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ORIGINAL RESEARCH

Solving Multi-order Nonlinear Caputo Fractional Differential Equations and Numerical Simulations

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Abstract

This paper addresses the challenging problem of establishing the existence and uniqueness of solutions for coupled multi-order fractional differential equations involving multiple fractional orders, a class of equations that arises in various scientific and engineering applications. By employing the powerful Banach contraction principle, we derive novel sufficient conditions that guarantee the existence and uniqueness of solutions. The theoretical findings are further substantiated through the presentation and analysis of a detailed illustrative example, highlighting the practical relevance of our results. To validate the theoretical results, we implement a modified Adams-Bashforth-Moulton predictor-corrector method for the numerical solution of the proposed multi-term Caputo fractional differential systems. In particular, three illustrative examples are presented to demonstrate the impact of fractional orders and memory effects on system dynamics. These simulations, covering nonlinear coupled oscillators, neural feedback processes, and epidemic models. The results confirm the robustness of the theoretical framework and the applicability of the method to real-world problems governed by fractional dynamics.

Keywords: Fractional derivative, fixed point technique, multi fractional order, boundary condition, numerical simulation.

Mathematics Subject Classification (2010): 47H10, 45D05, 34A08.

1 Introduction

Fractional differential equations (FDEs) provide a powerful generalization of classical differential equations by extending the notions of differentiation and integration to non-integer orders [1]. This extension allows for a more accurate and nuanced representation of complex phenomena in many scientific and engineering fields, particularly those characterized by memory effects and nonlocal interactions that integer-order models often fail to describe adequately [2]. As a result, FDEs have found wide applications in areas such as viscoelasticity, anomalous diffusion, control theory, signal processing, and biological systems, highlighting their effectiveness in modeling real-world processes with intricate dynamics [3]. The theoretical analysis of FDEs involves advanced analytical techniques, and because closed-form solutions are rarely available, the development of reliable and efficient numerical methods remains an important and active area of research [4]. Additional applications can be found in [5–7].

A variety of approaches have been proposed in the literature to define fractional integrals and derivatives, each developed to address different modeling needs. Among the earliest and most commonly used is the Riemann-Liouville formulation, which extends the classical Cauchy formula for repeated integration to fractional orders [2]. In contrast, the Caputo definition modifies this approach by applying the integer-order derivative prior to fractional integration. This formulation yields a zero derivative for constant functions and permits the use of initial conditions that closely resemble those of classical differential equations, making it particularly attractive for physical applications [8]. From a numerical standpoint, the Grünwald-Letnikov definition offers a discrete formulation based on finite differences and is well suited for computational implementations [9]. Another alternative is provided by the Hadamard fractional operators, which are defined through logarithmic kernels and are especially appropriate for scale-invariant problems posed on intervals originating at zero [10]. Beyond these, several other formulations such as the Miller-Ross and Weyl fractional derivatives have been introduced, each designed for specific classes of problems and offering distinct advantages in the modeling of complex systems [11].

FDEs involving multiple non-integer orders have gained increasing attention because they provide a more flexible and expressive framework for modeling complex system dynamics than single-order or classical formulations. By incorporating several fractional orders within a unified structure, multi-order FDEs are particularly effective in capturing intricate memory effects and nonlocal interactions acting across different temporal and spatial scales [12]. This enhanced modeling capability has led to successful applications in a wide range of fields, including control theory, where multi-order formulations can improve control performance [13], viscoelasticity, where multi-term fractional constitutive relations more accurately describe complex stress-strain behaviors and multi-stage relaxation processes [14], and anomalous diffusion, where multiple fractional operators allow the simultaneous characterization of distinct scaling regimes and propagation rates [15]. Beyond these areas, multi-order FDEs have also been employed to model biological systems with heterogeneous memory spans and electrical circuits featuring frequency-dependent components represented by multiple fractional impedances [16–22]. Despite their clear advantages, the theoretical analysis and numerical approximation of multi-order fractional differential equations remain challenging, and their further development continues to be an active and important area of research [2].

A substantial body of work, as reported in [3, 23], has produced numerous results on multi-order nonlinear fractional differential equations of the form

$$L(D)\Re = h\left(\rho, \Re\right).$$

Sangita and Varsha [24] considered the following class of nonlinear multi-order FDEs:

$$\begin{cases} L(D)\Re(\rho) = h(\varrho, \Re(\rho)), \ \rho \in [0, \rho], \ \rho \in [0,1] \\ 0 \le \beta_0 < \beta_1 < \cdots < \beta_{u-1} < \beta_u < 1, \end{cases} \tag{1.1}$$

where $L(D) = \mu_u\ {}^C D^{\beta_u} + \mu_{u-1}\ {}^C D^{\beta_u - 1} + \cdots + \mu_1\ {}^C D^{\beta_1} + \mu_0\ {}^C D^{\beta_0}$, $\mu_j \neq 0$ are real constants for $j = 0, 1, \ldots, u$, and ${}^C D^{\beta}$ is the Caputo fractional derivative of order β.

By employing double-parameter MittagLeffler functions, the authors derived the corresponding Green's functions for problem (1.1) under both periodic and anti-periodic boundary conditions. In addition, they established sufficient conditions for the existence and uniqueness of solutions to the associated fractional boundary value problem (BVP).

Fixed point (FP) techniques play a central role in the analysis of nonlinear multi-order FDEs, particularly in establishing the existence and uniqueness of solutions. Among these, the Banach FP theorem is widely used, as it guarantees the uniqueness of a solution by showing that the operator associated with the differential equation satisfies a contraction condition [25]. When uniqueness is not required, alternative results such as Schauder's and Krasnoselskii's FP theorems are commonly applied to demonstrate the existence of solutions, especially in settings involving compact or completely continuous operators [26]. In addition, ongoing extensions of classical FP principles and the introduction of new FP results have significantly expanded the class of nonlinear fractional problems that can be addressed, accommodating a wide range of boundary conditions and functional frameworks [27].

Sun and Zhao [28] employed a FP theorem to examine the existence of positive solutions for a class of multi-order FDEs:

$$L(D)\Re = h(\zeta, \Re), \ \zeta \in (0,1),$$

where $L(D) =^R D^{q_u} - b_{u-1}\ {}^R D^{q_u - 1} - \cdots - b_1\ {}^R D^{q_1}$, $0 < q_1 < \cdots < q_u < 1$, $b_j > 0$, and ${}^R D^{q_j}$ $(j = 1, 2, \cdots, u)$ are the Riemann-Louisville (RL) fractional derivatives.

Xiel et al. [29] studied the following class of multi-order FDEs:

$$\begin{cases} L(D)\Re(\rho) = h\left(\varrho, \Re(\rho), {}^R D_{0^+}^{\beta_u} \Re(\rho)\right), \ \zeta \in V = [0, +\infty), \\ \rho^{1-q_u} \Re(\rho)\big|_{\rho = 0} = 0, \end{cases}$$

where $L(D) =^R D_{0^+}^{q_u} - b_{q-1}\ {}^R D_{0^+}^{q_u - 1} - \cdots - b_1\ {}^R D_{0^+}^{q_1}$, $0 < q_1 < \cdots < q_u < 1$, $u \in \mathbb{N}$, $b_j \in \mathbb{R}$, $q_j + \beta_u < q_u$, ${}^R D_{0^+}^{q_j}$ $(j = 1, 2, \cdots, u)$ and ${}^R D_{0^+}^{\beta_u}$ are the RL fractional derivatives.

The growing interest in incommensurate, or multi-order FDEs stems from their effectiveness in modeling complex real-world processes where integer-order models fail. Developing a strong qualitative theory for this versatile framework is therefore essential for understanding and predicting such intricate dynamics. In this study, we focus on a particular class of coupled multi-order fractional differential equations defined on the infinite half-line $[0, +\infty)$, which can be expressed as:

$$\begin{cases} {}^R D_{0^+}^{r_u} \Re(\rho) - L(D) G(\rho, \Re(\rho), \Im(\rho))) = F\left(\rho, \Re(\rho), I_{0^+}^{\beta_u} \Im(\rho)\right), \ \rho \in J = [0, +\infty), \\ {}^R D_{0^+}^{t_u} \Im(\rho) - \widetilde{L}(D) \widetilde{G}(\rho, \Im(\rho), \Re(\rho))) = \widetilde{F}\left(\rho, \Im(\rho), I_{0^+}^{\kappa_u} \Re(\rho)\right), \ \rho \in J, \\ \rho^{1-r_u} \Re(\rho)\big|_{\rho = 0} = 0, \quad \rho^{1-t_u} \Im(\rho)\big|_{\rho = 0} = 0, \end{cases} \tag{1.2}$$

such that $r_j + \beta_u < r_u$, and $t_k + \kappa_u < t_u$, for $j, k = 1, 2, \cdots, u-1$, where

$$\begin{aligned} L(D) &= \sum_{j=1}^{u-1} b_j \ {}^{R}D_{0+}^{r_j}, \ 0 < r_1 < \cdots < r_u < 1, \ u \in \mathbb{N}, \ b_j \in \mathbb{R}, \\ \widetilde{L}(D) &= \sum_{k=1}^{u-1} c_k \ {}^{R}D_{0+}^{t_k}, \ 0 < t_1 < \cdots < t_u < 1, \ u \in \mathbb{N}, \ c_k \in \mathbb{R}, \end{aligned}$$

${}^{R}D_{0+}^{r_u}$, ${}^{R}D_{0+}^{t_u}$ are RL fractional derivatives and $I_{0+}^{\beta_u}, I_{0+}^{\kappa_u}$ RL fractional integrals with $r_u > \beta_u$ and $t_u > \kappa_u$. Moreover, $\rho^s F(\rho, \zeta, \xi), \rho^{s'} G(\rho, \zeta, \xi), \rho^w \widetilde{F}(\rho, \xi, \zeta), \rho^{w'} \widetilde{G}(\rho, \xi, \zeta) \in \mathbf{C}(J \times \Theta \times \Theta; \Theta)$, $s, s' \in [0, r_u)$, $w, w' \in [0, t_u)$, Θ is a Banach space (BS) under the norm $\|.\|$ and $\mathbf{C}(.,.)$ represents the space of all continuous functions.

The manuscript is organized as follows: Section 2 introduces the foundational definitions and essential preliminary results that form the basis for the subsequent analysis. Section 3 focuses on establishing the existence of solutions for the class of coupled multi-order FDEs defined in the equation (1.2). This analysis is conducted on the infinite interval $[0, +\infty)$ using the Banach FP theorem. Section 4 provides a concrete example to illustrate the practical relevance and effectiveness of the theoretical framework developed in the previous sections. This example helps clarify the results and demonstrates their applicability in real-world contexts. Section 5 presents numerical simulations supporting the theoretical findings. By employing a modified Adams-Bashforth-Moulton method, we investigate three nonlinear systems with multi-term Caputo derivatives: a memory-based oscillator, a neural-type model, and a fractional epidemic model. These simulations highlight the influence of memory effects on system behavior and verify the accuracy of the proposed numerical approach. Section 6 summarizes the main findings, emphasizes the contributions of this work, and discusses potential directions for future research. Additionally, a list of abbreviations is provided to avoid repeated terminology.

2 Preliminaries

This section provides a brief review of essential preliminaries. We recall key definitions and fundamental results to clarify notation and establish a common foundation before turning to the main body of the work.

Following [30–32], we consider the pair $\left(\mho_1, \|.\|_{\mho_1}\right)$ is a real BS, where

$$\mho_1 = \left\{ \zeta \in \mathbf{C}(J, \Theta) : \sup_{\rho \in J} \frac{\|\zeta(\rho)\|}{1 + \rho^\mu} < \infty, \ \mu > 1 \right\}$$

endowed with the norm $\|\zeta\|_{\mho_1} = \sup_{\rho \in J} \frac{\|\zeta(\rho)\|}{1+\rho^\mu}$. Assume that

$$\mho_2 = \left\{ \xi \in \mathbf{C}(J, \Theta) : \sup_{\rho \in J} \frac{\|\xi(\rho)\|}{1 + \rho^\mu} < \infty, \ \mu > 1 \right\},$$

equipped with the norm $\|\xi\|_{\mho_2} = \sup_{\rho \in J} \frac{\|\xi(\rho)\|}{1+\rho^\mu}$. Then, the pair $\left(\mho_2, \|.\|_{\mho_2}\right)$ is a BS.
Clearly, the product $(\mho = \mho_1 \times \mho_2, \|.\|_{\mho})$ is a BS under the norm $\|(\zeta + \xi)\|_{\mho} = \|\zeta\|_{\mho_1} + \|\xi\|_{\mho_2}$.

Definition 2.1. [3] For the function h:

(1) The RL fractional integral of order $\beta > 0$ can be described as

$$I_{b+}^{\beta}\left[h(\rho)\right] = \frac{1}{\Gamma\left(\beta\right)} \int_{b}^{\rho} \left(\rho - r\right)^{\beta - 1} h(r) dr, \;\; \rho > b,$$

, provided that the right integral exists.

(2) The RL fractional derivative of order $\beta > 0$ can be expressed as

$${}^{R}D_{b+}^{\beta}\left[h(\rho)\right] = \left(\frac{d}{d\rho}\right)^{u} I_{b+}^{u-\beta}\left[h(\rho)\right] = \frac{1}{\Gamma\left(u - \beta\right)} \left(\frac{d}{d\rho}\right)^{u} \int_{b}^{\rho} \left(\rho - r\right)^{u-\beta-1} h(r) dr, \;\; \rho > b,$$

where $u = [\beta] + 1$.

Remark 2.2. [3] Assume that the constants $\beta > 0$ and $\kappa > 0$. Then, the following properties are true:

(i) For $h\left(\rho\right) \in L(b, +\infty)$, $I_{b+}^{\beta} I_{b+}^{\kappa} h\left(\rho\right) = I_{b+}^{\beta+\kappa} h\left(\rho\right)$.

(ii) For $h\left(\rho\right) \in L^{1}(b, +\infty)$, ${}^{R}D_{b+}^{\beta} I_{b+}^{\beta} h\left(\rho\right) = h\left(\rho\right)$.

(iii) For $h\left(\rho\right) \in L^{1}(b, +\infty)$, ${}^{R}D_{b+}^{\beta} I_{b+}^{\kappa} h\left(\rho\right) = I_{b+}^{\beta-\kappa} h\left(\rho\right)$.

(iv) $I_{b+}^{\beta}\left(\rho - b\right)^{\kappa-1}\left(\zeta\right) = \frac{\Gamma(\kappa)}{\Gamma(\beta+\kappa)}\left(\zeta - b\right)^{\beta+\kappa-1}$.

Lemma 2.3. *[3] For $\beta > 0$, $\zeta \in L^{1}(b, +\infty)$, and ${}^{R}D_{b+}^{\beta}\zeta\left(\rho\right) \in L^{1}(b, +\infty)$, we obtain that*

$$I_{b+}^{\beta} \; {}^{R}D_{b+}^{\beta}\zeta\left(\rho\right) = \zeta\left(\rho\right) + a_{1}\left(\rho - b\right)^{\beta-1} + a_{2}\left(\rho - b\right)^{\beta-2} + \cdots + a_{u}\left(\rho - b\right)^{\beta-u},$$

for some $a_{j} \in \mathbb{R}$, $j \in \mathbb{N}$.

Theorem 2.4. *[33] Every contraction mapping from a BS Λ into itself has a unique FP.*

3 Existence and uniqueness results

This section is devoted to establishing the existence and uniqueness of solutions for the coupled mixed-order FDEs given in (1.2). We begin with the following Lemma:

Lemma 3.1. *Assume that $\Re : J \to \mho_{1}$ and $\Im : J \to \mho_{2}$ are continuous functions. The integral equation of the coupled mixed-order FDEs (1.2) can be expressed as*

$$\begin{aligned} \Re\left(\rho\right) &= \frac{1}{\Gamma\left(r_{u}\right)} \int_{0}^{\rho} \left(\rho - r\right)^{r_{u}-1} F\left(r, \Re\left(r\right), I_{0+}^{\beta_{u}}\Im\left(r\right)\right) dr \\ &\quad + \sum_{j=1}^{u-1} b_{j} \int_{0}^{\rho} \frac{\left(\rho - r\right)^{r_{u}-r_{j}-1}}{\Gamma\left(r_{u} - r_{j}\right)} G\left(r, \Re\left(r\right), \Im\left(r\right)\right)) dr, \end{aligned}$$

and

$$\Im\left(\rho\right) = \frac{1}{\Gamma\left(t_{u}\right)} \int_{0}^{\rho} \left(\rho - t\right)^{t_{u}-1} F\left(t, \Im\left(t\right), I_{0+}^{\beta_{u}}\Re\left(t\right)\right) dt$$

$$+\sum_{k=1}^{u-1} c_k \int_0^{\rho} \frac{(\rho-t)^{t_u-t_k-1}}{\Gamma(t_u-t_k)} G(t, \Im(t), \Re(t))\, dt,$$

for all $\rho \in J$.

Proof. Consider that

$$\begin{cases} {}^{R}D_{0+}^{r_u}\Re(\rho) = L(D)G(\rho, \Re(\rho), \Im(\rho)) + F\left(\rho, \Re(\rho), I_{0+}^{\beta_u}\Im(\rho)\right), & \rho \in J = [0, +\infty), \\ {}^{R}D_{0+}^{t_u}\Im(\rho) = \widetilde{L}(D)\widetilde{G}(\rho, \Im(\rho), \Re(\rho)) + \widetilde{F}\left(\rho, \Im(\rho), I_{0+}^{\kappa_u}\Re(\rho)\right), & \rho \in J. \end{cases}$$

Applying the operators $I_{0+}^{r_u}$ and $I_{0+}^{t_u}$, we have

$$I_{0+}^{r_u}\left({}^{R}D_{0+}^{r_u}\Re(\rho)\right) = I_{0+}^{r_u}\left(L(D)G(\rho, \Re(\rho), \Im(\rho)) + F\left(\rho, \Re(\rho), I_{0+}^{\beta_u}\Im(\rho)\right)\right), \tag{3.1}$$

and

$$I_{0+}^{t_u}\left({}^{R}D_{0+}^{t_u}\Im(\rho)\right) = I_{0+}^{t_u}\left(\widetilde{L}(D)\widetilde{G}(\rho, \Im(\rho), \Re(\rho)) + \widetilde{F}\left(\rho, \Im(\rho), I_{0+}^{\kappa_u}\Re(\rho)\right)\right). \tag{3.2}$$

Using Lemma 2.3, Remark 2.2, and the initial condition, it follows from (3.1) that

$$\begin{aligned} \Re(\rho) &= b_{u-1}I_{0+}^{r_u-r_{u-1}}G(\rho, \Re(\rho), \Im(\rho)) + \cdots + b_1 I_{0+}^{r_u-r_1}G(\rho, \Re(\rho), \Im(\rho)) \\ &\quad + I_{0+}^{r_u}F\left(\rho, \Re(\rho), I_{0+}^{\beta_u}\Im(\rho)\right) \\ &= \sum_{j=1}^{u-1} b_j I_{0+}^{r_u-r_j}G(\rho, \Re(\rho), \Im(\rho)) + I_{0+}^{r_u}F\left(\rho, \Re(\rho), I_{0+}^{\beta_u}\Im(\rho)\right) \\ &= \frac{1}{\Gamma(r_u)}\int_0^{\rho}(\rho-r)^{r_u-1}F\left(r, \Re(r), I_{0+}^{\beta_u}\Im(r)\right)dr \\ &\quad + \sum_{j=1}^{u-1} b_j \int_0^{\rho}\frac{(\rho-r)^{r_u-r_j-1}}{\Gamma(r_u-r_j)}G(r, \Re(r), \Im(r))\, dr, \end{aligned}$$

for all $\rho \in J$.

Similarly, for all $\rho \in J$, it follows from (3.2) that

$$\begin{aligned} \Im(\rho) &= \frac{1}{\Gamma(t_u)}\int_0^{\rho}(\rho-t)^{t_u-1}F\left(t, \Re(t), I_{0+}^{\beta_u}\Im(t)\right)dt \\ &\quad + \sum_{k=1}^{u-1} c_k \int_0^{\rho}\frac{(\rho-t)^{t_u-t_k-1}}{\Gamma(t_u-t_k)}G(t, \Re(t), \Im(t))\, dt. \end{aligned}$$

□

The following assumptions are introduced to ensure the existence and uniqueness of solutions for the proposed system and will be used throughout the subsequent analysis:

(A_1) For each $s, s' \in [0, r_u)$, and $w, w' \in [0, t_u)$, there exist differentiable functions $\zeta, \xi : \mho_1 \to J$ such that the functions $\rho^s F(\rho, \zeta, \xi), \rho^w \widetilde{F}(\rho, \xi, \zeta) : J \times \Theta \times \Theta \to \Theta$ and $\rho^{s'} G(\rho, \zeta, \xi), \rho^{w'} \widetilde{G}(\rho, \xi, \zeta) : J \times \Theta \times \Theta \to \Theta$ are continuous.

(A$_2$) For all $\rho \in J$, $\zeta, \xi : \mho_1 \to J$ and $\zeta^*, \xi^* : \mho_2 \to J$, there exist nonnegative continuous functions $B_1(.), B_2(.), ..., B_8(.)$ such that

$$\begin{cases} \left\| s^r \left[F\left(\rho, (1+\rho^\mu)\zeta, (1+\rho^\mu)\xi\right) - F\left(\rho, (1+\rho^\mu)\zeta^*, (1+\rho^\mu)\xi^*\right) \right] \right\| \\ \leq B_1(\rho) \left\| \zeta(\rho) - \zeta^*(\rho) \right\| + B_2(\rho) \left\| \xi(\rho) - \xi^*(\rho) \right\|, \\ \left\| s^w \left[\widetilde{F}\left(\rho, (1+\rho^\mu)\xi, (1+\rho^\mu)\zeta\right) - \widetilde{F}\left(\rho, (1+\rho^\mu)\xi^*, (1+\rho^\mu)\zeta^*\right) \right] \right\| \\ \leq B_3(\rho) \left\| \zeta(\rho) - \zeta^*(\rho) \right\| + B_4(\rho) \left\| \xi(\rho) - \xi^*(\rho) \right\|, \\ \left\| s^{r'} \left[G\left(\rho, (1+\rho^\mu)\zeta, (1+\rho^\mu)\xi\right) - G\left(\rho, (1+\rho^\mu)\zeta^*, (1+\rho^\mu)\xi^*\right) \right] \right\| \\ \leq B_5(\rho) \left\| \zeta(\rho) - \zeta^*(\rho) \right\| + B_6(\rho) \left\| \xi(\rho) - \xi^*(\rho) \right\|, \\ \left\| s^{w'} \left[\widetilde{G}\left(\rho, (1+\rho^\mu)\zeta, (1+\rho^\mu)\xi\right) - \widetilde{G}\left(\rho, (1+\rho^\mu)\zeta^*, (1+\rho^\mu)\xi^*\right) \right] \right\| \\ \leq B_7(\rho) \left\| \zeta(\rho) - \zeta^*(\rho) \right\| + B_8(\rho) \left\| \xi(\rho) - \xi^*(\rho) \right\| \end{cases}$$

where $s, s' \in [0, r_u)$, $w, w' \in [0, t_u)$.

(A$_3$) For all $\rho \in J$, there exist $K_i > 0$ $(i = 1, 2, \cdots, 8)$ with $\sum_{i=1}^{8} K_i \in (0,1)$ such that

$$\begin{cases} \sup\limits_{\rho \in J} \frac{1}{\Gamma(r_u)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-1}}{r^s} B_1(r)\, dr \leq K_1, \\ \sup\limits_{\rho \in J} \frac{1}{\Gamma(t_u)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-t)^{t_u-1}}{t^s} B_3(t)\, dt \leq K_2, \\ \sup\limits_{\rho \in J} \frac{1}{\Gamma(r_u)\Gamma(\beta_u+1)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-1}}{r^{s-\beta_u}} B_2(r)\, dr \leq K_3, \\ \sup\limits_{\rho \in J} \frac{1}{\Gamma(t_u)\Gamma(\kappa_u+1)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-t)^{t_u-1}}{t^{s-\kappa_u}} B_4(t)\, dt \leq K_4, \\ \sup\limits_{\rho \in J} \sum\limits_{j=1}^{u-1} |b_j| \frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}} B_5(r)\, dr \leq K_5, \\ \sup\limits_{\rho \in J} \sum\limits_{k=1}^{u-1} |c_k| \frac{1}{\Gamma(t_u-t_k)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-t)^{t_u-t_k-1}}{t^{s'}} B_7(t)\, dt \leq K_6, \\ \sup\limits_{\rho \in J} \sum\limits_{j=1}^{u-1} |b_j| \frac{1}{\Gamma(r_u-r_j)\Gamma(\beta_u+1)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-r_j-1}}{r^{s-\beta_u}} B_6(r)\, dr \leq K_7, \\ \sup\limits_{\rho \in J} \sum\limits_{k=1}^{u-1} |c_k| \frac{1}{\Gamma(t_u-t_k)\Gamma(\kappa_u+1)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-t)^{t_u-t_k-1}}{t^{s-\kappa_u}} B_8(t)\, dt \leq K_8. \end{cases}$$

(A$_4$) For all $\rho \in J$, there exist $L_k > 0$ $(k = 1, 2, \cdots, 4)$ such that

$$\begin{cases} \sup\limits_{\rho \in J} \frac{1}{\Gamma(r_u)(1+\rho^\mu)} \int_0^\rho (\rho-r)^{r_u-1} \left\| F(\rho,0,0) \right\| dr \leq L_1 < \infty, \\ \sup\limits_{\rho \in J} \frac{1}{\Gamma(t_u)(1+\rho^\mu)} \int_0^\rho (\rho-r)^{t_u-1} \left\| \widetilde{F}(\rho,0,0) \right\| dr \leq L_2 < \infty, \\ \sup\limits_{\rho \in J} \sum\limits_{j=1}^{u-1} |b_j| \frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)} \int_0^\rho (\rho-r)^{r_u-r_j-1} \left\| G(\rho,0,0) \right\| dr \leq L_3 < \infty, \\ \sup\limits_{\rho \in J} \sum\limits_{k=1}^{u-1} |c_k| \frac{1}{\Gamma(t_u-t_k)(1+\rho^\mu)} \int_0^\rho (\rho-r)^{t_u-t_k-1} \left\| \widetilde{G}(\rho,0,0) \right\| dr \leq L_4 < \infty. \end{cases}$$

We now present our main result in this section.

Theorem 3.2. *Under the hypotheses (A$_1$)-(A$_4$), the coupled mixed-order FDEs (1.2) has a unique solution.*

Proof. By the integral representation of problem (1.2), established in Lemma 3.1, we define the operator the operator $\widehat{\Upsilon}:\mho\rightarrow\mho$ as $\widehat{\Upsilon}\left(\Re,\Im\right)\left(\rho\right)=\left(\Upsilon_{1}\left(\Re,\Im\right)\left(\rho\right),\Upsilon_{2}\left(\Re,\Im\right)\left(\rho\right)\right)$, where

$$\begin{aligned}\Upsilon_{1}\left(\Re\left(\rho\right),\Im\left(\rho\right)\right) &= \frac{1}{\Gamma\left(r_{u}\right)}\int_{0}^{\rho}\left(\rho-r\right)^{r_{u}-1}F\left(r,\Re\left(r\right),I_{0^{+}}^{\beta_{u}}\Im\left(r\right)\right)dr\\ &\quad+\sum_{j=1}^{u-1}b_{j}\int_{0}^{\rho}\frac{\left(\rho-r\right)^{r_{u}-r_{j}-1}}{\Gamma\left(r_{u}-r_{j}\right)}G\left(r,\Re\left(r\right),\Im\left(r\right)\right))\,dr, \end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\Upsilon_{2}\left(\Im\left(\rho\right),\Re\left(\rho\right)\right) &= \frac{1}{\Gamma\left(t_{u}\right)}\int_{0}^{\rho}\left(\rho-t\right)^{t_{u}-1}F\left(t,\Im\left(t\right),I_{0^{+}}^{\beta_{u}}\Re\left(t\right)\right)dt\\ &\quad+\sum_{k=1}^{u-1}c_{k}\int_{0}^{\rho}\frac{\left(\rho-t\right)^{t_{u}-t_{k}-1}}{\Gamma\left(t_{u}-t_{k}\right)}G\left(t,\Im\left(t\right),\Re\left(t\right)\right))\,dt, \end{aligned}\tag{3.4}$$

for all $\rho\in J$ and all $\left(\Re\left(\rho\right),\Im\left(\rho\right)\right),\left(\Im\left(\rho\right),\Re\left(\rho\right)\right)\in\mho$. Taking the norm on (3.3), we have

$$\begin{aligned}\left\Vert\Upsilon_{1}\left(\Re,\Im\right)\right\Vert_{\mho_{1}} &\leq \frac{1}{\Gamma\left(r_{u}\right)}\int_{0}^{\rho}\left(\rho-r\right)^{r_{u}-1}\left\Vert F\left(r,\Re\left(r\right),I_{0^{+}}^{\beta_{u}}\Im\left(r\right)\right)\right\Vert dr\\ &\quad+\sum_{j=1}^{u-1}\left|b_{j}\right|\frac{1}{\Gamma\left(r_{u}-r_{j}\right)}\int_{0}^{\rho}\left(\rho-r\right)^{r_{u}-r_{j}-1}\left\Vert G\left(r,\Re\left(r\right),\Im\left(r\right)\right))\right\Vert dr, \end{aligned}$$

that is,

$$\begin{aligned}\frac{\left\Vert\Upsilon\left(\Re,\Im\right)\right\Vert_{\mho_{1}}}{\left(1+\rho^{\mu}\right)} &\leq \frac{1}{\Gamma\left(r_{u}\right)\left(1+\rho^{\mu}\right)}\int_{0}^{\rho}\frac{\left(\rho-r\right)^{r_{u}-1}}{r^{s}}\\ &\quad\times\left\Vert r^{s}\left[F\left(r,\Re\left(r\right),I_{0^{+}}^{\beta_{u}}\Im\left(r\right)\right)-F\left(r,0,0\right)+F\left(r,0,0\right)\right]\right\Vert dr\\ &\quad+\sum_{j=1}^{u-1}\left|b_{j}\right|\frac{1}{\Gamma\left(r_{u}-r_{j}\right)\left(1+\rho^{\mu}\right)}\int_{0}^{\rho}\frac{\left(\rho-r\right)^{r_{u}-r_{j}-1}}{r^{s'}}\\ &\quad\times\left\Vert r^{s'}\left[G\left(r,\Re\left(r\right),\Im\left(r\right)\right))-G(r,0,0)+G(r,0,0))\right]\right\Vert dr,\\ &\leq \frac{1}{\Gamma\left(r_{u}\right)\left(1+\rho^{\mu}\right)}\int_{0}^{\rho}\frac{\left(\rho-r\right)^{r_{u}-1}}{r^{s}}\left(B_{1}\left(\rho\right)\frac{\left\Vert\Re\left(r\right)\right\Vert}{\left(1+r^{\mu}\right)}+B_{2}\left(\rho\right)\frac{\left\Vert I_{0^{+}}^{\beta_{u}}\Im\left(r\right)\right\Vert}{\left(1+r^{\mu}\right)}\right)dr\\ &\quad+\frac{1}{\Gamma\left(r_{u}\right)\left(1+\rho^{\mu}\right)}\int_{0}^{\rho}\left(\rho-r\right)^{r_{u}-1}\left\Vert F\left(r,0,0\right)\right\Vert dr\\ &\quad+\sum_{j=1}^{u-1}\left|b_{j}\right|\frac{1}{\Gamma\left(r_{u}-r_{j}\right)\left(1+\rho^{\mu}\right)}\\ &\quad\times\int_{0}^{\rho}\frac{\left(\rho-r\right)^{r_{u}-r_{j}-1}}{r^{s'}}\left(B_{5}\left(\rho\right)\frac{\left\Vert\Re\left(r\right)\right\Vert}{\left(1+r^{\mu}\right)}+B_{6}\left(\rho\right)\frac{\left\Vert\Im\left(r\right)\right\Vert}{\left(1+r^{\mu}\right)}\right)dr\\ &\quad+\sum_{j=1}^{u-1}\left|b_{j}\right|\frac{1}{\Gamma\left(r_{u}-r_{j}\right)\left(1+\rho^{\mu}\right)}\int_{0}^{\rho}\left(\rho-r\right)^{r_{u}-r_{j}-1}\left\Vert G\left(r,0,0\right)\right\Vert dr. \end{aligned}\tag{3.5}$$

It should be noted that,

$$\frac{\left\Vert I_{0^{+}}^{\beta_{u}}\Im\left(r\right)\right\Vert}{\left(1+r^{\mu}\right)} \leq \frac{1}{\Gamma\left(\beta_{u}\right)\left(1+r^{\mu}\right)}\int_{0}^{r}\left(r-\varpi\right)^{\beta_{u}-1}\left\Vert\Im\left(\varpi\right)\right\Vert d\varpi$$

$$
\begin{aligned}
&\leq \frac{\|\Im\|_{\mho_2}}{\Gamma\left(\beta_u\right)} \int_0^r (r-\varpi)^{\beta_u-1} \frac{1+\varpi^\mu}{1+r^\mu} d\varpi \\
&\leq \frac{\|\Im\|_{\mho_2}}{\Gamma\left(\beta_u\right)} \int_0^r (r-\varpi)^{\beta_u-1} \\
&= \frac{r^{\beta_u}\|\Im\|}{\Gamma\left(\beta_u+1\right)}. \qquad (3.6)
\end{aligned}
$$

Applying (3.6) in (3.5), and using conditions (A$_3$) and (A$_4$), we can write

$$
\begin{aligned}
\frac{\|\Upsilon(\Re,\Im)\|_{\mho_1}}{(1+\rho^\mu)} \leq\ & \frac{1}{\Gamma(r_u)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-1}}{r^s(1+r^\mu)} B_1(\rho)\|\Re\|_{\mho_1} dr \\
&+\frac{1}{\Gamma(r_u)(1+\rho^\mu)\Gamma(\beta_u+1)} \int_0^\rho \frac{(\rho-r)^{r_u-1}}{r^{s-\beta_u}(1+r^\mu)} B_2(\rho)\|\Im\|_{\mho_2} dr + L_1 \\
&+\sum_{j=1}^{u-1}|b_j| \frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}(1+r^\mu)} B_5(\rho)\|\Re\|_{\mho_1} dr \\
&+\sum_{j=1}^{u-1}|b_j| \frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}(1+r^\mu)} B_6(\rho)\|\Im\|_{\mho_2} dr + L_3 \\
\leq\ & \frac{1}{\Gamma(r_u)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-1}}{r^s} B_1(\rho)\|\Re\|_{\mho_1} dr \\
&+\frac{1}{\Gamma(r_u)(1+\rho^\mu)\Gamma(\beta_u+1)} \int_0^\rho \frac{(\rho-r)^{r_u-1}}{r^{s-\beta_u}} B_2(\rho)\|\Im\|_{\mho_2} dr + L_1 \\
&+\sum_{j=1}^{u-1}|b_j| \frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}} B_5(\rho)\|\Re\|_{\mho_1} dr \\
&+\sum_{j=1}^{u-1}|b_j| \frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)} \int_0^\rho \frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}} B_6(\rho)\|\Im\|_{\mho_2} dr + L_3 \\
\leq\ & K_1\|\Re\|_{\mho_1} + K_3\|\Im\|_{\mho_2} + K_5\|\Re\|_{\mho_1} + K_7\|\Im\|_{\mho_2} + L_1 + L_3 \\
=\ & (K_1+K_5)\|\Re\|_{\mho_1} + (K_3+K_7)\|\Im\|_{\mho_2} + L_1 + L_3 < \infty.
\end{aligned}
$$

Similarly, one can write

$$
\frac{\|\Upsilon(\Im,\Re)\|_{\mho_2}}{(1+\rho^\mu)} \leq (K_2+K_6)\|\Im\|_{\mho_2} + (K_4+K_8)\|\Re\|_{\mho_1} + L_2 + L_4 < \infty.
$$

Hence,

$$
\begin{aligned}
&\frac{\left\|\widehat{\Upsilon}(\Im,\Re)\right\|_{\mho_2}}{(1+\rho^\mu)} \\
=\ & \frac{\|\Upsilon(\Re,\Im)\|_{\mho_1}}{(1+\rho^\mu)} + \frac{\|\Upsilon(\Im,\Re)\|_{\mho_2}}{(1+\rho^\mu)} \\
\leq\ & (K_1+K_4+K_5+K_8)\|\Re\|_{\mho_1} + (K_2+K_3+K_6+K_7)\|\Im\|_{\mho_2} + L_1 + L_3 + L_2 + L_4 < \infty.
\end{aligned}
$$

Therefore, for all $(\Re(\rho), \Im(\rho)) \in \mho$, we have $\widehat{\Upsilon}(\Re,\Im) \in \mho$.

Now, we prove that the mapping $\widehat{\Upsilon}: \mho \to \mho$ is a contraction mapping in $\mho$. Indeed, for all $\rho \in J$, assume that $(\Re,\Im), (\Xi,\hbar) \in \mho$, then, we have

$$
\frac{\|\Upsilon(\Re,\Im)(\rho) - \Upsilon(\Xi,\hbar)(\rho)\|_{\mho_1}}{(1+\rho^\mu)}
$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(r_u)(1+\rho^\mu)}\int_0^\rho(\rho-r)^{r_u-1}\left\|F\left(r,\Re(r),I_{0+}^{\beta_u}\Im(r)\right)-F\left(r,\Xi(r),I_{0+}^{\beta_u}\hbar(r)\right)\right\|dr\\
&\quad+\sum_{j=1}^{u-1}|b_j|\frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)}\int_0^\rho(\rho-r)^{r_u-r_j-1}\\
&\quad\times\|G(r,\Re(r),\Im(r)))-G(r,\Xi(r),\hbar(r)))\|\,dr\\
&\leq \frac{1}{\Gamma(r_u)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-1}}{r^s}\left\|r^s\left[F\left(r,\Re(r),I_{0+}^{\beta_u}\Im(r)\right)-F\left(r,\Xi(r),I_{0+}^{\beta_u}\hbar(r)\right)\right]\right\|dr\\
&\quad+\sum_{j=1}^{u-1}|b_j|\frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}}\\
&\quad\times\left\|r^{s'}\left[G(r,\Re(r),\Im(r)))-G(r,\Xi(r),\hbar(r)))\right]\right\|dr.
\end{aligned}$$

Applying hypotheses (A_1) and (A_2), we have

$$\begin{aligned}
\frac{\|\Upsilon(\Re,\Im)(\rho)-\Upsilon(\Xi,\hbar)(\rho)\|}{(1+\rho^\mu)}\leq\;&\frac{1}{\Gamma(r_u)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-1}}{r^s}\\
&\times\left[B_1(r)\frac{\|\Re(r)-\Xi(r)\|}{(1+r^\mu)}+B_2(r)\frac{\left\|I_{0+}^{\beta_u}\Im(r)-I_{0+}^{\beta_u}\hbar(r)\right\|}{(1+r^\mu)}\right]dr\\
&+\sum_{j=1}^{u-1}|b_j|\frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}}\\
&\times\left[B_5(r)\frac{\|\Re(r)-\Xi(r)\|}{(1+r^\mu)}+B_6(r)\frac{\|\Im(r)-\hbar(r)\|}{(1+r^\mu)}\right]dr. \qquad (3.7)
\end{aligned}$$

Using the fact (3.6), we deduce that

$$\frac{\left\|I_{0+}^{\beta_u}\Im(r)-I_{0+}^{\beta_u}\hbar(r)\right\|}{(1+r^\mu)}=\frac{\left\|I_{0+}^{\beta_u}[\Im(r)-\hbar(r)]\right\|}{(1+r^\mu)}\leq\frac{r^{\beta_u}\|\Im-\hbar\|_{\mho_2}}{\Gamma(\beta_u+1)}. \qquad (3.8)$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}
\frac{\|\Upsilon(\Re,\Im)(\rho)-\Upsilon(\Xi,\hbar)(\rho)\|_{\mho_1}}{(1+\rho^\mu)}\leq\;&\frac{\|\Re-\Xi\|_{\mho_1}}{\Gamma(r_u)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-1}}{r^s}B_1(r)\,dr\\
&+\frac{\|\Im-\hbar\|_{\mho_2}}{\Gamma(r_u)\Gamma(\beta_u+1)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-1}}{r^{s-\beta_u}}B_2(r)\,d(r)\\
&+\sum_{j=1}^{u-1}|b_j|\frac{\|\Re-\Xi\|_{\mho_1}}{\Gamma(r_u-r_j)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}(1+r^\mu)}B_5(r)\,dr\\
&+\sum_{j=1}^{u-1}|b_j|\frac{\|\Im-\hbar\|_{\mho_2}}{\Gamma(r_u-r_j)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}(1+r^\mu)}B_6(r)\,dr\\
\leq\;&\frac{\|\Re-\Xi\|_{\mho_1}}{\Gamma(r_u)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-1}}{r^s}B_1(r)\,dr\\
&+\frac{\|\Im-\hbar\|_{\mho_2}}{\Gamma(r_u)\Gamma(\beta_u+1)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-1}}{r^{s-\beta_u}}B_2(r)\,d(r)\\
&+\sum_{j=1}^{u-1}|b_j|\frac{\|\Re-\Xi\|_{\mho_1}}{\Gamma(r_u-r_j)(1+\rho^\mu)}\int_0^\rho\frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}}B_5(r)\,dr
\end{aligned}$$

$$+\sum_{j=1}^{u-1}|b_j|\,\frac{\|\Im-\hbar\|_{\mho_2}}{\Gamma\left(r_u-r_j\right)\left(1+\rho^{\mu}\right)}\int_0^{\rho}\frac{\left(\rho-r\right)^{r_u-r_j-1}}{r^{s'}}B_6\left(r\right)dr.$$

From the hypotheses (A_3) and (A_4), we get

$$\frac{\left\|\Upsilon\left(\Re,\Im\right)\left(\rho\right)-\Upsilon\left(\Xi,\hbar\right)\left(\rho\right)\right\|_{\mho_1}}{\left(1+\rho^{\mu}\right)}\leq\left(K_1+K_5\right)\left\|\Re-\Xi\right\|_{\mho_1}+\left(K_3+K_7\right)\left\|\Im-\hbar\right\|_{\mho_2}.$$

Similarly, one can write

$$\frac{\left\|\Upsilon\left(\Im,\Re\right)\left(\rho\right)-\Upsilon\left(\hbar,\Xi\right)\left(\rho\right)\right\|_{\mho_2}}{\left(1+\rho^{\mu}\right)}\leq\left(K_2+K_6\right)\left\|\Im-\hbar\right\|_{\mho_2}+\left(K_4+K_8\right)\left\|\Re-\Xi\right\|_{\mho_1}.$$

Hence,

$$\begin{aligned}
&\frac{\left\|\widehat{\Upsilon}\left(\Re,\Im\right)\left(\rho\right)-\widehat{\Upsilon}\left(\Xi,\hbar\right)\left(\rho\right)\right\|_{\mho}}{\left(1+\rho^{\mu}\right)}\\
=&\ \frac{\left\|\Upsilon\left(\Re,\Im\right)\left(\rho\right)-\Upsilon\left(\Xi,\hbar\right)\left(\rho\right)\right\|_{\mho_1}}{\left(1+\rho^{\mu}\right)}+\frac{\left\|\Upsilon\left(\Re,\Im\right)\left(\rho\right)-\Upsilon\left(\Xi,\hbar\right)\left(\rho\right)\right\|_{\mho_1}}{\left(1+\rho^{\mu}\right)}\\
=&\ \left(K_1+K_5+K_4+K_8\right)\left\|\Re-\Xi\right\|_{\mho_1}+\left(K_3+K_7+K_2+K_6\right)\left\|\Im-\hbar\right\|_{\mho_2}\\
\leq&\ \sum_{i=1}^{8}K_i\left\|\left(\Re,\Im\right)-\left(\Xi,\hbar\right)\right\|_{\mho}.
\end{aligned}$$

Because $\sum_{i=1}^{8}K_i<1$, the operator $\widehat{\Upsilon}$ is a contraction. Consequently, the Banach FP theorem guarantees a unique FP of $\widehat{\Upsilon}$, and hence a unique solution to the considered problem. □

4 An illustrative example

This section is devoted to a specific example that illustrates and validates our previously obtained results, offering clearer insight into their implications.

Consider the following problem:

$$\begin{cases}
{}^{R}D_{0^+}^{r_3}\Re\left(\rho\right)-L(D)\left(\frac{\rho^{\frac{1}{3}}}{\ln(81)}\cos\left(\Re\left(\rho\right)+\Im\left(\rho\right)\right)\right)=\frac{\rho^{\frac{1}{2}}}{9}\sin\left(\Re\left(\rho\right)\right)+\frac{\rho^{\frac{1}{20}}}{e^2}\tan^{-1}\left(I_{0^+}^{\beta_3}\Im\left(\rho\right)\right)+\rho^{\frac{1}{8}},\\
{}^{R}D_{0^+}^{t_3}\Im\left(\rho\right)-\widetilde{L}(D)\left(\frac{\rho^{\frac{1}{3}}}{\ln(1000)}\cos\left(\Re\left(\rho\right)+\Im\left(\rho\right)\right)\right)=\frac{\rho^{\frac{1}{2}}}{10}\sin\left(\Im\left(\rho\right)\right)+\frac{\rho^{\frac{1}{20}}}{e^2}\tan^{-1}\left(I_{0^+}^{\kappa_3}\Re\left(\rho\right)\right)+\rho^{\frac{1}{9}},\\
\rho^{1-r_3}\Re\left(\rho\right)\big|_{\rho=0}=0,\quad \rho^{1-t_3}\Im\left(\rho\right)\big|_{\rho=0}=0,\\
L(D)=\sum_{j=1}^{2}b_j\ {}^{R}D_{0^+}^{r_j},\ 0<r_1<r_2<r_3<1,\ b_1,b_2\in\mathbb{R},\\
\widetilde{L}(D)=\sum_{k=1}^{2}c_k\ {}^{R}D_{0^+}^{t_k},\ 0<t_1<t_2<t_3<1,\ c_1,c_2\in\mathbb{R},\\
r_j+\beta_u<r_u,\ t_k+\kappa_u<t_u,\ j,k=1,2,
\end{cases}\tag{4.1}$$

for all $\rho\in[0,+\infty)$, where $u=3$, $r_1=\frac{1}{27}$, $r_2=\frac{1}{9}$, $r_3=\frac{1}{3}$, $t_1=\frac{1}{16}$, $t_2=\frac{1}{8}$, $t_3=\frac{1}{4}$, $b_1=2$, $b_2=1$, $c_1=c_2=1$, $\beta_3=\frac{1}{9}$, $\kappa_3=\frac{1}{10}$. Also, we consider, $\mu=3$, $s=\frac{1}{4}$, $w=\frac{1}{2}$, $s'=\frac{1}{8}$, $w'=\frac{1}{4}$.

Clearly,

$$\begin{cases} G\left(\rho,\Re\left(\rho\right),\Im\left(\rho\right)\right))=\frac{\rho^{\frac{1}{3}}}{\ln(81)}\cos\left(\Re\left(\rho\right)+\Im\left(\rho\right)\right), \\ F\left(\rho,\Re\left(\rho\right),I_{0^+}^{\beta_u}\Im\left(\rho\right)\right)=\frac{\rho^{\frac{1}{2}}}{9}\sin\left(\Re\left(\rho\right)\right)+\frac{\rho^{\frac{1}{20}}}{\rho^2}\tan^{-1}\left(I_{0^+}^{\beta_3}\Im\left(\rho\right)\right)+\rho^{\frac{1}{8}} \\ \widetilde{G}\left(\rho,\Im\left(\rho\right),\Re\left(\rho\right)\right))=\frac{\rho^{\frac{1}{3}}}{\ln(1000)}\cos\left(\Re\left(\rho\right)+\Im\left(\rho\right)\right), \\ \widetilde{F}\left(\rho,\Im\left(\rho\right),I_{0^+}^{\kappa_u}\Re\left(\rho\right)\right)=\frac{\rho^{\frac{1}{2}}}{10}\sin\left(\Im\left(\rho\right)\right)+\frac{\rho^{\frac{1}{20}}}{\rho^2}\tan^{-1}\left(I_{0^+}^{\kappa_3}\Re\left(\rho\right)\right)+\rho^{\frac{1}{9}}. \end{cases}$$

Hence, $B_1\left(\rho\right)=\frac{\rho^{\frac{7}{12}}}{9}$, $B_2\left(\rho\right)=\frac{\rho^{\frac{3}{10}}}{e^2}$, $B_3\left(\rho\right)=\frac{\rho^{\frac{5}{6}}}{9}$, $B_4\left(\rho\right)=\frac{\rho^{\frac{11}{20}}}{e^2}$, $B_5\left(\rho\right)=B_6\left(\rho\right)=\frac{\rho^{\frac{11}{24}}}{\ln(81)}$, and $B_7\left(\rho\right)=B_8\left(\rho\right)=\frac{\rho^{\frac{7}{12}}}{\ln(1000)}$.

Therefore,

$$\begin{aligned} \frac{1}{\Gamma(r_u)(1+\rho^{\mu})}\int_0^{\rho}\frac{\left(\rho-r\right)^{r_u-1}}{r^s}B_1\left(r\right)dr &= \frac{1}{\Gamma(r_3)(1+\rho^3)}\int_0^{\rho}\frac{\left(\rho-r\right)^{r_3-1}}{r^{\frac{1}{4}}}\frac{r^{\frac{7}{12}}}{9}dr \\ &= \frac{1}{9\Gamma(\frac{1}{3})(1+\rho^3)}\int_0^{\rho}\left(\rho-r\right)^{\frac{1}{3}-1}r^{\frac{1}{3}}dr \\ &= \frac{\Gamma\left(\frac{1}{3}+1\right)}{9(1+\rho^3)\Gamma(\frac{1}{3}+\frac{1}{3}+1)}\rho^{\frac{1}{3}+\frac{1}{3}} \\ &= \frac{\Gamma\left(\frac{1}{3}+1\right)}{9\Gamma(\frac{1}{3}+\frac{1}{3}+1)}\frac{\rho^{\frac{2}{3}}}{1+\rho^3} \\ &\leq 0.10985\frac{\rho^{\frac{2}{3}}}{1+\rho^3}, \end{aligned}$$

it follows that

$$\sup_{\rho\in[0,+\infty)}\left(0.10985\frac{\rho^{\frac{2}{3}}}{1+\rho^3}\right)\leq 0.10985\times 0.53\approx 0.0582205=K_1.$$

Also,

$$\begin{aligned} \frac{1}{\Gamma(t_u)(1+\rho^{\mu})}\int_0^{\rho}\frac{\left(\rho-t\right)^{t_u-1}}{t^s}B_3\left(t\right)dt &= \frac{1}{9\Gamma(\frac{1}{4})(1+\rho^3)}\int_0^{\rho}\left(\rho-t\right)^{\frac{1}{4}-1}t^{\frac{7}{12}}dt \\ &= \frac{\Gamma\left(\frac{7}{12}+1\right)}{9\Gamma\left(\frac{1}{4}+\frac{7}{12}+1\right)}\frac{\rho^{\frac{5}{6}}}{(1+\rho^3)} \\ &\leq 0.10533\frac{\rho^{\frac{5}{6}}}{(1+\rho^3)}, \end{aligned}$$

which implies that

$$\sup_{\rho\in[0,+\infty)}\left(0.10533\frac{\rho^{\frac{5}{6}}}{(1+\rho^3)}\right)\leq 0.10533\times 0.51\approx 0.0537184=K_2$$

Again,

$$\begin{aligned} &\frac{1}{\Gamma(r_u)\Gamma\left(\beta_u+1\right)\left(1+\rho^{\mu}\right)}\int_0^{\rho}\frac{\left(\rho-r\right)^{r_u-1}}{r^{s-\beta_u}}B_2\left(r\right)dr \\ = \; &\frac{1}{\Gamma(\frac{1}{3})\Gamma\left(\frac{1}{9}+1\right)\left(1+\rho^3\right)}\int_0^{\rho}\frac{\left(\rho-r\right)^{\frac{1}{3}-1}}{r^{\frac{1}{4}-\frac{1}{9}}}\frac{r^{\frac{3}{10}}}{e^2}dr \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{e^2\Gamma(\frac{1}{3})\Gamma\left(\frac{1}{9}+1\right)(1+\rho^3)}\int_0^\rho (\rho-r)^{\frac{1}{3}-1}\, r^{\frac{29}{180}} dr \\
&= \frac{\Gamma\left(\frac{29}{180}+1\right)}{e^2\Gamma(\frac{1}{3}+\frac{29}{180}+1)\Gamma\left(\frac{1}{9}+1\right)}\frac{\rho^{\frac{89}{180}}}{(1+\rho^3)} \leq 0.14992\frac{\rho^{\frac{89}{180}}}{(1+\rho^3)}.
\end{aligned}
$$

Hence,

$$
\sup_{\rho\in[0,+\infty)}\left(0.14992\frac{\rho^{\frac{89}{180}}}{(1+\rho^3)}\right) \leq 0.14992\times 0.52 \approx 0.0779596 = K_3.
$$

To evalute K_4, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(t_u)\Gamma\left(\kappa_u+1\right)(1+\rho^\mu)}\int_0^\rho \frac{(\rho-t)^{t_u-1}}{t^{s-\kappa_u}} B_4\left(t\right) dt \\
&= \frac{1}{e^2\Gamma(\frac{1}{4})\Gamma\left(\frac{1}{10}+1\right)(1+\rho^3)}\int_0^\rho (\rho-t)^{\frac{1}{4}-1}\, t^{\frac{2}{5}} dt \\
&= \frac{\Gamma\left(\frac{2}{5}+1\right)}{e^2\Gamma\left(\frac{1}{4}+\frac{2}{5}+1\right)}\frac{\rho^{\frac{13}{20}}}{(1+\rho^3)} \leq 0.13340\frac{\rho^{\frac{13}{20}}}{(1+\rho^3)},
\end{aligned}
$$

taking $\sup_{\rho\in[0,+\infty)}$, we get

$$
\sup_{\rho\in[0,+\infty)}\left(0.13340\frac{\rho^{\frac{13}{20}}}{(1+\rho^3)}\right) \leq 0.13340\times 0.54 \approx 0.0720377 = K_4.
$$

To estimate K_5, we can write

$$
\begin{aligned}
&\sum_{j=1}^{u-1}|b_j|\frac{1}{\Gamma(r_u-r_j)(1+\rho^\mu)}\int_0^\rho \frac{(\rho-r)^{r_u-r_j-1}}{r^{s'}} B_5\left(r\right) dr \\
&= \sum_{j=1}^{2}|b_j|\frac{1}{\Gamma(r_3-r_j)(1+\rho^3)}\int_0^\rho \frac{(\rho-r)^{r_3-r_j-1}}{r^{s'}}\frac{r^{\frac{11}{24}}}{\ln(81)} dr \\
&= \frac{b_1}{\Gamma(r_3-r_1)(1+\rho^3)}\int_0^\rho \frac{(\rho-r)^{r_3-r_1-1}}{r^{s'}}\frac{r^{\frac{11}{24}}}{\ln(81)} dr \\
&\quad + b_2\frac{1}{\Gamma(r_3-r_2)(1+\rho^3)}\int_0^\rho \frac{(\rho-r)^{r_3-r_2-1}}{r^{s'}}\frac{r^{\frac{11}{24}}}{\ln(81)} dr \\
&= \frac{2}{\Gamma(\frac{1}{3}-\frac{1}{27})(1+\rho^3)}\int_0^\rho \frac{(\rho-r)^{\frac{1}{3}-\frac{1}{27}-1}}{r^{\frac{1}{8}}}\frac{r^{\frac{11}{24}}}{\ln(81)} dr \\
&\quad + \frac{1}{\Gamma(\frac{1}{3}-\frac{1}{9})(1+\rho^3)}\int_0^\rho \frac{(\rho-r)^{\frac{1}{3}-\frac{1}{9}-1}}{r^{\frac{1}{8}}}\frac{r^{\frac{11}{24}}}{\ln(81)} dr \\
&= \frac{2}{\Gamma(\frac{8}{27})(1+\rho^3)\ln(81)}\int_0^\rho (\rho-r)^{\frac{8}{27}-1}\, r^{\frac{1}{3}} dr + \frac{1}{\Gamma(\frac{2}{9})(1+\rho^3)\ln(81)}\int_0^\rho (\rho-r)^{\frac{2}{9}-1}\, r^{\frac{1}{3}} dr \\
&= \frac{2\Gamma(\frac{1}{3}+1)}{\ln(81)\Gamma(\frac{8}{27}+\frac{1}{3}+1)}\frac{\rho^{\frac{17}{27}}}{(1+\rho^3)} + \frac{\Gamma(\frac{1}{3}+1)}{\ln(81)\Gamma(\frac{2}{9}+\frac{1}{3}+1)}\frac{\rho^{\frac{5}{9}}}{(1+\rho^3)} \\
&\leq 0.45298\frac{\rho^{\frac{17}{27}}}{(1+\rho^3)} + 0.22850\frac{\rho^{\frac{5}{9}}}{(1+\rho^3)} = 0.68148\frac{\rho^{\frac{17}{27}}}{(1+\rho^3)},
\end{aligned}
$$

it follows that

$$
\sup_{\rho\in[0,+\infty)}\left(0.68148\frac{\rho^{\frac{17}{27}}}{(1+\rho^3)}\right) \leq 0.13340\times 0.55 \approx 0.07337 = K_5.
$$

To evaluate K_6, we obtain that

$$\begin{aligned}
&\sum_{k=1}^{2}|c_k|\frac{1}{\Gamma(t_u-t_k)(1+\rho^{\mu})}\int_0^{\rho}\frac{(\rho-t)^{t_u-t_k-1}}{t^{s'}}B_7\left(t\right)dt\\
=\;&\frac{c_1}{\Gamma(t_3-t_1)(1+\rho^3)}\int_0^{\rho}\frac{(\rho-t)^{t_3-t_1-1}}{t^{s'}}B_7\left(t\right)dt+\frac{c_2}{\Gamma(t_3-t_2)(1+\rho^3)}\int_0^{\rho}\frac{(\rho-t)^{t_3-t_2-1}}{t^{s'}}B_7\left(t\right)dt\\
=\;&\frac{1}{\Gamma(\frac{1}{4}-\frac{1}{16})(1+\rho^3)}\int_0^{\rho}\frac{(\rho-t)^{\frac{1}{4}-\frac{1}{16}-1}}{t^{\frac{1}{8}}}\frac{t^{\frac{7}{12}}}{\ln(1000)}dt\\
&+\frac{1}{\Gamma(\frac{1}{4}-\frac{1}{8})(1+\rho^3)}\int_0^{\rho}\frac{(\rho-t)^{\frac{1}{4}-\frac{1}{8}-1}}{t^{\frac{1}{8}}}\frac{t^{\frac{7}{12}}}{\ln(1000)}dt\\
=\;&\frac{\Gamma\left(\frac{11}{24}+1\right)}{\ln(1000)\Gamma(\frac{3}{16}+\frac{11}{24}+1)}\frac{\rho^{\frac{11}{24}}}{(1+\rho^3)}+\frac{\Gamma\left(\frac{11}{24}+1\right)}{\ln(1000)\Gamma(\frac{1}{8}+\frac{11}{24}+1)}\frac{\rho^{\frac{11}{24}}}{(1+\rho^3)}\\
=\;&0.27489\frac{\rho^{\frac{11}{24}}}{(1+\rho^3)},
\end{aligned}$$

taking $\sup_{\rho\in[0,+\infty)}$, we get

$$\sup_{\rho\in[0,+\infty)}\left(0.27489\frac{\rho^{\frac{11}{24}}}{(1+\rho^3)}\right)\leq 0.27489\times 0.51\approx 0.0785074=K_6.$$

Also, we can estimate K_7 as follows:

$$\begin{aligned}
&\sum_{j=1}^{2}|b_j|\frac{1}{\Gamma(r_u-r_j)\Gamma\left(\beta_u+1\right)(1+\rho^{\mu})}\int_0^{\rho}\frac{(\rho-r)^{r_u-r_j-1}}{r^{s-\beta_u}}B_6\left(r\right)dr\\
=\;&\frac{b_1}{\Gamma(r_3-r_1)\Gamma\left(\beta_3+1\right)(1+\rho^3)}\int_0^{\rho}\frac{(\rho-r)^{r_3-r_1-1}}{r^{s-\beta_3}}\frac{r^{\frac{7}{12}}}{\ln(1000)}dr\\
&+\frac{b_2}{\Gamma(r_3-r_2)\Gamma\left(\beta_3+1\right)(1+\rho^3)}\int_0^{\rho}\frac{(\rho-r)^{r_3-r_2-1}}{r^{s-\beta_3}}\frac{r^{\frac{7}{12}}}{\ln(1000)}dr\\
=\;&\frac{2}{\ln(1000)\Gamma(\frac{8}{27})\Gamma\left(\frac{1}{9}+1\right)(1+\rho^3)}\int_0^{\rho}(\rho-r)^{\frac{1}{3}-\frac{1}{27}-1}r^{\frac{4}{9}}dr\\
&+\frac{1}{\ln(1000)\Gamma(\frac{2}{9})\Gamma\left(\frac{1}{9}+1\right)(1+\rho^3)}\int_0^{\rho}(\rho-r)^{\frac{1}{3}-\frac{1}{9}-1}r^{\frac{4}{9}}dr\\
=\;&\frac{2\Gamma\left(\frac{4}{9}+1\right)}{\ln(1000)\Gamma(\frac{8}{27}+\frac{4}{9}+1)\Gamma\left(\frac{1}{9}+1\right)}\frac{\rho^{\frac{4}{9}}}{(1+\rho^3)}+\frac{\Gamma\left(\frac{4}{9}+1\right)}{\ln(1000)\Gamma(\frac{2}{9}+\frac{4}{9}+1)\Gamma\left(\frac{1}{9}+1\right)}\frac{\rho^{\frac{4}{9}}}{(1+\rho^3)}\\
=\;&0.44532\frac{\rho^{\frac{4}{9}}}{(1+\rho^3)},
\end{aligned}$$

which implies that

$$\sup_{\rho\in[0,+\infty)}\left(0.44532\frac{\rho^{\frac{4}{9}}}{(1+\rho^3)}\right)\leq 0.44532\times 0.51\approx 0.2271129=K_7.$$

Also, we have

$$\sum_{k=1}^{2}|c_k|\frac{1}{\Gamma(t_u-t_k)\Gamma\left(\kappa_u+1\right)(1+\rho^{\mu})}\int_0^{\rho}\frac{(\rho-t)^{t_u-t_k-1}}{t^{s-\kappa_u}}B_8\left(t\right)dt$$

$$
\begin{aligned}
&= \frac{c_1}{\Gamma(t_3-t_1)\Gamma\left(\kappa_3+1\right)\left(1+\rho^3\right)}\int_0^\rho \frac{(\rho-t)^{t_3-t_1-1}}{t^{s-\kappa_3}}B_8\left(t\right)dt \\
&\quad+\frac{c_2}{\Gamma(t_3-t_2)\Gamma\left(\kappa_3+1\right)\left(1+\rho^3\right)}\int_0^\rho \frac{(\rho-t)^{t_3-t_2-1}}{t^{s-\kappa_3}}B_8\left(t\right)dt \\
&= \frac{1}{\Gamma(\frac{1}{4}-\frac{1}{16})\Gamma\left(\frac{1}{10}+1\right)\left(1+\rho^3\right)}\int_0^\rho \frac{(\rho-t)^{\frac{1}{4}-\frac{1}{16}-1}}{t^{\frac{1}{4}-\frac{1}{10}}}\frac{t^{\frac{7}{12}}}{\ln(1000)}dt \\
&\quad+\frac{1}{\Gamma(\frac{1}{4}-\frac{1}{8})\Gamma\left(\frac{1}{10}+1\right)\left(1+\rho^3\right)}\int_0^\rho \frac{(\rho-t)^{\frac{1}{4}-\frac{1}{8}-1}}{t^{\frac{1}{4}-\frac{1}{10}}}\frac{t^{\frac{7}{12}}}{\ln(1000)}dt \\
&= \frac{1}{\ln(1000)\Gamma(\frac{3}{16})\Gamma\left(\frac{1}{10}+1\right)\left(1+\rho^3\right)}\int_0^\rho \left(\rho-t\right)^{\frac{1}{4}-\frac{1}{16}-1}t^{\frac{13}{30}}dt \\
&\quad+\frac{1}{\ln(1000)\Gamma(\frac{1}{8})\Gamma\left(\frac{1}{10}+1\right)\left(1+\rho^3\right)}\int_0^\rho \left(\rho-t\right)^{\frac{1}{4}-\frac{1}{8}-1}t^{\frac{13}{30}}dt \\
&= \frac{\Gamma\left(\frac{13}{30}+1\right)}{\ln(1000)\Gamma(\frac{3}{16}+\frac{13}{30}+1)\Gamma\left(\frac{1}{10}+1\right)}\frac{\rho^{\frac{13}{30}}}{\left(1+\rho^3\right)}+\frac{\Gamma\left(\frac{13}{30}+1\right)}{\ln(1000)\Gamma(\frac{1}{8}+\frac{13}{30}+1)\Gamma\left(\frac{1}{10}+1\right)}\frac{\rho^{\frac{13}{30}}}{\left(1+\rho^3\right)} \\
&= 0.30201\frac{\rho^{\frac{13}{30}}}{\left(1+\rho^3\right)},
\end{aligned}
$$

which yields,

$$
\sup_{\rho\in[0,+\infty)}\left(0.30201\frac{\rho^{\frac{13}{30}}}{\left(1+\rho^3\right)}\right)\leq 0.75448\times 0.52\approx 0.1570468=K_8.
$$

To estimate L_1, we can write

$$
\begin{aligned}
\frac{1}{\Gamma(r_u)(1+\rho^\mu)}\int_0^\rho \left(\rho-r\right)^{r_u-1}\left\|F(\rho,0,0)\right\|dr &= \frac{1}{\Gamma(\frac{1}{3})(1+\rho^3)}\int_0^\rho \left(\rho-r\right)^{\frac{1}{3}-1}r^{\frac{1}{8}}dr \\
&= \frac{\Gamma(\frac{1}{8}+1)}{\Gamma(\frac{1}{3}+\frac{1}{8}+1)}\frac{\rho^{\frac{1}{8}}}{\left(1+\rho^3\right)} \\
&= 1.0633921\frac{\rho^{\frac{1}{8}}}{\left(1+\rho^3\right)},
\end{aligned}
$$

it follows that

$$
\sup_{\rho\in[0,+\infty)}\left(1.0633921\frac{\rho^{\frac{1}{8}}}{\left(1+\rho^3\right)}\right)\leq 1.0633921\times 0.57\approx 0.6061335=L_1<\infty.
$$

To evaluate L_2, we have

$$
\begin{aligned}
\frac{1}{\Gamma(t_u)(1+\rho^\mu)}\int_0^\rho \left(\rho-r\right)^{t_u-1}\left\|\widetilde{F}(\rho,0,0)\right\|dr &= \frac{1}{\Gamma(\frac{1}{4})(1+\rho^3)}\int_0^\rho \left(\rho-r\right)^{\frac{1}{4}-1}r^{\frac{1}{9}}dr \\
&= \frac{\Gamma\left(\frac{1}{9}+1\right)}{\Gamma(\frac{1}{4}+\frac{1}{9}+1)(1+\rho^3)}\frac{\rho^{\frac{1}{9}}}{\left(1+\rho^3\right)} \\
&= 1.063904\frac{\rho^{\frac{1}{9}}}{\left(1+\rho^3\right)},
\end{aligned}
$$

hence,

$$
\sup_{\rho\in[0,+\infty)}\left(1.063904\frac{\rho^{\frac{1}{9}}}{\left(1+\rho^3\right)}\right)\leq 1.063904\times 0.58\approx 0.6170645=L_2<\infty.
$$

Also, we have

$$
\begin{aligned}
&\sum_{j=1}^{2}|b_j|\frac{1}{\Gamma(r_u-r_j)(1+\rho^{\mu})}\int_0^{\rho}(\rho-r)^{r_u-r_j-1}\left\|G(\rho,0,0)\right\|dr\\
=&\ \frac{b_1}{\Gamma(r_3-r_1)(1+\rho^3)}\int_0^{\rho}(\rho-r)^{r_3-r_1-1}\left\|G(\rho,0,0)\right\|dr\\
&+\frac{b_2}{\Gamma(r_3-r_2)(1+\rho^3)}\int_0^{\rho}(\rho-r)^{r_3-r_2-1}\left\|G(\rho,0,0)\right\|dr\\
=&\ \frac{2}{\Gamma(\frac{1}{3}-\frac{1}{27})(1+\rho^3)}\int_0^{\rho}(\rho-r)^{\frac{1}{3}-\frac{1}{27}-1}\frac{r^{\frac{1}{3}}}{\ln(81)}dr\\
&+\frac{1}{\Gamma(\frac{1}{3}-\frac{1}{9})(1+\rho^3)}\int_0^{\rho}(\rho-r)^{\frac{1}{3}-\frac{1}{9}-1}\frac{r^{\frac{1}{3}}}{\ln(81)}dr\\
=&\ \frac{2\Gamma\left(\frac{1}{3}+1\right)}{\ln(81)\Gamma(\frac{8}{27}+\frac{1}{3}+1)}\frac{\rho^{\frac{1}{3}}}{(1+\rho^3)}+\frac{\Gamma\left(\frac{1}{3}+1\right)}{\ln(81)\Gamma(\frac{2}{9}+\frac{1}{3}+1)}\frac{\rho^{\frac{1}{3}}}{(1+\rho^3)}\\
=&\ 0.681483\frac{\rho^{\frac{1}{3}}}{(1+\rho^3)},
\end{aligned}
$$

taking $\sup_{\rho\in[0,+\infty)}$, we can write

$$
\sup_{\rho\in[0,+\infty)}\left(0.681483\frac{\rho^{\frac{1}{3}}}{(1+\rho^3)}\right)\leq 0.681483\times 0.59\approx 0.4020749=L_3<\infty.
$$

Finally, to evaluate L_4, we have

$$
\begin{aligned}
&\sum_{k=1}^{2}|c_k|\frac{1}{\Gamma(t_u-t_k)(1+\rho^{\mu})}\int_0^{\rho}(\rho-r)^{t_u-t_k-1}\left\|\widetilde{G}(\rho,0,0)\right\|dr\\
=&\ \frac{c_1}{\Gamma(t_3-t_1)(1+\rho^3)}\int_0^{\rho}(\rho-r)^{t_3-t_1-1}\left\|\widetilde{G}(\rho,0,0)\right\|dr\\
&+\frac{c_2}{\Gamma(t_3-t_2)(1+\rho^3)}\int_0^{\rho}(\rho-r)^{t_3-t_2-1}\left\|\widetilde{G}(\rho,0,0)\right\|dr\\
=&\ \frac{1}{\Gamma(\frac{1}{4}-\frac{1}{16})(1+\rho^3)}\int_0^{\rho}(\rho-r)^{\frac{1}{16}-\frac{1}{8}-1}\frac{r^{\frac{1}{3}}}{\ln(1000)}dr\\
&+\frac{1}{\Gamma(\frac{1}{4}-\frac{1}{8})(1+\rho^3)}\int_0^{\rho}(\rho-r)^{\frac{1}{16}-\frac{1}{4}-1}\frac{r^{\frac{1}{3}}}{\ln(1000)}dr\\
=&\ \frac{\Gamma\left(\frac{1}{3}+1\right)}{\ln(1000)\Gamma(\frac{3}{16}+\frac{1}{3}+1)}\frac{\rho^{\frac{1}{3}}}{(1+\rho^3)}+\frac{\Gamma\left(\frac{1}{3}+1\right)}{\ln(1000)\Gamma(\frac{1}{8}+\frac{1}{3}+1)}\frac{\rho^{\frac{1}{3}}}{(1+\rho^3)}\\
=&\ 0.29169\frac{\rho^{\frac{1}{3}}}{(1+\rho^3)},
\end{aligned}
$$

it follows that

$$
\sup_{\rho\in[0,+\infty)}\left(0.29169\frac{\rho^{\frac{1}{3}}}{(1+\rho^3)}\right)\leq 0.29169\times 0.59\approx 0.1721014=L_4<\infty.
$$

Based on what has been calculated, we conclude that $\sum_{i=1}^{8}K_i=0.7980389<1$ and $L_k>0$, for $k=1,\ldots,4$. Therefore, all requirements of Theorem 3.2 are fulfilled. Hence, the problem (4.1) possesses a unique solution on the unbounded interval $[0,+\infty)$.

5 Numerical simulations

In this section, we present a modified Adams-Bashforth-Moulton predictor-corrector scheme [34–36] for the numerical solution of coupled system of nonlinear multi-term Caputo FDEs. The class of equations under consideration is given by:

$$\sum_{j=1}^{3} \lambda_{\Re_j}\, {}^{C}D_t^{\alpha_{\Re_j}} \Re(t) = f_{\Re}\left(t, \Re(t), \mathcal{I}_{\beta}[\Im](t)\right),$$

$$\sum_{j=1}^{3} \lambda_{\Im_j}\, {}^{C}D_t^{\alpha_{\Im_j}} \Im(t) = f_{\Im}\left(t, \Im(t), \mathcal{I}_{\kappa}[\Re](t)\right),$$

where ${}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order $0 < \alpha < 1$, and the memory-dependent terms are fractional integrals:

$$\mathcal{I}_{\beta}[\Im](t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \Im(\tau)\, d\tau, \quad \mathcal{I}_{\kappa}[\Re](t) = \frac{1}{\Gamma(\kappa)} \int_0^t (t-\tau)^{\kappa-1} \Re(\tau)\, d\tau,$$

Let $t_n = nh$, $n = 0, 1, \ldots, N$ be a uniform grid on $[0, T]$ with step size $h = \frac{T}{N}$. The fractional integrals are approximated using the trapezoidal quadrature rule:

$$\mathcal{I}_{\beta}[\Im](t_n) \approx \frac{h^{\beta}}{\Gamma(\beta)} \sum_{j=0}^{n-1} (t_n - t_j)^{\beta-1} \Im(t_j),$$

$$\mathcal{I}_{\kappa}[\Re](t_n) \approx \frac{h^{\kappa}}{\Gamma(\kappa)} \sum_{j=0}^{n-1} (t_n - t_j)^{\kappa-1} \Re(t_j).$$

The general multi-term Caputo equation:

$$\sum_{j=1}^{m} \lambda_j\, {}^{C}D_t^{\alpha_j} y(t) = f(t, y(t)), \quad y^{(k)}(0) = y_k,\ k = 0, \ldots, n-1,$$

is equivalent to the Volterra-type integral equation:

$$y(t) = \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k + \frac{1}{\lambda_1 \Gamma(\alpha_1)} \int_0^t (t-\tau)^{\alpha_1 - 1} \left[f(\tau, y(\tau)) - \sum_{j=2}^{m} \lambda_j\, {}^{C}D_{\tau}^{\alpha_j} y(\tau) \right] d\tau.$$

Let $F_j := f(t_j, y_j) - \sum_{i=2}^{m} \lambda_i\, {}^{C}D_t^{\alpha_i} y_j$. Then the Adams-Bashforth-Moulton predictorcorrector method proceeds in two stages:

Predictor (order 1):

$$\hat{y}_n = \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t_n^k + \frac{1}{\lambda_1 \Gamma(\alpha_1)} \sum_{j=0}^{n-1} b_{n-j}^{(\alpha_1)} F_j,$$

Corrector (order 2):

$$y_n = \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t_n^k + \frac{1}{\lambda_1 \Gamma(\alpha_1)} \left(a_0^{(\alpha_1)} F_n + \sum_{j=0}^{n-1} a_{n-j}^{(\alpha_1)} F_j \right),$$

with quadrature weights:

$$a_k^{(\alpha)} = h^\alpha \left[(k+1)^\alpha - k^\alpha \right], \quad b_k^{(\alpha)} = a_k^{(\alpha)}.$$

Caputo derivatives of order $\alpha \in (0,1)$ are approximated by discrete convolution:

$$^{C}D_t^\alpha y(t_n) \approx \frac{1}{h^\alpha} \sum_{k=0}^{n} w_k^{(\alpha)} y_{n-k},$$

where the weights $w_k^{(\alpha)}$ are computed recursively:

$$w_0^{(\alpha)} = 1, \quad w_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right) w_{k-1}^{(\alpha)}, \quad k \geq 1.$$

This scheme enables stable and accurate simulation of nonlinear coupled systems with memory effects, and is well-suited for fractional-order dynamics with multiple terms.

Example 5.1. Consider the problem (4.1) with the nonlinear source terms:

$$f_{\Re}(\rho, \Re, \mathcal{I}^{\beta_3}) = \rho^{1/2} \cdot 9 \sin(\Re) + \rho^{1/20} e^2 \cdot \tan^{-1}(\mathcal{I}^{\beta_3}(\Im)) + \rho^{1/8},$$
$$f_{\Im}(\rho, \Im, \mathcal{I}^{\kappa_3}) = \rho^{1/2} \cdot 10 \sin(\Im) + \rho^{1/20} e^2 \cdot \tan^{-1}(\mathcal{I}^{\kappa_3}(\Re)) + \rho^{1/9}.$$

The fractional derivative orders and coefficients are given by:

$$r = \left\{\tfrac{1}{27}, \tfrac{1}{9}, \tfrac{1}{3}\right\}, \quad \lambda_{\Re} = \{2, 1, 1\},$$
$$t = \left\{\tfrac{1}{16}, \tfrac{1}{8}, \tfrac{1}{4}\right\}, \quad \lambda_{\Im} = \{1, 1, 1\}.$$

In Figure 1 shows numerical results obtained for nonlinear coupled systems with memory effects for different fractional integral values of β_3 and κ_3, with initial conditions $\Re(0) = 0$ and $\Im(0) = 0$ during the time interval $\rho \in [0,1]$. From these results, we show that computational methods have been carried out to demonstrate the feasibility of the investigated theory.

Example 5.2. We consider a nonlinear system describing two interacting variables, $\Re(\rho)$ and $\Im(\rho)$, using multi-term Caputo fractional derivatives:

$$^{C}D_\rho^{0.3}\Re(\rho) + {}^{C}D_\rho^{0.6}\Re(\rho) + {}^{C}D_\rho^{0.9}\Re(\rho) = -a\Re(\rho)^3 + b\tanh(\Im(\rho)) + \rho^{0.2},$$

$$^{C}D_\rho^{0.2}\Im(\rho) + {}^{C}D_\rho^{0.4}\Im(\rho) + {}^{C}D_\rho^{0.7}\Im(\rho) = -c\Im(\rho)^2 + d\sin(\mathcal{I}^{\kappa_3}[\Re](\rho)) + \rho^{0.1},$$

where $a = 1.5, b = 2, c = 1, d = 2.5$. The model includes memory effects, nonlinear feedback, and external inputs. It can describe neural processes like adaptation and memory. See Teka et al. [37] for related models using fractional dynamics in neuroscience. The simulation results of this nonlinear system, shown in Figures 2, reveal how memory and nonlinear feedback influence the behavior of the variables $\Re(\rho)$ and $\Im(\rho)$ for different fractional integral values of κ_3, with initial conditions $\Re(0) = 0.1$

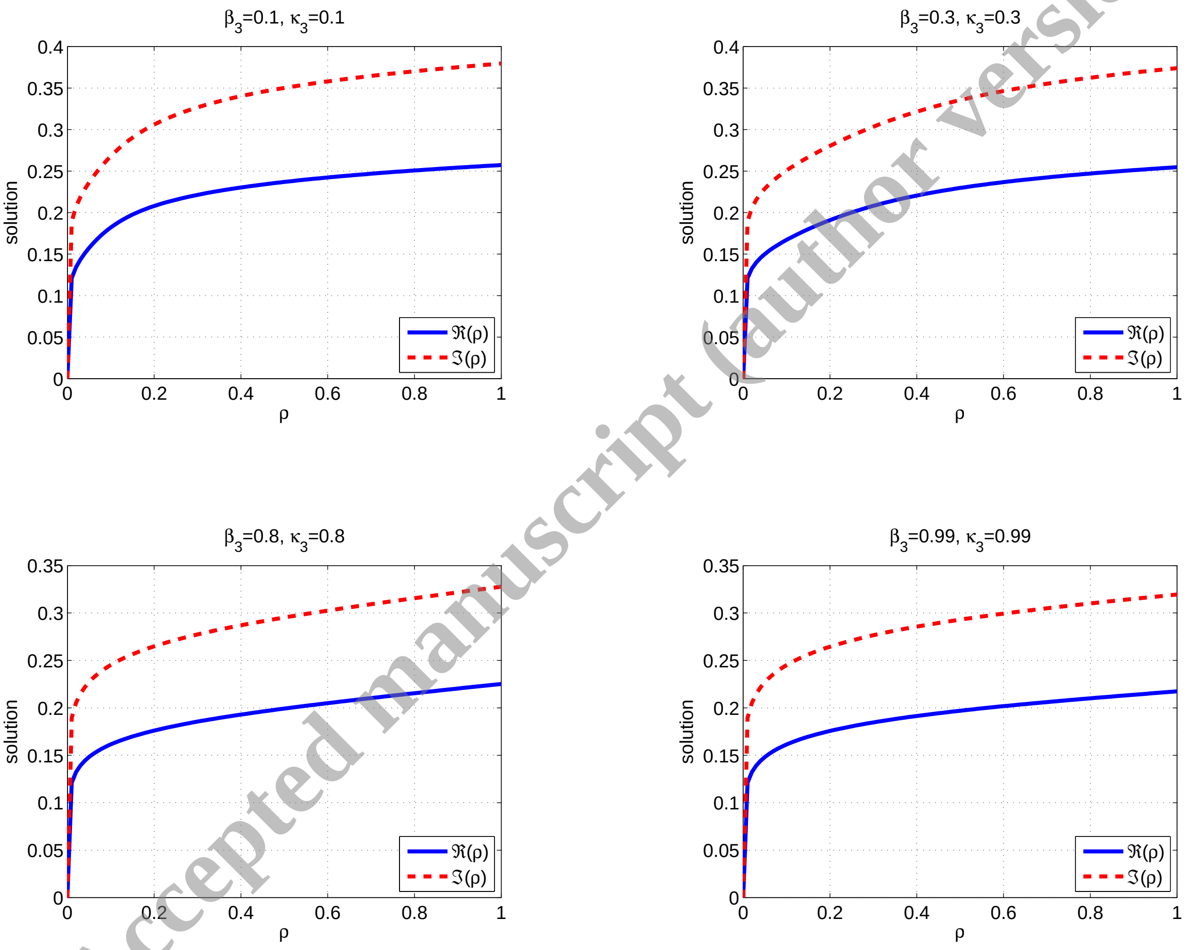

Figure 1: The behavior at β_3 and κ_3, with initial conditions $\Re(0) = 0$ and $\Im(0) = 0$ during the time interval $\rho \in [0, 1]$.

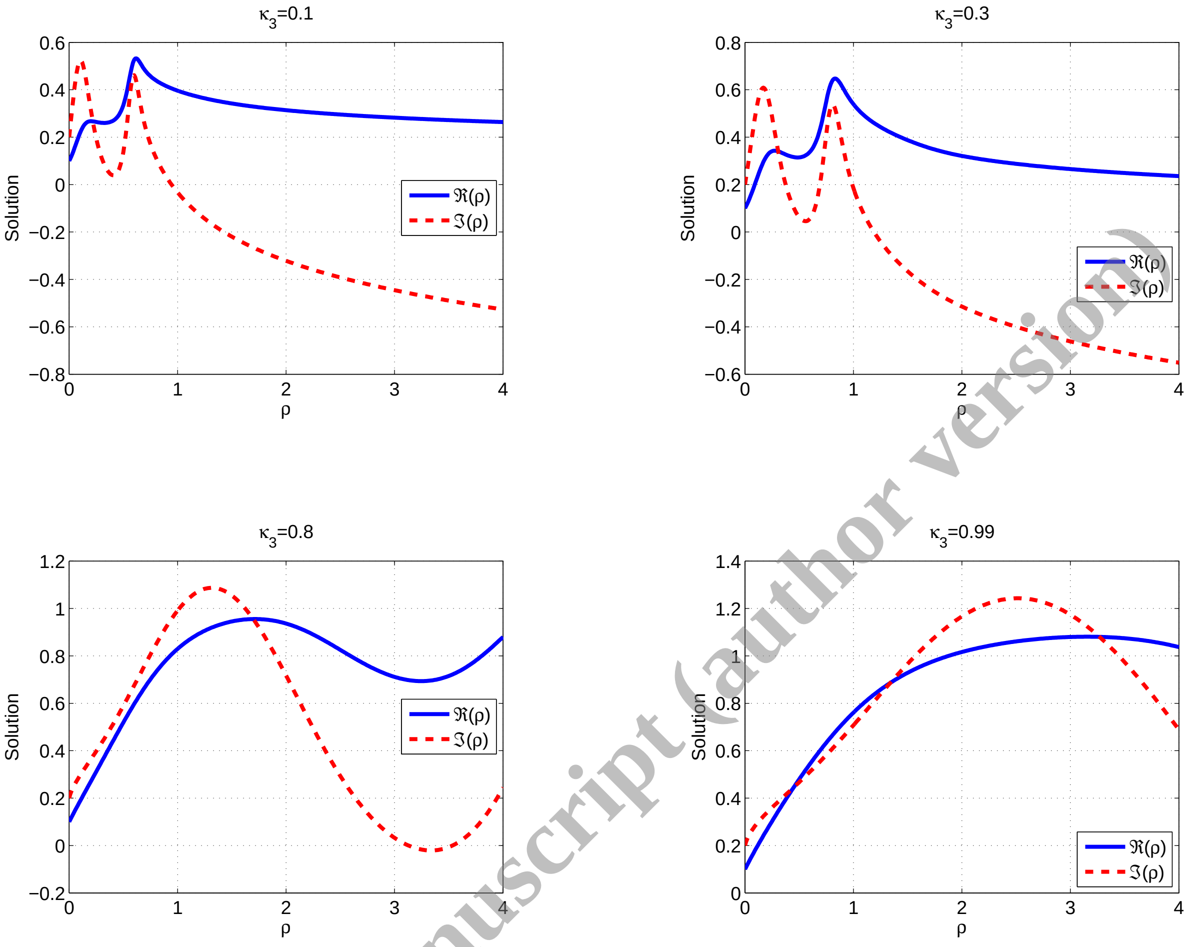

Figure 2: The behavior at κ_3, with initial conditions $\Re(0) = 0.1$ and $\Im(0) = 0.2$ during the time interval $\rho \in [0, 4]$.

and $\Im(0) = 0.2$, over the time interval $\rho \in [0, 4]$. The inclusion of multi-term fractional derivatives leads to gradual, memory-dependent changes, capturing nonlocal temporal effects. The cubic decay and hyperbolic tangent terms introduce saturation phenomena, while the fractional integral $\mathcal{I}_\gamma[\Re](\rho)$ enables the past behavior of $\Re$ to modulate the evolution of $\Im$. This dynamic coupling gives rise to complex temporal patterns, potentially resembling neural adaptation or delayed responses. Overall, the model demonstrates how fractional operators and memory terms can yield more realistic and adaptable system dynamics than traditional integer-order formulations.

Example 5.3. We consider the following system describing two interacting processes, $\Re(\rho)$ and $\Im(\rho)$, which may represent, for example, the infected and recovered populations in a fractional epidemic model:

$$^C D_\rho^{0.3} \Re(\rho) + {}^C D_\rho^{0.6} \Re(\rho) = -\alpha \Re(t) \Im(\rho) + \beta \mathcal{I}_3^\beta[\Re](\rho),$$

$$^C D_\rho^{0.4} \Im(\rho) + {}^C D_\rho^{0.8} \Im(\rho) = \gamma \Re(\rho) - \delta \Im(\rho) + \log\left(1 + |\mathcal{I}_3^\kappa[\Im](\rho)|\right),$$

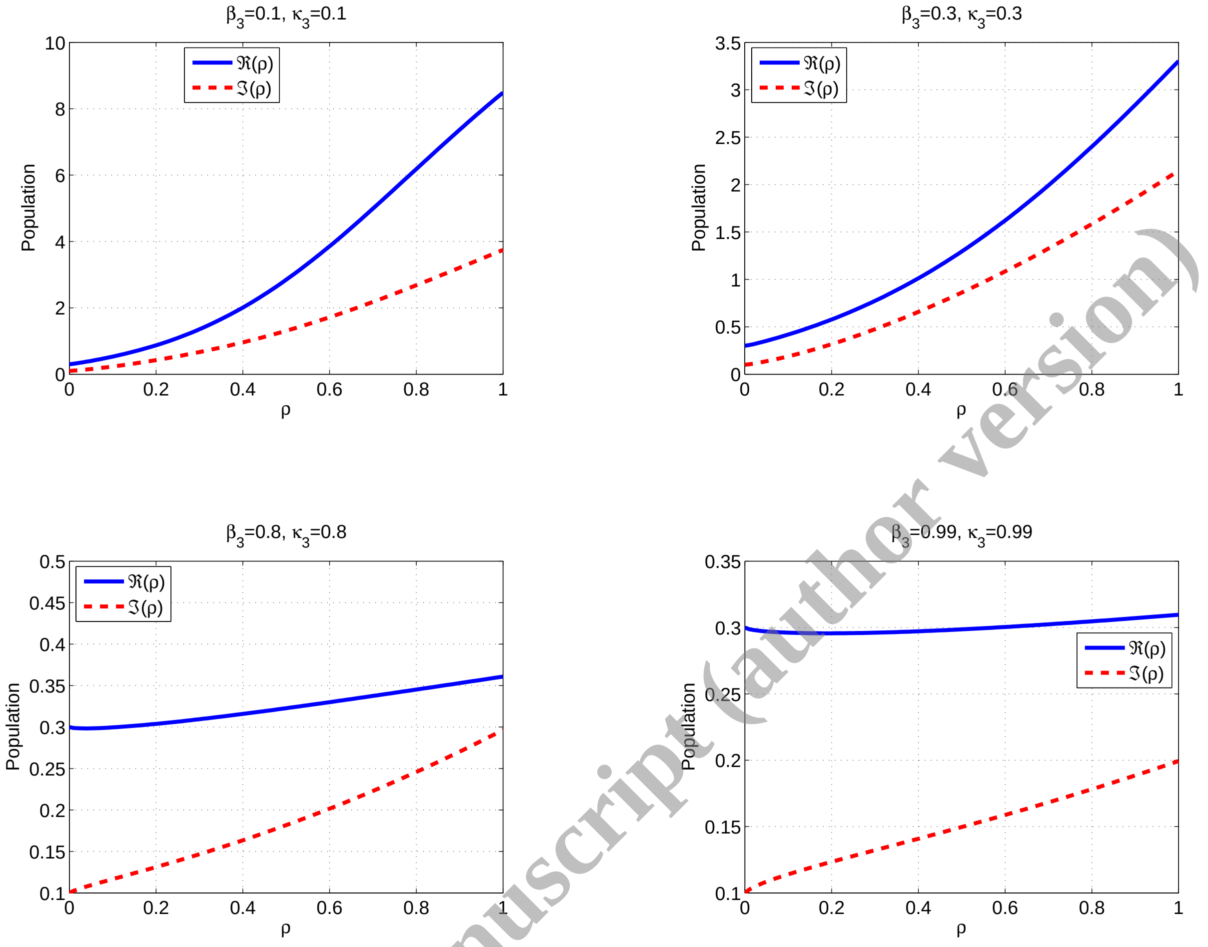

Figure 3: The behavior at β_3 and κ_3, with initial conditions $\Re(0) = 0.3$ and $\Im(0) = 0.1$ during the time interval $\rho \in [0, 1]$.

This system includes nonlinear infection coupling $(\Re, \Im)$, recovery, reinfection memory effects via integrals, and logarithmic feedbackmodeling population response, immunity, or long-term exposure [38, 39]. The simulation results in Figures 3 illustrate the evolution of the infected population, $\Re(\rho)$, and the recovered population, $\Im(\rho)$, over time under memory effects, for different fractional integral values of β_3 and κ_3, with initial conditions $\Re(0) = 0.3$ and $\Im(0) = 0.1$, within the time interval $\rho \in [0, 1]$. The infected group increases initially due to active transmission, then decreases as recovery and memory-based reinfection mechanisms take effect. The recovered population grows gradually, influenced by the infection rate and long-term memory. The fractional integrals $\mathcal{I}_\nu[\Re](\rho)$ and $\mathcal{I}_\kappa[\Im](\rho)$ play a significant role in shaping the system dynamics, demonstrating how past states affect present behavior.

6 Conclusion and future work

Coupled multi-order FDEs play a crucial role in a wide range of scientific and engineering applications due to their enhanced capability to model complex phenomena beyond the scope of classical integer-order systems. The incorporation of fractional derivatives enables the accurate representation of memory and hereditary effects that are intrinsic to many real-world processes, including viscoelastic behavior, anomalous diffusion, and biological interactions. In addition, the coupled structure of these systems allows for the modeling of interdependent processes, where the evolution of one state variable directly influences others. This feature makes them particularly well suited for describing complex networks arising in areas such as control theory, neural networks, and population dynamics. The multi-order formulation further enriches their modeling flexibility by permitting different system components to exhibit distinct memory characteristics, thereby yielding more realistic and nuanced descriptions in applications ranging from finance and signal processing to the behavior of complex materials.

In this work, we addressed the challenging problem of establishing the existence and uniqueness of solutions for coupled multi-order FDEs involving multiple fractional orders, a class of systems that naturally arises in many scientific and engineering contexts. By employing the Banach contraction principle, we derived new sufficient conditions that guarantee the existence and uniqueness of solutions. The theoretical results were supported by a detailed illustrative example, underscoring the practical applicability of the proposed framework. To further demonstrate its effectiveness, we implemented a modified Adams-Bashforth-Moulton predictorcorrector scheme to obtain numerical solutions. Three representative examples were examined, modeling oscillatory systems, neural feedback dynamics, and epidemic spread under memory effects. The numerical simulations corroborated the theoretical findings and clearly illustrated the impact of fractional-order dynamics on system stability and long-term behavior.

Future research on coupled multi-order fractional differential systems is expected to focus on the development of more advanced analytical techniques to address their inherent complexity. Important directions include the investigation of qualitative properties such as various stability notions, controllability, and observability for broader classes of systems. The design of efficient and accurate numerical methods, particularly for nonlinear problems, also remains a significant challenge. Moreover, application-oriented studies aimed at integrating real-time data into fractional-order models are likely to further enhance their relevance. Finally, exploring the effects of stochastic perturbations and time delays represents a promising avenue for advancing the theory and applications of coupled multi-order FDEs.

7 Abbreviations

FDE$\Longrightarrow$Fractional differential equation.
DE$\Longrightarrow$differential equation.
RL$\Longrightarrow$Riemann-Liouville.
BVP$\Longrightarrow$boundary value problem.
FP$\Longrightarrow$fixed point.
BS$\Longrightarrow$Banach space.

Data availability

Data available on request from the authors.

Conflicts of interest

The author declare that they have no conflicts of interest.

Author's contributions

All authors contributed equally to the writing of this article.

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