beta g'(x) g^{\beta-1}(x) e^{-(i+1)\frac{g^{\beta}(x)}{\lambda}}}{\lambda}\right).
\end{aligned}
$$

Appendix B. The score of Equation 2 equal to zero are represented as

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\lambda} &= -\frac{m}{\lambda}+\sum_{i=1}^{m}\frac{g^{\beta}(t_i)}{\lambda^{2}}+\sum_{i=1}^{m}\frac{2\theta g^{\beta}(t_i) e^{\frac{-g^{\beta}(t_i)}{\lambda}} e^{\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}{\lambda^{2}\left(2e^{\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}-1\right)}\\
&\quad+\sum_{i=1}^{m} r_i\frac{\theta g^{\beta}(t_i) e^{\frac{-g^{\beta}(t_i)}{\lambda}} e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}{\lambda^{2}\left(2-e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}\right)\ln\left(\frac{2-e^{-\theta}}{2-e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}\right)}=0,
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\beta} &= \frac{m}{\beta}+\sum_{i=1}^{m}\ln(g(t_i))-\sum_{i=1}^{m}\frac{g^{\beta}(t_i)\ln(g(t_i))}{\lambda}-\sum_{i=1}^{m}\frac{2\theta g^{\beta}(t_i)\ln(g(t_i)) e^{\frac{-g^{\beta}(t_i)}{\lambda}} e^{\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}{\lambda\left(2e^{\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}-1\right)}\\
&\quad-\sum_{i=1}^{m} r_i\frac{\theta g^{\beta}(t_i)\ln(g(t_i)) e^{\frac{-g^{\beta}(t_i)}{\lambda}} e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}{\lambda\left(2-e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}\right)\ln\left(\frac{2-e^{-\theta}}{2-e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}\right)}=0,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} &= \frac{m}{\theta}+\sum_{i=1}^{m}(r_i+1)\frac{e^{-\theta}}{(2-e^{-\theta})\ln(2-e^{-\theta})}-\sum_{i=1}^{m}\frac{2\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right) e^{\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}{2e^{\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}-1}\\
&\quad-\sum_{i=1}^{m} r_i\frac{\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right) e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}{\left(2-e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}\right)\ln\left(\frac{2-e^{-\theta}}{2-e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}\right)}=0.
\end{aligned}
\tag{9}
$$

From Equation 7, we have

$$
\hat{\lambda}(\beta,\theta)=\frac{1}{m}\left[\sum_{i=1}^{m} g^{\beta}(t_i)+\frac{2\theta g^{\beta}(t_i) e^{\frac{-g^{\beta}(t_i)}{\lambda}}}{2-e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}+\frac{r_i\theta g^{\beta}(t_i) e^{\frac{-g^{\beta}(t_i)}{\lambda}}}{\left(2e^{\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}-1\right)\ln\left(\frac{2-e^{-\theta}}{2-e^{-\theta\left(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}\right)}}\right)}\right],
\tag{10}
$$

by substituting Equation 10 into Equation 8, it will be reduced to the two-dimensional nonlinear score equation of β and

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θ as follows

$$\beta = m\Bigg(\sum_{i=1}^{m} -\big(\ln g(t_i)\big) + \frac{g^{\beta}(t_i)\ln(g(t_i))}{\lambda} + \frac{2\theta g^{\beta}(t_i)\ln(g(t_i))e^{\frac{-g^{\beta}(t_i)}{\lambda}}e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\lambda\big(2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}-1\big)}$$
$$+\frac{r_i\theta g^{\beta}(t_i)\ln(g(t_i))e^{\frac{-g^{\beta}(t_i)}{\lambda}}e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\lambda\big(2-e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}\big)\ln\Bigg(\dfrac{2-e^{-\theta}}{2-e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}\Bigg)}\Bigg)^{-1}. \tag{11}$$

Finally based on Equation 10 and Equation 11, the score equation of θ in Equation 9 leads to

$$\theta = m\Bigg(\sum_{i=1}^{m}\frac{-(r_i+1)e^{-\theta}}{(2-e^{-\theta})\ln(2-e^{-\theta})} + \frac{2(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}-1}$$
$$+\frac{r_i(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\big(2-e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}\big)\ln\Bigg(\dfrac{2-e^{-\theta}}{2-e^{-\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}\Bigg)}\Bigg)^{-1}.$$

Appendix C. The ML estimators of the vector parameters γ for complete sample X can be achieved by equating the score of Equation 3 to zero, as below

$$\frac{\partial\mathcal{L}_c(\gamma,X)}{\partial\lambda} = -\frac{n}{\lambda} + \sum_{i=1}^{m}\frac{g^{\beta}(t_i)}{\lambda^2} + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{g^{\beta}(z_{ik})}{\lambda^2} + \sum_{i=1}^{m}\frac{2\theta g^{\beta}(t_i)e^{\frac{-g^{\beta}(t_i)}{\lambda}}e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\lambda^2\Big(2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}-1\Big)}$$
$$+\sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{2\theta g^{\beta}(z_{ik})e^{\frac{-g^{\beta}(z_{ik})}{\lambda}}e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}{\lambda^2\Big(2e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}-1\Big)} = 0, \tag{12}$$

$$\frac{\partial\mathcal{L}_c(\gamma,X)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{m}\ln g(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln g(z_{ik}) - \sum_{i=1}^{m}\frac{g^{\beta}(t_i)\ln(g(t_i))}{\lambda}$$
$$-\sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{g^{\beta}(z_{ik})\ln(g(z_{ik}))}{\lambda} - \sum_{i=1}^{m}\frac{2\theta g^{\beta}(t_i)\ln(g(t_i))e^{\frac{-g^{\beta}(t_i)}{\lambda}}e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\lambda\Big(2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}-1\Big)}$$
$$-\sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{2\theta g^{\beta}(z_{ik})\ln(g(z_{ik}))e^{\frac{-g^{\beta}(z_{ik})}{\lambda}}e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}{\lambda\Big(2e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}-1\Big)} = 0, \tag{13}$$

$$\frac{\partial\mathcal{L}_c(\gamma,X)}{\partial\theta} = \frac{n}{\theta} - \frac{ne^{-\theta}}{(2-e^{-\theta})\ln(2-e^{-\theta})} - \sum_{i=1}^{m}\frac{2(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}-1}$$
$$-\sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{2(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}{2e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}-1} = 0. \tag{14}$$

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Equation 12–Equation 14 can be represented as

$$\frac{\partial \mathcal{L}_c(\gamma, X)}{\partial \lambda} = -\frac{n}{\lambda} + \sum_{i=1}^{m} \frac{g^{\beta}(t_i)}{\lambda^2} + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \frac{E(g^{\beta}(z_{ik})|Z_{ik} > t_i)}{\lambda^2} \tag{15}$$
$$+ \sum_{i=1}^{m} \frac{2\theta g^{\beta}(t_i) e^{\frac{-g^{\beta}(t_i)}{\lambda}} e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\lambda^2\Big(2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})} - 1\Big)}$$
$$+ \sum_{i=1}^{m}\sum_{k=1}^{r_i} E\Bigg(\frac{2\theta g^{\beta}(z_{ik}) e^{\frac{-g^{\beta}(z_{ik})}{\lambda}} e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}{\lambda^2\Big(2e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})} - 1\Big)}\Bigg|Z_{ik} > t_i\Bigg) = 0,$$

$$\frac{\partial \mathcal{L}_c(\gamma, X)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{m} \ln g(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} E(\ln g(z_{ik})|Z_{ik} > t_i) - \sum_{i=1}^{m} \frac{g^{\beta}(t_i)\ln(g(t_i))}{\lambda} \tag{16}$$
$$- \sum_{i=1}^{m}\sum_{k=1}^{r_i} E\Bigg(\frac{g^{\beta}(z_{ik})\ln(g(z_{ik}))}{\lambda}\Bigg|Z_{ik} > t_i\Bigg) - \sum_{i=1}^{m} \frac{2\theta g^{\beta}(t_i)\ln(g(t_i)) e^{\frac{-g^{\beta}(t_i)}{\lambda}} e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{\lambda\Big(2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})} - 1\Big)}$$
$$- \sum_{i=1}^{m}\sum_{k=1}^{r_i} E\Bigg(\frac{2\theta g^{\beta}(z_{ik})\ln(g(z_{ik})) e^{\frac{-g^{\beta}(z_{ik})}{\lambda}} e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}{\lambda\Big(2e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})} - 1\Big)}\Bigg|Z_{ik} > t_i\Bigg) = 0,$$

$$\frac{\partial \mathcal{L}_c(\gamma, X)}{\partial \theta} = \frac{n}{\theta} - \frac{ne^{-\theta}}{(2-e^{-\theta})\ln(2-e^{-\theta})} - \sum_{i=1}^{m} \frac{2(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}}) e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})}}{2e^{\theta(1-e^{\frac{-g^{\beta}(t_i)}{\lambda}})} - 1} \tag{17}$$
$$- \sum_{i=1}^{m}\sum_{k=1}^{r_i} E\Bigg(\frac{2(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}}) e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}{2e^{\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})} - 1}\Bigg|Z_{ik} > t_i\Bigg) = 0.$$

Let $A = \Big(\ln(2-e^{-\theta}) - \ln(2-e^{-\theta(1-e^{-\frac{g^{\beta}(t_i)}{\lambda}})})\Big)^{-1}$ and based on Equation 4, the conditional expectations in Equation 15–Equation 17 can be computed as follows

$$E_{1i}(t, \beta, \lambda, \theta) = E(g^{\beta}(z_{ik})|Z_{ik} > t) = A\int_{t}^{\infty} \frac{\theta\beta g^{2\beta-1}(z_{ik}) g'(z_{ik}) e^{-\frac{g^{\beta}(z_{ik})}{\lambda}} e^{-\theta(1-e^{-\frac{g^{\beta}(z_{ik})}{\lambda}})}}{\lambda\Big(2-e^{-\theta(1-e^{-\frac{g^{\beta}(z_{ik})}{\lambda}})}\Big)} dz_{ik}.$$

Now consider $u = -\frac{g^{\beta}(z_{ik})}{\lambda}$, so $du = \frac{-\beta}{\lambda} g^{\beta-1}(z_{ik}) g'(z_{ik})$ and

$$E_{1i}(t, \beta, \lambda, \theta) = A\theta \int_{-\infty}^{-\frac{g^{\beta}(t)}{\lambda}} \frac{e^{u} e^{-\theta(1-e^{u})}}{(2-e^{-\theta(1-e^{u})})}(-u\lambda)du = -A\theta\lambda \int_{-\infty}^{-\frac{g^{\beta}(t)}{\lambda}} \sum_{k=1}^{\infty} \frac{e^{-k\theta(1-e^{u})}}{2^k} ue^{u}\, du$$
$$= -A\theta\lambda \sum_{k=1}^{\infty}\sum_{i=0}^{\infty} \frac{e^{-k\theta}(\theta k)^i}{2^k\, i!} \int_{-\infty}^{-\frac{g^{\beta}(t)}{\lambda}} e^{(i+1)u} u\, du = -A\theta\lambda \sum_{k=1}^{\infty}\sum_{i=0}^{\infty} \frac{e^{-k\theta}(\theta k)^i\, \Gamma(2, \frac{(i+1)g^{\beta}(t)}{\lambda})}{2^k\, i!(i+1)^2},$$

where $\Gamma(.,.)$ is the upper incomplete Gamma function.

By the same arguments, we compute the following expectations

$$
\begin{aligned}
E_{2i}(t,\beta,\lambda,\theta) &= E\Bigg(\frac{e^{\frac{-g^{\beta}(z_{ik})}{\lambda}}g^{\beta}(z_{ik})}{2-e^{-\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}\Big|Z_{ik}>t_i\Bigg) = A\theta\lambda\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{ke^{-k\theta}(\theta k)^i\,\Gamma(2,\frac{(i+2)g^{\beta}(t)}{\lambda})}{2^{k+1}\,i!(i+2)^2},\\
E_{3i}(t,\beta,\lambda,\theta) &= E(\ln(g(z_{ik}))|Z_{ik}>t)\\
&= \frac{A\theta}{\beta}\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{e^{-k\theta}(\theta k)^i}{2^k\,(i+1)!}\Big[e^{-\frac{(i+1)g^{\beta}(t)}{\lambda}}\Big(\ln(\lambda)+\ln(\frac{g^{\beta}(t)}{\lambda})\Big)-\mathrm{Ei}\big(-\frac{(i+1)g^{\beta}(t)}{\lambda}\big)\Big],\\
E_{4i}(t,\beta,\lambda,\theta) &= E(\ln(g(z_{ik}))g^{\beta}(z_{ik})|Z_{ik}>t)\\
&= \frac{-A\theta\lambda}{\beta}\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{e^{-k\theta}(\theta k)^i}{2^k\,i!}\Bigg[-\frac{\ln(\lambda)\,\Gamma(2,\frac{(i+1)g^{\beta}(t)}{\lambda})}{(i+1)^2}-\ln(i+1)\Gamma(2,\frac{(i+1)g^{\beta}(t)}{\lambda})\\
&\quad+\ln(\frac{(i+1)g^{\beta}(t)}{\lambda})e^{-\frac{(i+1)g^{\beta}(t)}{\lambda}}(\frac{(i+1)g^{\beta}(t)}{\lambda}+1)-\mathrm{Ei}(-\frac{(i+1)g^{\beta}(t)}{\lambda})+e^{-\frac{(i+1)g^{\beta}(t)}{\lambda}}\Bigg],\\
E_{5i}(t,\beta,\lambda,\theta) &= E\Bigg(\frac{\ln(g(z_{ik}))g^{\beta}(z_{ik})e^{\frac{-g^{\beta}(z_{ik})}{\lambda}}}{2-e^{-\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}\Big|Z_{ik}>t_i\Bigg)\\
&= \frac{A\theta\lambda}{\beta}\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{ke^{-k\theta}(\theta k)^i}{2^{k+1}\,i!(i+2)^2}\Bigg[\ln(\frac{\lambda}{i+2})\Gamma(2,\frac{(i+2)g^{\beta}(t)}{\lambda})\\
&\quad+\ln(\frac{(i+2)g^{\beta}(t)}{\lambda})e^{-\frac{(i+2)g^{\beta}(t)}{\lambda}}(\frac{(i+2)g^{\beta}(t)}{\lambda}+1)-\mathrm{Ei}(-\frac{(i+2)g^{\beta}(t)}{\lambda})+e^{-\frac{(i+2)g^{\beta}(t)}{\lambda}}\Bigg],\\
E_{6i}(t,\beta,\lambda,\theta) &= E\Bigg(\frac{1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}}}{2-e^{-\theta(1-e^{\frac{-g^{\beta}(z_{ik})}{\lambda}})}}\Big|Z_{ik}>t_i\Bigg)\\
&= A\theta\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{ke^{-k\theta}(\theta k)^i}{2^{k+1}\,i!}\Bigg[\frac{e^{-\frac{(i+1)g^{\beta}(t)}{\lambda}}}{i+1}-\frac{e^{-\frac{(i+2)g^{\beta}(t)}{\lambda}}}{i+2}\Bigg],
\end{aligned}
$$

where Ei(.) is the Exponential integral as $\mathrm{Ei}(-z) = -\int_z^{\infty}\frac{e^{-t}}{t}dt$.

Appendix D. In the M-step, at $(k+1)$-th iteration, the value of $\hat{\lambda}^{(k+1)}$ of the EM algorithm is obtained by solving the following equation based on $(\hat{\lambda}^{(k)},\hat{\beta}^{(k)},\hat{\theta}^{(k)})$

$$
\begin{aligned}
\hat{\lambda}^{(k+1)} &= \frac{1}{n}\Bigg[\sum_{i=1}^{m}g^{\hat{\beta}^{(k)}}(t_i)+\sum_{i=1}^{m}r_iE_1(t,\hat{\beta}^{(k)},\hat{\lambda}^{(k)},\hat{\theta}^{(k)})+\sum_{i=1}^{m}\frac{2\hat{\theta}^{(k)}g^{\hat{\beta}^{(k)}}(t_i)e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k)}}}}{\Big(2-e^{-\hat{\theta}^{(k)}(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k)}}})}\Big)}\\
&\quad+2\sum_{i=1}^{m}r_i\hat{\theta}^{(k)}E_2(t,\hat{\beta}^{(k)},\hat{\lambda}^{(k)},\hat{\theta}^{(k)})\Bigg],
\end{aligned}
\tag{18}
$$

then $\hat{\beta}^{(k+1)}$ is obtained by solving the equation below based on $(\hat{\lambda}^{(k+1)},\hat{\beta}^{(k)},\hat{\theta}^{(k)})$

$$
\begin{aligned}
\frac{n}{\hat{\beta}^{(k+1)}} &= \sum_{i=1}^{m}\ln g(t_i)-\sum_{i=1}^{m}r_iE_3(t,\hat{\beta}^{(k)},\hat{\lambda}^{(k+1)},\hat{\theta}^{(k)})+\sum_{i=1}^{m}\frac{g^{\hat{\beta}^{(k)}}(t_i)\ln(g(t_i))}{\hat{\lambda}^{(k+1)}}\\
&\quad+\sum_{i=1}^{m}\frac{r_iE_4(t,\hat{\beta}^{(k)},\hat{\lambda}^{(k+1)},\hat{\theta}^{(k)})}{\hat{\lambda}^{(k+1)}}+\sum_{i=1}^{m}\frac{2\hat{\theta}^{(k)}g^{\hat{\beta}^{(k)}}(t_i)\ln(g(t_i))e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k+1)}}}}{\hat{\lambda}^{(k+1)}\Big(2-e^{-\hat{\theta}^{(k)}(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k+1)}}})}\Big)}\\
&\quad+\sum_{i=1}^{m}r_i\frac{2\hat{\theta}^{(k)}E_5(t,\hat{\beta}^{(k)},\hat{\lambda}^{(k+1)},\hat{\theta}^{(k)})}{\hat{\lambda}^{(k+1)}},
\end{aligned}
\tag{19}
$$

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Finally, the estimation of θ at iteration $(k+1)$ is computed based on $(\hat{\lambda}^{(k+1)}, \hat{\beta}^{(k+1)}, \hat{\theta}^{(k)})$ as below

$$\begin{aligned} \frac{n}{\hat{\theta}^{(k+1)}} &= \frac{n e^{-\hat{\theta}^{(k)}}}{(2-e^{-\hat{\theta}^{(k)}})\ln(2-e^{-\hat{\theta}^{(k)}})} + \sum_{i=1}^{m} \frac{2\left(1-e^{\frac{-g^{\hat{\beta}^{(k+1)}}(t_i)}{\hat{\lambda}^{(k+1)}}}\right)e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k+1)}}(t_i)}{\hat{\lambda}^{(k+1)}}}\right)}}{2e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k+1)}}(t_i)}{\hat{\lambda}^{(k+1)}}}\right)}-1} \\ &\quad +2\sum_{i=1}^{m} r_i E_6(t, \hat{\beta}^{(k+1)}, \hat{\lambda}^{(k+1)}, \hat{\theta}^{(k)}). \end{aligned} \tag{20}$$

$(\hat{\lambda}^{(k+1)}, \hat{\beta}^{(k+1)}, \hat{\theta}^{(k+1)})$ is implemented as a new value of (λ, β, θ) in the subsequent iteration and the steps are repeated until convergence is reached. The iterative procedures in Equation 18–Equation 20 are stopped on reaching the convergence as $|\hat{\lambda}^{(k+1)} - \hat{\lambda}^{(k)}| + |\hat{\beta}^{(k+1)} - \hat{\beta}^{(k)}| + |\hat{\theta}^{(k+1)} - \hat{\theta}^{(k)}| < \varepsilon$, where $\varepsilon > 0$ is an arbitrary small value.

Appendix E. Using Equation 12–Equation 14, the estimators of γ at the $(k+1)$-step of the algorithm can be obtained as

$$\begin{aligned} \hat{\lambda}^{(k+1)} &= \frac{1}{n}\Bigg[\sum_{i=1}^{m} g^{\hat{\beta}^{(k)}}(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} g^{\hat{\beta}^{(k)}}(z_{ik}) + \sum_{i=1}^{m} \frac{2\hat{\theta}^{(k)} g^{\hat{\beta}^{(k)}}(t_i) e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k)}}} e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k)}}}\right)}}{2e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k)}}}\right)}-1} \\ &\quad + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \frac{2\hat{\theta}^{(k)} g^{\hat{\beta}^{(k)}}(z_{ik}) e^{\frac{-g^{\hat{\beta}^{(k)}}(z_{ik})}{\hat{\lambda}^{(k)}}} e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(z_{ik})}{\hat{\lambda}^{(k)}}}\right)}}{2e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(z_{ik})}{\hat{\lambda}^{(k)}}}\right)}-1}\Bigg], \\ \frac{n}{\hat{\beta}^{(k+1)}} &= -\sum_{i=1}^{m} \ln g(t_i) - \sum_{i=1}^{m}\sum_{k=1}^{r_i} \ln g(z_{ik}) + \sum_{i=1}^{m} \frac{g^{\hat{\beta}^{(k)}}(t_i)\ln(g(t_i))}{\hat{\lambda}^{(k+1)}} \\ &\quad + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \frac{g^{\hat{\beta}^{(k)}}(z_{ik})\ln(g(z_{ik}))}{\hat{\lambda}^{(k+1)}} + \sum_{i=1}^{m} \frac{2\hat{\theta}^{(k)} g^{\hat{\beta}^{(k)}}(t_i)\ln(g(t_i)) e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k+1)}}} e^{\theta\left(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k+1)}}}\right)}}{\hat{\lambda}^{2(k+1)}\left(2e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(t_i)}{\hat{\lambda}^{(k+1)}}}\right)}-1\right)} \\ &\quad + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \frac{2\hat{\theta}^{(k)} g^{\hat{\beta}^{(k)}}(z_{ik})\ln(g(z_{ik})) e^{\frac{-g^{\hat{\beta}^{(k)}}(z_{ik})}{\hat{\lambda}^{(k+1)}}} e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(z_{ik})}{\hat{\lambda}^{(k)}}}\right)}}{\hat{\lambda}^{2(k+1)}\left(2e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k)}}(z_{ik})}{\hat{\lambda}^{(k)}}}\right)}-1\right)}, \\ \frac{n}{\hat{\theta}^{(k+1)}} &= \frac{n e^{-\hat{\theta}^{(k)}}}{(2-e^{-\hat{\theta}^{(k)}})\ln(2-e^{-\hat{\theta}^{(k)}})} + \sum_{i=1}^{m} \frac{2\left(1-e^{\frac{-g^{\hat{\beta}^{(k+1)}}(t_i)}{\hat{\lambda}^{(k+1)}}}\right)e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k+1)}}(t_i)}{\hat{\lambda}^{(k+1)}}}\right)}}{2e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k+1)}}(t_i)}{\hat{\lambda}^{(k+1)}}}\right)}-1} \\ &\quad + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \frac{2\left(1-e^{\frac{-g^{\hat{\beta}^{(k+1)}}(z_{ik})}{\hat{\lambda}^{(k+1)}}}\right)e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k+1)}}(z_{ik})}{\hat{\lambda}^{(k+1)}}}\right)}}{2e^{\hat{\theta}^{(k)}\left(1-e^{\frac{-g^{\hat{\beta}^{(k+1)}}(z_{ik})}{\hat{\lambda}^{(k+1)}}}\right)}-1}. \end{aligned}$$

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