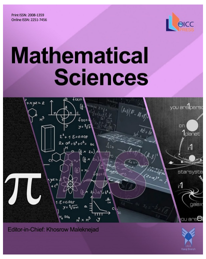

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Research Article

Approximate Null Controllability of Atangana–Baleanu Fractional Stochastic Evolution Equations with Poisson Jumps and Fractional Brownian Motion

Yazid Alhojilan*

Department of Mathematics, College of Science, Qassim University, Saudi Arabia

*Corresponding author: yhjielan@qu.edu.sa

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Abstract:

This paper investigates the approximate null controllability of a class of Atangana–Baleanu fractional stochastic evolution equations in a separable Hilbert space, driven simultaneously by a fractional Brownian motion and a Poisson random measure. The system is governed by the Caputo-type Atangana–Baleanu fractional derivative of order $\alpha \in (1/2, 1)$, a linear operator generating an α-resolvent family, a bounded linear control operator, and nonlinear terms satisfying suitable growth and integrability conditions. Assuming that the associated linear system is approximately null controllable and imposing standard Carathéodory-type assumptions on the nonlinearities, we derive sufficient conditions ensuring the approximate null controllability of the full nonlinear stochastic system in the mean-square sense. The analysis relies on the variation-of-constants formula associated with fractional resolvent families, mean-square estimates for stochastic integrals with respect to fractional Brownian motion and compensated Poisson random measures, and a Schauder fixed point argument in an appropriate space of stochastic processes. An illustrative example involving a controlled Atangana–Baleanu fractional stochastic partial differential equation with Dirichlet boundary conditions is provided to demonstrate the applicability of the abstract results.

Keywords: Fractional Derivatives; Approximate Null Controllability; Stochastic Evolution Equations; Fractional Brownian Motion; Poisson Jumps; Stochastic Systems

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1. Introduction

Fractional differential equations have become an important tool for modeling dynamical systems with memory and hereditary properties, since they capture nonlocal temporal effects and long-range dependence. In particular, fractional *stochastic* evolution equations in infinite-dimensional spaces arise naturally when the dynamics are influenced by random perturbations, such as Gaussian noises (Brownian or fractional Brownian) and discontinuous jump effects modeled by Poisson random measures.

Among the various fractional operators, the Atangana–Baleanu (A–B) derivative has attracted substantial attention because it is defined through a non-singular Mittag–Leffler kernel and therefore yields models with nonlocal but non-singular memory effects; see the seminal work of Atangana and Baleanu [1] and subsequent developments in deterministic and stochastic settings. Several contributions have addressed existence, uniqueness, and stability of solutions for deterministic and stochastic evolution equations involving the A–B derivative, as well as different notions of controllability and optimal control in that framework; see, for instance, [2, 3, 4, 5, 6]. From a modeling point of view, the use of Atangana–Baleanu fractional operators with non-singular kernels allows one to describe memory effects in a realistic manner without introducing singular behavior at the initial time, which is relevant in many physical, engineering, and biological systems.

Controllability is a fundamental concept in control theory. In the context of stochastic evolution equations, various forms of controllability have been investigated, including exact null controllability, approximate controllability, and trajectory controllability, for systems driven by Brownian motion, fractional Brownian motion, or Lévy-type noises; see, e.g., [7, 8, 9, 10]. In particular, null controllability of Atangana–Baleanu fractional stochastic systems with Poisson jumps and fractional Brownian motion in a Hilbert space has been studied in [11], where sufficient conditions ensuring that the state can be steered exactly to zero at a prescribed terminal time were derived under appropriate assumptions on the linear part, the control operator, and nonlinear perturbations. Fractional Brownian motion is introduced to capture long-range dependence and temporal correlation in stochastic perturbations, which cannot be modeled by classical Brownian motion. In contrast, Poisson random measures account for sudden and discontinuous random effects, such as abrupt shocks or rare events. The combination of fractional Brownian motion and Poisson jumps therefore provides a more comprehensive stochastic framework.

On the other hand, approximate controllability is often a more realistic and flexible objective, especially in infinite-dimensional systems where exact null controllability may fail or require restrictive spectral hypotheses. Approximate controllability for various classes of Atangana–Baleanu fractional stochastic systems (including neutral, Sobolev-type, and delay equations) has been considered in several works, typically under conditions guaranteeing that the reachable set is dense in the state space and that the nonlinear terms satisfy suitable growth and continuity assumptions; see [3, 2, 5, 6, 12] and the references therein. Recent studies have further extended approximate controllability analysis to related fractional stochastic frameworks. For instance, approximate controllability results for Atangana–Baleanu fractional neutral stochastic systems and integrodifferential models have been reported in [13, 14], while Hilfer fractional stochastic systems driven by fractional-type noises and Poisson jumps have been investigated in [15, 16]. These contributions highlight the ongoing interest in controllability problems for fractional stochastic systems with memory and jump effects.

Relation to the null controllability result in [11]. The paper [11] analyzes null controllability for Atangana–Baleanu fractional stochastic systems with Poisson jumps and fractional Brownian motion in a Hilbert space, under a Gramian-type condition for the associated linear system and Carathéodory-type assumptions on the nonlinearities. The focus there is on exact steering to the origin at time T for the underlying linear system and for the coupled nonlinear system, using a resolvent representation and a fixed point argument.

In contrast, the present paper is concerned with *approximate* null controllability for a similar class of Atangana–Baleanu fractional stochastic evolution equations. From the modeling point of view, approximate controllability is typically more natural in infinite-dimensional settings where exact null controllability may fail or be too restrictive. From the analytical point of view, our approach combines:

- the approximate null controllability of the associated linear system (used here as a standing hypothesis), and
- a Schauder fixed point argument on a suitable ball in $C([0,T]; L^2(\Omega;\mathcal{K}))$,

together with mean-square estimates for the stochastic convolutions driven by a fractional Brownian motion and a compensated Poisson random measure. The motivation of the present work is threefold: (i) to study approximate null controllability for stochastic systems involving a non-singular Atangana–Baleanu fractional derivative, (ii) to address systems driven simultaneously by fractional Brownian motion and Poisson jumps, and (iii) to extend linear approximate controllability results to nonlinear stochastic systems through a fixed point approach.

More precisely, we assume approximate null controllability of the linear system (rather than deriving it from Gramian conditions as in [11]) and prove that, under an explicit smallness condition associated with the quadratic bounds of the nonlinearities, the full nonlinear stochastic system is approximately null controllable in the *mean-square* sense. The proof strategy freezes a control that approximately steers the linear part and compensates the nonlinear effects via a compact fixed point argument, supported by suitable deterministic and stochastic convolution estimates.

Instead of emphasizing maximal novelty claims within the broad literature, we focus on presenting a clear and verifiable set of sufficient conditions under a concrete and widely used abstract framework (fractional resolvent families, mean-square estimates for integrals with respect to fractional Brownian motion, and compensated Poisson integrals), and on clarifying how approximate steering of the linear part can be combined with a fixed point argument for the nonlinear stochastic system. The contribution of this paper is therefore methodological rather than comparative, aiming to provide a unified analytical framework applicable to a wide class of Atangana–Baleanu fractional stochastic evolution equations.

We investigate approximate null controllability for a class of Atangana–Baleanu fractional stochastic evolution equations driven jointly by a fractional Brownian motion and a Poisson random measure in a separable Hilbert space. Specifically, we consider the nonlinear controlled system

$$\begin{cases} {}^{ABC}D^{\alpha}_{0+}x(t) = Ax(t) + Bu(t) + F(t,x(t)) \\ \qquad\qquad + G(t,x(t))\,dB^H(t) \quad t \in (0,T] \\ \qquad\qquad + \displaystyle\int_Z \Gamma(t,x(t^-),z)\,\tilde{N}(dt,dz) \\ x(0) = x_0 \in \mathcal{K}, \end{cases} \tag{1}$$

where ${}^{ABC}D^{\alpha}_{0+}$ denotes the A–B Caputo fractional derivative of order $\alpha \in (1/2,1)$, A is a (possibly un-

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bounded) linear operator generating an α-resolvent family on $\mathcal{K}$, B is a bounded linear control operator, F, G, Γ are nonlinear terms, B^H is a fractional Brownian motion with Hurst index $H \in (1/2, 1)$, and $\tilde{N}$ is a compensated Poisson random measure on a measurable space Z; see [1, 11, 7, 9].

We assume that the corresponding linear system obtained by removing the nonlinear terms F, G, Γ is approximately null controllable at time T, in the sense that the reachable set at time T is dense in $\mathcal{K}$. Under this assumption and suitable continuity and growth hypotheses on F, G, Γ, we derive sufficient conditions ensuring approximate null controllability of the full nonlinear system in the mean-square sense. Our analysis is based on the mild solution concept associated with the A–B fractional resolvent family, mean-square estimates for stochastic integrals with respect to fractional Brownian motion and compensated Poisson random measures, and a Schauder fixed point argument in a closed, convex subset of $C([0,T]; L^2(\Omega; \mathcal{K}))$ endowed with a natural norm.

The paper is organized as follows. In Section 2, we recall basic definitions and results on the Atangana–Baleanu fractional derivative, fractional resolvent families, and stochastic integrals with respect to fractional Brownian motion and Poisson random measures in Hilbert spaces. Section 3 introduces the controlled Atangana–Baleanu fractional stochastic evolution equation, the notion of mild solution, and the approximate null controllability of the associated linear system. In Section 4, we state and prove our main theorem on approximate null controllability for the nonlinear system under suitable assumptions. Section 5 presents an illustrative example involving a stochastic fractional partial differential equation with Dirichlet boundary conditions. Finally, Section 6 contains some concluding remarks and possible directions for future work.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a right-continuous filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions. Let $\mathcal{K}$ be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote by $L(\mathcal{K})$ the space of all bounded linear operators on $\mathcal{K}$.

2.1 Atangana–Baleanu fractional derivative and integral

We recall the Caputo-type Atangana–Baleanu fractional derivative, which is defined in terms of the Mittag–Leffler kernel; see [1, 11].

Definition 2.1 Let $\alpha \in (0,1)$ and let $g \in H^1(a,b)$, where $a < b$. The Atangana–Baleanu fractional derivative of Caputo type of order α of g is defined by

$$
{}^{ABC}D^{\alpha}_{a+}g(t) = \frac{\varpi(\alpha)}{1-\alpha}\int_a^t g'(s)\, M_\alpha(-\theta(t-s)^\alpha)\, ds \quad t \in (a,b] \tag{2}
$$

where $\theta = \alpha/(1-\alpha)$, M_α denotes the Mittag–Leffler function

$$
M_\alpha(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(n\alpha+1)} \quad \xi \in \mathbb{R},
$$

and the normalization function $\varpi(\alpha)$ is given by

$$
\varpi(\alpha) = (1-\alpha) + \frac{\alpha}{\Gamma(\alpha)}.
$$

Remark 2.2 Throughout the paper, the Atangana–Baleanu fractional derivative is understood in the Caputo sense and denoted by ${}^{ABC}D^{\alpha}_{0+}$. The notation ${}^{AB}I^{\alpha}_{0+}$ is used only for the corresponding Atangana–Baleanu fractional integral operator.

The corresponding Atangana–Baleanu fractional integral operator is given by

$$
{}^{AB}I^{\alpha}_{a+}g(t) = \frac{1-\alpha}{\varpi(\alpha)}g(t) + \frac{\alpha}{\varpi(\alpha)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)\, ds \quad t \in [a,b]. \tag{3}
$$

2.2 Fractional resolvent families

Let $A : D(A) \subset \mathcal{K} \to \mathcal{K}$ be a closed linear operator. We adopt the notion of fractional resolvent families associated with sectorial operators, which provides an appropriate framework for Atangana–Baleanu type evolution equations; see [11, 17].

Definition 2.3 Let $\alpha \in (0,1)$. A family $\{S_\alpha(t)\}_{t\geq 0} \subset L(\mathcal{K})$ is called an α-resolvent family generated by A if:

1. $S_\alpha(0) = I$ and $S_\alpha(t)$ is strongly continuous for $t \geq 0$;
2. $S_\alpha(t)$ commutes with A, i.e. $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$;
3. For all $x \in D(A)$ and $t \geq 0$,
$$
{}^{ABC}D^{\alpha}_{0+}S_\alpha(t)x = AS_\alpha(t)x,
$$
where ${}^{ABC}D^{\alpha}_{0+}$ is understood in the sense of the Atangana–Baleanu Caputo derivative.

In many applications, the generator A is assumed to be sectorial so that the α-resolvent family admits an integral representation involving the resolvent of A (see, e.g., [17]). We shall assume that the following uniform bounds hold: there exist positive constants C_1, C_2 such that

$$
\|S_\alpha(t)\| \leq C_1, \qquad \|Q_\alpha(t)\| \leq C_2\, t^{\alpha-1} \quad t > 0, \tag{4}
$$

where $\{Q_\alpha(t)\}_{t>0}$ is an auxiliary family associated with S_α and both S_α and Q_α are compact operators for each $t > 0$; see [11].

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2.3 Fractional Brownian motion and Poisson random measure

Let Y be a separable Hilbert space and let $Q \in L(Y)$ be a non-negative, symmetric, trace-class operator with eigenvalues $\{b_n\}_{n\geq 1}$ and an orthonormal basis of eigenvectors $\{\tau_n\}_{n\geq 1}$. We consider a Q-fractional Brownian motion $B^H(t) = B^H_Q(t)$ on Y with Hurst parameter $H \in (1/2, 1)$, which can be represented as

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{b_n}\, \tau_n \beta_n^H(t) \quad t \in [0,T],$$

where $\{\beta_n^H\}_{n\geq 1}$ are independent real-valued fractional Brownian motions with the same Hurst index H; see [7].

We denote by $L_2^0(Y,\mathcal{K})$ the space of all Q-Hilbert–Schmidt operators from Y to $\mathcal{K}$, endowed with the norm

$$\|\eta\|^2_{L_2^0} = \sum_{n=1}^{\infty} b_n \|\eta\tau_n\|^2 \quad \eta \in L_2^0(Y,\mathcal{K}).$$

In the sequel, stochastic integrals with respect to B^H will be understood in this Q-Hilbert–Schmidt setting.

Remark 2.4 The restriction $\alpha \in (1/2, 1)$ is technical and ensures integrability of the kernels (e.g. $(t-s)^{2\alpha-2}$) that appear in the mean-square estimates for the stochastic convolution terms (with respect to fractional Brownian motion and compensated Poisson random measures). In particular, $2\alpha - 2 > -1$ guarantees that these kernels are integrable near $s = t$. This restriction is essential for the validity of the stochastic integral estimates used throughout the paper. When $\alpha \leq 1/2$, the kernel $(t-s)^{2\alpha-2}$ fails to be integrable near $s = t$, and the mean-square estimates for stochastic convolution terms are no longer valid under the present assumptions. On the other hand, the case $\alpha \in (1,2)$ corresponds to higher-order fractional dynamics and requires a different formulation of the Atangana–Baleanu derivative, together with additional initial conditions and a substantially different resolvent framework. Such extensions involve higher regularity assumptions and are beyond the scope of the present work. For these reasons, we restrict our analysis to $\alpha \in (1/2, 1)$, which represents the natural setting where both the Atangana–Baleanu fractional structure and the stochastic integral estimates are simultaneously well-defined.

We shall use the following estimate for stochastic integrals with respect to fractional Brownian motion; see, e.g., [7].

Lemma 2.5 Let $\eta : [0,T] \to L_2^0(Y,\mathcal{K})$ be such that $\int_0^T \|\eta(s)\|^2_{L_2^0}\, ds < \infty$. Then, for any $t \in [0,T]$,

$$\mathbb{E}\left\|\int_0^t \eta(s) dB^H(s)\right\|^2 \leq 2H t^{2H-1} \int_0^t \|\eta(s)\|^2_{L_2^0}\, ds \leq C_{H,T} \int_0^t \|\eta(s)\|^2_{L_2^0}\, ds, \quad (5)$$

where one may take $C_{H,T} := 2HT^{2H-1}$.

Remark 2.6 (On the fBm integral estimate) For the purposes of the subsequent fixed point estimates, only a bound of the form

$$\mathbb{E}\left\|\int_0^t \eta(s) dB^H(s)\right\|^2 \leq C_{H,T} \int_0^t \|\eta(s)\|^2_{L_2^0}\, ds, \qquad t \in [0,T],$$

is needed, where $C_{H,T} > 0$ depends only on H and T. Thus, the explicit dependence on t (e.g. the factor t^{2H-1}) is not essential for the arguments below. This estimate is consistent with the general mean-square inequality for stochastic integrals with respect to fractional Brownian motion in Hilbert spaces; see, for example, [7], Lemma 2.1. In that reference, the bound is stated in the form $\mathbb{E}\| \int_0^t \eta(s)\, dB^H(s)\|^2 \leq C_{H,T} \int_0^t \|\eta(s)\|^2_{L_2^0}\, ds$, with an unspecified constant $C_{H,T} > 0$ depending only on H and T. The explicit choice $C_{H,T} = 2HT^{2H-1}$ used here is one convenient realization of this general bound.

We also consider a Poisson random measure N on $(0,T] \times Z$ with intensity measure $\lambda \otimes \nu$, where $(Z, \mathcal{Z})$ is a measurable space and ν is a σ-finite measure on Z. The compensated Poisson random measure $\tilde{N}$ is defined by

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)\, dt.$$

For suitable predictable integrands $\Gamma(s,z)$ taking values in $\mathcal{K}$, the stochastic integral

$$\int_0^t \int_Z \Gamma(s,z)\, \tilde{N}(ds,dz)$$

is well-defined and satisfies, under standard square-integrability conditions,

$$\mathbb{E}\left\|\int_0^t \int_Z \Gamma(s,z)\, \tilde{N}(ds,dz)\right\|^2 = \int_0^t \int_Z \mathbb{E}\,\|\Gamma(s,z)\|^2\, \nu(dz)\, ds. \quad (6)$$

Remark 2.7 (On the Poisson integral estimate) Identity (6) is the standard isometry for stochastic integrals with respect to compensated Poisson random measures in Hilbert spaces. It is a classical result in stochastic analysis and forms the basis for mean-square estimates of jump-driven stochastic convolution terms; see, for example, [9] and the references therein.

3. The controlled system and mild solutions

Let $T > 0$ be fixed and set $J = [0,T]$. We consider the following controlled Atangana–Baleanu fractional stochastic evolution equation in $\mathcal{K}$:

$$\begin{cases} {}^{ABC}D^{\alpha}_{0+}x(t) = Ax(t) + Bu(t) + F(t,x(t)) \\ \qquad + G(t,x(t))\, dB^H(t) \\ \qquad + \displaystyle\int_Z \Gamma\big(t, x(t^-), z\big)\, \tilde{N}(dt,dz), \qquad t \in (0,T], \\ x(0) = x_0 \in \mathcal{K}. \end{cases} \quad (7)$$

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Remark 3.1 (Interpretation of (7)) Equation (7) is written in a compact differential form. Throughout this work, the dynamics are understood through the corresponding mild formulation (8), in which the stochastic terms are interpreted as mean-square stochastic integrals with respect to the fractional Brownian motion B^H and the compensated Poisson random measure $\tilde{N}$.

Here:

- $\alpha \in (1/2, 1)$ is the fractional order;
- $A : D(A) \subset \mathcal{K} \to \mathcal{K}$ is a sectorial operator generating an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ and an associated family $\{Q_\alpha(t)\}_{t>0}$ satisfying (4);
- $u(\cdot)$ is the control function taking values in a separable Hilbert space $\mathcal{U}$ of admissible controls, and $B \in L(\mathcal{U}, \mathcal{K})$ is a bounded linear control operator;
- $F : J \times L^2(\Omega; \mathcal{K}) \to L^2(\Omega; \mathcal{K})$, $G : J \times L^2(\Omega; \mathcal{K}) \to L_2^0(Y, \mathcal{K})$, and $\Gamma : J \times L^2(\Omega; \mathcal{K}) \times Z \to \mathcal{K}$ are nonlinear mappings;
- B^H is a Q-fractional Brownian motion on Y with Hurst index $H \in (1/2, 1)$;
- $\tilde{N}$ is a compensated Poisson random measure on $(0, T] \times Z$ with intensity measure $\lambda \otimes \nu$.

Remark 3.2 (On the jump term $x(t^-)$) The notation $x(t^-)$ denotes the left limit of the (generally càdlàg) process $x(\cdot)$ at time t, which is standard in stochastic equations with Poisson jumps.

We work in the Banach space

$$\bar{C} = C\big(J; L^2(\Omega; \mathcal{K})\big),$$

endowed with the norm

$$\|x\|_{\bar{C}} = \sup_{t\in J} \left(\mathbb{E}\,\|x(t)\|^2\right)^{1/2}.$$

Definition 3.3 *A process $x(\cdot) \in \bar{C}$ is called a mild solution of (7) corresponding to a control $u \in L^2(J; \mathcal{U})$ if, for all $t \in J$,*

$$\begin{aligned}
x(t) &= S_\alpha(t)x_0 \\
&\quad + K_1 \int_0^t (t-s)^{\alpha-1}\Big[F(s, x(s)) + Bu(s)\Big]\, ds \\
&\quad + K_1 \int_0^t (t-s)^{\alpha-1} G(s, x(s))\, dB^H(s) \\
&\quad + K_1 \int_0^t (t-s)^{\alpha-1} \int_Z \Gamma(s, x(s^-), z)\, \tilde{N}(ds, dz) \\
&\quad + K_2 \int_0^t Q_\alpha(t-s)\Big[F(s, x(s)) + Bu(s)\Big]\, ds \\
&\quad + K_2 \int_0^t Q_\alpha(t-s) G(s, x(s))\, dB^H(s) \\
&\quad + K_2 \int_0^t Q_\alpha(t-s) \int_Z \Gamma(s, x(s^-), z)\, \tilde{N}(ds, dz),
\end{aligned} \tag{8}$$

where $K_1, K_2 > 0$ are constants determined by the Atangana–Baleanu fractional structure (depending on α and $\varpi(\alpha)$). All stochastic integrals are understood in the mean-square sense described in Section 2; see [1, 11].

Remark 3.4 (Justification of the mild solution formulation.) Definition 3.3 follows from the standard variation-of-constants formula associated with Atangana–Baleanu fractional evolution equations. Applying the Atangana–Baleanu fractional integral operator ${}^{AB}I_{0+}^{\alpha}$ to system (7) and using the α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ together with the auxiliary family $\{Q_\alpha(t)\}_{t>0}$ yields an equivalent Volterra-type integral equation. The constants K_1 and K_2 arise from the decomposition of the Atangana–Baleanu fractional integral into an instantaneous part and a fractional convolution part and coincide with those used in [1, 11]. The stochastic integrals with respect to B^H and $\tilde{N}$ are interpreted in the $L^2(\Omega; \mathcal{K})$-sense using Lemma 2.5 and the Poisson isometry (6).

Remark 3.5 Measurability/predictability of stochastic integrands)
In the mild formulation (8), the stochastic integrals with respect to B^H and $\tilde{N}$ are understood for admissible integrands (in particular, predictable for the Poisson integral) satisfying the corresponding square-integrability conditions in Section 2.

Remark 3.6 (Origin of the constants K_1 and K_2)
The constants $K_1, K_2 > 0$ appearing in the mild solution formula (8) arise from the decomposition of the Atangana–Baleanu fractional integral operator ${}^{AB}I_{0+}^{\alpha}$ into its instantaneous part and its fractional convolution part. More precisely, these constants are determined by the normalization function $\varpi(\alpha)$ in (2) and coincide with the coefficients appearing in the variation-of-constants formula for Atangana–Baleanu fractional evolution equations; see, for example, [1, 11]. In the present work, only the positivity of K_1 and K_2 and their dependence on α are used in the subsequent estimates.

Remark 3.7 Throughout the paper, the initial condition x_0 is assumed to be deterministic. It is naturally identified with a constant random variable in $L^2(\Omega; \mathcal{K})$. Treating random initial conditions would require a different controllability formulation and additional assumptions, which are beyond the scope of the present work.

3.1 Approximate null controllability of the linear system

We first consider the associated linear system

$$\begin{cases} {}^{ABC}D_{0+}^{\alpha} y(t) = Ay(t) + Bu(t), & t \in (0, T], \\ y(0) = y_0 \in \mathcal{K}. \end{cases} \tag{9}$$

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Its mild solution is given by

$$\begin{aligned} y(t) &= S_\alpha(t)y_0 \\ &+K_1\int_0^t (t-s)^{\alpha-1}Bu(s)\,ds \\ &+K_2\int_0^t Q_\alpha(t-s)Bu(s)\,ds. \end{aligned} \tag{10}$$

In particular, at time T we have

$$y(T) = S_\alpha(T)y_0 + \mathcal{L}_0^T u,$$

where $\mathcal{L}_0^T$ is defined by (11).

Define the linear control operator

$$\mathcal{L}_0^T : L^2(J;\mathcal{U}) \to \mathcal{K}$$

by

$$\begin{aligned} \mathcal{L}_0^T u &= K_1\int_0^T (T-s)^{\alpha-1}Bu(s)\,ds \\ &+K_2\int_0^T Q_\alpha(T-s)Bu(s)\,ds. \end{aligned} \tag{11}$$

The reachable set of the linear system at time T from the origin is

$$\mathcal{R}_0(T) = \{\mathcal{L}_0^T u : u \in L^2(J;\mathcal{U})\}.$$

Remark 3.8 Under the standing bounds on B and the families $\{S_\alpha(t)\}$ and $\{Q_\alpha(t)\}$, the mapping $\mathcal{L}_0^T$ defined in (11) is a well-defined bounded linear operator from $L^2(J;\mathcal{U})$ into $\mathcal{K}$.

Definition 3.9 The linear system (9) is said to be *approximately null controllable* on J if

$$\overline{\mathcal{R}_0(T)} = \mathcal{K},$$

where the closure is taken with respect to the norm of $\mathcal{K}$. Equivalently, for every $y_0 \in \mathcal{K}$ and every $\varepsilon > 0$, there exists a control $u \in L^2(J;\mathcal{U})$ such that the corresponding mild solution satisfies $\|y(T)\| < \varepsilon$.

Remark 3.10 In this work, controllability for the nonlinear stochastic system is formulated and proved in the *mean-square sense*, that is, with terminal condition

$$\left(\mathbb{E}\,\|x(T)\|^2\right)^{1/2} < \varepsilon.$$

No almost sure controllability result is claimed. The deterministic space $\mathcal{K}$ is identified with constant random variables in $L^2(\Omega;\mathcal{K})$, which is standard in stochastic controllability problems.

A classical characterization states that approximate null controllability of the linear system holds if and only if the adjoint operator $(\mathcal{L}_0^T)^*$ is injective, equivalently $\ker(\mathcal{L}_0^T)^* = \{0\}$; see, for example, [8, 5, 12]. In the present work, approximate null controllability of the linear system (9) is assumed as a standing hypothesis.

4. Main assumptions and controllability result

We now state the structural assumptions on the operators and nonlinear mappings in the stochastic setting. Throughout, $\bar{C} = C(J; L^2(\Omega;\mathcal{K}))$ is endowed with the norm

$$\|x\|_{\bar{C}} := \sup_{t\in J}\left(\mathbb{E}\|x(t)\|^2\right)^{1/2}.$$

Assumption 4.1 The operator $A : D(A) \subset \mathcal{K} \to \mathcal{K}$ generates an α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ and an associated family $\{Q_\alpha(t)\}_{t>0}$ satisfying the bounds (4). Moreover, $S_\alpha(t)$ and $Q_\alpha(t)$ are compact operators for each $t > 0$.

Assumption 4.2 The linear system (9) is approximately null controllable on J, that is,

$$\overline{\mathcal{R}_0(T)} = \mathcal{K},$$

equivalently, the adjoint operator $(\mathcal{L}_0^T)^*$ is injective.

Assumption 4.3 The mapping $F : J \times L^2(\Omega;\mathcal{K}) \to L^2(\Omega;\mathcal{K})$ satisfies:

1. For each $x \in L^2(\Omega;\mathcal{K})$, the mapping $t \mapsto F(t,x)$ is strongly measurable; for each $t \in J$, the mapping $x \mapsto F(t,x)$ is continuous.

2. For every $R > 0$, there exists a nonnegative function $N_R \in L^1(J)$ such that

$$\sup_{\|x\|_{L^2(\Omega;\mathcal{K})}\le R} \mathbb{E}\|F(t,x)\|^2 \le N_R(t) \quad \text{for a.e. } t \in J,$$

   and

$$\sup_{t\in J}\int_0^t (t-s)^{\alpha-1}N_R(s)\,ds < \infty.$$

Assumption 4.4 The mapping $G : J \times L^2(\Omega;\mathcal{K}) \to L_2^0(Y,\mathcal{K})$ satisfies:

1. For each $x \in L^2(\Omega;\mathcal{K})$, the mapping $t \mapsto G(t,x)$ is strongly measurable; for each $t \in J$, the mapping $x \mapsto G(t,x)$ is continuous.

2. For every $R > 0$, there exists a nonnegative function $g_R \in L^1(J)$ such that

$$\sup_{\|x\|_{L^2(\Omega;\mathcal{K})}\le R} \mathbb{E}\|G(t,x)\|^2_{L_2^0} \le g_R(t) \quad \text{for a.e. } t \in J,$$

   and

$$\sup_{t\in J}\int_0^t (t-s)^{2\alpha-2}g_R(s)\,ds < \infty.$$

Assumption 4.5 The mapping $\Gamma : J\times L^2(\Omega;\mathcal{K})\times Z \to \mathcal{K}$ satisfies:

1. For each $x \in L^2(\Omega;\mathcal{K})$, the mapping $(t,z) \mapsto \Gamma(t,x,z)$ is measurable; for each $(t,z) \in J\times Z$, the mapping $x \mapsto \Gamma(t,x,z)$ is continuous in $L^2(\Omega;\mathcal{K})$.

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2. For every $R > 0$, there exists a nonnegative function $\chi_R \in L^1(J)$ such that
$$\sup_{\|x\|_{L^2(\Omega;\mathcal{K})} \le R} \int_Z \mathbb{E}\|\Gamma(t,x,z)\|^2 \, \nu(dz) \le \chi_R(t) \quad (12)$$
$$\text{for a.e. } t \in J,$$
and
$$\sup_{t \in J} \int_0^t (t-s)^{2\alpha-2} \chi_R(s)\, ds < \infty.$$

Remark 4.6 Assumptions 4.3–4.5 are Carathéodory-type hypotheses formulated directly on $L^2(\Omega;\mathcal{K})$. They ensure that all deterministic and stochastic convolution terms in (8) are well-defined in $\bar{C}$ and admit uniform bounds on balls in $\bar{C}$.

Remark 4.7 (On the $L^2(\Omega;\mathcal{K})$-formulation of the nonlinearities)
The mappings F, G, and Γ are formulated in Assumptions 4.3–4.5 as operators acting on $J \times L^2(\Omega;\mathcal{K})$. This is consistent with the common pathwise setting where the nonlinearities act on $\mathcal{K}$ and are then lifted to $L^2(\Omega;\mathcal{K})$ via composition with the stochastic process $x(\cdot)$. In the illustrative example of Section 5, the nonlinearities are defined pointwise in the spatial variable, which recovers the standard concrete form used in stochastic partial differential equations.

Theorem 4.8 Suppose that Assumptions 4.1–4.5 hold and that the linear system (9) is approximately null controllable on J (Assumption 4.2). Assume in addition that for each $\varepsilon > 0$ there exists $R > 0$ such that
$$C(T,\alpha,H)\,\Theta_R < \min\{R/2, \varepsilon/2\}, \qquad (13)$$
where
$$\begin{aligned}\Theta_R := {}& \sup_{t\in J}\left(\int_0^t (t-s)^{\alpha-1} N_R(s)\, ds\right)^{1/2} \\ & + \sup_{t\in J}\left(\int_0^t (t-s)^{2\alpha-2} g_R(s)\, ds\right)^{1/2} \\ & + \sup_{t\in J}\left(\int_0^t (t-s)^{2\alpha-2} \chi_R(s)\, ds\right)^{1/2},\end{aligned}$$
and $C(T,\alpha,H) > 0$ depends only on T, α, H, $\|B\|$, the constants in (4), and K_1, K_2. Then system (7) is approximately null controllable on J in the mean-square sense.

Remark 4.9 (Origin of the constant $C(T,\alpha,H)$) The constant $C(T,\alpha,H)$ in Theorem 4.8 is obtained by collecting the constants arising in the deterministic convolution estimates associated with the families $\{S_\alpha(t)\}_{t\ge0}$ and $\{Q_\alpha(t)\}_{t>0}$, together with the mean-square bounds for the stochastic integrals with respect to the fractional Brownian motion (Lemma 2.5–Remark 2.6) and the compensated Poisson random measure (6). No explicit numerical value is required; only its dependence on T, α, H and the structural bounds in (4) and on K_1, K_2 is used in the fixed point argument.

Remark 4.10 The square-root structure in the definition of Θ_R naturally arises from the mean-square estimates of the stochastic convolution terms and the definition of the $\bar{C}$-norm. It does not impose any additional restriction on the nonlinearities beyond Assumptions 4.3–4.5, but is merely a convenient way to collect the bounds obtained from the $L^2(\Omega;\mathcal{K})$ estimates.

Remark 4.11 (On the role of the smallness condition)
Condition (13) is a sufficient (not necessary) condition ensuring that the fixed point operator maps a suitable closed ball of $\bar{C}$ into itself and that the terminal mean-square estimate achieves the required accuracy. No optimality is claimed. In particular, (13) may be satisfied by taking T sufficiently small or by assuming that the growth coefficients in F, G, and Γ are small enough so that the quantities defining Θ_R remain controlled on a suitable ball.

Proof: Fix $x_0 \in \mathcal{K}$ (deterministic) and $\varepsilon > 0$.

Step 1: Linear approximate steering. By Assumption 4.2, there exists $u \in L^2(J;\mathcal{U})$ such that the mild solution $y(\cdot)$ of (9) satisfies
$$\|y(T)\| < \varepsilon/2.$$
Fix such a control u.

Step 2: Fixed point operator. Define $\Phi : \bar{C} \to \bar{C}$ by the right-hand side of (8) and set
$$B_R := \{x \in \bar{C} : \ \|x\|_{\bar{C}} \le R\}.$$

Step 3: Invariance of B_R. Using the bounds in Assumptions 4.1–4.5, together with Lemma 2.5 and the Poisson isometry (6), one obtains
$$\sup_{t\in J} \|\Phi x(t)\|_{L^2(\Omega;\mathcal{K})} \le M_0 + C(T,\alpha,H)\,\Theta_R,$$
where $M_0 > 0$ depends only on x_0 and the fixed control u. Choose $R > 0$ such that $R \ge 2M_0$ and (13) holds. Then, for any $x \in B_R$,
$$\|\Phi x\|_{\bar{C}} \le M_0 + C(T,\alpha,H)\Theta_R \le R/2 + R/2 = R,$$
so $\Phi(B_R) \subset B_R$.

Step 4: Complete continuity of Φ. We show that $\Phi : B_R \to \bar{C}$ is completely continuous.

(i) Continuity. Let $x_n \to x$ in $\bar{C}$ with $x_n \in B_R$. By Assumptions 4.3–4.5, we have $F(\cdot, x_n(\cdot)) \to F(\cdot, x(\cdot))$ in $L^2(\Omega;\mathcal{K})$ a.e. on J, and similarly for G and Γ. Moreover, the integrable functions N_R, g_R, χ_R yield domination for the deterministic and stochastic convolution terms. Therefore, by dominated convergence together with Lemma 2.5 and (6), we obtain $\|\Phi x_n(t) - \Phi x(t)\|_{L^2(\Omega;\mathcal{K})} \to 0$ for each $t \in J$. Taking the supremum over $t \in J$ and using the same uniform bounds as in Step 3 gives $\|\Phi x_n - \Phi x\|_{\bar{C}} \to 0$. Hence Φ is continuous on B_R.

(ii) Relative compactness of $\Phi(B_R)$. Let $x_n \in B_R$ and set $v_n = \Phi x_n$. Fix $t \in (0,T]$ and $\delta \in (0,t)$. Split

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each convolution integral defining $v_n(t)$ into integrals over $[0, t-\delta]$ and $[t-\delta, t]$. On $[0, t-\delta]$, the operators $S_\alpha(t-s)$ and $Q_\alpha(t-s)$ are compact, hence the corresponding images of bounded sets are relatively compact in $L^2(\Omega; \mathcal{K})$. On $[t-\delta, t]$, the contributions are uniformly small (uniformly in n) because the kernels $(t-s)^{\alpha-1}$ and $(t-s)^{2\alpha-2}$ are integrable and the integrands are uniformly dominated by N_R, g_R, χ_R. The same decomposition applies to the stochastic convolution terms using Lemma 2.5 and (6). Consequently, for each fixed $t \in (0, T]$, the set $\{v_n(t)\}$ is relatively compact in $L^2(\Omega; \mathcal{K})$.

(iii) Equicontinuity. Using the strong continuity of $S_\alpha(\cdot)$ and $Q_\alpha(\cdot)$ and the uniform bounds from Step 3, we obtain that $\{v_n\}$ is equicontinuous in $C(J; L^2(\Omega; \mathcal{K}))$. By the (vector-valued) Arzelà–Ascoli theorem, $\{v_n\}$ has a convergent subsequence in $\bar{C}$. Hence $\Phi(B_R)$ is relatively compact in $\bar{C}$, and therefore Φ is completely continuous on B_R.

Step 5: Terminal estimate. Let $y(\cdot)$ be the mild solution of the linear system driven by the same control u. Then $\|y(T)\| < \varepsilon/2$ by Step 1. Moreover, using the same estimates as in Step 3 and the definition of Θ_R, we have

$$\left(\mathbb{E}\|x(T) - y(T)\|^2\right)^{1/2} \le C(T, \alpha, H)\, \Theta_R < \varepsilon/2,$$

where $C(T, \alpha, H)$ is the same constant as in Step 3. Hence, by the triangle inequality in $L^2(\Omega; \mathcal{K})$,

$$\left(\mathbb{E}\|x(T)\|^2\right)^{1/2} \le \left(\mathbb{E}\|x(T) - y(T)\|^2\right)^{1/2} + \|y(T)\| < \varepsilon.$$

This proves approximate null controllability in the mean-square sense.

Remark 4.12 The constant $C(T, \alpha, H)$ can be written explicitly in terms of (4), Lemma 2.5, (6), and K_1, K_2. In particular, for sufficiently small T or sufficiently small nonlinear coefficients, condition (13) may hold for some $R > 0$.

5. Illustrative example

In this section, we present an example of a controlled Atangana–Baleanu fractional stochastic partial differential equation fitting into the abstract framework above. We consider the one-dimensional spatial interval $(0, 1)$ and set $\mathcal{K} = Y = \mathcal{U} = L^2(0, 1)$.

Let A be the Dirichlet Laplacian operator

$$A\varphi = \frac{\partial^2 \varphi}{\partial \xi^2} \quad \xi \in (0, 1)$$

with domain

$$D(A) = \left\{\varphi \in H^2(0, 1) \cap H_0^1(0, 1)\right\}.$$

It is well known that A generates an analytic C_0-semigroup $\{S(t)\}_{t \ge 0}$ on $\mathcal{K}$ and that $S(t)$ is compact for each $t > 0$. Moreover, under the standard sectorial framework, A generates an α-resolvent family $\{S_\alpha(t)\}_{t \ge 0}$ and an associated family $\{Q_\alpha(t)\}_{t > 0}$ satisfying (4) and the compactness property; see [17, 11].

Remark on the linear controllability assumption. When the control acts in a distributed way on the whole spatial domain (as in the present setting with $\mathcal{U} = \mathcal{K} = L^2(0, 1)$ and $B = I$), approximate null controllability is a natural hypothesis. In this illustrative example, we keep it as an explicit standing assumption, consistently with Assumption 4.2.

Consider the fractional stochastic PDE:

$$\begin{cases} {}^{ABC}D_{0+}^{\alpha} x(t, \xi) = \dfrac{\partial^2 x(t, \xi)}{\partial \xi^2} + u(t, \xi) + F\big(t, x(t, \xi)\big) \\ \qquad\qquad + G\big(t, x(t, \xi)\big)\, dB^H(t) \\ \qquad\qquad + \displaystyle\int_Z \Gamma\big(t, x(t, \xi), z\big)\, \tilde{N}(dt, dz) \\ \qquad\qquad\qquad t \in (0, T], \qquad \xi \in (0, 1), \\ x(t, 0) = 0, \qquad x(t, 1) = 0, \qquad t \in [0, T], \\ x(0, \xi) = x_0(\xi), \qquad \xi \in [0, 1]. \end{cases} \tag{14}$$

We interpret $x(t)(\xi) = x(t, \xi)$ and $u(t)(\xi) = u(t, \xi)$. We take the control operator $B : \mathcal{U} \to \mathcal{K}$ to be the identity, i.e. $Bu = u$. Moreover, we define the nonlinearities pointwise by

$$\begin{aligned} (F(t, x))(\xi) &= F(t, x(\xi)), \\ (G(t, x))(\xi) &= G(t, x(\xi)), \\ (\Gamma(t, x, z))(\xi) &= \Gamma(t, x(\xi), z). \end{aligned}$$

Assuming that F, G, and Γ satisfy the structural conditions of Assumptions 4.3–4.5, and that the associated linear system

$${}^{ABC}D_{0+}^{\alpha} y(t) = Ay(t) + Bu(t), \qquad y(0) = y_0,$$

is approximately null controllable on J (Assumption 4.2), it follows from Theorem 4.8 that the nonlinear PDE (14) is approximately null controllable on J. In particular, for each deterministic initial datum $x_0 \in L^2(0, 1)$ (identified with a constant random variable in $L^2(\Omega; L^2(0, 1))$) and each $\varepsilon > 0$, there exists a control $u \in L^2(J; L^2(0, 1))$ such that the corresponding mild solution satisfies

$$\left(\mathbb{E}\, \|x(T)\|^2_{L^2(0,1)}\right)^{1/2} < \varepsilon.$$

Verification of the hypotheses. We briefly verify that the abstract assumptions imposed in Sections 3–4 are satisfied for system (14). The Dirichlet Laplacian A on $L^2(0, 1)$ is densely defined, self-adjoint, and sectorial, and it generates an analytic semigroup which is compact for each $t > 0$. Consequently, A admits an α-resolvent family $\{S_\alpha(t)\}_{t \ge 0}$ and an associated family $\{Q_\alpha(t)\}_{t > 0}$ satisfying the bounds (4), and $S_\alpha(t)$ and $Q_\alpha(t)$ are compact for $t > 0$; see [17, 11]. Hence Assumption 4.1 holds in this setting.

The control operator B is the identity on $L^2(0, 1)$ and is therefore bounded. Under the distributed-control setting, we explicitly *assume* that the corresponding linear

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system satisfies approximate null controllability on J (Assumption 4.2), as required in Theorem 4.8.

Finally, the nonlinear terms F, G, and Γ are assumed to satisfy the Carathéodory-type conditions and local quadratic bounds stated in Assumptions 4.3–4.5. Under these conditions, all hypotheses of Theorem 4.8 are fulfilled, and the approximate null controllability result applies to system (14).

6. Conclusion

In this paper, we have investigated the approximate null controllability of a class of Atangana–Baleanu fractional stochastic evolution equations in a separable Hilbert space, driven jointly by a fractional Brownian motion with Hurst index $H \in (1/2, 1)$ and a compensated Poisson random measure. Assuming that the associated linear system is approximately null controllable and that the nonlinear terms satisfy suitable Carathéodory-type conditions with integrable quadratic bounds, we derived sufficient conditions ensuring approximate null controllability of the nonlinear system in the mean-square sense.

From a broader perspective, the present framework complements and unifies several recent approximate controllability results for fractional stochastic systems, including those in [13, 14, 15, 16], by treating Atangana–Baleanu fractional dynamics under the joint influence of fractional Brownian motion and Poisson jumps.

The analysis relies on the variation-of-constants formula associated with an α-resolvent family, mean-square estimates for stochastic integrals with respect to fractional Brownian motion and Poisson jumps, and a Schauder fixed point argument on a closed, convex subset of $C([0,T]; L^2(\Omega; \mathcal{K}))$. An illustrative example involving a controlled Atangana–Baleanu fractional stochastic partial differential equation with Dirichlet boundary conditions was presented to demonstrate the applicability of the abstract theoretical results. It is worth emphasizing that the controllability conditions obtained in this work are sufficient in nature and are not claimed to be sharp. This reflects the methodological focus of the paper, which aims at providing a robust and verifiable analytical framework rather than optimal or necessary conditions. Possible directions for future research include the investigation of approximate boundary controllability for Atangana–Baleanu fractional stochastic systems, extensions to multivalued or differential inclusion models involving Clarke's subdifferential, and the study of optimal control problems within the Atangana–Baleanu fractional stochastic framework.

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Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.
Conflict of interests
The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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