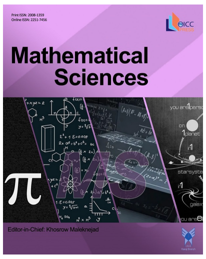

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Research Article

Solvability and Stability for a Tripled System of Fractional Integro-differential Equations under Integral Boundary Conditions

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Abstract:

This article investigates the solvability and stability of a nonlinear tripled system of fractional integro-differential equations via integral boundary conditions. We prove the fundamental existence and uniqueness results for the solution using powerful methods from nonlinear functional analysis, including the Banach fixed-point theorem. Crucially, we gauge the system's resistance to small perturbations using Ulam-Hyers stability analysis. Our results confirm that approximate solutions are close to the true solution, providing an essential resilience measure for complex models in which memory and hereditary traits are captured by fractional derivatives. Finally, an illustrative example has been presented to validate the stated theoretical results.

Keywords: Fractional derivatives; Green function; Fixed point technique; Ulam-Hyers stability; Integro-differential equation

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1. Introduction

Fractional derivatives are often considered more accurate than ordinary (integer-order) derivatives for modeling complex, real-world systems primarily because they inherently incorporate memory and hereditary properties into the model, a crucial feature that classical integer-order models typically neglect [1]. While ordinary derivatives are local operators whose value depends only on the immediate neighborhood of a point, fractional derivatives are non-local operators whose definition involves an integral, meaning the current state depends on the entire history of the function's process [2]. This long-term memory capability allows fractional differential equations (FDEs) to more adequately describe phenomena like anomalous diffusion, viscoelasticity, and biological kinetics, often providing a better fit to experimental data with fewer parameters compared to high-order classic models [3].

The application of FDEs has recently intensified across multidisciplinary fields, leveraging their non-local nature to model systems that exhibit memory effects and long-range dependence with high fidelity. In biomathematics, FDEs are increasingly used to model complex biological phenomena, such as the spread of infectious diseases (epidemiology) and the movement of biomolecules in crowded cellular environments, capturing the anomalous (non-Fickian) diffusion more accurately than classical models [4]. Furthermore, in engineering and control theory, new research focuses on using fractional operators for the enhanced design of dynamic systems, including robust fractional-order control algorithms and the precise modeling of materials like viscoelastic polymers and electrical circuits with fractal, where the memory of past stresses or charges significantly influences the current behavior [5, 6]. This trend is facilitated by the development of novel analytical and numerical methods, often based on integral transforms, to solve complex fractional-order equations that were previously intractable [7]. For more applications, see [8, 9, 10].

The mathematical notion known as Ulam's type stability was first proposed by Stanislaw Ulam in 1940 and subsequently refined by Donald Hyers. This concept addresses the resilience of solutions to differential equations, specifically describing their behavior when subject to small disturbances (perturbations) in the initial conditions or the equation itself. In essence, it serves to quantify a system's stability and responsiveness to these minor errors [11, 12]. The work of Ulam and Hyers focused on the stability properties of various integer-order differential equations, resulting in the formalized notion of Hyers-Ulam stability for Cauchy functional equations. Their findings provided stronger conclusions regarding key functions like polynomials, isometric mappings, and convex functions. Following their foundational contributions, countless researchers have extended and generalized Hyer's framework. The subsequent literature now features extensive investigations into many generalized stability concepts of Ulam type [13, 14, 15, 16, 17, 18, 19, 20].

Tripled fractional derivatives and tripled fixed points play an important role in advancing modern mathematical modeling and analysis, particularly for complex systems with memory, nonlocal interactions, and multi-stage dynamics. A tripled fractional derivative extends classical fractional calculus by capturing three interdependent rates of change, allowing more accurate descriptions of phenomena such as viscoelastic materials, anomalous diffusion, and coupled biological or economic processes. Similarly, tripled fixed points generalize fixed-point theory to settings where three mappings or variables interact simultaneously, providing powerful tools for proving existence, uniqueness, and stability of solutions to nonlinear systems, for more details, see [21, 22, 23, 24, 25].

Integro-differential equations are essential mathematical tools for modeling complex systems where the rate of change depends on both the current state and a historical accumulation of data, a feature particularly prevalent in viscoelasticity, population dynamics, and epidemiology [26, 27]. Recent advancements have seen integro-differential equations applied to modern frontiers such as neuroscience, where fractional integro-differential equations are used to model the long-term memory effects of EEG signals for emotion recognition [28], and stochastic viral modeling, where partial integro-differential equations approximate the unpredictable spread of infections in compartmental systems. These equations typically require specific initial or boundary conditions such as the Caputo fractional derivative to maintain physically interpretable integer-order initial states to ensure the existence and uniqueness of solutions in non-local environments [29, 30]. By incorporating integral operators, these models effectively capture hereditary properties and time lags that standard differential equations often overlook, making them indispensable for high-fidelity simulations in fluid mechanics, bio-chemical reactions, and neural network dynamics [31].

The practical implications of these findings extend across various disciplines that rely on complex differential systems constrained by integral boundary conditions. In the realm of hydrogeology, for example, fractional integro-differential equations serve as a critical tool for modeling the complexities of subsurface water flow [32], while chemical engineers utilize similar frameworks to investigate reaction-diffusion processes [33, 34]. Because the utility of these mathematical models in fields such as population dynamics or circulatory system analysis depends on their predictive reliability, establishing the absolute stability of solutions is essential for their real-world application [35].

Integral boundary conditions are crucial in mathematical modeling because they allow the constraints of a physical system to be defined not just at discrete points but by requiring that an integral quantity of the unknown function (or its derivatives) over a specified domain must satisfy a certain value. This approach is essential in fields where conservation laws or cumulative effects govern the system's behavior. For instance, integral boundary conditions are commonly used in phase field models to describe electric or thermal transport phenomena and appear naturally in the study of certain second-order differential equations related to steady-state diffusion with reactions. Furthermore, the study of the existence and uniqueness of solutions to problems involving FDEs frequently incorporates integral boundary conditions, making them a significant focus in contemporary analysis [36, 37].

To see the practical importance of integral boundary conditions, consider this real-world example

$$\begin{cases} \theta'' + g(\psi)\, h(\psi, \theta) = 0, \\ \theta(c_1) = 0,\ \alpha\theta'(c_2) - \theta(\gamma) = 0, \end{cases}$$

where $c_1 < \psi < c_2$, $0 < \gamma \leq 1$, $\alpha > 0$ and $c_1, c_2 \in \mathbb{R}$. This is a thermoregulator model for the one-dimensional heat equation. It has well-defined solutions for a heated bar where a control system modulates the heat input based on temperature measurements from a sensor at position γ. The formulation extends the heat equation to include nonlinear gradient terms and time-varying sources. In this more general setup, a controller at γ regulates the heated bar's heat output using temperature data from multiple sensors distributed along its length. It can be stated as

$$\begin{cases} \theta'' - h(\psi, \theta, \theta') = 0, \\ \theta(c_1) = 0,\ \theta'(c_2) = \int\limits_{c_1}^{c_2} \theta(r)\, dk(r)\,(\gamma) = 0. \end{cases}$$

Building on prior work concerning integro-systems with fractional order derivatives, we aim to extend the results of [20, 38]. Integro-differential equations are

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critical tools, as they model complex systems where current states depend on both instantaneous rates of change and accumulated past effects. Despite their importance, studies investigating Ulam-type stability for integro-differential equations with fractional derivatives are scarce. Furthermore, while extensive research exists on FDEs, Ulam's stability for tripled systems subject to integral boundary conditions remains an open area. The proposed system is as follows:

$$\begin{cases} {}^cD^{\kappa_1}u(\zeta) = \phi_1(\zeta, v(\zeta), w(\zeta), \Omega^w(\zeta)), \\ \qquad\qquad\qquad\qquad \zeta \in V = [0,1], \\ {}^cD^{\kappa_2}v(\zeta) = \phi_2(\zeta, w(\zeta), u(\zeta), \Omega^u(\zeta)), \ \zeta \in V, \\ {}^cD^{\kappa_3}w(\zeta) = \phi_3(\zeta, u(\zeta), v(\zeta), \Omega^v(\zeta)), \ \zeta \in V, \\ \varpi u(0) + \tau u'(0) = \delta \int_0^1 l(u(r))\,dr, \\ \varpi u(1) + \tau u'(1) = \delta \int_0^1 m(u(r))\,dr, \\ \widetilde{\varpi} v(0) + \widetilde{\tau} v'(0) = \widetilde{\delta} \int_0^1 \widetilde{l}(v(r))\,dr, \\ \widetilde{\varpi} v(1) + \widetilde{\tau} v'(1) = \widetilde{\delta} \int_0^1 \widetilde{m}(v(r))\,dr, \\ \widehat{\varpi} w(0) + \widehat{\tau} w'(0) = \widehat{\delta} \int_0^1 \widehat{l}(w(r))\,dr, \\ \widehat{\varpi} w(1) + \widehat{\tau} w'(1) = \widehat{\delta} \int_0^1 \widehat{m}(w(r))\,dr, \end{cases} \tag{1}$$

where ${}^cD^{\kappa_1}$, ${}^cD^{\kappa_2}$ and ${}^cD^{\kappa_3}$ are the Caputo fractional derivatives with orders $\kappa_1, \kappa_2, \kappa_3 \in (0,1]$, respectively. Moreover, $l, \widetilde{l}, \widehat{l}, m, \widetilde{m}, \widehat{m} : \mathbb{R} \to \mathbb{R}$, $\varpi, \widetilde{\varpi}, \widehat{\varpi}, \delta, \widetilde{\delta}, \widehat{\delta} > 0$ and $\tau, \widetilde{\tau}, \widehat{\tau} \geq 0$ are real numbers. Furthermore, $\phi_1, \phi_2, \phi_3 : V \times \mathbb{R}^3 \to \mathbb{R}$ and $\Omega^w, \Omega^u, \Omega^v : V \times V \to [0, \infty)$ such that

$$\begin{cases} \Omega^w(\zeta) = \int_0^\zeta \varphi(\zeta, r)\, w(r)\,dr, \\ \Omega^u(\zeta) = \int_0^\zeta \widetilde{\varphi}(\zeta, r)\, u(r)\,dr, \\ \Omega^v(\zeta) = \int_0^\zeta \widehat{\varphi}(\zeta, r)\, v(r)\,dr. \end{cases}$$

The motivations of this manuscript is summarized as follows:

(1) This study investigates a system of nonlinear fractional integro-differential equations (1) governed by integral boundary conditions, addressing systems where memory and hereditary properties are central.

(2) By leveraging the Banach fixed-point theorem in the framework of nonlinear functional analysis, we establish rigorous proof for the existence and uniqueness of the solution.

(3) We systematically apply Ulam-Hyers stability criteria to quantify the system's resilience to perturbations. This analysis ensures a high degree of closeness between approximate and exact solutions.

(4) The stability results confirm the model's well-posedness and fundamental dependability, ensuring it remains robust under the fluctuating conditions typical of complex physical systems.

(5) The theoretical framework is substantiated through a series of illustrative applications, demonstrating a high degree of correlation between our numerical results and the established analytical findings.

2. Basic facts and hypotheses

In this section, we lay the groundwork for our investigation by introducing essential concepts, notations, and preliminary results. These foundations are critical for maintaining clarity and rigor throughout the following derivations and proofs. Specifically, we state the necessary lemmas that will be directly utilized in the subsequent parts to establish our main theorems.

Definition 2.1 [24]

(i) For the function $v : (0,1) \to \mathbb{R}$, the Caputo fractional derivative with order $\kappa > 0$ is described as

$$ {}^cD^\kappa v(\zeta) = \frac{1}{\Gamma(\sigma - \kappa)} \int_0^\zeta (\zeta - r)^{\sigma - \kappa - 1} v^{(\sigma)}(r)\,dr, \quad \sigma = [\kappa] + 1.$$

(ii) For the function $v : (0,1) \to \mathbb{R}$, the Riemann-Liouville fractional derivative with order $\kappa > 0$ is given by

$$ I^\kappa v(\zeta) = \frac{1}{\Gamma(\sigma - \kappa)} \left(\frac{d}{d\zeta}\right)^\sigma \int_0^\zeta (\zeta - r)^{\sigma - \kappa - 1} v(r)\,dr, \quad \sigma = [\kappa] + 1,$$

provided the integral exists, where $[\kappa]$ stands for the integer part of κ and $\Gamma(.)$ is the Euler gamma function.

Lemma 2.2 [24] For any real number $\kappa > 0$,

(i) the solution of the FDE

$$ {}^cD^\kappa v(\zeta) = 0$$

is

$$ v(\zeta) = \rho_0 + \rho_1\zeta + \rho_2\zeta^2 + \cdots + \rho_{\sigma-1}\zeta^{\sigma-1},$$

(ii) the solution of the FDE

$$ {}^cD^\kappa v(\zeta) = \hbar(\zeta)$$

is given by

$$ I^\kappa({}^cD^\kappa v(\zeta)) = I^\kappa v(\zeta) + \rho_0 + \rho_1\zeta + \rho_2\zeta^2 + \cdots + \rho_{\sigma-1}\zeta^{\sigma-1},$$

where $\sigma = [\kappa] + 1$, $\rho_i \in \mathbb{R}$, $i = 1, 2, \cdots, \sigma - 1$.

Lemma 2.3 [25] Assume that $\hbar, \varphi_1, \varphi_2 \in C([V, \mathbb{R}])$. The fractional problem

$$\begin{cases} {}^cD^\kappa v(\zeta) = \hbar(\zeta), \ \zeta \in V = [0,1], \\ \varpi u(0) + \tau u'(0) = \delta \int_0^1 \varphi_1(r)\,dr, \\ \varpi u(1) + \tau u'(1) = \delta \int_0^1 \varphi_2(r)\,dr, \end{cases}$$

has the following unique solution:

$$ v(\zeta) = \int_0^1 F_\kappa(\zeta, r)\,\hbar(r)\,dr + \frac{\delta}{\varpi^2}\Big[(\varpi(1 - \zeta) + \tau) \int_0^1 \varphi_1(r)\,dr + (\tau + \varpi\zeta) \int_0^1 \varphi_2(r)\,dr\Big],$$

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where $F_{\kappa}(\zeta, r)$ is the Green's function defined by

$$F_{\kappa}(\zeta,r)=\begin{cases}\frac{\varpi(\zeta-r)^{\kappa-1}+(\tau-\varpi\zeta)(1-r)^{\kappa-1}}{\varpi\Gamma(\kappa)} & r\le\zeta,\\ +\frac{\tau(\tau-\varpi\zeta)(1-r)^{\kappa-2}}{\varpi^{2}\Gamma(\kappa-1)}, & \\ & \\ \frac{(\tau-\varpi\zeta)(1-r)^{\kappa-1}}{\varpi\Gamma(\kappa)} & \zeta\le r.\\ +\frac{\tau(\tau-\varpi\zeta)(1-r)^{\kappa-2}}{\varpi^{2}\Gamma(\kappa-1)}, & \end{cases}$$

Now, assume that $A = \{v(\zeta) : v \in C(V)\}$ is a Banach space under the norm $\|v\|_A = \max_{\zeta\in V} |v(\zeta)|$. Then, the norm on product space is described as $\|(u, v, w)\|_{\Theta=A\times A\times A} = \|u\|_A + \|v\|_A + \|w\|_A$. Clearly, $(\Theta, \|(.,.,.)\|_{\Theta})$ is a Banach space too. Furthermore, the cone $B \subset \Theta$ is given by

$$B = (u, v, w) \in \Theta : u(\zeta) \ge 0, v(\zeta) \ge 0, \text{ and } w(\zeta) \ge 0.$$

Now, to transform our problem to a fixed point problem, we define the operator $\Im : \Theta \longrightarrow \Theta$ as

$$\Im(u,v,w)(\zeta)=\begin{pmatrix}\Im_{\kappa_1}(u,v,w)(\zeta)\\ \Im_{\kappa_2}(v,w,u)(\zeta)\\ \Im_{\kappa_3}(w,u,v)(\zeta)\end{pmatrix}$$

$$=\begin{cases}\int_0^1 F_{\kappa_1}(\zeta,r)\,\phi_1(r,v(r),w(r),\Omega^{w}(r))\,dr\\ +\frac{\delta}{\varpi^2}\Big[(\varpi(1-\zeta)+\tau)\int_0^1 l(u(r))\,dr\\ +(\tau+\varpi\zeta)\int_0^1 m(u(r))\,dr\Big],\\ \\ \int_0^1 F_{\kappa_2}(\zeta,r)\,\phi_2(r,w(r),u(r),\Omega^{u}(r))\,dr\\ +\frac{\widetilde{\delta}}{\widetilde{\varpi}^2}\Big[(\widetilde{\varpi}(1-\zeta)+\widetilde{\tau})\int_0^1 \widetilde{l}(v(r))\,dr\\ +(\widetilde{\tau}+\widetilde{\varpi}\zeta)\int_0^1 \widetilde{m}(v(r))\,dr\Big],\\ \\ \int_0^1 F_{\kappa_3}(\zeta,r)\,\phi_3(r,u(r),v(r),\Omega^{v}(r))\,dr\\ +\frac{\widehat{\delta}}{\widehat{\varpi}^2}\Big[(\widehat{\varpi}(1-\zeta)+\widehat{\tau})\int_0^1 \widehat{l}(w(r))\,dr\\ +(\widehat{\tau}+\widehat{\varpi}\zeta)\int_0^1 \widehat{m}(w(r))\,dr\Big].\end{cases}$$

The fixed point of the operator $\Im$ is equivalent to the solution of the tripled system (1). To establish the existence and uniqueness of this solution, we need the subsequent hypotheses:

(H_1) For $\zeta \in V = [0,1]$, there exists $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, e_1, e_2, e_3, e_4 \in C(V, \mathbb{R}^+)$ such that

$$\begin{cases}|\phi_1(\zeta,v(\zeta),w(\zeta),\Omega^{w}(\zeta))|\le a_1(\zeta)\\ +a_2(\zeta)|v(\zeta)|+a_3(\zeta)|w(\zeta)|+a_4(\zeta)|\Omega^{w}(\zeta)|,\\ \\ |\phi_2(\zeta,w(\zeta),u(\zeta),\Omega^{u}(\zeta))|\le b_1(\zeta)\\ +b_2(\zeta)|w(\zeta)|+b_3(\zeta)|u(\zeta)|+b_4(\zeta)|\Omega^{u}(\zeta)|,\\ \\ |\phi_3(\zeta,u(\zeta),v(\zeta),\Omega^{v}(\zeta))|\le e_1(\zeta)\\ +e_2(\zeta)|u(\zeta)|+e_3(\zeta)|v(\zeta)|+e_4(\zeta)|\Omega^{v}(\zeta)|,\end{cases}$$

for $u(\zeta), v(\zeta), w(\zeta) \in C(V, \mathbb{R})$ with

$$a_i^*=\sup_{\zeta\in V} a_i(\zeta),\ b_i^*=\sup_{\zeta\in V} b_i(\zeta),$$
$$e_i^*=\sup_{\zeta\in V} e_i(\zeta),\quad i=1,2,3,4.$$

(H_2) For $\zeta \in V$, there exist $N_l, N_m, N_{\widetilde{l}}, N_{\widetilde{m}}, N_{\widehat{l}}$, and $N_{\widehat{m}}$ such that

$$\begin{aligned}&|l(u(\zeta))|\le N_l|u(\zeta)|,\\ &|m(u(\zeta))|\le N_m|u(\zeta)|,\\ &\left|\widetilde{l}(u(\zeta))\right|\le N_{\widetilde{l}}|u(\zeta)|,\\ &|\widetilde{m}(u(\zeta))|\le N_{\widetilde{m}}|u(\zeta)|,\\ &\left|\widehat{l}(u(\zeta))\right|\le N_{\widehat{l}}|u(\zeta)|,\\ &|m(u(\zeta))|\le N_{\widehat{m}}|u(\zeta)|,\end{aligned}$$

for all $u(\zeta) \in C(V, \mathbb{R}^+)$. In addition, there exist positive real constants $\Upsilon_l, \Upsilon_m, \Upsilon_{\widetilde{l}}, \Upsilon_{\widetilde{m}}, \Upsilon_{\widehat{l}}$, and $\Upsilon_{\widehat{m}}$ such that

$$\begin{aligned}&|l(u)-l(u^*)|\le\Upsilon_l|u-u^*|,\\ &|m(u)-m(u^*)|\le\Upsilon_m|u-u^*|,\\ &\left|\widetilde{l}(u)-\widetilde{l}(u^*)\right|\le\Upsilon_{\widetilde{l}}|u-u^*|,\\ &|\widetilde{m}(u)-\widetilde{m}(u^*)|\le\Upsilon_{\widetilde{m}}|u-u^*|,\\ &\left|\widehat{l}(u)-\widehat{l}(u^*)\right|\le\Upsilon_{\widehat{l}}|u-u^*|,\\ &|\widetilde{m}(u)-\widetilde{m}(u^*)|\le\Upsilon_{\widehat{m}}|u-u^*|,\end{aligned}$$

for each $\zeta \in V$ and all $u, u^* \in C(V, \mathbb{R})$.

(H_3) For $\zeta \in V$, there exist N_{ρ_1}, N_{ρ_2}, and N_{ρ_3} such that

$$\begin{aligned}&\int_0^1|\varphi(\zeta,r)|\,dr\le N_{\rho_1},\\ &\int_0^1|\widetilde{\varphi}(\zeta,r)|\,dr\le N_{\rho_2},\\ &\int_0^1|\widehat{\varphi}(\zeta,r)|\,dr\le N_{\rho_3}.\end{aligned}$$

Moreover, we assume

$$\begin{aligned}D_1&=\int_0^1\left|F_{\kappa_1}(\zeta,r)\right|a_1(r)dr<\infty,\\ E_1&=\int_0^1\left|F_{\kappa_1}(\zeta,r)\right|a_2(r)dr\\ &\quad+\int_0^1\left|F_{\kappa_1}(\zeta,r)\right|a_3(r)dr\\ &\quad+N_{\rho_1}\int_0^1\left|F_{\kappa_1}(\zeta,r)\right|a_4(r)dr\\ &\quad+\left(\frac{\delta(\varpi+\tau)(N_l+N_m)}{\varpi^2}\right)<\frac{1}{3},\end{aligned}$$

$$\begin{aligned}D_2&=\int_0^1\left|F_{\kappa_2}(\zeta,r)\right|b_1(r)dr<\infty,\\ E_2&=\int_0^1\left|F_{\kappa_2}(\zeta,r)\right|b_2(r)dr\\ &\quad+\int_0^1\left|F_{\kappa_2}(\zeta,r)\right|b_3(r)dr\\ &\quad+N_{\rho_2}\int_0^1\left|F_{\kappa_2}(\zeta,r)\right|b_4(r)dr\\ &\quad+\left(\frac{\widetilde{\delta}(\widetilde{\varpi}+\widetilde{\tau})(N_{\widetilde{l}}+N_{\widetilde{m}})}{\widetilde{\varpi}^2}\right)<\frac{1}{3},\end{aligned}$$

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and

$$D_3 = \int_0^1 \left|F_{\kappa_3}(\zeta, r)\right| e_1(r) dr < \infty,$$

$$E_3 = \int_0^1 \left|F_{\kappa_3}(\zeta, r)\right| e_2(r) dr + \int_0^1 \left|F_{\kappa_3}(\zeta, r)\right| e_3(r) dr + N_{\rho_3} \int_0^1 \left|F_{\kappa_3}(\zeta, r)\right| e_4(r) dr + \left(\frac{\widehat{\delta}(\widehat{\varpi}+\widehat{\tau})\left(N_{\widehat{l}}+N_{\widehat{m}}\right)}{\widehat{\varpi}^2}\right) < \frac{1}{3}.$$

(H$_4$) For each $\zeta \in V$, there exist constants $\Upsilon_{\phi_1}, \Upsilon_{\phi_2}, \Upsilon_{\phi_3}, \widetilde{\Upsilon}_{\phi_1}, \widetilde{\Upsilon}_{\phi_2}, \widetilde{\Upsilon}_{\phi_3}, \widehat{\Upsilon}_{\phi_1}, \widehat{\Upsilon}_{\phi_2} \widehat{\Upsilon}_{\phi_3} > 0$ such that

$$\begin{aligned}&|\phi_1(\zeta, v(\zeta), w(\zeta), \Omega^w(\zeta)) - \phi_1\left(\zeta, v^*(\zeta), w^*(\zeta), \Omega^{w^*}(\zeta)\right)| \\ &\leq \Upsilon_{\phi_1}|v(\zeta)-v^*(\zeta)| + \widetilde{\Upsilon}_{\phi_1}|w(\zeta)-w^*(\zeta)| + \widehat{\Upsilon}_{\phi_1}\left|\Omega^w(\zeta)-\Omega^{w^*}(\zeta)\right|,\end{aligned}$$

$$\begin{aligned}&|\phi_2(\zeta, w(\zeta), u(\zeta), \Omega^u(\zeta)) - \phi_2\left(\zeta, w^*(\zeta), u^*(\zeta), \Omega^{u^*}(\zeta)\right)| \\ &\leq \Upsilon_{\phi_2}|w(\zeta)-w^*(\zeta)| + \widetilde{\Upsilon}_{\phi_2}|u(\zeta)-u^*(\zeta)| + \widehat{\Upsilon}_{\phi_2}\left|\Omega^u(\zeta)-\Omega^{u^*}(\zeta)\right|,\end{aligned}$$

and

$$\begin{aligned}&|\phi_3(\zeta, u(\zeta), v(\zeta), \Omega^v(\zeta)) - \phi_3\left(\zeta, u^*(\zeta), v^*(\zeta), \Omega^{v^*}(\zeta)\right)| \\ &\leq \Upsilon_{\phi_3}|u(\zeta)-u^*(\zeta)| + \widetilde{\Upsilon}_{\phi_3}|v(\zeta)-v^*(\zeta)| + \widehat{\Upsilon}_{\phi_3}\left|\Omega^v(\zeta)-\Omega^{v^*}(\zeta)\right|,\end{aligned}$$

for all $u, v, w, u^*, v^*, w^* \in C(V, \mathbb{R})$.

(H$_5$) We assume that

(i) $\mathfrak{R}_1 = G_{\kappa_3 \widehat{\varpi}\widehat{\tau}}\left(\Upsilon_{\phi_3} + \widetilde{\Upsilon}_{\phi_3} + \widehat{\Upsilon}_{\phi_3} N_{\rho_3}\right) + \Upsilon_{\widehat{\varpi}\widehat{\tau}}$, where $\Upsilon_{\widehat{\varpi}\widehat{\tau}} = \frac{\widehat{\delta}(\widehat{\varpi}+\widehat{\tau})(\Upsilon_{\widehat{l}}+\Upsilon_{\widehat{m}})}{\widehat{\varpi}^2}$ and

$$G_{\kappa_3 \widehat{\varpi}\widehat{\tau}} = \max_{\zeta \in [0,1]} \left|\int_0^1 F_{\kappa_3}(\zeta, r) dr\right| = \frac{1}{\Gamma(1+\kappa_3)} + \frac{2(\widehat{\varpi}+\widehat{\tau})}{\Gamma(1+\kappa_3)} + \frac{2\left(\widehat{\tau}^2 + \widehat{\varpi}\widehat{\tau}\right)}{\widehat{\varpi}^2 \Gamma(\kappa_3)},$$

(ii) $\mathfrak{R}_2 = G_{\kappa_1 \widetilde{\varpi}\widetilde{\tau}}\left(\Upsilon_{\phi_1} + \widetilde{\Upsilon}_{\phi_1} + \widehat{\Upsilon}_{\phi_1} N_{\rho_1}\right) + \Upsilon_{\varpi\tau}$, where $\Upsilon_{\varpi\tau} = \frac{\delta(\varpi+\tau)(\Upsilon_l+\Upsilon_m)}{\varpi^2}$ and

$$G_{\kappa_1 \widetilde{\varpi}\widetilde{\tau}} = \max_{\zeta \in [0,1]} \left|\int_0^1 F_{\kappa_1}(\zeta, r) dr\right| = \frac{1}{\Gamma(1+\kappa_1)} + \frac{2(\widetilde{\varpi}+\widetilde{\tau})}{\Gamma(1+\kappa_1)} + \frac{2\left(\widetilde{\tau}^2 + \widetilde{\varpi}\widetilde{\tau}\right)}{\widetilde{\varpi}^2 \Gamma(\kappa_1)},$$

(iii) $\mathfrak{R}_3 = G_{\kappa_2 \varpi\tau}\left(\Upsilon_{\phi_2} + \widetilde{\Upsilon}_{\phi_2} + \widehat{\Upsilon}_{\phi_2} N_{\rho_2}\right) + \Upsilon_{\widetilde{\varpi}\widetilde{\tau}}$, where $\Upsilon_{\widetilde{\varpi}\widetilde{\tau}} = \frac{\widetilde{\delta}(\widetilde{\varpi}+\widetilde{\tau})(\Upsilon_{\widetilde{l}}+\Upsilon_{\widetilde{m}})}{\widetilde{\varpi}^2}$ and

$$G_{\kappa_2 \varpi\tau} = \max_{\zeta \in [0,1]} \left|\int_0^1 F_{\kappa_2}(\zeta, r) dr\right| = \frac{1}{\Gamma(1+\kappa_2)} + \frac{2(\varpi+\tau)}{\Gamma(1+\kappa_2)} + \frac{2\left(\tau^2 + \varpi\tau\right)}{\varpi^2 \Gamma(\kappa_2)}.$$

3. Existence theorem

This section establishes the existence and uniqueness of the solution for the system (1) under hypotheses (H$_1$)-(H$_5$).

Theorem 3.1 *Via the hypotheses (H$_1$)-(H$_5$), the fractional problem (1) has a unique solution, provided that*

$$\mathfrak{R} = \max\{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3\} < 1.$$

Proof. Assume that λ is a positive number such that

$$\lambda = \max\left\{\frac{2D_1}{1-2E_1}, \frac{2D_2}{1-2E_2}, \frac{2D_3}{1-2E_3}\right\}.$$

Define a set U as

$$U = \{(u, v, w) \in \Theta : \|(u, v, w)\|_\Theta \leq \lambda\}.$$

We begin with prove that the $\mathfrak{I}$ maps U into itself. Indeed, for $\zeta \in V$ and by the assertions (H$_1$)-(H$_3$), one has

$$\begin{aligned}&\left|\mathfrak{I}_{\kappa_1}(u, v, w)(\zeta)\right| \\ \leq& \int_0^1 \left|F_{\kappa_1}(\zeta, r)\right| |\phi_1(r, v(r), w(r), \Omega^w(r))| dr \\ &+ \frac{\delta}{\varpi^2}\left[|(\varpi(1-\zeta)+\tau)| \int_0^1 |l(u(r))| dr + |(\tau+\varpi\zeta)| \int_0^1 |m(u(r))| dr\right] \\ \leq& \int_0^1 \left|F_{\kappa_1}(\zeta, r)\right| a_1(r) dr \\ &+ \int_0^1 \left|F_{\kappa_1}(\zeta, r)\right| [a_2(r)|v(r)|] dr \\ &+ \int_0^1 \left|F_{\kappa_1}(\zeta, r)\right| [a_3(r)|w(r)|] dr \\ &+ \int_0^1 \left|F_{\kappa_1}(\zeta, r)\right| [a_4(r)|\Omega^w(\zeta)|] dr \\ &+ \frac{\delta}{\varpi^2}\left[|(\varpi+\tau)| \int_0^1 |l(u(r))| dr + |(\tau+\varpi)| \int_0^1 |m(u(r))| dr\right]\end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_0^1 \left|F_{\kappa_1}(\zeta,r)\right| a_1(r)\,dr \\
& +\lambda \left[\int_0^1 \left|F_{\kappa_1}(\zeta,r)\right| a_2(r)\,dr \right.\\
& + \int_0^1 \left|F_{\kappa_1}(\zeta,r)\right| a_3(r)\,dr \\
& + N_{\rho_1} \int_0^1 \left|F_{\kappa_1}(\zeta,r)\right| a_4(r)\,dr \\
& \left. + \frac{\delta(\varpi+\tau)(N_l+N_m)}{\varpi^2}\right] \\
=\ & D_1 + \lambda E_1.
\end{aligned}
$$

Maximizing both sides of the inequality over the set $\mathfrak{I}$ gives

$$\left\|\mathfrak{I}_{\kappa_1}(u,v,w)\right\|_A \leq \frac{\lambda}{3}.$$

Similarly, we have

$$\left\|\mathfrak{I}_{\kappa_2}(v,w,u)(\zeta)\right\|_A \leq \frac{\lambda}{3} \text{ and } \left\|\mathfrak{I}_{\kappa_3}(w,u,v)(\zeta)\right\|_A \leq \frac{\lambda}{3}.$$

Hence,

$$
\begin{aligned}
\|\mathfrak{I}(u,v,w)\|_\Theta &= \left\|\mathfrak{I}_{\kappa_1}(u,v,w)\right\|_A \\
&+ \left\|\mathfrak{I}_{\kappa_2}(v,w,u)(\zeta)\right\|_A + \left\|\mathfrak{I}_{\kappa_3}(w,u,v)(\zeta)\right\|_A \leq \lambda.
\end{aligned}
$$

This proves that, $\mathfrak{I}$ is maps U into itself.

Next, we show that $\mathfrak{I}$ is a contraction mapping. For each $\zeta \in V$, by the assumptions (H_2)-(H_5), we can write

$$
\begin{aligned}
& \left|\mathfrak{I}_{\kappa_1}(u,v,w)(\zeta) - \mathfrak{I}_{\kappa_1}(u^*,v^*,w^*)(\zeta)\right| \qquad (2)\\
&\leq \int_0^1 \left|F_{\kappa_1}(\zeta,r)\right| \left|\phi_1(r,v(r),w(r),\Omega^w(r))\right.\\
&\left. -\phi_1\left(r,v^*(r),w^*(r),\Omega^{w^*}(r)\right)\right| dr \\
&+\frac{\delta}{\varpi^2}\left[\left|(\varpi(1-\zeta)+\tau)\right| \int_0^1 \left|l(u(r)) - l(u^*(r))\right| dr\right.\\
&\left. + \left|(\tau+\varpi\zeta)\right| \int_0^1 \left|m(u(r)) - m(u^*(r))\right| dr\right] \\
&\leq \int_0^1 \left|F_{\kappa_1}(\zeta,r)\right| \left(\Upsilon_{\phi_1}\left|v(\zeta)-v^*(\zeta)\right|\right.\\
&\left. +\widetilde{\Upsilon}_{\phi_1}\left|w(\zeta)-w^*(\zeta)\right| + \widehat{\Upsilon}_{\phi_1}\left|\Omega^w(\zeta)-\Omega^{w^*}(\zeta)\right|\right) \\
&+\frac{\delta(\varpi+\tau)}{\varpi^2}\left(\Upsilon_l \int_0^1 \left|u(r)-u^*(r)\right| dr\right.\\
&\left. +\Upsilon_m \int_0^1 \left|u(r)-u^*(r)\right| dr\right) \\
&\leq G_{\kappa_1\varpi\tau}\left(\Upsilon_{\phi_1}\left|v(r)-v^*(r)\right|\right.\\
&\left. +\widetilde{\Upsilon}_{\phi_1}\left|w(r)-w^*(r)\right| + \widehat{\Upsilon}_{\phi_1}\left|\Omega^w(\zeta)-\Omega^{w^*}(\zeta)\right|\right) \\
&+\frac{\delta(\varpi+\tau)(\Upsilon_l+\Upsilon_m)}{\varpi^2}\left|u(r)-u^*(r)\right|.
\end{aligned}
$$

Since

$$
\begin{aligned}
& \left|\Omega^w(\zeta) - \Omega^{w^*}(\zeta)\right| \qquad (3)\\
&= \left|\int_0^\zeta \varphi(\zeta,r)\,w(r)\,dr - \int_0^\zeta \varphi(\zeta,r)\,w^*(r)\,dr\right| \\
&\leq \int_0^\zeta \left|\varphi(\zeta,r)\right| \left|w(r)-w^*(r)\right| dr \\
&\leq N_{\rho_1}\left|w(r)-w^*(r)\right|.
\end{aligned}
$$

It follows from (2) and (3) that

$$
\begin{aligned}
& \left|\mathfrak{I}_{\kappa_1}(u,v,w)(\zeta) - \mathfrak{I}_{\kappa_1}(u^*,v^*,w^*)(\zeta)\right| \\
&\leq G_{\kappa_1\varpi\tau}\left(\Upsilon_{\phi_1}\left|v(r)-v^*(r)\right|\right.\\
&\left. +\left(\widetilde{\Upsilon}_{\phi_1}+\widehat{\Upsilon}_{\phi_1}N_{\rho_1}\right)\left|w(r)-w^*(r)\right|\right) \\
&+\frac{\delta(\varpi+\tau)(\Upsilon_l+\Upsilon_m)}{\varpi^2}\left|u(r)-u^*(r)\right|.
\end{aligned}
$$

Maximizing both sides of the inequality over $\mathfrak{I}_{\kappa_1}$ yields,

$$
\begin{aligned}
& \left\|\mathfrak{I}_{\kappa_1}(u,v,w) - \mathfrak{I}_{\kappa_1}(u^*,v^*,w^*)\right\|_A \qquad (4)\\
&\leq G_{\kappa_1\varpi\tau}\Upsilon_{\phi_1}\left\|v-v^*\right\|_A \\
&+G_{\kappa_1\varpi\tau}\left(\widetilde{\Upsilon}_{\phi_1}+\widehat{\Upsilon}_{\phi_1}N_{\rho_1}\right)\left\|w-w^*\right\|_A \\
&+\frac{\delta(\varpi+\tau)(\Upsilon_l+\Upsilon_m)}{\varpi^2}\left\|u-u^*\right\|_A.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \left\|\mathfrak{I}_{\kappa_2}(v,w,u) - \mathfrak{I}_{\kappa}(v^*,w^*,u^*)\right\|_A \\
&\leq G_{\kappa_2\varpi\tau}\Upsilon_{\phi_2}\left\|w-w^*\right\|_A \\
&+G_{\kappa_2\varpi\tau}\left(\widetilde{\Upsilon}_{\phi_2}+\widehat{\Upsilon}_{\phi_2}N_{\rho_2}\right)\left\|u-u^*\right\|_A \\
&+\frac{\widehat{\delta}(\widehat{\varpi}+\widehat{\tau})\left(\Upsilon_{\widehat{l}}+\Upsilon_{\widehat{m}}\right)}{\widehat{\varpi}^2}\left\|v-v^*\right\|_A,
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\|\mathfrak{I}_{\kappa_3}(w,u,v) - \mathfrak{I}_{\kappa}(w^*,u^*,v^*)\right\|_A \qquad (5)\\
&\leq G_{\kappa_3\widetilde{\varpi}\widetilde{\tau}}\Upsilon_{\phi_3}\left\|u-u^*\right\|_A \\
&+G_{\kappa_3\widetilde{\varpi}\widetilde{\tau}}\left(\widetilde{\Upsilon}_{\phi_3}+\widehat{\Upsilon}_{\phi_3}N_{\rho_3}\right)\left\|v-v^*\right\|_A \\
&+\frac{\widetilde{\delta}(\widetilde{\varpi}+\widetilde{\tau})\left(\Upsilon_{\widetilde{l}}+\Upsilon_{\widetilde{m}}\right)}{\widetilde{\varpi}^2}\left\|w-w^*\right\|_A.
\end{aligned}
$$

From (4)-(5), we get

$$
\begin{aligned}
& \left\|\mathfrak{I}(u,v,w) - \mathfrak{I}(u^*,v^*,w^*)\right\|_\Theta \\
&\leq \mathfrak{R}\left\|(u,v,w)-(u^*,v^*,w^*)\right\|_\Theta,
\end{aligned}
$$

where $\Theta = A \times A \times A$. Since $\mathfrak{R} < 1$, $\mathfrak{I}$ is a contraction mapping. By employing Banach's fixed-point theorem, we conclude that $\mathfrak{I}$ has a unique fixed point, which is a unique solution to the tripled problem (1).

4. Stability results

This section is devoted to investigating the Ulam-type stability characteristics of the proposed coupled system (1). Specifically, we aim to determine the conditions under which small perturbations in the system's equations lead only to small changes in its solutions, confirming its robustness.

Assume the following inequality holds for some real number $\ell = \max\left\{\ell_{\kappa_1}, \ell_{\kappa_2}, \ell_{\kappa_3}\right\} > 0$:

$$\left\{\begin{array}{l} |{}^{c}D^{\kappa_1}u(\zeta) - \phi_1(\zeta, v(\zeta), w(\zeta), \Omega^{w}(\zeta))| \leq \ell_{\kappa_1},\ \zeta \in V, \\ |{}^{c}D^{\kappa_2}v(\zeta) - \phi_2(\zeta, w(\zeta), u(\zeta), \Omega^{u}(\zeta))| \leq \ell_{\kappa_2},\ \zeta \in V, \\ |{}^{c}D^{\kappa_3}w(\zeta) - \phi_3(\zeta, u(\zeta), v(\zeta), \Omega^{v}(\zeta))| \leq \ell_{\kappa_3},\ \zeta \in V. \end{array}\right. \tag{6}$$

Motivated by the definitions of Ulam stability in [20], we introduce the definitions below:

Definition 4.1 The tripled system (1) exhibits Ulam-Hyers stability if, for every solution $(u, v, w) \in \Theta$ of the inequality (1), there exists a unique exact solution $(\overline{u}, \overline{v}, \overline{w}) \in \Theta$ of the problem (6) such that

$$|(u, v, w)(\zeta) - (\overline{u}, \overline{v}, \overline{w})(\zeta)| \leq G_{\kappa_1\kappa_2\kappa_3\varpi\tau\widetilde{\varpi}\widetilde{\tau}\widehat{\varpi}\widehat{\tau}}\ell,\ \zeta \in V.$$

Definition 4.2 The tripled system (1) exhibits generalized Ulam-Hyers stability if, for every solution $(u, v, w) \in \Theta$ of the inequality (1), there exists a function $J(\zeta) \in C(\mathbb{R}\mathbb{R}^{+})$ with $J(0) = 0$ and there exists a unique exact solution $(\overline{u}, \overline{v}, \overline{w}) \in \Theta$ of (6) such that

$$|(u, v, w)(\zeta) - (\overline{u}, \overline{v}, \overline{w})(\zeta)| \leq J(\ell),\ \zeta \in V.$$

Remark 4.3 A trio $(, v, w)$ is considered a solution of the system (6) if there exist functions $K(u), L(v), M(w) \in C(V, \mathbb{R})$ such that

(i) For all $\zeta \in V$, $|K(\zeta)| \leq \ell_{\kappa_1}$, $|M(\zeta)| \leq \ell_{\kappa_2}$ and $|L(\zeta)| \leq \ell_{\kappa_3}$;

(ii)

$$\left\{\begin{array}{l} {}^{c}D^{\kappa_1}u(\zeta) = \phi_1(\zeta, v(\zeta), w(\zeta), \Omega^{w}(\zeta)) + K(\zeta),\ \zeta \in V, \\ {}^{c}D^{\kappa_2}v(\zeta) = \phi_2(\zeta, w(\zeta), u(\zeta), \Omega^{u}(\zeta)) + M(\zeta),\ \zeta \in V, \\ {}^{c}D^{\kappa_3}w(\zeta) = \phi_3(\zeta, u(\zeta), v(\zeta), \Omega^{v}(\zeta)) + L(\zeta),\ \zeta \in V. \end{array}\right.$$

Lemma 4.4 For any solution $(u, v, w) \in \Theta$ to the problem (6), the following inequalities are fulfilled:

$$\left\{\begin{array}{l} \Big|u(\zeta) - \int_0^1 F_{\kappa_1}(\zeta, r)\,\phi_1(r, v(r), w(r), \Omega^{w}(r))\,dr \\ \quad - \frac{\delta}{\varpi^2}\Big[(\varpi(1-\zeta) + \tau)\int_0^1 l(u(r))\,dr \\ \quad +(\tau + \varpi\zeta)\int_0^1 m(u(r))\,dr\Big]\Big| \leq G_{\kappa_1\varpi\tau}\ell_{\kappa_1}, \\ \\ \Big|v(\zeta) - \int_0^1 F_{\kappa_2}(\zeta, r)\,\phi_2(r, w(r), u(r), \Omega^{u}(r))\,dr \\ \quad - \frac{\widetilde{\delta}}{\widetilde{\varpi}^2}\Big[(\widetilde{\varpi}(1-\zeta) + \widetilde{\tau})\int_0^1 \widetilde{l}(v(r))\,dr \\ \quad +(\widetilde{\tau} + \widetilde{\varpi}\zeta)\int_0^1 \widetilde{m}(v(r))\,dr\Big]\Big| \leq G_{\kappa_2\widetilde{\varpi}\widetilde{\tau}}\ell_{\kappa_2}, \\ \\ \Big|w(\zeta) - \int_0^1 F_{\kappa_3}(\zeta, r)\,\phi_3(r, u(r), v(r), \Omega^{v}(r))\,dr \\ \quad - \frac{\widehat{\delta}}{\widehat{\varpi}^2}\Big[(\widehat{\varpi}(1-\zeta) + \widehat{\tau})\int_0^1 \widehat{l}(w(r))\,dr \\ \quad +(\widehat{\tau} + \widehat{\varpi}\zeta)\int_0^1 \widehat{m}(w(r))\,dr\Big]\Big| \leq G_{\kappa_3\widehat{\varpi}\widehat{\tau}}\ell_{\kappa_3},\ \zeta \in V. \end{array}\right.$$

Proof. It follows from Remark 4.3 (ii) that

$$\left\{\begin{array}{l} {}^{c}D^{\kappa_1}u(\zeta) = \\ \quad \phi_1(\zeta, v(\zeta), w(\zeta), \Omega^{w}(\zeta)) + K(\zeta),\ \zeta \in V, \\ {}^{c}D^{\kappa_2}v(\zeta) = \\ \quad \phi_2(\zeta, w(\zeta), u(\zeta), \Omega^{u}(\zeta)) + M(\zeta),\ \zeta \in V, \\ {}^{c}D^{\kappa_3}w(\zeta) = \\ \quad \phi_3(\zeta, u(\zeta), v(\zeta), \Omega^{v}(\zeta)) + L(\zeta),\ \zeta \in V, \\ \varpi u(0) + \tau u'(0) = \delta\int_0^1 l(u(r))\,dr, \\ \varpi u(1) + \tau u'(1) = \delta\int_0^1 m(u(r))\,dr, \\ \widetilde{\varpi}v(0) + \widetilde{\tau}v'(0) = \widetilde{\delta}\int_0^1 \widetilde{l}(v(r))\,dr, \\ \widetilde{\varpi}v(1) + \widetilde{\tau}v'(1) = \widetilde{\delta}\int_0^1 \widetilde{m}(v(r))\,dr, \\ \widehat{\varpi}w(0) + \widehat{\tau}w'(0) = \widehat{\delta}\int_0^1 \widehat{l}(w(r))\,dr, \\ \widehat{\varpi}w(1) + \widehat{\tau}w'(1) = \widehat{\delta}\int_0^1 \widehat{m}(w(r))\,dr. \end{array}\right. \tag{7}$$

By Lemma 2.3, the solution of (7) is given by

$$\left\{\begin{array}{l} u(\zeta) = \int_0^1 F_{\kappa_1}(\zeta, r)\,\phi_1(r, v(r), w(r), \Omega^{w}(r))\,dr \\ \qquad + \int_0^1 F_{\kappa_1}(\zeta, r)\,K(r)dr \\ \qquad + \frac{\delta}{\varpi^2}\Big[(\varpi(1-\zeta) + \tau)\int_0^1 l(u(r))\,dr \\ \qquad + (\tau + \varpi\zeta)\int_0^1 m(u(r))\,dr\Big],\ \zeta \in V, \\ \\ v(\zeta) = \int_0^1 F_{\kappa_2}(\zeta, r)\,\phi_2(r, w(r), u(r), \Omega^{u}(r))\,dr \\ \qquad + \int_0^1 F_{\kappa_2}(\zeta, r)\,M(r)dr \\ \qquad + \frac{\widetilde{\delta}}{\widetilde{\varpi}^2}\Big[(\widetilde{\varpi}(1-\zeta) + \widetilde{\tau})\int_0^1 \widetilde{l}(v(r))\,dr \\ \qquad + (\widetilde{\tau} + \widetilde{\varpi}\zeta)\int_0^1 \widetilde{m}(v(r))\,dr\Big],\ \zeta \in V, \\ \\ w(\zeta) = \int_0^1 F_{\kappa_3}(\zeta, r)\,\phi_3(r, u(r), v(r), \Omega^{v}(r))\,dr \\ \qquad + \int_0^1 F_{\kappa_3}(\zeta, r)\,L(r)dr \\ \qquad + \frac{\widehat{\delta}}{\widehat{\varpi}^2}\Big[(\widehat{\varpi}(1-\zeta) + \widehat{\tau})\int_0^1 \widehat{l}(w(r))\,dr \\ \qquad + (\widehat{\tau} + \widehat{\varpi}\zeta)\int_0^1 \widehat{m}(w(r))\,dr\Big]\ \zeta \in V. \end{array}\right. \tag{8}$$

Applying Remark 4.3 (i) in the first equation of (8) and using the hypothesis (H_5), we can write

$$\begin{array}{l} \Big|u(\zeta) - \int_0^1 F_{\kappa_1}(\zeta, r)\,\phi_1(r, v(r), w(r), \Omega^{w}(r))\,dr \\ \quad - \frac{\delta}{\varpi^2}\Big[(\varpi(1-\zeta) + \tau)\int_0^1 l(u(r))\,dr \\ \quad + (\tau + \varpi\zeta)\int_0^1 m(u(r))\,dr\Big]\Big| \\ \quad \leq \Big|\int_0^1 F_{\kappa_1}(\zeta, r)\,K(r)dr\Big| \\ \quad \leq \int_0^1 \Big|F_{\kappa_1}(\zeta, r)\,K(r)\Big|dr \leq G_{\kappa_1\varpi\tau}\ell_{\kappa_1}. \end{array}$$

Similarly, we have

$$\begin{array}{l} \Big|v(\zeta) - \int_0^1 F_{\kappa_2}(\zeta, r)\phi_2(r, w(r), u(r), \Omega^{u}(r))\,dr \\ \quad - \frac{\widetilde{\delta}}{\widetilde{\varpi}^2}\Big[(\widetilde{\varpi}(1-\zeta) + \widetilde{\tau})\int_0^1 \widetilde{l}(v(r))\,dr \\ \quad + (\widetilde{\tau} + \widetilde{\varpi}\zeta)\int_0^1 \widetilde{m}(v(r))\,dr\Big]\Big| \leq G_{\kappa_2\widetilde{\varpi}\widetilde{\tau}}\ell_{\kappa_2}, \end{array}$$

and

$$\begin{array}{l} \Big|w(\zeta) - \int_0^1 F_{\kappa_3}(\zeta, r)\phi_3(r, u(r), v(r), \Omega^{v}(r))dr \\ \quad - \frac{\widehat{\delta}}{\widehat{\varpi}^2}\Big[(\widehat{\varpi}(1-\zeta) + \widehat{\tau})\int_0^1 \widehat{l}(w(r))\,dr \\ \quad + (\widehat{\tau} + \widehat{\varpi}\zeta)\int_0^1 \widehat{m}(w(r))\,dr\Big]\Big| \leq G_{\kappa_3\widehat{\varpi}\widehat{\tau}}\ell_{\kappa_3}. \end{array}$$

for each $\zeta \in V$. This finishes the required.

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Theorem 4.5 The fractional order tripled system (1) is Ulam-Hyers stable and also generalized Ulam-Hyers stable, provided that the hypotheses (H_3)-(H_5) are satisfied and

$$\begin{aligned}\Lambda \;=\; & (1-\Upsilon_{\varpi\tau})\left[(1-\Upsilon_{\widetilde{\varpi}\widetilde{\tau}})(1-\Upsilon_{\widehat{\varpi}\widehat{\tau}})-\Upsilon_{\kappa_2\widetilde{\varpi}\widehat{\tau}}\Upsilon_{\kappa_2\varpi\tau}\right]\\ & -\Upsilon_{\kappa_1\widetilde{\varpi}\tau}\left[\Upsilon_{\kappa_2\widehat{\varpi}\widetilde{\tau}}(1-\Upsilon_{\widehat{\varpi}\widetilde{\tau}})+\Upsilon_{\kappa_2\widehat{\varpi}\widetilde{\tau}}\Upsilon_{\kappa_3\varpi\tau}\right]\\ & -\Upsilon_{\kappa_1\widetilde{\varpi}\widetilde{\tau}}\left[\Upsilon_{\kappa_2\widehat{\varpi}\widetilde{\tau}}\Upsilon_{\kappa_2\varpi\tau}+(1-\Upsilon_{\widetilde{\varpi}\widetilde{\tau}})\Upsilon_{\kappa_3\varpi\tau}\right]\neq 0.\end{aligned}$$

Proof. Suppose that $(u,v,w)\in\Theta$ is a unique solution to the fractional system (1), and $(u^*,v^*,w^*)\in\Theta$ is a unique solution to the following problem:

$$\begin{cases} {}^cD^{\kappa_1}u^*(\zeta)=\phi_1\left(\zeta,v^*(\zeta),w^*(\zeta),\Omega^{w^*}(\zeta)\right),\\ \qquad\qquad\qquad\qquad \zeta\in V=[0,1],\\ {}^cD^{\kappa_2}v^*(\zeta)=\phi_2(\zeta,w^*(\zeta),u^*(\zeta),\Omega^{u^*}(\zeta)),\ \zeta\in V,\\ {}^cD^{\kappa_3}w^*(\zeta)=\phi_3(\zeta,u^*(\zeta),v^*(\zeta),\Omega^{v^*}(\zeta)),\ \zeta\in V,\\ \varpi u^*(0)+\tau u^{*\prime}(0)=\delta\int_0^1 l(u^*(r))\,dr,\\ \varpi u^*(1)+\tau u^{*\prime}(1)=\delta\int_0^1 m(u^*(r))\,dr,\\ \widetilde{\varpi}v^*(0)+\widetilde{\tau}v^{*\prime}(0)=\widetilde{\delta}\int_0^1\widetilde{l}(v^*(r))\,dr,\\ \widetilde{\varpi}v^*(1)+\widetilde{\tau}v^{*\prime}(1)=\widetilde{\delta}\int_0^1\widetilde{m}(v^*(r))\,dr,\\ \widehat{\varpi}w^*(0)+\widehat{\tau}w^{*\prime}(0)=\widehat{\delta}\int_0^1\widehat{l}(w^*(r))\,dr,\\ \widehat{\varpi}w^*(1)+\widehat{\tau}w^{*\prime}(1)=\widehat{\delta}\int_0^1\widehat{m}(w^*(r))\,dr.\end{cases}$$

By Lemmas 2.3 and 4.4, the solution of the above system can be expressed as

$$\begin{cases} u^*\zeta)=\int_0^1 F_{\kappa_1}(\zeta,r)\phi_1(r,v^*(r),w^*(r),\Omega^{w^*}(r))dr\\ \qquad +\frac{\delta}{\varpi^2}\left[(\varpi(1-\zeta)+\tau)\int_0^1 l(u^*(r))\,dr\right.\\ \qquad \left.+(\tau+\varpi\zeta)\int_0^1 m(u^*(r))\,dr\right],\\[2ex] v^*(\zeta)=\int_0^1 F_{\kappa_2}(\zeta,r)\phi_2(r,w^*(r),u^*(r),\Omega^{u^*}(r))dr\\ \qquad +\frac{\widetilde{\delta}}{\widetilde{\varpi}^2}\left[(\widetilde{\varpi}(1-\zeta)+\widetilde{\tau})\int_0^1\widetilde{l}(v^*(r))\,dr\right.\\ \qquad \left.+(\widetilde{\tau}+\widetilde{\varpi}\zeta)\int_0^1\widetilde{m}(v^*(r))\,dr\right],\\[2ex] w^*(\zeta)=\int_0^1 F_{\kappa_3}(\zeta,r)\phi_3(r,u^*(r),v^*(r),\Omega^{v^*}(r))dr\\ \qquad +\frac{\widehat{\delta}}{\widehat{\varpi}^2}\left[(\widehat{\varpi}(1-\zeta)+\widehat{\tau})\int_0^1\widehat{l}(w^*(r))\,dr\right.\\ \qquad \left.+(\widehat{\tau}+\widehat{\varpi}\zeta)\int_0^1\widehat{m}(w^*(r))\,dr\right],\ \zeta\in V.\end{cases}$$

Hence, we get

$$\begin{aligned}&|u(\zeta)-u^*(\zeta)|\\ &=\left|u(\zeta)-\int_0^1 F_{\kappa_1}(\zeta,r)\phi_1(r,v^*(r),w^*(r),\Omega^{w^*}(r))dr\right.\\ &\quad-\frac{\delta}{\varpi^2}\left[(\varpi(1-\zeta)+\tau)\int_0^1 l(u^*(r))\,dr\right.\\ &\quad\left.\left.+(\tau+\varpi\zeta)\int_0^1 m(u^*(r))\,dr\right]\right|\\ &\le\left|u(\zeta)-\int_0^1 F_{\kappa_1}(\zeta,r)\phi_1(r,v(r),w(r),\Omega^{w}(r))dr\right.\\ &\quad-\frac{\delta}{\varpi^2}\left[(\varpi(1-\zeta)+\tau)\int_0^1 l(u(r))\,dr\right.\\ &\quad\left.\left.+(\tau+\varpi\zeta)\int_0^1 m(u(r))\,dr\right]\right|\\ &\quad+\left|\int_0^1 F_{\kappa_1}(\zeta,r)\left[\phi_1(r,v(r),w(r),\Omega^{w}(r))\right.\right.\\ &\quad\left.\left.-\phi_1\left(r,v^*(r),w^*(r),\Omega^{w^*}(r)\right)\right]dr\right|\\ &\quad+\left|\frac{\delta}{\varpi^2}\left[(\varpi(1-\zeta)+\tau)\int_0^1[l(u(r))-l(u^*(r))]\,dr\right.\right.\\ &\quad\left.\left.+(\tau+\varpi\zeta)\int_0^1[m(u(r))-m(u^*(r))]\,dr\right]\right|\\ &\le G_{\kappa_1\varpi\tau}\ell_{\kappa_1}+\int_0^1\left|F_{\kappa_1}(\zeta,r)\right|\left(\Upsilon_{\phi_1}|v(\zeta)-v^*(\zeta)|\right.\\ &\quad\left.+\widetilde{\Upsilon}_{\phi_1}|w(\zeta)-w^*(\zeta)|+\widehat{\Upsilon}_{\phi_1}\left|\Omega^{w}(\zeta)-\Omega^{w^*}(\zeta)\right|\right)\\ &\quad+\frac{\delta(\varpi+\tau)}{\varpi^2}\left(\Upsilon_l\int_0^1|u(r)-u^*(r)|\,dr\right.\\ &\quad\left.+\Upsilon_m\int_0^1|u(r)-u^*(r)|\,dr\right)\\ &\le G_{\kappa_1\varpi\tau}\ell_{\kappa_1}+G_{\kappa_1\widetilde{\varpi}\widetilde{\tau}}\left(\Upsilon_{\phi_1}|v(r)-v^*(r)|+\right.\\ &\quad\left.\left(\widetilde{\Upsilon}_{\phi_1}+\widehat{\Upsilon}_{\phi_1}N_{\rho_1}\right)|w(r)-w^*(r)|\right)\\ &\quad+\frac{\delta(\varpi+\tau)(\Upsilon_l+\Upsilon_m)}{\varpi^2}|u(r)-u^*(r)|\\ &\le G_{\kappa_1\varpi\tau}\ell_{\kappa_1}+\Upsilon_{\kappa_1\widetilde{\varpi}\widetilde{\tau}}|v(r)-v^*(r)|\\ &\quad+\Upsilon_{\kappa_1\widetilde{\varpi}\widetilde{\tau}}|w(r)-w^*(r)|+\Upsilon_{\varpi\tau}|u(r)-u^*(r)|,\end{aligned}$$

where $\Upsilon_{\kappa_1\widetilde{\varpi}\widetilde{\tau}}=G_{\kappa_1\widetilde{\varpi}\widetilde{\tau}}\left(\Upsilon_{\phi_1}+\widetilde{\Upsilon}_{\phi_1}+\widehat{\Upsilon}_{\phi_1}N_{\rho_1}\right)$ and $\Upsilon_{\varpi\tau}=\frac{\delta(\varpi+\tau)(\Upsilon_l+\Upsilon_m)}{\varpi^2}$. Therefore, one has

$$\begin{aligned}(1-\Upsilon_{\varpi\tau})\|u-u^*\|_A &\le G_{\kappa_1\varpi\tau}\ell_{\kappa_1}\\ &\quad+\Upsilon_{\kappa_1\widetilde{\varpi}\widetilde{\tau}}\|v-v^*\|_A+\Upsilon_{\kappa_1\widetilde{\varpi}\widetilde{\tau}}\|w-w^*\|_A.\end{aligned}\tag{9}$$

Similarly, we have

$$\begin{aligned}(1-\Upsilon_{\widetilde{\varpi}\widetilde{\tau}})\|v-v^*\|_A &\le G_{\kappa_2\widetilde{\varpi}\widetilde{\tau}}\ell_{\kappa_2}\\ &\quad+\Upsilon_{\kappa_2\widehat{\varpi}\widetilde{\tau}}\|w-w^*\|_A+\Upsilon_{\kappa_2\widehat{\varpi}\widetilde{\tau}}\|u-u^*\|_A,\end{aligned}\tag{10}$$

and

$$\begin{aligned}(1-\Upsilon_{\widehat{\varpi}\widehat{\tau}})\|w-w^*\|_A &\le G_{\kappa_3\widehat{\varpi}\widehat{\tau}}\ell_{\kappa_3}\\ &\quad+\Upsilon_{\kappa_3\varpi\tau}\|u-u^*\|_A+\Upsilon_{\kappa_2\varpi\tau}\|v-v^*\|_A,\end{aligned}\tag{11}$$

where $\Upsilon_{\kappa_2\widehat{\varpi}\widetilde{\tau}}=G_{\kappa_2\widehat{\varpi}\widetilde{\tau}}\left(\Upsilon_{\phi_2}+\widetilde{\Upsilon}_{\phi_2}+\widehat{\Upsilon}_{\phi_2}N_{\rho_2}\right)$, $\Upsilon_{\kappa_3\varpi\tau}=G_{\kappa_3\varpi\tau}\left(\Upsilon_{\phi_3}+\widetilde{\Upsilon}_{\phi_3}+\widehat{\Upsilon}_{\phi_3}N_{\rho_3}\right)$, $\Upsilon_{\widetilde{\varpi}\widetilde{\tau}}=\frac{\widetilde{\delta}(\widetilde{\varpi}+\widetilde{\tau})(\Upsilon_{\widetilde{l}}+\Upsilon_{\widetilde{m}})}{\widetilde{\varpi}^2}$ and $\Upsilon_{\widehat{\varpi}\widehat{\tau}}=\frac{\widehat{\delta}(\widehat{\varpi}+\widehat{\tau})(\Upsilon_{\widehat{l}}+\Upsilon_{\widehat{m}})}{\widehat{\varpi}^2}$.

Equations (9)-(11) allow us to write

$$\begin{cases} (1-\Upsilon_{\varpi\tau})\,\|u-u^*\|_A - \Upsilon_{\kappa_1\widetilde{\varpi\tau}}\,\|v-v^*\|_A \\ \qquad -\Upsilon_{\kappa_1\widetilde{\varpi\tau}}\,\|w-w^*\|_A \le G_{\kappa_1\varpi\tau}\ell_{\kappa_1}, \\ (1-\Upsilon_{\widetilde{\varpi\tau}})\,\|v-v^*\|_A - \Upsilon_{\kappa_2\widehat{\varpi\tau}}\,\|w-w^*\|_A \\ \qquad -\Upsilon_{\kappa_2\widehat{\varpi\tau}}\,\|u-u^*\|_A \le G_{\kappa_2\widetilde{\varpi\tau}}\ell_{\kappa_2}, \\ (1-\Upsilon_{\widehat{\varpi\tau}})\,\|w-w^*\|_A - \Upsilon_{\kappa_3\varpi\tau}\,\|u-u^*\|_A \\ \qquad -\Upsilon_{\kappa_2\varpi\tau}\,\|v-v^*\|_A \le G_{\kappa_3\widehat{\varpi\tau}}\ell_{\kappa_3}. \end{cases}$$

The equation can be expressed in a matrix form as

$$\begin{pmatrix} (1-\Upsilon_{\varpi\tau}) & -\Upsilon_{\kappa_1\widetilde{\varpi\tau}} & -\Upsilon_{\kappa_1\widetilde{\varpi\tau}} \\ -\Upsilon_{\kappa_2\widehat{\varpi\tau}} & (1-\Upsilon_{\widetilde{\varpi\tau}}) & -\Upsilon_{\kappa_2\widehat{\varpi\tau}} \\ -\Upsilon_{\kappa_3\varpi\tau} & -\Upsilon_{\kappa_2\varpi\tau} & (1-\Upsilon_{\widehat{\varpi\tau}}) \end{pmatrix} \times \begin{pmatrix} \|u-u^*\|_A \\ \|v-v^*\|_A \\ \|w-w^*\|_A \end{pmatrix} \le \begin{pmatrix} G_{\kappa_1\varpi\tau}\ell_{\kappa_1} \\ G_{\kappa_2\widetilde{\varpi\tau}}\ell_{\kappa_2} \\ G_{\kappa_3\widehat{\varpi\tau}}\ell_{\kappa_3} \end{pmatrix}.$$

Using the inverse of the matrix, one can write

$$\begin{pmatrix} \|u-u^*\|_A \\ \|v-v^*\|_A \\ \|w-w^*\|_A \end{pmatrix} \le \frac{1}{\Lambda}\begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \Lambda_4 & \Lambda_5 & \Lambda_6 \\ \Lambda_7 & \Lambda_8 & \Lambda_9 \end{pmatrix}\begin{pmatrix} G_{\kappa_1\varpi\tau}\ell_{\kappa_1} \\ G_{\kappa_2\widetilde{\varpi\tau}}\ell_{\kappa_2} \\ G_{\kappa_3\widehat{\varpi\tau}}\ell_{\kappa_3} \end{pmatrix}, \tag{12}$$

where

$$\begin{cases} \Lambda = (1-\Upsilon_{\varpi\tau})\left[(1-\Upsilon_{\widetilde{\varpi\tau}})(1-\Upsilon_{\widehat{\varpi\tau}}) \right. \\ \left. -\Upsilon_{\kappa_2\widehat{\varpi\tau}}\Upsilon_{\kappa_2\varpi\tau}\right] \\ -\Upsilon_{\kappa_1\widetilde{\varpi\tau}}\left[\Upsilon_{\kappa_2\widehat{\varpi\tau}}(1-\Upsilon_{\widehat{\varpi\tau}}) + \Upsilon_{\kappa_2\widehat{\varpi\tau}}\Upsilon_{\kappa_3\varpi\tau}\right] \\ -\Upsilon_{\kappa_1\widetilde{\varpi\tau}}\left[\Upsilon_{\kappa_2\widehat{\varpi\tau}}\Upsilon_{\kappa_2\varpi\tau} + (1-\Upsilon_{\widetilde{\varpi\tau}})\Upsilon_{\kappa_3\varpi\tau}\right] \neq 0, \\ \Lambda_1 = (1-\Upsilon_{\widetilde{\varpi\tau}})(1-\Upsilon_{\widehat{\varpi\tau}}) - \Upsilon_{\kappa_2\varpi\tau}\Upsilon_{\kappa_2\widehat{\varpi\tau}}, \\ \Lambda_2 = -\Upsilon_{\kappa_1\widetilde{\varpi\tau}}(1-\Upsilon_{\widehat{\varpi\tau}}) - \Upsilon_{\kappa_1\widetilde{\varpi\tau}}\Upsilon_{\kappa_2\widehat{\varpi\tau}}, \\ \Lambda_3 = \Upsilon_{\kappa_1\widetilde{\varpi\tau}}\Upsilon_{\kappa_2\widehat{\varpi\tau}} + \Upsilon_{\kappa_2\widehat{\varpi\tau}}(1-\Upsilon_{\widetilde{\varpi\tau}}), \\ \Lambda_4 = -\Upsilon_{\kappa_2\widehat{\varpi\tau}}(1-\Upsilon_{\widehat{\varpi\tau}}) - \Upsilon_{\kappa_2\widehat{\varpi\tau}}\Upsilon_{\kappa_3\varpi\tau} \\ \Lambda_5 = (1-\Upsilon_{\varpi\tau})(1-\Upsilon_{\widehat{\varpi\tau}}) - \Upsilon_{\kappa_1\widetilde{\varpi\tau}}\Upsilon_{\kappa_3\varpi\tau}, \\ \Lambda_6 = -\Upsilon_{\kappa_2\widehat{\varpi\tau}}(1-\Upsilon_{\varpi\tau}) - \Upsilon_{\kappa_1\widetilde{\varpi\tau}}\Upsilon_{\kappa_2\widehat{\varpi\tau}} \\ \Lambda_7 = \Upsilon_{\kappa_2\widehat{\varpi\tau}}\Upsilon_{\kappa_2\varpi\tau} + \Upsilon_{\kappa_3\varpi\tau}(1-\Upsilon_{\widetilde{\varpi\tau}}), \\ \Lambda_8 = -\Upsilon_{\kappa_2\varpi\tau}(1-\Upsilon_{\varpi\tau}) - \Upsilon_{\kappa_1\widetilde{\varpi\tau}}\Upsilon_{\kappa_3\varpi\tau}, \\ \Lambda_9 = (1-\Upsilon_{\varpi\tau})(1-\Upsilon_{\widehat{\varpi\tau}}) - \Upsilon_{\kappa_1\widetilde{\varpi\tau}}\Upsilon_{\kappa_2\widehat{\varpi\tau}}. \end{cases}$$

From (12), we can write

$$\begin{cases} \|u-u^*\|_A \le \dfrac{\Lambda_1 G_{\kappa_1\varpi\tau}\ell_{\kappa_1}}{\Lambda} + \dfrac{\Lambda_2 G_{\kappa_2\widetilde{\varpi\tau}}\ell_{\kappa_2}}{\Lambda} + \dfrac{\Lambda_3 G_{\kappa_3\widehat{\varpi\tau}}\ell_{\kappa_3}}{\Lambda}, \\ \|v-v^*\|_A \le \dfrac{\Lambda_4 G_{\kappa_1\varpi\tau}\ell_{\kappa_1}}{\Lambda} + \dfrac{\Lambda_5 G_{\kappa_2\widetilde{\varpi\tau}}\ell_{\kappa_2}}{\Lambda} + \dfrac{\Lambda_6 G_{\kappa_3\widehat{\varpi\tau}}\ell_{\kappa_3}}{\Lambda}, \\ \|w-w^*\|_A \le \dfrac{\Lambda_7 G_{\kappa_1\varpi\tau}\ell_{\kappa_1}}{\Lambda} + \dfrac{\Lambda_8 G_{\kappa_2\widetilde{\varpi\tau}}\ell_{\kappa_2}}{\Lambda} + \dfrac{\Lambda_9 G_{\kappa_3\widehat{\varpi\tau}}\ell_{\kappa_3}}{\Lambda}. \end{cases}$$

Therefore

$$\begin{aligned} \|u-u^*\|_A + \|v-v^*\|_A + \|w-w^*\|_A &\le \frac{G_{\kappa_1\varpi\tau}\ell_{\kappa_1}}{\Lambda}(\Lambda_1+\Lambda_4+\Lambda_7) \\ &+ \frac{G_{\kappa_2\widetilde{\varpi\tau}}\ell_{\kappa_2}}{\Lambda}(\Lambda_2+\Lambda_5+\Lambda_8) \\ &+ \frac{G_{\kappa_3\widehat{\varpi\tau}}\ell_{\kappa_3}}{\Lambda}(\Lambda_3+\Lambda_6+\Lambda_9). \end{aligned}$$

It follows that

$$\|(u,v,w)-(u^*,v^*,w^*)\| \le G_{\kappa_1\kappa_2\kappa_3\varpi\tau\widetilde{\varpi\tau}\widehat{\varpi\tau}}\,\ell,$$

where $\ell = \max\left\{\ell_{\kappa_1},\ell_{\kappa_2},\ell_{\kappa_3}\right\}$ and

$$G_{\kappa_1\kappa_2\kappa_3\varpi\tau\widetilde{\varpi\tau}\widehat{\varpi\tau}} = \frac{G_{\kappa_1\varpi\tau}(\Lambda_1+\Lambda_4+\Lambda_7)}{\Lambda} + \frac{G_{\kappa_2\widetilde{\varpi\tau}}(\Lambda_2+\Lambda_5+\Lambda_8)}{\Lambda} + \frac{G_{\kappa_3\widehat{\varpi\tau}}(\Lambda_3+\Lambda_6+\Lambda_9)}{\Lambda}.$$

Consequently, the preceding inequality implies the Ulam-Hyers stability of the tripled system (1). Moreover, the inequality can be succinctly expressed as:

$$\|(u,v,w)-(u^*,v^*,w^*)\| \le J(\ell), \text{ with } J(0)=0.$$

This inequality confirms the generalized Ulam-Hyers stability of the proposed tripled problem (1).

5. Illustrative applications

In this section, we transition from theoretical derivation to practical application by presenting a series of illustrative cases that demonstrate the utility of our model. To validate the core findings regarding existence, uniqueness, and Ulam-Hyers stability, we provide a detailed numerical example that substantiates our analytical results with concrete data.

The significance of a tripled system of fractional integro-differential equations under integral boundary conditions lies in its ability to model complex, interconnected phenomena that possess memory and non-local interactions as follows:

- Biological Memory and Heredity: Unlike classical calculus, fractional derivatives capture the history of a biological process. In a tripled system, this represents the co-evolution of three interacting species or biomarkers where the rate of change depends on past states, such as in multi-strain viral dynamics or nutrient-phytoplankton-zooplankton cycles.

- Physical Anomalous Diffusion: In physics, these systems describe anomalous transport in porous or fractal media, where the tripled nature accounts for the coupling between temperature, pressure, and concentration, and the fractional order represents the sub-diffusive movement of particles that don't follow standard Brownian motion.

- Non-local Feedback: The use of integral boundary conditions is physically significant because it implies the system is governed by a weighted average or total mass/energy constraint over a spatial domain rather than just fixed values at the edges. This is crucial for modeling closed-loop biological systems or thermal equilibrium in a rod where the total heat energy is regulated.

- Viscoelastic Synergy: In materials science, a tripled system can represent the interaction between stress, strain, and internal damping in complex polymers, where integro-differential terms accurately reflect the material's ability to dissipate energy over time.

Now, to validate the the theoretical results, we introduced the following example:

Example 5.1 Assume that the following fractional tripled system:

$$\begin{cases} {}^cD^{\frac{1}{2}}u(\zeta) = \frac{1}{90\zeta^2}\left[\frac{|v(\zeta)|}{1+|v(\zeta)|}+\frac{|w(\zeta)|}{1+|w(\zeta)|}\right] \\ \qquad\qquad +\int\limits_0^{\zeta}\frac{\sin(r\zeta)}{50}w(r)dr, \\ {}^cD^{\frac{1}{3}}v(\zeta) = \frac{e^{-\zeta}}{50}\left[\frac{|w(\zeta)|}{1+|w(\zeta)|}+\frac{|v(\zeta)|}{1+|v(\zeta)|}\right] \\ \qquad\qquad +\int\limits_0^{\zeta}\frac{2}{r^4+89}v(r)\,dr, \\ {}^cD^{\frac{1}{4}}w(\zeta) = \frac{1}{75}\left[\zeta\sin u(\zeta)+\zeta\cos u(\zeta)\right] \\ \qquad\qquad +\int\limits_0^{\zeta}\frac{e^{-(r-\zeta)}}{40}u(r)\,dr, \\ u(0)+u'(0) = \int_0^1\frac{|u(r)|}{14+|u(r)|}dr, \\ u(1)+u'(1) = \int_0^1\frac{|u(r)|}{16+|u(r)|}dr, \\ v(0)+v'(0) = \frac{1}{20}\int_0^1\sin v(r)\,dr, \\ v(1)+v'(1) = \frac{1}{25}\int_0^1\sin v(r)\,dr, \\ w(0)+w'(0) = \frac{1}{30}\int_0^1\cos w(r)\,dr, \\ w(1)+w'(1) = \frac{1}{35}\int_0^1\cos w(r)\,dr, \end{cases} \tag{13}$$

for $\zeta\in V=[0,1]$. Clearly, system (13) is another form of the system (1) with $\kappa_1=\frac{1}{2}$, $\kappa_2=\frac{1}{3}$, $\kappa_3=\frac{1}{4}$, $\varpi=\widetilde{\varpi}=\widehat{\varpi}=1$, $\delta=\widetilde{\delta}=\widehat{\delta}=1$, $\tau=\widetilde{\tau}=\widehat{\tau}=1$, $l(u(r))=\frac{|u(r)|}{14+|u(r)|}$, $m(u(r))=\frac{|u(r)|}{16+|u(r)|}$, $\widetilde{l}(v(r))=\frac{1}{20}\sin v(r)$, $\widetilde{m}(v(r))=\frac{1}{25}\sin v(r)$, $\widehat{l}(w(r))=\frac{1}{30}\cos w(r)$, $\widehat{m}(w(r))=\frac{1}{35}\cos w(r)$, $\Omega^w(\zeta)=\int\limits_0^{\zeta}\frac{\sin(r\zeta)}{50}w(r)\,dr$, $\Omega^w(\zeta)=\int\limits_0^{\zeta}\frac{2}{r^4+89}v(r)\,dr$, $\Omega^v(\zeta)=\int\limits_0^{\zeta}\frac{e^{-(r-\zeta)}}{40}v(r)\,dr$ and

$$\begin{cases} \phi_1(\zeta,v(\zeta),w(\zeta),\Omega^w(\zeta)) \\ \quad = \frac{1}{90\zeta^2}\left[\frac{|v(\zeta)|}{1+|v(\zeta)|}+\frac{|w(\zeta)|}{1+|w(\zeta)|}\right]+\int\limits_0^{\zeta}\frac{\sin(r\zeta)}{50}w(r)\,dr, \\ \phi_2(\zeta,w(\zeta),u(\zeta),\Omega^u(\zeta)) \\ \quad = \frac{e^{-\zeta}}{50}\left[\frac{|w(\zeta)|}{1+|w(\zeta)|}+\frac{|u(\zeta)|}{1+|v(\zeta)|}\right]+\int\limits_0^{\zeta}\frac{2}{r^4+89}u(r)\,dr, \\ {}^c\phi_3(\zeta,u(\zeta),v(\zeta),\Omega^v(\zeta)) \\ \quad = \frac{1}{75}\left[\zeta\sin u(\zeta)+\zeta\cos v(\zeta)\right]+\int\limits_0^{\zeta}\frac{e^{-(r-\zeta)}}{40}v(r)\,dr. \end{cases}$$

Now, from the condition (H_2), we have

$$\begin{aligned} |l(u(\zeta))| &\le \frac{|u(r)|}{14+|u(r)|}\le\frac{1}{14}|u(\zeta)|, \\ |m(u(\zeta))| &\le \frac{|u(r)|}{16+|u(r)|}\le\frac{1}{16}|u(\zeta)|, \\ \left|\widetilde{l}(v(\zeta))\right| &\le \frac{1}{20}|\sin v(r)|\le\frac{1}{20}|v(r)|, \\ |\widetilde{m}(u(\zeta))| &\le \frac{1}{25}|v(r)|, \\ \left|\widehat{l}(w(\zeta))\right| &\le \frac{1}{30}|w(\zeta)|, \\ |m(u(\zeta))| &\le N_{\widehat{m}}|u(\zeta)|\frac{1}{35}|w(\zeta)|, \end{aligned}$$

and

$$\begin{aligned} |l(u)-l(u^*)| &= \left|\frac{|u(r)|}{14+|u(r)|}-\frac{|u^*(r)|}{14+|u^*(r)|}\right| \\ &\le \frac{1}{14}\Upsilon_l|u-u^*|, \end{aligned}$$

$$\begin{aligned} |m(u)-m(u^*)| &\le \frac{1}{16}|u-u^*|, \\ \left|\widetilde{l}(v)-\widetilde{l}(v^*)\right| &\le \frac{1}{20}|\sin v(r)-\sin v^*(r)| \\ &\le \frac{1}{20}|v-v^*|, \end{aligned}$$

$$|\widetilde{m}(v)-\widetilde{m}(v^*)|\le\frac{1}{25}|v-v^*|,$$

$$\begin{aligned} \left|\widehat{l}(w)-\widehat{l}(w^*)\right| &\le \frac{1}{30}|\cos w(r)-\cos w^*(r)| \\ &\le \frac{1}{30}|w-w^*|, \end{aligned}$$

$$|\widehat{m}(w)-\widehat{m}(w^*)|\le\frac{1}{35}|w-w^*|,$$

Thus, $N_l=\Upsilon_l=\frac{1}{14}$, $N_m=\Upsilon_m=\frac{1}{16}$, $N_{\widetilde{l}}=\Upsilon_{\widetilde{l}}=\frac{1}{20}$, $N_{\widetilde{m}}=\Upsilon_{\widetilde{m}}=\frac{1}{25}$, $N_{\widehat{l}}=\Upsilon_{\widehat{l}}=\frac{1}{30}$, and $N_{\widehat{m}}=\Upsilon_{\widehat{m}}=\frac{1}{35}$.

From the hypothesis (H_3), one can write

$$\begin{aligned} \int_0^1|\varphi(\zeta,r)|\,dr &= \int_0^1\frac{|\sin(r\zeta)|}{50}dr\le\frac{1}{50}, \\ \int_0^1|\widetilde{\varphi}(\zeta,r)|\,dr &= \int_0^1\left|\frac{2}{r^4+98}\right|dr\le\frac{1}{45}, \\ \int_0^1|\widehat{\varphi}(\zeta,r)|\,dr &= \int_0^1\left|\frac{e^{-(r-\zeta)}}{40}\right|dr\le\frac{1}{40}. \end{aligned}$$

Thus, $N_{\rho_1}=\frac{1}{50}$, $N_{\rho_2}=\frac{1}{45}$, and $N_{\rho_3}=\frac{1}{40}$.

From the hypothesis (H_4), we get

$$\begin{aligned} &\left|\phi_1(\zeta,v(\zeta),w(\zeta),\Omega^w(\zeta))\right. \\ &\qquad \left.-\phi_1\left(\zeta,v^*(\zeta),w^*(\zeta),\Omega^{w^*}(\zeta)\right)\right| \\ &= \left|\frac{1}{90\zeta^2}\left[\frac{|v(\zeta)|}{1+|v(\zeta)|}+\frac{|w(\zeta)|}{1+|w(\zeta)|}\right]\right. \\ &\qquad +\int\limits_0^{\zeta}\frac{\sin(r\zeta)}{50}w(r)\,dr \\ &\qquad -\frac{1}{90\zeta^2}\left[\frac{|v^*(\zeta)|}{1+|v^*(\zeta)|}+\frac{|w^*(\zeta)|}{1+|w^*(\zeta)|}\right] \\ &\qquad \left.+\int\limits_0^{\zeta}\frac{\sin(r\zeta)}{50}w^*(r)\,dr\right| \\ &\le \frac{1}{90}|v(\zeta)-v^*(\zeta)|+\frac{1}{90}|w(\zeta)-w^*(\zeta)| \\ &\qquad +\frac{1}{50}\left|\Omega^w(\zeta)-\Omega^{w^*}(\zeta)\right| \end{aligned}$$

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$$\begin{aligned}
&\left|\phi_2\left(\zeta, w(\zeta), u(\zeta), \Omega^u(\zeta)\right)\right. \\
&\qquad \left.-\phi_2\left(\zeta, w^*(\zeta), u^*(\zeta), \Omega^{u^*}(\zeta)\right)\right| \\
&\leq \left|\frac{e^{-\zeta}}{50}\left[\frac{|w(\zeta)|}{1+|w(\zeta)|}+\frac{|u(\zeta)|}{1+|v(\zeta)|}\right]\right. \\
&\quad +\int_0^{\zeta} \frac{2}{r^4+89} u(r)\, dr \\
&\quad -\frac{e^{-\zeta}}{50}\left[\frac{|w(\zeta)|}{1+|w(\zeta)|}+\frac{|u(\zeta)|}{1+|v(\zeta)|}\right] \\
&\quad \left.+\int_0^{\zeta} \frac{2}{r^4+89} u(r)\, dr\right| \\
&\leq \frac{1}{50}\left|w(\zeta)-w^*(\zeta)\right|+\frac{1}{50}\left|u(\zeta)-u^*(\zeta)\right| \\
&\quad +\frac{1}{45}\left|\Omega^u(\zeta)-\Omega^{u^*}(\zeta)\right| \\
&\leq \Upsilon_{\phi_2}\left|w(\zeta)-w^*(\zeta)\right|+\widetilde{\Upsilon}_{\phi_2}\left|u(\zeta)-u^*(\zeta)\right| \\
&\quad +\widehat{\Upsilon}_{\phi_2}\left|\Omega^u(\zeta)-\Omega^{u^*}(\zeta)\right|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\phi_3\left(\zeta, u(\zeta), v(\zeta), \Omega^v(\zeta)\right)\right. \\
&\qquad \left.-\phi_3\left(\zeta, u^*(\zeta), v^*(\zeta), \Omega^{v^*}(\zeta)\right)\right| \\
&= \left|\frac{1}{75}\left[\zeta \sin u(\zeta)+\zeta \cos v(\zeta)\right]\right. \\
&\quad +\int_0^{\zeta} \frac{e^{-(r-\zeta)}}{40} v(r)\, dr \\
&\quad -\frac{1}{75}\left[\zeta \sin u(\zeta)+\zeta \cos v(\zeta)\right] \\
&\quad \left.+\int_0^{\zeta} \frac{e^{-(r-\zeta)}}{40} v(r)\, dr\right| \\
&\leq \frac{1}{75}\left|u(\zeta)-u^*(\zeta)\right|+\frac{1}{75}\left|v(\zeta)-v^*(\zeta)\right| \\
&\quad +\frac{1}{40}\left|\Omega^v(\zeta)-\Omega^{v^*}(\zeta)\right|,
\end{aligned}$$

Thus, $\Upsilon_{\phi_1}=\widetilde{\Upsilon}_{\phi_1}=\frac{1}{90}$, $\widehat{\Upsilon}_{\phi_1}=\frac{1}{50}$, $\Upsilon_{\phi_2}=\widetilde{\Upsilon}_{\phi_2}=\frac{1}{50}$, $\widehat{\Upsilon}_{\phi_2}=\frac{1}{45}$, $\Upsilon_{\phi_3}=\widetilde{\Upsilon}_{\phi_3}=\frac{1}{75}$, and $\widehat{\Upsilon}_{\phi_3}=\frac{1}{40}$.

From the hypothesis (H_5), we have

$$\begin{aligned}
G_{\kappa_3 \widehat{\varpi}\widehat{\tau}} &= \frac{1}{\Gamma(1+\kappa_3)}+\frac{2(\widehat{\varpi}+\widehat{\tau})}{\Gamma(1+\kappa_3)}+\frac{2\left(\widehat{\tau}^2+\widehat{\varpi}\widehat{\tau}\right)}{\widehat{\varpi}^2 \Gamma(\kappa_3)} \\
&\approx 6.6196, \\
G_{\kappa_1 \widetilde{\varpi}\widetilde{\tau}} &= \frac{5}{\Gamma\left(1+\frac{1}{2}\right)}+\frac{4}{\Gamma\left(\frac{1}{2}\right)}=\frac{5}{0.88622}+\frac{4}{\sqrt{\pi}} \\
&\approx 7.8987, \\
G_{\kappa_2 \varpi\tau} &= \frac{1}{\Gamma(1+\kappa_2)}+\frac{2(\varpi+\tau)}{\Gamma(1+\kappa_2)}+\frac{2\left(\tau^2+\varpi\tau\right)}{\varpi^2 \Gamma(\kappa_2)} \\
&\approx 7.0924.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Re_1 &= G_{\kappa_3 \widehat{\varpi}\widehat{\tau}}(\Upsilon_{\phi_3}+\widetilde{\Upsilon}_{\phi_3} \\
&\quad +\widehat{\Upsilon}_{\phi_3} N_{\rho_3})+\frac{\widetilde{\delta}(\widetilde{\varpi}+\widetilde{\tau})\left(\Upsilon_{\widetilde{l}}+\Upsilon_{\widetilde{m}}\right)}{\widetilde{\varpi}^2} \\
&= 6.6196\left(\frac{1}{75}+\frac{1}{75}+\frac{1}{40}\frac{1}{40}\right)+2\left(\frac{1}{20}+\frac{1}{25}\right) \\
&\approx 0.3607<1,
\end{aligned}$$

$$\begin{aligned}
\Re_2 &= G_{\kappa_1 \widetilde{\varpi}\widetilde{\tau}}\left(\Upsilon_{\phi_1}+\widetilde{\Upsilon}_{\phi_1}+\widehat{\Upsilon}_{\phi_1} N_{\rho_1}\right) \\
&\quad +\frac{\delta(\varpi+\tau)\left(\Upsilon_l+\Upsilon_m\right)}{\varpi^2} \\
&= 7.8987\left(\frac{1}{90}+\frac{1}{90}+\frac{1}{50}\frac{1}{50}\right)+2\left(\frac{1}{14}+\frac{1}{16}\right) \\
&\approx 0.4465<1,
\end{aligned}$$

and

$$\begin{aligned}
\Re_3 &= G_{\kappa_2 \varpi\tau}\left(\Upsilon_{\phi_2}+\widetilde{\Upsilon}_{\phi_2}+\widehat{\Upsilon}_{\phi_2} N_{\rho_2}\right) \\
&\quad +\frac{\widehat{\delta}(\widehat{\varpi}+\widehat{\tau})\left(\Upsilon_{\widehat{l}}+\Upsilon_{\widehat{m}}\right)}{\widehat{\varpi}^2} \\
&= 7.0924\left(\frac{1}{50}+\frac{1}{50}+\frac{1}{45}\frac{1}{45}\right)+2\left(\frac{1}{30}+\frac{1}{35}\right) \\
&\approx 0.4110<1.
\end{aligned}$$

Thus, $\Re=\max\{\Re_1, \Re_2, \Re_3\}=0.4465<1$. Therefore, all requirements of Theorem 3.1 are fulfilled and the tripled system (13) possesses a unique solution. Additionally, it is not difficult to see that $\Lambda \neq 0$. Hence, by Theorem 4.5, the tripled fractional problem (13) is both Ulam-Hyers stability and generalized Ulam-Hyers stability.

6. Conclusion and future work

By leveraging the Banach and Schauder fixed-point theorems, the solvability of nonlinear systems of fractional integro-differential equations under integral boundary conditions is recast as a fixed-point problem within a suitable function space. Our investigation first established existence and uniqueness through the Banach fixed-point theorem and then utilized Ulam-Hyers stability analysis to quantify the system's resilience, confirming that approximate solutions remain close to exact ones and ensuring the model is well-posed and physically meaningful. Despite these theoretical successes, significant challenges remain in managing computational complexity and the strictness of required conditions, such as the difficulty of maintaining a contraction constant less than 1 in highly coupled systems or constructing Green's functions for non-local integral boundary conditions. Furthermore, the memory effect inherent in fractional derivatives complicates stability for tripled systems, as errors can propagate across all coupled variables. Consequently, future research is expected to shift toward complex coupling dynamics,including non-instantaneous impulses and stochastic perturbations while exploring generalized operators like Hilfer or Katugampola derivatives and hybrid fixed-point theorems. Ultimately, the field is moving toward prioritizing Ulam-Hyers-Rassias stability in higher dimensions and developing optimized numerical schemes, such as fractional-order neural networks, to address these computationally demanding problems in real-time.

Authors contributions
All authors contributed equally in the writing and editing of this article.
Availability of data and materials
Data available on request from the authors.
Conflict of interests
TThe author declares that he does not have any conflict of interest.

References

1. Almeida R, Bastos N, and Monteiro M. Modeling some real phenomena by fractional differential equations. 2015. Available from: https://doi.org/10.1002/mma.3818
2. Kaikousidis C and Dokoumetzidis A. Implementation of non-linear mixed effects models defined by fractional differential equations. J. Pharmacokinet Pharmacodyn 2023; 50:283–95
3. Kumar D and Baleanu D. Fractional Calculus and Its Applications in Physics. Vol. 7. Frontiers in Physics, 2019. Available from: https://doi.org/10.3389/fphy.2019.00081
4. Gómez-Aguilar J and Atangana A. Applications of Fractional Calculus to Modeling in Dynamics and Chaos. New York, 2022. Available from: https://doi.org/10.1201/9781003006244
5. Wang S. The Ulam stability of fractional differential equation with the Caputo-Fabrizio derivative. J. Function Spaces 2022; 2022:Article ID 7268518, 9 pages. Available from: https://doi.org/10.1155/2022/7268518
6. Uddin A, Juyal P, Shah R, Kailasavalli S, Bharani J, and K H. A Comprehensive Review of Fractional Calculus in Modeling Real-World Phenomena. Commun. Appl. Nonlinear Anal. 2025; 23. Available from: https://doi.org/10.52783/cana.v32.4182
7. Akgül A, Ülgül E, Sakar N, Bilgi B, and Eke A. New applications of the new general integral transform method with different fractional derivatives. Alex. Eng. J. 2023; 80:498–505
8. Bassiouny E. Mathematical model for hyperbolic two temperature fractional-order thermoelastic materials subjected to thermal loading. Appl. Math. Inf. Sci. 2021; 15:23–9
9. Senol M and Kasmaei H. Perturbation-iteration algorithm for systems of fractional differential equations and convergence analysis. Prog. Fract. Diff. Appl. 2017; 03:271–9
10. Mushtaq A. Lone SMMKOKOOzRLTM. Conversed fractional programming approach goal programming: A mathematical programming approach. J. Stat. Appl. Prob. 2022; 11:811–8
11. Ulam S. A collection of the mathematical problems. New York: Interscience, 1960
12. Hyers D. On the stability of the linear functional equation. Proc Natl Acad Sci U S A 1941; 27:222–4
13. Mahdy AMS. Stability, existence, and uniqueness for solving fractional glioblastoma multiforme using a Caputo-Fabrizio derivative. Mathematical Methods in the Applied Sciences 2023; 46:1–15
14. Jung SM. Hyers-Ulam stability of linear differential equations of the first order II. Applied Mathematics Letters 2006; 19:854–8
15. Mehmood N, Abbas A, Akgül A, Abdeljawad T, and Alqudah MA. Existence and stability results for coupled system of fractional differential equations involving AB-Caputo derivative. Fractals 2023; 31:2340023
16. Hammad HA, Qasymeh M, and Abdel-Aty M. Existence and stability results for a Langevin system with Caputo–Hadamard fractional operators. International Journal of Geometric Methods in Modern Physics 2024; 21:2450218
17. Hammad HA, Rashwan RA, Nafea A, Samei ME, and Sen M De la. Stability and existence of solutions for a tripled problem of fractional hybrid delay differential equations. Symmetry 2022; 14:2579
18. Chaurasiya BK and Kumar A. A novel approach for the stability analysis of a system of mnon linear fractional differential equations of arbitrary order. Journal of Mathematics and Computer Science 2025; 39:116–30
19. Dhawan K, Vats RK, Kumar S, and Kumar A. Existence and stability analysis for nonlinear boundary value problem involving Caputo fractional derivative. Discrete Dynamics in Nature and Society 2023; 30:107–21

20. Chalishajar D and Kumar A. Existence, uniqueness and Ulam's stability of solutions for a coupled system of fractional differential equations with integral boundary conditions. Mathematics 2018; 6:96
21. Ahmad B, Hamdan S, Alsaedi S, and Ntouyas SK. A study of a nonlinear coupled system of three fractional differential equations with nonlocal coupled boundary conditions. Advances in Difference Equations 2021 :278
22. Kumar MS, Deepa M, Kavitha J, and Sadhasivam V. Existence theory of fractional order three-dimensional differential system at resonance. Mathematical Modelling and Control 2023; 3:127–38
23. Ibnelazyz L. Tripled system of fractional Langevin equations with tripled multipoint boundary conditions. Filomat 2024; 38:11401–32
24. Gassem F, Ali A, Aldwoah K, Egami RH, Osman O, Younis B, and Touati A. Qualitative analysis of a three-dimensional dynamical system of fractal-fractional-order evolution differential equations with terminal boundary conditions. Fractal and Fractional 2025; 9:259
25. Aljoudi S, Alamri H, and Alotaibi A. On Hilfer–Hadamard tripled system with symmetric nonlocal Riemann–Liouville integral boundary conditions. Symmetry 2025; 17:1867
26. Iskakova N. On the solvability of a boundary value problem for delay integro-differential equation. Mathematical Methods in the Applied Sciences 2025 :1–13
27. Noori SRM and Taghizadeh N. Modified differential transform method for solving linear and nonlinear pantograph type of differential and Volterra integro-differential equations with proportional delays. Advances in Difference Equations 2020; 2020:649
28. Unknown. A numerical method for Ψ-fractional integro-differential equations by Bell polynomials. Applied Numerical Mathematics 2025; 207:244–53
29. Kechar C, Hamoud AA, Ardjouni A, Emadifar H, Karim K, and Ahmed A. Solution of fractional integro-differential equations with some existence and stability results. Boundary Value Problems 2026; 2026:16
30. Kattan DA and Hammad HA. Solving fractional integro-differential equations with delay and relaxation impulsive terms by fixed point techniques. Boundary Value Problems 2024; 2024:146
31. Hetmaniok E, Pleszczyński M, and Khan Y. Solving the integral differential equations with delayed argument by using the DTM method. Sensors 2022; 22:4124
32. Jafari H, Mehdinejadiani B, and Baleanu D. Fractional calculus for modeling unconfined groundwater: applications in engineering, life and social sciences. Mathematics 2019; 7:119–38
33. Oldham KB. Fractional differential equations in electrochemistry. Advances in Engineering Software 2010; 41:9–12
34. Jannelli A. Numerical solutions of fractional differential equations arising in engineering sciences. Mathematics 2020; 8:215
35. Ionescu C, Lopes A, Copot D, Machado JT, and Bates JH. The role of fractional calculus in modeling biological phenomena: a review. Communications in Nonlinear Science and Numerical Simulation 2017; 51:141–59
36. Sweis H, Shawagfeh N, and Abu Arqub O. Fractional crossover delay differential equations of MittagLeffler kernel: existence, uniqueness, and numerical solutions using the Galerkin algorithm based on shifted Legendre polynomials. Results in Physics 2022; 41:105891
37. Miller KS and Ross B. An introduction to fractional calculus and fractional differential equations. New York: Wiley Interscience, 1993
38. Alam M, Zada A, and Abdeljawad T. Stability analysis of an implicit fractional integro-differential equation via integral boundary conditions. Alexandria Engineering Journal 2024; 87:501–14
39. Kilbas AA, Srivastava HM, and Trujillo JJ. Theory and applications of fractional differential equations. New York: Elsevier Science, 2006
40. Ahmad B and Nieto JJ. Existence results for nonlinear boundary value problems of fractional integro-differential equations with integral boundary conditions. Boundary Value Problems 2009 :1–11
41. Rus IA. Ulam stabilities of ordinary differential equations in a Banach space. Carpathian Journal of Mathematics 2010; 26:103–7

10.57647/mathsci.2026.2002.13