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ORIGINAL RESEARCH

On the Existence and Stability of Solutions for Implicit Coupled Neutral Fractional Differential Systems

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Abstract

This paper investigated the existence, uniqueness, and stability behavior of a class of nonlinear implicit coupled neutral fractional differential equations governed by the Caputo-Atangana-Baleanu derivative. The existence of solutions was obtained by applying Krasnoselskii's fixed point theorem, whereas the Banach contraction principle was employed to establish both exis-tence and uniqueness results. Furthermore, several important stability concepts were analyzed, namely Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and generalized Ulam-Hyers-Rassias stability. To illustrate the applicability of the developed theory, suitable examples were presented, validating the solvability and stability results for the proposed system.

Keywords: Fractional derivative; fixed point; stability analysis; neutral fractional differential equation

MSC (2020): 35A23, 47H10, 47H09, 26A33

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1 Introduction

Fractional calculus, a field of mathematics that generalizes the operations of differentiation and integration to non-integer orders, offers a powerful framework for modeling complex systems with memory and non-local behavior. Unlike classical calculus, which is inherently local, fractional calculus incorporates the entire history of a function to determine its future state, making it particularly well-suited for describing phenomena in fields such as physics, engineering, and finance [1,2]. The theory's origins can be traced back to a 1695 letter between Guillaume de l'Hôpital and Gottfried Leibniz, where the question of a half-order derivative was first posed [3]. Since then, various definitions of fractional derivatives and integrals have been developed, with the Riemann-Liouville (RL) and Caputo definitions being among the most widely used [4,5]. These tools have found practical applications in modeling viscoelastic materials, anomalous diffusion, and the control of dynamical systems, providing a more accurate description of real-world phenomena that traditional integer-order models fail to capture [6,7].

The Atangana-Baleanu-Caputo (ABC) derivative is a significant contribution to fractional calculus, distinguished by its use of the non-local and non-singular Mittag-Leffler kernel. This unique property allows the ABC derivative to model systems with memory and hereditary effects more effectively than traditional fractional derivatives that use a singular power-law kernel. Its applications span various scientific and engineering disciplines. For example, it has been used to model complex phenomena in finance, such as asset flow dynamics, by more accurately representing long-term market behavior and investor decisions. In epidemiology, ABC fractional models have been developed to study the spread of infectious diseases like polio and influenza, providing a more realistic understanding of disease dynamics by incorporating memory and population-level effects. Furthermore, its applications extend to biomechanics for modeling processes like bone remodeling, and in image processing where ABC-fractional masks have been used for image denoising. This derivative has also been applied to model physical processes such as heat transfer and to analyze the stability of dynamic systems, demonstrating its versatility and effectiveness in capturing the intricate, non-local behavior of real-world systems [8–13].

Neutral type equations are a cornerstone of fractional differential equation (FDE) theory, with broad applications across natural sciences and industry. These equations, which involve the derivative of the unknown function at the current time and the derivative of the unknown function with a time delay, are particularly challenging to analyze due to their unique qualities. Despite these complexities, they have proven instrumental in modeling diverse processes in biology, engineering, and physics. Examples of their use include modeling the dynamics of infectious diseases like in epidemiology, controlling satellite orbits and electromechanical systems, and analyzing cellular neural networks, stock exchanges, and population dynamics. Their utility also extends to fields such as artificial intelligence and pharmacokinetics. In recent years, the mathematical community has shown a significant and growing interest in these impulsive equations, reflecting their importance and the ongoing effort to overcome the analytical hurdles they present [14–16].

Researchers have made significant strides in the analysis of FDEs, particularly in understanding their existence, uniqueness, and stability under various conditions. Mahto et al. [17], for example, have investigated Caputo impulsive FDEs and their applications, while Guo et al. [18] have also contributed to the study of impulsive FDEs. Further extending this area, authors in references [19–21] have delved into neutral type impulsive FDEs, detailing their unique conditions.

A major direction of contemporary research has centered on the study of implicit fractional differential equations, owing to their ability to describe complex dynamical processes in which the derivative term appears in an implicit form and the system behavior depends strongly on memory effects. In this context, several authors have employed the Caputo fractional derivative to establish fundamental qualitative results, including the existence and uniqueness of solutions as well as the determination of suitable intervals for solution existence for different classes of such problems [22, 23]. Extending these developments, Asma et al. [24] investigated the existence and stability of implicit fractional differential equations by means of the Atangana–Baleanu–Caputo derivative, which is characterized by a non-singular and non-local kernel, thereby offering a more realistic framework for certain physical and engineering applications. Subsequently, Sowbakiya et al. [25] advanced this line of research by studying the existence, uniqueness, and stability of a class of nonlinear implicit neutral fractional differential equations involving the same ABC derivative. These contributions collectively highlight the increasing interest in employing modern fractional operators with generalized kernels to address more sophisticated nonlinear models and to obtain deeper qualitative insights into their solution structure and long-term behavior. For further related developments and complementary results, we refer the reader to [26–33].

Kucche et al. [34] significantly advanced the study of nonlinear implicit FDEs by proving the existence of solutions for these complex systems. Their work uniquely employed the ABC derivative, which is characterized by its non-singular Mittag-Leffler kernel. This approach is particularly valuable for modeling systems with memory and hereditary properties. The study established these existence results under specific boundary and non-local initial conditions, which are critical for accurately representing real-world physical and engineering phenomena. This research, alongside other key contributions [24, 25, 34], provides a robust theoretical framework for analyzing the behavior of intricate systems, demonstrating the effectiveness of advanced fractional calculus in addressing complex problems.

Building upon the foundational work discussed above, this study is motivated to investigate the non-linear implicit coupled neutral FDEs. Specifically, we focus on systems that incorporate the Atangana–Baleanu derivative in the Caputo sense, leveraging its non-singular kernel to more accurately model complex phenomena. This approach allows us to consider a system of the following form:

$$\begin{cases} {}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right] \\ =\Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right),\ s\in\Upsilon=[0,1], \\ {}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right] \\ =\Re_2\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right), \\ \psi(0)=\psi_0,\ \phi(0)=\phi_0. \end{cases} \tag{1.1}$$

where ${}^{ABC}D_s^{\rho}$ represents the Atangana-Baleanu derivative in the Caputo sense with order $\rho\in(0,1)$, $\psi(s)\in C(\Upsilon,\mathbb{R})$, $\hbar_1,\hbar_2\in C(\Upsilon\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\Re_1,\Re_2\in C\left(\Upsilon\times\mathbb{R}^3,\mathbb{R}\right)$ $\left(\mathbb{R}^3=\mathbb{R}\times\mathbb{R}\times\mathbb{R}\right)$ are continuous functions and $\psi_0,\phi_0\in\mathbb{R}_+$. Moreover, the non-linear functions $\hbar_1(0,\psi(0),\phi(0))=0$, $\hbar_2(0,\psi(0),\phi(0))=0$,

$$\begin{cases} \hbar_1(0,\psi(0),\phi(0))=0, \\ \hbar_2(0,\psi(0),\phi(0))=0, \\ \Re_1\left(0,\psi(0),\phi(0),{}^{ABC}D_s^{\rho}\left[\psi(0)-\hbar_1(0,\psi(0),\phi(0))\right]\right)=0, \\ \Re_2\left(0,\psi(0),\phi(0),{}^{ABC}D_s^{\rho}\left[\psi(0)-\hbar_2(0,\psi(0),\phi(0))\right]\right)=0. \end{cases}$$

The motivation of this paper arises from the growing importance of fractional differential equations in modeling complex dynamical phenomena with memory and hereditary effects, where classical integer-order models often fail to provide an accurate description. In particular, the fractional equation associated with system (1.1) offers a flexible framework for representing such processes, which calls for a rigorous theoretical investigation of its solvability and stability behavior. Motivated by these challenges, this work develops a comprehensive analytical study of the considered problem. The main innovations of the paper lie in employing the Banach fixed point theorem to establish existence and uniqueness of solutions, while Krasnoselskii's fixed point theorem is utilized to derive further existence results under more general assumptions. Moreover, a detailed stability analysis is presented through several important notions of Ulam stability, including Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability, which clarify the sensitivity of solutions to small perturbations. Finally, an illustrative example is included to confirm the applicability and effectiveness of the obtained theoretical results.

2 Basic facts

This section lays out the theoretical foundation for our study. We start by defining key terms and concepts, then present our main propositions and theorems. These essential results are the building blocks for the analysis and conclusions that follow.

Definition 2.1. [1] For the function $\psi \in C(\Upsilon, \mathbb{R})$, the left RL integral of order $\rho > 0$ can be written as

$$^{RL}I_s^{\rho}\psi(s) = \frac{1}{\Gamma(\rho)}\int_0^s (s-\zeta)^{\rho-1}\psi(\zeta)\,d\zeta.$$

Definition 2.2. [35] For $\psi' \in H^1(a,b)$ and $a \leq b$, the ABC derivative of order $\rho \in [0,1]$ is given by

$$^{ABC}D_s^{\rho}[\psi(s)] = \frac{\Lambda(\rho)}{1-\rho}\int_0^s \psi'(\zeta)\varphi_\rho\left[\frac{-\rho(s-\zeta)^{\rho}}{1-\rho}\right]d\zeta,$$

where $\Lambda(\rho)$ is a normalizing positive function fulfills $\Lambda(0) = \Lambda(1) = 1$ and φ_ρ is the Mittag–Leffler function.

Definition 2.3. [35] For the function $\psi \in C(\Upsilon, \mathbb{R})$, the Atangana–Baleanu fractional integral of order $\rho > 0$ is defined by

$$^{AB}I_s^{\rho}\psi(s) = \frac{1-\rho}{\Lambda(\rho)}\psi(s) + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s (s-\zeta)^{\rho-1}\psi(\zeta)\,d\zeta.$$

Lemma 2.4. *[36] If $0 < \rho < 1$, then*

$$^{AB}I_s^{\rho}\left(^{ABC}D_s^{\rho}\right)(\psi(s)) \quad = \quad \psi(s) - \psi(0).$$

Theorem 2.5. *[37] (**Ascoli–Arzelá's theorem**) A subset ψ of continuous functions on a compact Hausdorff metric space Θ is relatively compact precisely when it is both uniformly bounded and equicontinuous.*

Theorem 2.6. *[38] (**Banach's FP theorem**) On a Banch space* $(\Theta, \|.\|)$, *every contraction mapping has a unique FP. Moreover, if*

$$\|\Im\varphi_1 - \Im\varphi_2\| \leq \lambda \|\varphi_1 - \varphi_2\|, \text{ for all } \varphi_1, \varphi_2 \in \Theta,$$

then the mapping $\Im : \Theta \to \Theta$ *has a FP, that is, there exists* $\varphi_0 \in \Theta$ *such that* $\varphi_0 = \Im\varphi_0$.

Definition 2.7. [39] An element $(\varphi_1, \varphi_2) \in \Theta \times \Theta$ is said to be a coupled FP of a mapping $\Im : \Theta \times \Theta \to \Theta$ if $\varphi_1 = \Im(\varphi_1, \varphi_2)$ and $\varphi_2 = \Im(\varphi_2, \varphi_1)$.

Theorem 2.8. *[38] (**Krasnoselskii's FP theorem**) In a Banach space* Θ, *let* U *be a bounded, closed, and convex subset. Consider two mappings,* $\Im_1, \Im_2 : U \to \Theta$ *such that*

(i) $\Im_1\psi + \Im_2\phi \in U$, *for all* $\psi, \phi \in U$,

(ii) $\Im_1$ *is a contraction,*

(iii) $\Im_2$ *is completely continuous,*

then, the equation $\psi = \Im_1\psi + \Im_2\psi$ *has a solution on* U.

3 Assertions and auxiliary lemma

In this section, we have transformed the original equation (1.1) into an equivalent fractional integral equation under certain hypotheses. This is a crucial step, as it allows us to leverage powerful tools from fractional calculus to analyze the existence and uniqueness of its solutions, which we will present in the next section.

Assume that $\Theta = C(\Upsilon) \times C(\Upsilon)$ the Banach space equipped with the norm

$$\|(\psi, \phi)\|_\Theta = \|\psi\|_{C(\Upsilon)} + \|\phi\|_{C(\Upsilon)}$$

where $C(\Upsilon)$ is the space of continuous-real values functions on $\Upsilon = [0, 1]$ and $\|\psi\|_{C(\Upsilon)} = \sup\{|\psi(s)| : s \in \Upsilon\}$.

(A$_1$) The functions $\Re_i \in C(\Upsilon \times \mathbb{R}^3, \mathbb{R})$ and for $\psi, \phi, \xi, \widehat{\psi}, \widehat{\phi}, \widehat{\xi} \in C(\Upsilon, \mathbb{R})$, there exists a constant $N_{\Re_i} > 0$ such that

$$\begin{aligned} &\left\|\Re_i(s, \psi, \phi, \xi) - \Re_i\left(s, \widehat{\psi}, \widehat{\phi}, \widehat{\xi}\right)\right\| \\ \leq\; & N_{\Re_i}\left[\left\|\psi(s) - \widehat{\psi}(s)\right\| + \left\|\phi(s) - \widehat{\phi}(s)\right\| + \left\|\xi(s) - \widehat{\xi}(s)\right\|\right], \end{aligned}$$

for $s \in \Upsilon$ and $i = 1, 2$.

(A$_2$) The functions $\Re_i \in C(\Upsilon \times \mathbb{R}^3, \mathbb{R})$ and for $\psi, \phi, \xi \in C(\Upsilon, \mathbb{R})$, there exist positive constants $N_{\Re_{i,1}}, N_{\Re_{i,2}}, N_{\Re_{i,3}}$, and $N_{\Re_{i,4}}$ such that

$$\|\Re_i(s, \psi, \phi, \xi)\| \leq N_{\Re_{i,1}} + N_{\Re_{i,2}}\|\psi\| + N_{\Re_{i,3}}\|\phi\| + N_{\Re_{i,4}}\|\xi\|, \ i = 1, 2,$$

for $s \in \Upsilon$.

(A$_3$) The functions $\hbar_i \in C(\Upsilon \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and for $\psi, \phi, \widehat{\psi}, \widehat{\phi} \in C(\Upsilon, \mathbb{R})$, there exist positive constants $M_{\hbar_{i,1}}$, $M_{\hbar_{i,2}}$, and $M_{\hbar_i}$ such that

$$\left\|\hbar_i(s,\psi,\phi) - \hbar_i\left(s,\widehat{\psi},\widehat{\phi}\right)\right\| \le M_{\hbar_{i,1}}\left(\left\|\psi(s) - \widehat{\psi}(s)\right\| + \left\|\phi(s) - \widehat{\phi}(s)\right\|\right),$$

$$M_{\hbar_{i,2}} = \max_{s\in\Upsilon}\|\hbar_i(s,0,0)\|,$$

and $M_{\hbar_i} = \max\left\{M_{\hbar_{i,1}}, M_{\hbar_{i,2}}\right\}$, for $i = 1, 2$.

(A$_4$) For $r > 0$, define the set A_r as $A_r = \{(\psi,\phi) \in \Theta : \|(\psi,\phi)\|_\Theta \le r\}$ such that

$$r \ge \frac{\|(\psi_0,\phi_0)\| + N_1^*\left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)}\right)}{1 - \left(M_{\hbar_1} + M_{\hbar_2} + N_2^*\left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)}\right)\right)},$$

where

$$N_1^* = \left(N_{\Re_{1,1}} + N_{\Re_{2,1}} + \frac{N_{\Re_{1,4}}N_{\Re_{1,1}}}{1 - N_{\Re_{1,4}}} + \frac{N_{\Re_{2,4}}N_{\Re_{2,1}}}{1 - N_{\Re_{2,4}}}\right),$$

and

$$N_2^* = N_{\Re_{1,2}} + N_{\Re_{2,2}} + N_{\Re_{1,3}} + N_{\Re_{2,3}} + \frac{N_{\Re_{1,4}}\left(N_{\Re_{1,2}} + N_{\Re_{1,3}}\right)}{1 - N_{\Re_{1,4}}} + \frac{N_{\Re_{2,4}}\left(N_{\Re_{2,2}} + N_{\Re_{2,3}}\right)}{1 - N_{\Re_{2,4}}}.$$

Lemma 3.1. *Assume that $\hbar_i \in C(\Upsilon \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $\Re_i \in C\left(\Upsilon \times \mathbb{R}^3, \mathbb{R}\right)$ $(i = 1, 2)$ are continuous functions, the pair $(\psi, \phi) \in \Theta$ is a solution to the problem (1.1) if*

$$\begin{cases} \psi(s) = \psi_0 + \hbar_1(s,\psi(s),\phi(s)) + \frac{1-\rho}{\Lambda(\rho)}\Re_1\left(s,\psi(s),\phi(s), {}^{ABC}D_s^\rho[\psi(s) - \hbar_1(s,\psi(s),\phi(s))]\right) \\ +\frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s (s-\xi)^{\rho-1}\Re_1\left(\xi,\psi(\xi),\phi(\xi), {}^{ABC}D_s^\rho[\psi(\xi) - \hbar_1(\xi,\psi(\xi),\phi(\xi))]\right)d\xi, \\ \phi(s) = \phi_0 + \hbar_2(s,\psi(s),\phi(s)) + \frac{1-\rho}{\Lambda(\rho)}\Re_2\left(s,\psi(s),\phi(s), {}^{ABC}D_s^\rho[\psi(s) - \hbar_2(s,\psi(s),\phi(s))]\right) \\ +\frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s (s-\xi)^{\rho-1}\Re_2\left(\xi,\psi(\xi),\phi(\xi), {}^{ABC}D_s^\rho[\psi(\xi) - \hbar_2(\xi,\psi(\xi),\phi(\xi))]\right)d\xi. \end{cases} \tag{3.1}$$

Proof. By affecting the first side of the equation (1.1) by ${}^{AB}I_s^\rho$, we have

$$\begin{aligned} & {}^{AB}I_s^\rho\left({}^{ABC}D_s^\rho[\psi(s) - \hbar_1(s,\psi(s),\phi(s))]\right) \\ = & {}^{AB}I_s^\rho\left(\Re_1\left(s,\psi(s),\phi(s), {}^{ABC}D_s^\rho[\psi(s) - \hbar_1(s,\psi(s),\phi(s))]\right)\right). \end{aligned}$$

From Lemma 2.4, we can write

$$\begin{aligned} & (\psi(s) - \hbar_1(s,\psi(s),\phi(s))) - (\psi(0) - \hbar_1(0,\psi(0),\phi(0))) \\ = & {}^{AB}I_s^\rho\left(\Re_1\left(s,\psi(s),\phi(s), {}^{ABC}D_s^\rho[\psi(s) - \hbar_1(s,\psi(s),\phi(s))]\right)\right), \end{aligned}$$

hence

$$\begin{aligned} & \psi(s) - \hbar_1(s,\psi(s),\phi(s)) - \psi_0 \\ = & {}^{AB}I_s^\rho\left(\Re_1\left(s,\psi(s),\phi(s), {}^{ABC}D_s^\rho[\psi(s) - \hbar_1(s,\psi(s),\phi(s))]\right)\right). \end{aligned}$$

From Definition 2.3, we have

$$\psi(s) - \hbar_1(s,\psi(s),\phi(s)) - \psi_0$$

$$
\begin{aligned}
= & \frac{1-\rho}{\Lambda(\rho)} \Re_1\left(s, \psi(s), \phi(s), {}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s, \psi(s), \phi(s))\right]\right) \\
& + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} \Re_1\left(\xi, \psi(\xi), \phi(\xi), {}^{ABC}D_s^{\rho}\left[\psi(\xi)-\hbar_1(\xi, \psi(\xi), \phi(\xi))\right]\right) d\xi,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
\psi(s) = & \psi_0 + \hbar_1(s, \psi(s), \phi(s)) + \frac{1-\rho}{\Lambda(\rho)} \Re_1\left(s, \psi(s), \phi(s), {}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s, \psi(s), \phi(s))\right]\right) \\
& + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} \Re_1\left(\xi, \psi(\xi), \phi(\xi), {}^{ABC}D_s^{\rho}\left[\psi(\xi)-\hbar_1(\xi, \psi(\xi), \phi(\xi))\right]\right) d\xi,
\end{aligned}
$$

Similarly, one has

$$
\begin{aligned}
\phi(s) = & \phi_0 + \hbar_2(s, \psi(s), \phi(s)) + \frac{1-\rho}{\Lambda(\rho)} \Re_2\left(s, \psi(s), \phi(s), {}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_2(s, \psi(s), \phi(s))\right]\right) \\
& + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} \Re_2\left(\xi, \psi(\xi), \phi(\xi), {}^{ABC}D_s^{\rho}\left[\psi(\xi)-\hbar_2(\xi, \psi(\xi), \phi(\xi))\right]\right) d\xi.
\end{aligned}
$$

This completes the proof. □

4 Existence and uniqueness results

This section presents a comprehensive analysis of the existence and uniqueness of a solution for the problem under consideration.

Theorem 4.1. *Let $\hbar_i \in C(\Upsilon \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $\Re_i \in C\left(\Upsilon \times \mathbb{R}^3, \mathbb{R}\right)$ $(i=1,2)$ are continuous functions and the pair $(\psi, \phi) \in \Theta$ is a solution to the problem (1.1). Then, the problem (1.1) has a unique solution, provided that the assertions (A_1)-(A_4) hold and*

$$
\Psi_i = M_{\hbar_i} + \frac{1-\rho}{\Lambda(\rho)}\left(N_{\Re_i} + \frac{N_{\Re_i}^2}{1-N_{\Re_i}}\right) + \frac{1}{\Lambda(\rho)\Gamma(\rho)}\left(M_{\hbar_{i,1}} + \frac{M_{\hbar_{i,1}} N_{\Re_i}}{1-N_{\Re_i}}\right) < 1,
$$

for $i=1,2$.

Proof. Define the operator $B: A_r \to A_r$ as

$$
B(\psi, \phi)(s) = \left(B_1(\psi, \phi)(s), B_2(\phi, \psi)(s)\right)
$$

where

$$
\begin{cases}
B_1(\psi, \phi)(s) = \psi(s) = \psi_0 + \hbar_1(s, \psi(s), \phi(s)) \\
+\frac{1-\rho}{\Lambda(\rho)} \Re_1\left(s, \psi(s), \phi(s), {}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s, \psi(s), \phi(s))\right]\right) \\
+\frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} \Re_1\left(\xi, \psi(\xi), \phi(\xi), {}^{ABC}D_s^{\rho}\left[\psi(\xi)-\hbar_1(\xi, \psi(\xi), \phi(\xi))\right]\right) d\xi, \\
B_2(\phi, \psi)(s) = \phi(s) = \phi_0 + \hbar_2(s, \psi(s), \phi(s)) \\
+\frac{1-\sigma}{\Lambda(\sigma)} \Re_2\left(s, \psi(s), \phi(s), {}^{ABC}D_s^{\sigma}\left[\psi(s)-\hbar_2(s, \psi(s), \phi(s))\right]\right) \\
+\frac{\sigma}{\Lambda(\sigma)\Gamma(\sigma)} \int_0^s (s-\xi)^{\sigma-1} \Re_2\left(\xi, \psi(\xi), \phi(\xi), {}^{ABC}D_s^{\sigma}\left[\psi(\xi)-\hbar_2(\xi, \psi(\xi), \phi(\xi))\right]\right) d\xi.
\end{cases}
$$

It follows from the assertions (A_2) and (A_3) that

$$
\left\|B_1(\psi, \phi)(s)\right\|
$$

$$
\begin{aligned}
&\leq \|\psi_0\| + \|\hbar_1(s,\psi(s),\phi(s)) - \hbar_1(s,0,0)\| + \|\hbar_1(s,0,0)\| \\
&\quad + \frac{1-\rho}{\Lambda(\rho)} \left|\Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi(s) - \hbar_1(s,\psi(s),\phi(s))\right]\right)\right| \\
&\quad + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} \left|\Re_1\left(\xi,\psi(\xi),\phi(\xi),{}^{ABC}D_s^{\rho}\left[\psi(\xi) - \hbar_1(\xi,\psi(\xi),\phi(\xi))\right]\right)\right| d\xi \\
&\leq \|\psi_0\| + M_{\hbar_{1,1}}(\|\psi(s)\| + \|\phi(s)\|) + M_{\hbar_{1,2}} \\
&\quad + \frac{1-\rho}{\Lambda(\rho)}\left[N_{\Re_{1,1}} + N_{\Re_{1,2}}|\psi(s)| + N_{\Re_{1,3}}|\phi(s)| + N_{\Re_{1,4}}\left|{}^{ABC}D_s^{\rho}\left[\psi(s) - \hbar_1(s,\psi(s),\phi(s))\right]\right|\right] \\
&\quad + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} \\
&\quad \times \left[N_{\Re_{1,1}} + N_{\Re_{1,2}}|\psi(\xi)| + N_{\Re_{1,3}}|\phi(\xi)| + N_{\Re_{1,4}}\left|{}^{ABC}D_s^{\rho}\left[\psi(\xi) - \hbar_1(\xi,\psi(\xi),\phi(\xi))\right]\right|\right] d\xi. \quad (4.1)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\left|{}^{ABC}D_s^{\rho}\left[\psi(s) - \hbar_1(s,\psi(s),\phi(s))\right]\right| \\
&= \left|\Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi(s) - \hbar_1(s,\psi(s),\phi(s))\right]\right)\right| \\
&\leq \left[N_{\Re_{1,1}} + N_{\Re_{1,2}}|\psi(s)| + N_{\Re_{1,3}}|\phi(s)| + N_{\Re_{1,4}}\left|{}^{ABC}D_s^{\rho}\left[\psi(s) - \hbar_1(s,\psi(s),\phi(s))\right]\right|\right],
\end{aligned}
$$

which implies that

$$
\left|{}^{ABC}D_s^{\rho}\left[\psi(s) - \hbar_1(s,\psi(s),\phi(s))\right]\right| \leq \frac{N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}} + \frac{N_{\Re_{1,2}}}{1-N_{\Re_{1,4}}}|\psi(s)| + \frac{N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}}|\phi(s)|. \quad (4.2)
$$

From (4.2) in (4.1), we get

$$
\begin{aligned}
&\|B_1(\psi,\phi)(s)\| \\
&\leq \|\psi_0\| + M_{\hbar_{1,1}}(\|\psi(s)\| + \|\phi(s)\|) + M_{\hbar_{1,2}} \\
&\quad + \frac{1-\rho}{\Lambda(\rho)}\Big\{N_{\Re_{1,1}} + N_{\Re_{1,2}}|\psi(s)| + N_{\Re_{1,3}}|\phi(s)| \\
&\quad + N_{\Re_{1,4}}\left(\frac{N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}} + \frac{N_{\Re_{1,2}}}{1-N_{\Re_{1,4}}}|\psi(s)| + \frac{N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}}|\phi(s)|\right)\Big\} \\
&\quad + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1}\Big\{N_{\Re_{1,1}} + N_{\Re_{1,2}}|\psi(s)| + N_{\Re_{1,3}}|\phi(s)| \\
&\quad + N_{\Re_{1,4}}\left(\frac{N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}} + \frac{N_{\Re_{1,2}}}{1-N_{\Re_{1,4}}}|\psi(s)| + \frac{N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}}|\phi(s)|\right)\Big\} d\xi \\
&\leq \|\psi_0\| + M_{\hbar_1}\|(\psi,\phi)\| + \frac{1-\rho}{\Lambda(\rho)}\Big\{N_{\Re_{1,1}} + \left(N_{\Re_{1,2}} + N_{\Re_{1,3}}\right)\|(\psi,\phi)\| \\
&\quad + \left(\frac{N_{\Re_{1,4}}N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}} + \frac{N_{\Re_{1,4}}N_{\Re_{1,2}} + N_{\Re_{1,4}}N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}}\|(\psi,\phi)\|\right)\Big\} \\
&\quad + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1}\Big\{N_{\Re_{1,1}} + \left(N_{\Re_{1,2}} + N_{\Re_{1,3}}\right)\|(\psi,\phi)\| \\
&\quad + \left(\frac{N_{\Re_{1,4}}N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}} + \frac{N_{\Re_{1,4}}N_{\Re_{1,2}} + N_{\Re_{1,4}}N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}}\|(\psi,\phi)\|\right)\Big\} d\xi.
\end{aligned}
$$

where $M_{\hbar_1} = \max\left\{M_{\hbar_{1,1}}, M_{\hbar_{1,2}}\right\}$. By the assertion ($A_4$), one has

$$
\|B_1(\psi,\phi)(s)\|
$$

$$
\begin{aligned}
&\leq \|\psi_0\| + M_{\hbar_1} r + \frac{1-\rho}{\Lambda(\rho)} \left\{ N_{\Re_{1,1}} + \left(N_{\Re_{1,2}} + N_{\Re_{1,3}}\right) r \right. \\
&\quad \left. + \left(\frac{N_{\Re_{1,4}} N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}} + \frac{N_{\Re_{1,4}} N_{\Re_{1,2}} + N_{\Re_{1,4}} N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}} r \right) \right\} \\
&\quad + \frac{1^{\rho}}{\Lambda(\rho)\Gamma(\rho+1)} \left\{ N_{\Re_{1,1}} + \left(N_{\Re_{1,2}} + N_{\Re_{1,3}}\right) r \right. \\
&\quad \left. + \left(\frac{N_{\Re_{1,4}} N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}} + \frac{N_{\Re_{1,4}} N_{\Re_{1,2}} + N_{\Re_{1,4}} N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}} r \right) \right\} \\
&\leq \|\psi_0\| + \left(N_{\Re_{1,1}} + \frac{N_{\Re_{1,4}} N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}} \right) \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)} \right) \\
&\quad + \left\{ M_{\hbar_1} + \left(\left(N_{\Re_{1,2}} + N_{\Re_{1,3}}\right) + \frac{N_{\Re_{1,4}} N_{\Re_{1,2}} + N_{\Re_{1,4}} N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}} \right) \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)} \right) \right\} r. \quad (4.3)
\end{aligned}
$$

Similarly, one can write

$$
\begin{aligned}
&\left\| B_2(\phi,\psi)(s) \right\| \\
&\leq \|\phi_0\| + \left(N_{\Re_{2,1}} + \frac{N_{\Re_{2,4}} N_{\Re_{2,1}}}{1-N_{\Re_{2,4}}} \right) \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho+1)} \right) \\
&\quad + \left\{ M_{\hbar_2} + \left(\left(N_{\Re_{2,2}} + N_{\Re_{2,3}}\right) + \frac{N_{\Re_{2,4}} N_{\Re_{2,2}} + N_{\Re_{2,4}} N_{\Re_{2,3}}}{1-N_{\Re_{2,4}}} \right) \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho+1)} \right) \right\} r \quad (4.4)
\end{aligned}
$$

where $M_{\hbar_2} = \max\left\{ M_{\hbar_{2,1}}, M_{\hbar_{2,2}} \right\}$. It follows from (4.3) in (4.4) that

$$
\begin{aligned}
&\left\| B(\psi,\phi)(s) \right\| \\
&\leq \left\| B_1(\psi,\phi)(s) \right\| + \left\| B_2(\phi,\psi)(s) \right\| \\
&\leq \|(\psi_0,\phi_0)\| + \left(N_{\Re_{1,1}} + N_{\Re_{2,1}} + \frac{N_{\Re_{2,4}} N_{\Re_{2,1}}}{1-N_{\Re_{2,4}}} + \frac{N_{\Re_{1,4}} N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}} \right) \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho+1)} \right) \\
&\quad + \left\{ M_{\hbar_1} + M_{\hbar_2} + \left[\left(N_{\Re_{1,2}} + N_{\Re_{2,2}} + N_{\Re_{1,3}} + N_{\Re_{2,3}}\right) + \frac{N_{\Re_{1,4}} N_{\Re_{1,2}} + N_{\Re_{1,4}} N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}} \right. \right. \\
&\quad \left. \left. + \frac{N_{\Re_{2,4}} N_{\Re_{2,2}} + N_{\Re_{2,4}} N_{\Re_{2,3}}}{1-N_{\Re_{2,4}}} \right] \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)} \right) \right\} r \\
&= \|(\psi_0,\phi_0)\| + N_1^* \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)} \right) \\
&\quad + \left(M_{\hbar_1} + M_{\hbar_2} + N_2^* \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)} \right) \right) r \\
&\leq r.
\end{aligned}
$$

Therefore, $B(A_r) \subset A_r$ for $r > 0$. Now, we prove that B is a contraction. Indeed, by the assertions (A_1) and (A_3), we have

$$
\begin{aligned}
&\left\| B_i(\psi,\phi)(s) - B_i\left(\widehat{\psi},\widehat{\phi}\right)(s) \right\| \\
&\leq \left\| \hbar_i(s,\psi(s),\phi(s)) - \hbar_i\left(s,\widehat{\psi}(s),\widehat{\phi}(s)\right) \right\| \\
&\quad + \frac{1-\rho}{\Lambda(\rho)} \left| \Re_i\left(s,\psi(s),\phi(s), {}^{ABC}D_s^{\rho}\left[\psi(s) - \hbar_i(s,\psi(s),\phi(s))\right]\right) \right. \\
&\quad \left. - \Re_i\left(s,\widehat{\psi}(s),\widehat{\phi}(s), {}^{ABC}D_s^{\rho}\left[\widehat{\psi}(s) - \hbar_i\left(s,\widehat{\psi}(s),\widehat{\phi}(s)\right)\right]\right) \right|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s (s-\xi)^{\rho-1}\left|\Re_i\left(\xi,\psi\left(\xi\right),\phi(\xi),^{ABC}D_s^{\rho}\left[\psi\left(\xi\right)-\hbar_i\left(\xi,\psi\left(\xi\right),\phi(\xi)\right)\right]\right)\right.\\
&\left.-\Re_i\left(\xi,\widehat{\psi}\left(\xi\right),\widehat{\phi}(\xi),^{ABC}D_s^{\rho}\left[\widehat{\psi}\left(\xi\right)-\hbar_1\left(\xi,\widehat{\psi}\left(\xi\right),\widehat{\phi}(\xi)\right)\right]\right)\right|d\xi\\
\leq\; & M_{\hbar_{i,1}}\left(\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right)\\
&+\frac{1-\rho}{\Lambda(\rho)}N_{\Re_i}\left[\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right.\\
&\left.+\left|^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_i\left(s,\psi\left(s\right),\phi(s)\right)\right]-^{ABC}D_s^{\rho}\left[\widehat{\psi}\left(s\right)-\hbar_i\left(s,\widehat{\psi}\left(s\right),\widehat{\phi}(s)\right)\right]\right|\right]\\
&+\frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s (s-\xi)^{\rho-1}M_{\hbar_{i,1}}\left[\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right.\\
&\left.+\left|^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_i\left(s,\psi\left(s\right),\phi(s)\right)\right]-^{ABC}D_s^{\rho}\left[\widehat{\psi}\left(s\right)-\hbar_i\left(s,\widehat{\psi}\left(s\right),\widehat{\phi}(s)\right)\right]\right|\right]d\xi. \qquad (4.5)
\end{aligned}
$$

for $i=1,2$. On the other hand, by (1.1), we get

$$
\begin{aligned}
&\left|^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_i\left(s,\psi\left(s\right),\phi(s)\right)\right]-^{ABC}D_s^{\rho}\left[\widehat{\psi}\left(s\right)-\hbar_i\left(s,\widehat{\psi}\left(s\right),\widehat{\phi}(s)\right)\right]\right|\\
=\; & \left|\Re_i\left(s,\psi\left(s\right),\phi(s),^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_i\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)\right.\\
&\left.-\Re_i\left(s,\widehat{\psi}\left(s\right),\widehat{\phi}(s),^{ABC}D_s^{\rho}\left[\widehat{\psi}\left(s\right)-\hbar_i\left(s,\widehat{\psi}\left(s\right),\widehat{\phi}(s)\right)\right]\right)\right|\\
\leq\; & N_{\Re_i}\left[\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right.\\
&\left.+\left|^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_i\left(s,\psi\left(s\right),\phi(s)\right)\right]-^{ABC}D_s^{\rho}\left[\widehat{\psi}\left(s\right)-\hbar_i\left(s,\widehat{\psi}\left(s\right),\widehat{\phi}(s)\right)\right]\right|\right],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left|^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_i\left(s,\psi\left(s\right),\phi(s)\right)\right]-^{ABC}D_s^{\rho}\left[\widehat{\psi}\left(s\right)-\hbar_i\left(s,\widehat{\psi}\left(s\right),\widehat{\phi}(s)\right)\right]\right|\\
\leq\; & \frac{N_{\Re_i}}{1-N_{\Re_i}}\left(\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right). \qquad (4.6)
\end{aligned}
$$

From (4.6) in (4.5), we obtain for $i=1,2$,

$$
\begin{aligned}
&\left\|B_i\left(\psi,\phi\right)(s)-B_i\left(\widehat{\psi},\widehat{\phi}\right)(s)\right\|\\
\leq\; & M_{\hbar_i}\left(\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right)\\
&+\frac{1-\rho}{\Lambda(\rho)}N_{\Re_i}\left[\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right.\\
&\left.+\frac{N_{\Re_i}}{1-N_{\Re_i}}\left(\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right)\right]\\
&+\frac{1^{\rho}}{\Lambda(\rho)\Gamma(\rho)}M_{\hbar_{i,1}}\left[\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right.\\
&\left.+\frac{N_{\Re_i}}{1-N_{\Re_i}}\left(\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right)\right]\\
\leq\; & \left(M_{\hbar_i}+\frac{1-\rho}{\Lambda(\rho)}\left(N_{\Re_i}+\frac{N_{\Re_i}^2}{1-N_{\Re_i}}\right)\right.\\
&\left.+\frac{1}{\Lambda(\rho)\Gamma(\rho)}\left(M_{\hbar_{i,1}}+\frac{M_{\hbar_{i,1}}N_{\Re_i}}{1-N_{\Re_i}}\right)\right)\left\|(\psi,\phi)-\left(\widehat{\psi},\widehat{\phi}\right)\right\|\\
=\; & \Psi_i\left\|(\psi,\phi)-\left(\widehat{\psi},\widehat{\phi}\right)\right\|. \qquad (4.7)
\end{aligned}
$$

Since $\Psi_i < 1$ $(i = 1, 2)$, we can use Theorem 2.6 and get $B(\psi,\phi) = (\psi,\phi)$, for $(\psi,\phi) \in A_r$. Hence, the operator B has a unique coupled FP, which is a unique solution to the problem (3.1). □

Our next goal in this part is to study the existence of a solution for the system (1.1) by applying Krasnoselskii's FP theorem.

Theorem 4.2. *Under assertions (A_1)-(A_4), the problem (1.1) has a solution provided that*

$$2\left(M_{\hbar_1} + M_{\hbar_2}\right) < 1.$$

Proof. Define the operators $B_1, B_2 : A_r \to A_r$ as

$$\begin{aligned} B_1(\psi,\phi) &= \left(B_{11}(\psi,\phi), B_{12}(\phi,\psi)\right), \\ B_2(\psi,\phi) &= \left(B_{21}(\psi,\phi), B_{22}(\phi,\psi)\right) \end{aligned}$$

$$\begin{cases} B_{11}(\psi,\phi)(s) = \psi_0 + \hbar_1\left(s, \psi(s), \phi(s)\right), \\ B_{12}(\phi,\psi)(s) = \phi_0 + \hbar_2\left(s, \psi(s), \phi(s)\right), \\ B_{21}(\psi,\phi)(s) = \frac{1-\rho}{\Lambda(\rho)} \Re_1\left(s, \psi(s), \phi(s), {}^{ABC}D_s^\rho\left[\psi(s) - \hbar_1\left(s, \psi(s), \phi(s)\right)\right]\right) \\ + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} \Re_1\left(\xi, \psi(\xi), \phi(\xi), {}^{ABC}D_s^\rho\left[\psi(\xi) - \hbar_1\left(\xi, \psi(\xi), \phi(\xi)\right)\right]\right) d\xi, \\ B_{22}(\phi,\psi)(s) = \frac{1-\rho}{\Lambda(\rho)} \Re_2\left(s, \psi(s), \phi(s), {}^{ABC}D_s^\rho\left[\psi(s) - \hbar_2\left(s, \psi(s), \phi(s)\right)\right]\right) \\ + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\sigma-1} \Re_2\left(\xi, \psi(\xi), \phi(\xi), {}^{ABC}D_s^\rho\left[\psi(\xi) - \hbar_2\left(\xi, \psi(\xi), \phi(\xi)\right)\right]\right) d\xi. \end{cases} \tag{4.8}$$

we can finish the proof if we can prove that the pair $(\psi,\phi) \in A_{r_0}$ is a solution to the equation $(\psi,\phi) = B_1(\psi,\phi) + B_2(\psi,\phi)$, where $A_r = \{(\psi,\phi) \in \Theta : \|(\psi,\phi)\|_\Theta \le r\}$ is a closed bounded and convex subset of Θ. We split the proof onto the following steps:

Step 1: Claim that for $(\psi,\phi) \in A_r$, $B_1(\psi,\phi) + B_2(\psi,\phi) \in A_r$. Using the assertion ($A_3$) and (4.8), we can write

$$\begin{aligned} \|B_1(\psi,\phi)\| &= \left\|\left(B_{11}(\psi,\phi), B_{12}(\phi,\psi)\right)\right\| \\ &\le \|(\psi_0,\phi_0)\| + \left\|\hbar_1\left(s,\psi(s),\phi(s)\right)\right\| + \left\|\hbar_2\left(s,\psi(s),\phi(s)\right)\right\| \\ &\le \|(\psi_0,\phi_0)\| + \left\|\hbar_1\left(s,\psi(s),\phi(s)\right) - \hbar_1(s,0,0)\right\| + \left\|\hbar_1(s,0,0)\right\| \\ &\quad + \left\|\hbar_2\left(s,\psi(s),\phi(s)\right) - \hbar_2(s,0,0)\right\| + \left\|\hbar_2(s,0,0)\right\| \\ &\le \|(\psi_0,\phi_0)\| + \left(M_{\hbar_{1,1}} + M_{\hbar_{2,1}}\right)\left(\|\psi(s)\| + \|\phi(s)\|\right) + M_{\hbar_{1,2}} + M_{\hbar_{2,2}} \\ &\le \|(\psi_0,\phi_0)\| + \left(M_{\hbar_1} + M_{\hbar_2}\right)\|(\psi,\phi)\| + \left(M_{\hbar_1} + M_{\hbar_2}\right). \end{aligned} \tag{4.9}$$

where $M_{\hbar_1} = \max\left\{M_{\hbar_{1,1}}, M_{\hbar_{1,2}}\right\}$ and $M_{\hbar_2} = \max\left\{M_{\hbar_{2,1}}, M_{\hbar_{2,2}}\right\}$.Again by (4.8) and the hypothesis (A_2), we have

$$\begin{aligned} & \|B_2(\psi,\phi)\| = \left\|\left(B_{21}(\psi,\phi), B_{22}(\phi,\psi)\right)\right\| \\ = \;& \|B_{21}(\psi,\phi)\| + \|B_{22}(\phi,\psi)\| \\ \le \;& \frac{1-\rho}{\Lambda(\rho)} \left|\Re_1\left(s, \psi(s), \phi(s), {}^{ABC}D_s^\rho\left[\psi(s) - \hbar_1\left(s, \psi(s), \phi(s)\right)\right]\right)\right| \\ & + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} \left|\Re_1\left(\xi, \psi(\xi), \phi(\xi), {}^{ABC}D_s^\rho\left[\psi(\xi) - \hbar_1\left(\xi, \psi(\xi), \phi(\xi)\right)\right]\right)\right| d\xi \end{aligned}$$

$$
\begin{aligned}
&+\frac{1-\rho}{\Lambda(\rho)}\left|\Re_2\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)\right| \\
&+\frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\left|\int_0^s\left(s-\xi\right)^{\sigma-1}\Re_2\left(\xi,\psi\left(\xi\right),\phi(\xi),{}^{ABC}D_s^{\rho}\left[\psi\left(\xi\right)-\hbar_2\left(\xi,\psi\left(\xi\right),\phi(\xi)\right)\right]\right)\right|d\xi \\
\leq\ &\frac{1-\rho}{\Lambda(\rho)}\left\{N_{\Re_{1,1}}+N_{\Re_{1,2}}+\left(N_{\Re_{1,2}}+N_{\Re_{2,2}}\right)\|\psi\|+\left(N_{\Re_{1,3}}+N_{\Re_{2,3}}\right)\|\phi\|\right. \\
&\left.+N_{\Re_{1,4}}\left|D_s^{\rho}\left[\psi\left(\xi\right)-\hbar_1\left(\xi,\psi\left(\xi\right),\phi(\xi)\right)\right]\right|+N_{\Re_{2,4}}\left|D_s^{\rho}\left[\psi\left(\xi\right)-\hbar_2\left(\xi,\psi\left(\xi\right),\phi(\xi)\right)\right]\right|\right\} \\
&+\frac{1^{\rho}}{\Lambda(\rho)\Gamma(\rho)}\left\{N_{\Re_{1,1}}+N_{\Re_{1,2}}+\left(N_{\Re_{1,2}}+N_{\Re_{2,2}}\right)\|\psi\|+\left(N_{\Re_{1,3}}+N_{\Re_{2,3}}\right)\|\phi\|\right. \\
&\left.+N_{\Re_{1,4}}\left|D_s^{\rho}\left[\psi\left(\xi\right)-\hbar_1\left(\xi,\psi\left(\xi\right),\phi(\xi)\right)\right]\right|+N_{\Re_{2,4}}\left|D_s^{\rho}\left[\psi\left(\xi\right)-\hbar_2\left(\xi,\psi\left(\xi\right),\phi(\xi)\right)\right]\right|\right\}
\end{aligned}
\tag{4.10}
$$

Using (4.2), we obtain

$$
\left|D_s^{\rho}\left[\psi\left(\xi\right)-\hbar_1\left(\xi,\psi\left(\xi\right),\phi(\xi)\right)\right]\right|\leq\frac{N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}}+\frac{N_{\Re_{1,2}}}{1-N_{\Re_{1,4}}}\left|\psi\left(s\right)\right|+\frac{N_{\Re_{1,3}}}{1-N_{\Re_{1,4}}}\left|\phi\left(s\right)\right| \tag{4.11}
$$

Similarly

$$
\left|D_s^{\rho}\left[\psi\left(\xi\right)-\hbar_2\left(\xi,\psi\left(\xi\right),\phi(\xi)\right)\right]\right|\leq\frac{N_{\Re_{2,1}}}{1-N_{\Re_{2,4}}}+\frac{N_{\Re_{2,2}}}{1-N_{\Re_{2,4}}}\left|\psi\left(s\right)\right|+\frac{N_{\Re_{2,3}}}{1-N_{\Re_{2,4}}}\left|\phi\left(s\right)\right| \tag{4.12}
$$

It follows from (4.11)-(4.12) that

$$
\begin{aligned}
\|B_2\left(\psi,\phi\right)\|\ \leq\ &\left\{N_{\Re_{1,1}}+N_{\Re_{1,2}}+\frac{N_{\Re_{1,4}}N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}}+\frac{N_{\Re_{2,4}}N_{\Re_{2,1}}}{1-N_{\Re_{2,4}}}\right. \\
&+\left(N_{\Re_{1,2}}+N_{\Re_{2,2}}+N_{\Re_{1,3}}+N_{\Re_{2,3}}+\frac{N_{\Re_{1,4}}\left(N_{\Re_{1,2}}+N_{\Re_{1,3}}\right)}{1-N_{\Re_{1,4}}}\right. \\
&\left.\left.+\frac{N_{\Re_{2,4}}\left(N_{\Re_{2,2}}+N_{\Re_{2,3}}\right)}{1-N_{\Re_{2,4}}}\right)\|(\psi,\phi)\|\right\}\left(\frac{1-\rho}{\Lambda(\rho)}+\frac{1^{\rho}}{\Lambda(\rho)\Gamma(\rho)}\right).
\end{aligned}
\tag{4.13}
$$

From (4.9), (4.13) and the hypotheses (A$_4$), we have

$$
\begin{aligned}
&\|B_1\left(\psi,\phi\right)+B_2\left(\psi,\phi\right)\| \\
\leq\ &\|B_1\left(\psi,\phi\right)\|+\|B_2\left(\psi,\phi\right)\| \\
\leq\ &\|(\psi_0,\phi_0)\| \\
&+\left(M_{\hbar_1}+M_{\hbar_2}+N_{\Re_{1,1}}+N_{\Re_{1,2}}+\frac{N_{\Re_{1,4}}N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}}+\frac{N_{\Re_{2,4}}N_{\Re_{2,1}}}{1-N_{\Re_{2,4}}}\right)\left(\frac{1-\rho}{\Lambda(\rho)}+\frac{1}{\Lambda(\rho)\Gamma(\rho)}\right) \\
&+\left(M_{\hbar_1}+M_{\hbar_2}+N_{\Re_{1,2}}+N_{\Re_{2,2}}+N_{\Re_{1,3}}+N_{\Re_{2,3}}+\frac{N_{\Re_{1,4}}\left(N_{\Re_{1,2}}+N_{\Re_{1,3}}\right)}{1-N_{\Re_{1,4}}}\right) \\
&\times\left(\frac{1-\rho}{\Lambda(\rho)}+\frac{1}{\Lambda(\rho)\Gamma(\rho)}\right)\|(\psi,\phi)\| \\
\leq\ &\|(\psi_0,\phi_0)\|+N_1^*\left(\frac{1-\rho}{\Lambda(\rho)}+\frac{1}{\Lambda(\rho)\Gamma(\rho)}\right)+M_{\hbar_1}+M_{\hbar_2}+N_2^*\left(\frac{1-\rho}{\Lambda(\rho)}+\frac{1}{\Lambda(\rho)\Gamma(\rho)}\right)r \\
\leq\ &r,
\end{aligned}
$$

which implies that $B_1\left(\psi,\phi\right)+B_2\left(\psi,\phi\right)\in A_r$ for $\psi,\phi\in A_r$, $r>0$, where N_1^* and N_2^* are defined in the hypotheses (A$_4$).

Step 2: Show that B_1 is a contraction mapping. Indeed, for $\psi,\phi\in A_r$, by the assertion (A_3), we have

$$\begin{aligned}
& \left\|B_1(\psi,\phi)-B_1\left(\widehat{\psi},\widehat{\phi}\right)\right\| \\
= & \left\|\left(B_{11}(\psi,\phi)+B_{12}(\phi,\psi)\right)-\left(B_{11}\left(\widehat{\psi},\widehat{\phi}\right)+B_{12}\left(\widehat{\phi},\widehat{\psi}\right)\right)\right\| \\
\leq & \left\|B_{11}(\psi,\phi)-B_{11}\left(\widehat{\psi},\widehat{\phi}\right)\right\|+\left\|B_{12}(\phi,\psi)-B_{12}\left(\widehat{\phi},\widehat{\psi}\right)\right\| \\
= & \left\|\hbar_1(s,\psi(s),\phi(s))-\hbar_1\left(s,\widehat{\psi}(s),\widehat{\phi}(s)\right)\right\| \\
& +\left\|\hbar_2(s,\psi(s),\phi(s))-\hbar_2\left(s,\widehat{\psi}(s),\widehat{\phi}(s)\right)\right\| \\
\leq & M_{\hbar_{1,1}}\left(\left\|\psi(s)-\widehat{\psi}(s)\right\|+\left\|\phi(s)-\widehat{\phi}(s)\right\|\right) \\
& +M_{\hbar_{2,1}}\left(\left\|\psi(s)-\widehat{\psi}(s)\right\|+\left\|\phi(s)-\widehat{\phi}(s)\right\|\right) \\
\leq & 2\left(M_{\hbar_1}+M_{\hbar_2}\right)\left\|\left(\psi(s),\phi(s)\right)-\left(\widehat{\psi}(s),\widehat{\phi}(s)\right)\right\|.
\end{aligned}$$

where $M_{\hbar_1}=\max\left\{M_{\hbar_{1,1}},M_{\hbar_{1,2}}\right\}$ and $M_{\hbar_2}=\max\left\{M_{\hbar_{2,1}},M_{\hbar_{2,2}}\right\}$. Since $2\left(M_{\hbar_1}+M_{\hbar_2}\right)<1$, we conclude that B_1 is a contraction.

Step 3: Prove that the operator B_2 is completely continuous. For this, we show firstly that B_2 is continuous on A_r. Assume that $\psi_n,\phi_n,\psi,\phi\in A_r$ such that $\lim_{n\to\infty}(\psi_n,\phi_n)=(\psi,\phi)$, that is, $\lim_{n\to\infty}\psi_n=\psi$ and $\lim_{n\to\infty}\phi_n=\phi$.

According to the assertion (A_1), one has

$$\begin{aligned}
& \lim_{n\to\infty}\Re_1\left(s,\psi_n(s),\phi_n(s),{}^{ABC}D_s^\rho\left[\psi_n(s)-\hbar_1(s,\psi_n(s),\phi_n(s))\right]\right) \\
= & \Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^\rho\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right),
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{n\to\infty}\Re_2\left(s,\psi_n(s),\phi_n(s),{}^{ABC}D_s^\rho\left[\phi_n(s)-\hbar_2(s,\psi_n(s),\phi_n(s))\right]\right) \\
= & \Re_2\left(s,\psi(s),\phi(s),{}^{ABC}D_s^\rho\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right).
\end{aligned}$$

Also, for $s\in\Upsilon$, we have

$$\begin{aligned}
& \sup_{s\in\Upsilon}\left|\Re_1\left(s,\psi_n(s),\phi_n(s),{}^{ABC}D_s^\rho\left[\psi_n(s)-\hbar_1(s,\psi_n(s),\phi_n(s))\right]\right)\right. \\
& \left.-\Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^\rho\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right)\right| \\
= & 0,
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{s\in\Upsilon}\left|\Re_2\left(s,\psi_n(s),\phi_n(s),{}^{ABC}D_s^\rho\left[\phi_n(s)-\hbar_2(s,\psi_n(s),\phi_n(s))\right]\right)\right. \\
& \left.-\Re_2\left(s,\psi(s),\phi(s),{}^{ABC}D_s^\rho\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right)\right| \\
= & 0.
\end{aligned}$$

On the other hand, for $s\in\Upsilon$, we can write

$$\left\|B_2(\psi_n,\phi_n)(s)-B_2(\psi,\phi)(s)\right\|$$

$$
\begin{aligned}
&= \left\|\left(B_{21}\left(\psi_n,\phi_n\right)(s)+B_{22}\left(\phi_n,\psi_n\right)(s)\right)-\left(B_{21}\left(\psi,\phi\right)(s)+B_{22}\left(\phi,\psi\right)(s)\right)\right\| \\
&\leq \left\|B_{21}\left(\psi_n,\phi_n\right)(s)-B_{21}\left(\psi,\phi\right)(s)\right\|+\left\|B_{22}\left(\phi_n,\psi_n\right)(s)-B_{22}\left(\phi,\psi\right)(s)\right\| \\
&\leq \frac{1-\rho}{\Lambda(\rho)}\sup_{s\in\Upsilon}\left|\Re_1\left(s,\psi_n(s),\phi_n(s),{}^{ABC}D_s^\rho\left[\psi_n(s)-\hbar_1\left(s,\psi_n(s),\phi_n(s)\right)\right]\right)\right. \\
&\quad \left.-\Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^\rho\left[\psi(s)-\hbar_1\left(s,\psi(s),\phi(s)\right)\right]\right)\right| \\
&\quad +\frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s(s-\xi)^{\rho-1} \\
&\quad \times\sup_{s\in\Upsilon}\left|\Re_1\left(\xi,\psi_n(\xi),\phi_n(\xi),{}^{ABC}D_s^\rho\left[\psi_n(\xi)-\hbar_1\left(\xi,\psi_n(\xi),\phi_n(\xi)\right)\right]\right)\right| \\
&\quad -\Re_1\left(\xi,\psi_n(\xi),\phi_n(\xi),{}^{ABC}D_s^\rho\left[\psi_n(\xi)-\hbar_1\left(\xi,\psi_n(\xi),\phi_n(\xi)\right)\right]\right)\Big|\,d\xi \\
&\quad +\frac{1-\rho}{\Lambda(\rho)}\sup_{s\in\Upsilon}\left|\Re_2\left(s,\psi_n(s),\phi_n(s),{}^{ABC}D_s^\rho\left[\phi_n(s)-\hbar_{12}\left(s,\psi_n(s),\phi_n(s)\right)\right]\right)\right. \\
&\quad \left.-\Re_2\left(s,\psi(s),\phi(s),{}^{ABC}D_s^\rho\left[\phi(s)-\hbar_2\left(s,\psi(s),\phi(s)\right)\right]\right)\right| \\
&\quad +\frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s(s-\xi)^{\rho-1} \\
&\quad \times\sup_{s\in\Upsilon}\left|\Re_2\left(\xi,\psi_n(\xi),\phi_n(\xi),{}^{ABC}D_s^\rho\left[\phi_n(\xi)-\hbar_2\left(\xi,\psi_n(\xi),\phi_n(\xi)\right)\right]\right)\right| \\
&\quad -\Re_2\left(\xi,\psi_n(\xi),\phi_n(\xi),{}^{ABC}D_s^\rho\left[\phi_n(\xi)-\hbar_2\left(\xi,\psi_n(\xi),\phi_n(\xi)\right)\right]\right)\Big|\,d\xi \\
&\leq \left(\frac{1-\rho}{\Lambda(\rho)}+\frac{1^\rho}{\Lambda(\rho)\Gamma(\rho)}\right) \\
&\quad \times\left\{\sup_{s\in\Upsilon}\left|\Re_1\left(s,\psi_n(s),\phi_n(s),{}^{ABC}D_s^\rho\left[\psi_n(s)-\hbar_1\left(s,\psi_n(s),\phi_n(s)\right)\right]\right)\right.\right. \\
&\quad \left.-\Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^\rho\left[\psi(s)-\hbar_1\left(s,\psi(s),\phi(s)\right)\right]\right)\right| \\
&\quad +\sup_{s\in\Upsilon}\left|\Re_2\left(s,\psi_n(s),\phi_n(s),{}^{ABC}D_s^\rho\left[\phi_n(s)-\hbar_2\left(s,\psi_n(s),\phi_n(s)\right)\right]\right)\right. \\
&\quad \left.\left.-\Re_2\left(s,\psi(s),\phi(s),{}^{ABC}D_s^\rho\left[\phi(s)-\hbar_2\left(s,\psi(s),\phi(s)\right)\right]\right)\right|\right\}.
\end{aligned}
$$

Similar to (4.13), we have

$$
\begin{aligned}
&\left\|B_2\left(\psi_n,\phi_n\right)(s)-B_2\left(\psi,\phi\right)(s)\right\| \\
\leq\; &\left(\frac{1-\rho}{\Lambda(\rho)}+\frac{1^\rho}{\Lambda(\rho)\Gamma(\rho)}\right)\left\{N_{\Re_{1,1}}+N_{\Re_{1,2}}+\frac{N_{\Re_{1,4}}N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}}+\frac{N_{\Re_{2,4}}N_{\Re_{2,1}}}{1-N_{\Re_{2,4}}}\right. \\
&\left.+\left(N_{\Re_{1,2}}+N_{\Re_{2,2}}+N_{\Re_{1,3}}+N_{\Re_{2,3}}+\frac{N_{\Re_{1,4}}\left(N_{\Re_{1,2}}+N_{\Re_{1,3}}\right)}{1-N_{\Re_{1,4}}}+\frac{N_{\Re_{2,4}}\left(N_{\Re_{2,2}}+N_{\Re_{2,3}}\right)}{1-N_{\Re_{2,4}}}\right)\right\} \\
&\times\left\|\left(\psi_n,\phi_n\right)-\left(\psi,\phi\right)\right\| \\
\to\; &0 \text{ as } n\to\infty.
\end{aligned}
\tag{4.14}
$$

Therefore, B_2 is continuous on A_r.

Next, we prove that B_2 is a relatively compact. It is sufficient to prove that B_2 is a uniformly bounded and equicontinuous. From (4.13), and the definition of A_r we get

$$
\begin{aligned}
\left\|B_2\left(\psi,\phi\right)\right\| \leq\; &\left(\frac{1-\rho}{\Lambda(\rho)}+\frac{1^\rho}{\Lambda(\rho)\Gamma(\rho)}\right)\left\{N_{\Re_{1,1}}+N_{\Re_{1,2}}+\frac{N_{\Re_{1,4}}N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}}+\frac{N_{\Re_{2,4}}N_{\Re_{2,1}}}{1-N_{\Re_{2,4}}}\right. \\
&+\left(N_{\Re_{1,2}}+N_{\Re_{2,2}}+N_{\Re_{1,3}}+N_{\Re_{2,3}}+\frac{N_{\Re_{1,4}}\left(N_{\Re_{1,2}}+N_{\Re_{1,3}}\right)}{1-N_{\Re_{1,4}}}\right.
\end{aligned}
$$

$$+ \frac{N_{\Re_{2,4}}\left(N_{\Re_{2,2}}+N_{\Re_{2,3}}\right)}{1-N_{\Re_{2,4}}}\Bigg) r\Bigg\}$$
$$\leq \; r.$$

Hence, B_2 is a uniformly bounded. Now, we prove that for $(\psi,\phi)\in\Theta$, the mapping B_2 is equicontinuous. Let $0\leq s_1\leq s_2\leq 1$, then by (4.14), we have

$$\begin{aligned}
& \left|B_2(\psi,\phi)(s_2)-B_2(\psi,\phi)(s_1)\right| \\
\leq\; & \left(\frac{1-\rho}{\Lambda(\rho)}+\frac{(s_2-s_1)^{\rho}}{\Lambda(\rho)\Gamma(\rho)}\right)\Bigg\{N_{\Re_{1,1}}+N_{\Re_{1,2}}+\frac{N_{\Re_{1,4}}N_{\Re_{1,1}}}{1-N_{\Re_{1,4}}}+\frac{N_{\Re_{2,4}}N_{\Re_{2,1}}}{1-N_{\Re_{2,4}}} \\
& +\Bigg(N_{\Re_{1,2}}+N_{\Re_{2,2}}+N_{\Re_{1,3}}+N_{\Re_{2,3}}+\frac{N_{\Re_{1,4}}\left(N_{\Re_{1,2}}+N_{\Re_{1,3}}\right)}{1-N_{\Re_{1,4}}} \\
& +\frac{N_{\Re_{2,4}}\left(N_{\Re_{2,2}}+N_{\Re_{2,3}}\right)}{1-N_{\Re_{2,4}}}\Bigg)\Bigg\}\left|(\psi,\phi)(s_2)-(\psi,\phi)(s_1)\right| \\
\to\; & 0 \text{ as } s_2\to s_1.
\end{aligned}$$

Hence, B_2 is equicontinuous for $\psi,\phi\in A_r$. by Ascoli–Arzela's theorem, we conclude that B_2 is a relatively compact. Therefore, all requirements of Theorem 2.8 are fulfilled. Then, there exists the pair $(\psi,\phi)\in\Theta$ such that $(\psi,\phi)=B_1(\psi,\phi)+B_2(\psi,\phi)$, which means (ψ,ϕ) is a coupled FP of the operator B, which is a solution to the problem (1.1)

□

5 Stability results

Stability analysis has become one of the central themes in the theory of fractional differential equations, since it provides essential information about the sensitivity, robustness, and long-term behavior of solutions under perturbations in initial data or system parameters. Owing to the memory-dependent nature of fractional operators, stability results for such systems are often richer and more delicate than those for classical differential equations. Recent studies have established various forms of stability, including Lyapunov stability, Mittag–Leffler stability, finite-time stability, and several Ulam-type stabilities for equations involving Caputo, Caputo–Fabrizio, Atangana–Baleanu, Hilfer, and generalized fractional derivatives. These investigations have been carried out for diverse classes of models such as delay systems, impulsive problems, stochastic equations, and integro-differential systems, demonstrating the broad applicability of fractional methods in engineering, physics, biology, and control theory. In particular, recent contributions in [41–44] have provided new criteria and refined techniques for analyzing stability properties of nonlinear fractional systems, further enriching the theoretical framework and motivating continued research in this rapidly developing area.

Motivated to the recent results [41–44], in this section, we conduct a detailed stability analysis of the system described by the equation (1.1) as follows:

Definition 5.1. We define the solution (ψ,ϕ) of the equation (1.1) to be UH-stable if we can find a constant $\wp>0$ and $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}>0$ ledas to, for all solution $(\psi,\phi)\in\Theta$,

$$\begin{cases} \left|{}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right. \\ \left.-\Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right)\right|\le\varepsilon_1, \\ \left|{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right. \\ \left.-\Re_2\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right)\right|\le\varepsilon_2, \end{cases} \tag{5.1}$$

for $s\in\Upsilon=[0,1]$, we have

$$\|(\psi,\phi)-(\psi^*,\phi^*)\|\le\wp\varepsilon,$$

where (ψ^*,ϕ^*) is a unique solution of the problem (1.1). Moreover, a solution (ψ,ϕ) of the equation (1.1) is defined as generalized UH-stable if there exists a function $Q:(0,\infty)\to(0,\infty)$ with $Q(0)=0$ such that

$$\|(\psi,\phi)-(\psi^*,\phi^*)\|\le\wp Q(\varepsilon).$$

Definition 5.2. We define the solution (ψ,ϕ) of the equation (1.1) to be UHR-stable for a continuous function $\theta=\max\{\theta_1,\theta_2\}\in\Theta$ if we can find a constant $\wp>0$ and $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}>0$ implies that, for each solution $(\psi,\phi)\in\Theta$,

$$\begin{cases} \left|{}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right. \\ \left.-\Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right)\right|\le\theta_1(s)\,\varepsilon_1, \\ \left|{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right. \\ \left.-\Re_2\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right)\right|\le\theta_2(s)\,\varepsilon_2, \end{cases} \tag{5.2}$$

for $s\in\Upsilon$, we have

$$\|(\psi,\phi)-(\psi^*,\phi^*)\|\le\wp\theta(s)\,\varepsilon,$$

where (ψ^*,ϕ^*) is a unique solution of the problem (1.1).

Definition 5.3. We define the solution (ψ,ϕ) of the problem (1.1) to be generalized UHR-stable for a continuous function $\theta=\max\{\theta_1,\theta_2\}\in\Theta$ if we can find a constant $\wp>0$ such that, for each solution $(\psi,\phi)\in\Theta$ as follows:

$$\begin{cases} \left|{}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right. \\ \left.-\Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right)\right|\le\theta_1(s), \\ \left|{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right. \\ \left.-\Re_2\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right)\right|\le\theta_2(s), \end{cases} \tag{5.3}$$

for $s\in\Upsilon$, we have

$$\|(\psi,\phi)-(\psi^*,\phi^*)\|\le\wp\theta(s),$$

where (ψ^*,ϕ^*) is a unique solution to the problem (1.1).

Remark 5.4. The pair $(\psi,\phi)\in\Theta$ is a solution of (5.1) if and only if there exists a function $p=\max\{p_1,p_2\}\in C(\Upsilon)$ depending on ψ, and ϕ such that for $s\in\Upsilon$,

(i) $|p_1(s)|\le\varepsilon_1$ and $|p_2(s)|\le\varepsilon_2$,

(ii)

$$\begin{cases} {}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_1\left(s,\psi\left(s\right),\phi(s)\right)\right] \\ -\Re_1\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_1\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)-p_1(s)=0, \\ {}^{ABC}D_s^{\rho}\left[\phi\left(s\right)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right] \\ -\Re_2\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)\big|-p_2(s)=0. \end{cases}$$

Remark 5.5. The pair $(\phi,\psi)\in\Theta$ is a solution of (5.1) if and only if there exists the function $p=\max\{p_1,p_2\}\in C(\Upsilon)$ depending on ψ, and ϕ such that for $s\in\Upsilon$,

(i) $|p_1(s)|\leq\theta_1(s)\varepsilon_1$ and $|p_2(s)|\leq\theta_2(s)\varepsilon_2$,

(ii)

$$\begin{cases} {}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_1\left(s,\psi\left(s\right),\phi(s)\right)\right] \\ -\Re_1\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_1\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)-p_1(s)=0, \\ {}^{ABC}D_s^{\rho}\left[\phi\left(s\right)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right] \\ -\Re_2\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)\big|-p_2(s)=0. \end{cases}$$

Remark 5.6. The pair $(\phi,\psi)\in\Theta$ is a solution of (5.1) if and only if exists the function $p=\max\{p_1,p_2\}\in C(\Upsilon)$ depending on ψ, and ϕ such that for $s\in\Upsilon$,

(i) $|p_1(s)|\leq\theta_1(s)$ and $|p_2(s)|\leq\theta_2(s)$,

(ii)

$$\begin{cases} {}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_1\left(s,\psi\left(s\right),\phi(s)\right)\right] \\ -\Re_1\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_1\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)-p_1(s)=0, \\ {}^{ABC}D_s^{\rho}\left[\phi\left(s\right)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right] \\ -\Re_2\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)\big|-p_2(s)=0. \end{cases}$$

Lemma 5.7. *Assume that $(\psi,\phi)\in\Theta=C(\Upsilon)\times C(\Upsilon)$ is a solution to the problem*

$$\begin{cases} {}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_1\left(s,\psi\left(s\right),\phi(s)\right)\right] \\ =\Re_1\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_1\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)+p_1(s),\ \ s\in\Upsilon, \\ {}^{ABC}D_s^{\rho}\left[\phi\left(s\right)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right] \\ =\Re_2\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)+p_1(s), \\ \psi\left(0\right)=\psi_0,\ \ \phi\left(0\right)=\phi_0. \end{cases} \tag{5.4}$$

Then, the pair (ψ,ϕ) satisfies the following relations:

$$\|\psi-B_1(\psi,\phi)\|\leq\Omega_{p_1}\varepsilon_1 \ \ \textit{and} \ \ \|\psi-B_2(\phi,\psi)\|\leq\Omega_{p_2}\varepsilon_2$$

where

$$\Omega_{p_1}=\Omega_{p_2}=\left(\frac{1-\rho}{\Lambda(\rho)}+\frac{1}{\Lambda(\rho)\Gamma(\rho)}\right).$$

Proof. Assume that $(\psi,\phi)\in\Theta$ is a solution to (1.1),

$$\psi\left(s\right)=B_1\left(\psi,\phi\right)+\frac{1-\rho}{\Lambda(\rho)}p_1\left(s\right)+\frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s\left(s-\xi\right)^{\rho-1}p_1\left(s\right)d\xi,$$

and

$$\phi(s) = B_2(\phi,\psi) + \frac{1-\rho}{\Lambda(\rho)} p_2(s) + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} p_2(s)\, d\xi.$$

Using Remark 5.4, one has

$$\begin{aligned} |\psi(s) - B_1(\psi,\phi)| &\leq \frac{1-\rho}{\Lambda(\rho)} \varepsilon_1 + \frac{\rho\varepsilon_1}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} d\xi \\ &\leq \frac{1-\rho}{\Lambda(\rho)} \varepsilon_1 + \frac{\varepsilon_1}{\Lambda(\rho)\Gamma(\rho)} \\ &= \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)}\right) \varepsilon_1 \\ &= \Omega_{p_1}\varepsilon_1, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} |\phi(s) - B_2(\phi,\psi)| &\leq \frac{1-\rho}{\Lambda(\rho)} \varepsilon_2 + \frac{\rho\varepsilon_2}{\Lambda(\rho)\Gamma(\rho)} \int_0^s (s-\xi)^{\rho-1} d\xi \\ &\leq \frac{1-\rho}{\Lambda(\rho)} \varepsilon_2 + \frac{\varepsilon_2}{\Lambda(\rho)\Gamma(\rho)} \\ &= \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)}\right) \varepsilon_2 \\ &= \Omega_{p_2}\varepsilon_2. \end{aligned} \tag{5.6}$$

□

Theorem 5.8. *Via the assertions (A_1)-(A_4), the system (1.1) is UH-stable and generalized UH-stable, provided that $\Psi_i < 1$ $(i = 1, 2)$, where Ψ_i is defined by Theorem 4.1.*

Proof. Assume that $(\psi,\phi) \in \Theta$ is a solution and $(\widehat{\psi},\widehat{\phi}) \in \Theta$ is a unique solution of (1.1), then

$$\begin{aligned} \left\|(\psi,\phi) - (\widehat{\psi},\widehat{\phi})\right\| &= \left\|(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\| \\ &= \left\|(\psi,\phi) - B(\psi,\phi) + B(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\| \\ &\leq \|(\psi,\phi) - B(\psi,\phi)\| + \left\|B(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\| \\ &= \|(\psi,\phi) - (B_1(\psi,\phi), B_2(\phi,\psi))\| + \left\|B(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\| \\ &\leq \|\psi - B_1(\psi,\phi)\| + \|\phi - B_2(\phi,\psi)\| + \left\|B(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\|. \end{aligned} \tag{5.7}$$

It follows from Theorem 4.1 (Inequality (4.7)), Lemma 5.7 (inequalities (5.5) and(5.6)) and (5.7) that

$$\begin{aligned} \left\|(\psi,\phi) - (\widehat{\psi},\widehat{\phi})\right\| &\leq \Omega_{p_1}\varepsilon_1 + \Omega_{p_2}\varepsilon_2 + \Psi \left\|(\psi,\phi) - (\widehat{\psi},\widehat{\phi})\right\| \\ &\leq 2\Omega_p\varepsilon + \Psi \left\|(\psi,\phi) - (\widehat{\psi},\widehat{\phi})\right\|, \end{aligned}$$

which implies that

$$\left\|(\psi,\phi) - (\widehat{\psi},\widehat{\phi})\right\| \leq \frac{2\Omega_p}{1-\Psi}\varepsilon,$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and $p = \max\{p_1, p_2\}$. Take $\wp_1 = \frac{2\Omega_p}{1-\Psi}$, the solution to the problem (1.1) is proven to be UH-stable. Additionally, if we set the function $Q(\varepsilon) = \varepsilon$, the solution satisfies the criteria for generalized UH-stable as defined in Definition 5.1. □

Lemma 5.9. *Under the conditions in Remark 5.5, for the solution of (5.4), the following is true*

$$\|\psi - B_1(\psi,\phi)\| \leq \Omega_{p_1}\theta_1(s)\varepsilon_1 \text{ and } \|\psi - B_2(\phi,\psi)\| \leq \Omega_{p_2}\theta_1(s)\varepsilon_2$$

where

$$\Omega_{p_1} = \Omega_{p_2} = \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)}\right).$$

Proof. Let $(\psi,\phi) \in \Theta$ is a solution to (1.1),

$$\psi(s) = B_1(\psi,\phi) + \frac{1-\rho}{\Lambda(\rho)}p_1(s) + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s (s-\xi)^{\rho-1} p_1(s)\, d\xi,$$

and

$$\phi(s) = B_2(\phi,\psi) + \frac{1-\rho}{\Lambda(\rho)}p_2(s) + \frac{\rho}{\Lambda(\rho)\Gamma(\rho)}\int_0^s (s-\xi)^{\rho-1} p_2(s)\, d\xi.$$

From Remark 5.5, we have

$$\begin{aligned} |\psi(s) - B_1(\psi,\phi)| &\leq \frac{(1-\rho)\,\theta_1(s)}{\Lambda(\rho)}\varepsilon_1 + \frac{\rho\varepsilon_1}{\Lambda(\rho)\Gamma(\rho)}\int_0^s (s-\xi)^{\rho-1}\theta_1(s)d\xi \\ &\leq \frac{(1-\rho)\,\theta_1(s)}{\Lambda(\rho)}\varepsilon_1 + \frac{\varepsilon_1\theta_1(s)}{\Lambda(\rho)\Gamma(\rho)} \\ &= \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)}\right)\theta_1(s)\varepsilon_1 \\ &= \Omega_{p_1}\theta_1(s)\varepsilon_1, \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} |\phi(s) - B_2(\phi,\psi)| &\leq \frac{(1-\rho)\,\theta_2(s)}{\Lambda(\rho)}\varepsilon_2 + \frac{\rho\varepsilon_2\theta_2(s)}{\Lambda(\rho)\Gamma(\rho)}\int_0^s (s-\xi)^{\rho-1} d\xi \\ &\leq \frac{(1-\rho)\,\theta_2(s)}{\Lambda(\rho)}\varepsilon_2 + \frac{\varepsilon_2\theta_2(s)}{\Lambda(\rho)\Gamma(\rho)} \\ &= \left(\frac{1-\rho}{\Lambda(\rho)} + \frac{1}{\Lambda(\rho)\Gamma(\rho)}\right)\theta_2(s)\varepsilon_2 \\ &= \Omega_{p_2}\theta_2(s)\varepsilon_2. \end{aligned} \tag{5.9}$$

□

Theorem 5.10. *Under the hypotheses (A_1)-(A_4), the problem (1.1) is UHR-stable and generalized UHR-stable, whenever $\Psi_i < 1$ $(i = 1,2)$, where Ψ_i is described as Theorem 4.1.*

Proof. Assume that $(\psi,\phi) \in \Theta$ and $(\widehat{\psi},\widehat{\phi}) \in \Theta$ is a unique solution of (1.1), then, we get

$$\begin{aligned} \left\|(\psi,\phi) - (\widehat{\psi},\widehat{\phi})\right\| &= \left\|(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\| \\ &= \left\|(\psi,\phi) - B(\psi,\phi) + B(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\| \\ &\leq \|(\psi,\phi) - B(\psi,\phi)\| + \left\|B(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\| \\ &= \|(\psi,\phi) - (B_1(\psi,\phi), B_2(\phi,\psi))\| + \left\|B(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\| \\ &\leq \|\psi - B_1(\psi,\phi)\| + \|\phi - B_2(\phi,\psi)\| + \left\|B(\psi,\phi) - B(\widehat{\psi},\widehat{\phi})\right\|. \end{aligned} \tag{5.10}$$

From Theorem 4.1 (Inequality (4.7)) and Lemma 5.9 (inequalities (5.8) and (5.9)) and (5.10), it follows that

$$\begin{aligned}\left\|(\psi,\phi)-(\widehat{\psi},\widehat{\phi})\right\| &\leq \Omega_{p_1}\theta_1(s)\varepsilon_1+\Omega_{p_2}\theta_2(s)\varepsilon_2+\Psi\left\|(\psi,\phi)-(\widehat{\psi},\widehat{\phi})\right\| \\ &\leq 2\Omega_p\theta(s)\varepsilon+\Psi\left\|(\psi,\phi)-(\widehat{\psi},\widehat{\phi})\right\|,\end{aligned}$$

which implies that

$$\left\|(\psi,\phi)-(\widehat{\psi},\widehat{\phi})\right\| \leq \frac{2\Omega_p}{1-\Psi}\theta(s)\varepsilon,$$

for $s \in \Upsilon$, where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, $p = \max\{p_1, p_2\}$ and $\theta = \max\{\theta_1, \theta_2\}$. Let $\wp = \frac{2\Omega_p}{1-\Psi}$, the solution to the problem (1.1) is UHR-stable by Definition 5.2. Furthermore, if we set $\varepsilon = 1$, by Remark 5.6 and Definition 5.3, the solution satisfies the criteria to be generalized UHR-stable. □

6 Applications under illustrative examples

To validate the claims of existence and uniqueness of solutions, this section provides some illustrative examples. These cases act as tangible proof, showing how the theoretical principles discussed earlier ensure a solution exists and is the only possible one. They reinforce the validity of the concepts by connecting theory to concrete applications.

Example 6.1. Implicit coupled neutral FDEs are a highly specialized area of mathematics that model complex systems with memory, or hereditary properties. While specific applications are still an active area of research, these equations are used in various fields where traditional integer-order differential equations are insufficient to capture the full dynamics [1–7].

- In engineering and control systems, these equations are particularly useful for modeling complex engineering systems. They can describe control systems with time delays and memory, where the current state depends on past states.
- In physics and mechanics, they are applied to model phenomena in viscoelastic materials, which exhibit both viscous and elastic characteristics. the "neutral" term in the equation accounts for the dependence of the system's behavior on the derivative of a delayed state, which is crucial for these materials.
- In biological systems, processes often have non-local memory effects. Coupled neutral FDEs can model complex interactions like cellular systems and population dynamics, where the growth of one population is influenced by the past states of another.
- In economics and finance, these equations can be used to model economic systems with long-term memory effects. They are particularly useful for predicting asset price volatility and other financial phenomena where past behavior influences future trends.
- In thermoelasticity and heat transfer, they are used to model heat transfer in materials with memory. The fractional derivative can account for the non-local nature of heat conduction, where the heat flux at a given point is influenced by the temperature history of the entire body.

Example 6.2. Consider the following problem:

$$\begin{cases} {}^{ABC}D_s^{\frac{1}{3}}\left[\psi(s)-\frac{e^{-s}}{50}\left(\psi\left(\frac{s}{4}\right)+\phi(\frac{s}{4})\right)\right]=\frac{s}{4}+\frac{\ln(2-s)}{80}\psi(s)+\frac{\sin(s)}{30}\phi(s) \\ +\frac{\cos(s)}{35}{}^{ABC}D_s^{\frac{1}{3}}\left[\psi(s)-\frac{e^{-s}}{50}\left(\psi\left(\frac{s}{4}\right)+\phi(\frac{s}{4})\right)\right], \\ \\ {}^{ABC}D_s^{\frac{1}{3}}\left[\phi(s)-\frac{e^{-\cos(s)}}{75}\left(\psi\left(\frac{s}{5}\right)+\phi(\frac{s}{5})\right)\right]=\frac{s}{5}+\frac{\sqrt{4-s}}{100}\psi(s)+\frac{e^{-\sin(s)}}{40}\phi(s) \\ +\frac{\tan^{-1}(s)}{70}{}^{ABC}D_s^{\frac{1}{3}}\left[\phi(s)-\frac{e^{-\cos(s)}}{75}\left(\psi\left(\frac{s}{5}\right)+\phi(\frac{s}{5})\right)\right], \\ \\ \psi(0)=0,\ \phi(0)=0, \end{cases} \tag{6.1}$$

where $s=[0,1]$. Clearly, the problem (6.1) is a special case from the problem (1.1) with $\rho=\frac{1}{3}$ and

$$\begin{cases} \hbar_1(s,\psi(s),\phi(s))=\frac{e^{-s}}{50}\left(\psi\left(\frac{s}{4}\right)+\phi(\frac{s}{4})\right), \\ \hbar_2(s,\psi(s),\phi(s))=\frac{e^{-\cos(s)}}{75}\left(\psi\left(\frac{s}{5}\right)+\phi(\frac{s}{5})\right), \\ \Re_1\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi(s)-\hbar_1(s,\psi(s),\phi(s))\right]\right)=\frac{s}{4}+\frac{\ln(2-s)}{80}\psi(s) \\ +\frac{\sin(s)}{30}\phi(s)+\frac{\cos(s)}{35}{}^{ABC}D_s^{\frac{1}{3}}\left[\psi(s)-\frac{e^{-s}}{50}\left(\psi\left(\frac{s}{4}\right)+\phi(\frac{s}{4})\right)\right], \\ {}^{ABC}D_s^{\frac{1}{3}}\left[\phi(s)-\frac{e^{-\cos(s)}}{75}\left(\psi\left(\frac{s}{5}\right)+\phi(\frac{s}{5})\right)\right]=\frac{s}{5}+\frac{\sqrt{4-s}}{100}\psi(s)+\frac{e^{-\sin(s)}}{40}\phi(s) \\ +\frac{\tan^{-1}(s)}{70}{}^{ABC}D_s^{\frac{1}{3}}\left[\phi(s)-\frac{e^{-\cos(s)}}{75}\left(\psi\left(\frac{s}{5}\right)+\phi(\frac{s}{5})\right)\right], \\ \Re_2\left(s,\psi(s),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2(s,\psi(s),\phi(s))\right]\right)=\frac{s}{5}+\frac{\sqrt{4-s}}{100}\psi(s) \\ +\frac{e^{-\sin(s)}}{40}\phi(s)+\frac{\tan^{-1}(s)}{70}{}^{ABC}D_s^{\frac{1}{3}}\left[\phi(s)-\frac{e^{-\cos(s)}}{75}\left(\psi\left(\frac{s}{5}\right)+\phi(\frac{s}{5})\right)\right]. \end{cases}$$

According to the condition (A_3), we can write

$$\begin{aligned} &\left\|\hbar_1(s,\psi,\phi)-\hbar_1\left(s,\widehat{\psi},\widehat{\phi}\right)\right\| \\ =&\left\|\frac{e^{-s}}{50}\left(\psi\left(\frac{s}{4}\right)+\phi(\frac{s}{4})\right)-\frac{e^{-s}}{50}\left(\widehat{\psi}\left(\frac{s}{4}\right)+\widehat{\phi}(\frac{s}{4})\right)\right\| \\ \leq&\ \frac{1}{50}\left(\left\|\psi(s)-\widehat{\psi}(s)\right\|+\left\|\phi(s)-\widehat{\phi}(s)\right\|\right), \end{aligned}$$

and

$$\begin{aligned} &\left\|\hbar_2(s,\psi,\phi)-\hbar_2\left(s,\widehat{\psi},\widehat{\phi}\right)\right\| \\ =&\left\|\frac{e^{-\cos(s)}}{75}\left(\psi\left(\frac{s}{5}\right)+\phi(\frac{s}{5})\right)-\frac{e^{-\cos(s)}}{75}\left(\psi\left(\frac{s}{5}\right)+\phi(\frac{s}{5})\right)\right\| \\ \leq&\ \frac{1}{75}\left(\left\|\psi(s)-\widehat{\psi}(s)\right\|+\left\|\phi(s)-\widehat{\phi}(s)\right\|\right), \end{aligned}$$

which implies that $M_{\hbar_1}=M_{\hbar_{1,1}}=\frac{1}{50}$, $M_{\hbar_2}=M_{\hbar_{2,1}}=\frac{1}{75}$. Also, using the assertions (A_1) and (A_3), we have

$$\begin{aligned} &\left\|\Re_1(s,\psi,\phi,\xi)-\Re_1\left(s,\widehat{\psi},\widehat{\phi},\widehat{\xi}\right)\right\| \\ =&\left\|\frac{s}{4}+\frac{\ln(2-s)}{80}\psi(s)+\frac{\sin(s)}{30}\phi(s)+\frac{\cos(s)}{35}{}^{ABC}D_s^{\frac{1}{3}}\left[\psi(s)-\frac{e^{-s}}{50}\left(\psi\left(\frac{s}{4}\right)+\phi(\frac{s}{4})\right)\right]\right. \end{aligned}$$

$$
\begin{aligned}
& -\frac{s}{4}+\frac{\ln(2-s)}{80}\widehat{\psi}(s)+\frac{\sin(s)}{30}\widehat{\phi}(s)+\frac{\cos(s)}{35}{}^{ABC}D_s^{\frac{1}{3}}\left[\widehat{\psi}(s)-\frac{e^{-s}}{50}\left(\widehat{\psi}\left(\frac{s}{4}\right)+\widehat{\phi}(\frac{s}{4})\right)\right]\Bigg\| \\
\leq\ & \frac{1}{40}\left\|\psi(s)-\widehat{\psi}(s)\right\|+\frac{1}{30}\left\|\phi(s)-\widehat{\phi}(s)\right\|+\frac{1}{35}\left\|\psi(s)-\widehat{\psi}(s)\right\|+\frac{1}{50}\left\|\xi(s)-\widehat{\xi}(s)\right\| \\
\leq\ & \frac{1}{30}\left[\left\|\psi(s)-\widehat{\psi}(s)\right\|+\left\|\phi(s)-\widehat{\phi}(s)\right\|+\left\|\xi(s)-\widehat{\xi}(s)\right\|\right],
\end{aligned}
$$

similarly,

$$
\begin{aligned}
& \left\|\Re_2(s,\psi,\phi,\xi)-\Re_2\left(s,\widehat{\psi},\widehat{\phi},\widehat{\xi}\right)\right\| \\
\leq\ & \frac{1}{40}\left[\left\|\psi(s)-\widehat{\psi}(s)\right\|+\left\|\phi(s)-\widehat{\phi}(s)\right\|+\left\|\xi(s)-\widehat{\xi}(s)\right\|\right],
\end{aligned}
$$

where $\xi=\xi(\psi,\phi)$ and $\widehat{\xi}=\widehat{\xi}\left(\widehat{\psi},\widehat{\phi}\right)$. Hence, $N_{\Re_1}=\frac{1}{30}$ and $N_{\Re_2}=\frac{1}{40}$. Moreover,

$$
\begin{aligned}
& \|\Re_1(s,\psi,\phi,\xi)\| \\
=\ & \left\|\frac{s}{4}+\frac{\ln(2-s)}{80}\psi(s)+\frac{\sin(s)}{30}\phi(s)+\frac{\cos(s)}{35}{}^{ABC}D_s^{\frac{1}{3}}\left[\psi(s)-\frac{e^{-s}}{50}\left(\phi\left(\frac{s}{4}\right)+\phi(\frac{s}{4})\right)\right]\right\| \\
\leq\ & \frac{1}{4}+\frac{1}{40}\|\psi\|+\frac{1}{30}\|\phi\|+\frac{1}{35}\|\xi\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\Re_2(s,\psi,\phi,\xi)\| =\ & \Bigg\|\frac{s}{5}+\frac{\sqrt{4-s}}{100}\psi(s)+\frac{e^{-\sin(s)}}{40}\phi(s)+\frac{e^{-\sin(s)}}{40}\phi(s) \\
& +\frac{\tan^{-1}(s)}{70}{}^{ABC}D_s^{\frac{1}{4}}\left[\phi(s)-\frac{e^{-\cos(s)}}{75}\left(\psi\left(\frac{s}{5}\right)+\phi(\frac{s}{5})\right)\right]\Bigg\| \\
\leq\ & \frac{1}{5}+\frac{1}{50}\|\psi\|+\frac{1}{40}\|\phi\|+\frac{1}{70}\|\xi\|.
\end{aligned}
$$

where $\xi=\xi(\psi,\phi)$. Hence, $N_{\Re_{1,1}}=\frac{1}{4}$, $N_{\Re_{2,1}}=\frac{1}{5}$, $N_{\Re_{1,2}}=\frac{1}{40}$, $N_{\Re_{2,2}}=\frac{1}{50}$, $N_{\Re_{1,3}}=\frac{1}{30}$, $N_{\Re_{2,3}}=\frac{1}{40}$, $N_{\Re_{1,4}}=\frac{1}{35}$, and $N_{\Re_{2,4}}=\frac{1}{70}$. From the assertion ($A_4$), we can evaluate $N_1^*\approx 0.46025$ and $N_2^*\approx 0.10570$. Choose $\Lambda(\rho)=1$, then $r\geq 0.558669>0$. Under the date above, we have $\Psi_1\approx 0.126573$ and $\Psi_2\approx 0.035532$. Thus, $\Psi_i<1$ for $i=1,2$. Therefore all requirements of Theorem 4.1 are fulfilled. Then, the problem (6.1) has a unique solution. Additionally, since $\Psi_i<1$ for $i=1,2$, the solution to problem (6.1) exhibits UH, generalized UH, UHR, and generalized UHR stability, as noted in Theorems 5.8 and 5.10.

For more analysis, we give the following example:

Example 6.3. Consider the following system:

$$
\begin{cases}
{}^{ABC}D_s^{\frac{1}{2}}\left[\psi(s)-\frac{s^4}{20+20s^4}(\sin\psi(s)+\cos\phi(s))\right]=\frac{se^{-\sqrt{s}}}{\ln 10}+\frac{\tan^{-1}\psi(s)}{18(1+s^2)}\psi(s)+\frac{s}{15+s^3}\ln(1+\phi(s)) \\
+\frac{1}{20(1+\sqrt{s})}{}^{ABC}D_s^{\frac{1}{2}}\left[\psi(s)-\frac{s^4}{20+20s^4}(\sin\psi(s)+\cos\phi(s))\right], \\
\\
{}^{ABC}D_s^{\frac{1}{2}}\left[\phi(s)-\frac{\sqrt{s}}{8+7\sqrt{s}}(\psi(s)+\sin(\phi(1-s)))\right]=\frac{s}{\sqrt{2}+2s}+\frac{\cos\sqrt{s}}{25(1+s)}\psi(s)+\frac{1-s}{18}\phi(s) \\
+\frac{\sin(s)}{30}{}^{ABC}D_s^{\frac{1}{2}}\left[\phi(s)-\frac{\sqrt{s}}{8+7\sqrt{s}}(\psi(s)+\sin(\phi(1-s)))\right], \\
\\
\psi(0)=0,\ \phi(0)=0,
\end{cases}
\tag{6.2}
$$

where $s=[0,1]$. It is clear that, the system (6.1) is another form of the system (1.1) with $\rho=\frac{1}{2}$ and

$$\begin{cases} \hbar_1\left(s,\psi\left(s\right),\phi(s)\right)=\frac{s^4}{20+20s^4}\left(\sin\psi\left(s\right)+\cos\phi(s)\right), \\ \hbar_2\left(s,\psi\left(s\right),\phi(s)\right)=\frac{\sqrt{s}}{8+7\sqrt{s}}\left(\psi\left(s\right)+\sin\left(\phi\left(1-s\right)\right)\right), \\ \Re_1\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\psi\left(s\right)-\hbar_1\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)=\frac{se^{-\sqrt{s}}}{\ln 10}+\frac{\tan^{-1}\psi(s)}{18(1+s^2)} \\ +\frac{s}{15+s^3}\ln\left(1+\phi\left(s\right)\right)+\frac{1}{20(1+\sqrt{s})}{}^{ABC}D_s^{\frac{1}{2}}\left[\psi\left(s\right)-\frac{s^4}{20+20s^4}\left(\sin\psi\left(s\right)+\cos\phi(s)\right)\right], \\ \Re_2\left(s,\psi\left(s\right),\phi(s),{}^{ABC}D_s^{\rho}\left[\phi(s)-\hbar_2\left(s,\psi\left(s\right),\phi(s)\right)\right]\right)=\frac{s}{\sqrt{2}+2s}+\frac{\cos\sqrt{s}}{25(1+s)}\psi\left(s\right)+\frac{1-s}{18}\phi\left(s\right) \\ +\frac{\sin(s)}{30}{}^{ABC}D_s^{\frac{1}{2}}\left[\phi\left(s\right)-\frac{\sqrt{s}}{8+7\sqrt{s}}\left(\psi\left(s\right)+\sin\left(\phi\left(1-s\right)\right)\right)\right]. \end{cases}$$

From the condition (A_3), we have

$$\begin{aligned} &\left\|\hbar_1\left(s,\psi,\phi\right)-\hbar_1\left(s,\widehat{\psi},\widehat{\phi}\right)\right\| \\ =\ &\left\|\frac{s^4}{20+20s^4}\left(\sin\psi\left(s\right)+\cos\phi(s)\right)-\frac{s^4}{20+20s^4}\left(\sin\widehat{\psi}\left(s\right)+\cos\widehat{\phi}(s)\right)\right\| \\ \leq\ &\frac{1}{40}\left(\left\|\sin\psi\left(s\right)-\sin\widehat{\psi}\left(s\right)\right\|+\left\|\cos\phi(s)-\cos\widehat{\phi}(s)\right\|\right) \\ \leq\ &\frac{1}{40}\left(\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right), \end{aligned}$$

and

$$\begin{aligned} &\left\|\hbar_2\left(s,\psi,\phi\right)-\hbar_2\left(s,\widehat{\psi},\widehat{\phi}\right)\right\| \\ =\ &\left\|\frac{\sqrt{s}}{8+7\sqrt{s}}\left(\psi\left(s\right)+\sin\left(\phi\left(1-s\right)\right)\right)-\frac{\sqrt{s}}{8+7\sqrt{s}}\left(\psi\left(s\right)+\sin\left(\phi\left(1-s\right)\right)\right)\right\| \\ \leq\ &\frac{1}{15}\left(\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\sin\left(\phi\left(1-s\right)\right)-\widehat{\phi}\left(s\right)\right\|\right), \\ \leq\ &\frac{1}{15}\left(\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|\right), \end{aligned}$$

which implies that $M_{\hbar_1}=M_{\hbar_{1,1}}=\frac{1}{40}$, $M_{\hbar_2}=M_{\hbar_{2,1}}=\frac{1}{15}$. Moreover, using hypotheses (A_1) and (A_3), one has

$$\begin{aligned} &\left\|\Re_1\left(s,\psi,\phi,\xi\right)-\Re_1\left(s,\widehat{\psi},\widehat{\phi},\widehat{\xi}\right)\right\| \\ =\ &\left\|\frac{se^{-\sqrt{s}}}{\ln 10}+\frac{\tan^{-1}\psi\left(s\right)}{18(1+s^2)}+\frac{s}{15+s^3}\ln\left(1+\phi\left(s\right)\right)\right. \\ &+\frac{1}{20(1+\sqrt{s})}{}^{ABC}D_s^{\frac{1}{2}}\left[\psi\left(s\right)-\frac{s^4}{20+20s^4}\left(\sin\psi\left(s\right)+\cos\phi(s)\right)\right] \\ &-\left(\frac{se^{-\sqrt{s}}}{\ln 10}+\frac{\tan^{-1}\widehat{\psi}\left(s\right)}{18(1+s^2)}+\frac{s}{15+s^3}\ln\left(1+\widehat{\phi}\left(s\right)\right)\right. \\ &\left.\left.+\frac{1}{20(1+\sqrt{s})}{}^{ABC}D_s^{\frac{1}{2}}\left[\psi\left(s\right)-\frac{s^4}{20+20s^4}\left(\sin\widehat{\psi}\left(s\right)+\cos\widehat{\phi}(s)\right)\right]\right)\right\| \\ \leq\ &\frac{1}{36}\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\frac{1}{16}\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|+\frac{1}{40}\left\|\xi\left(s\right)-\widehat{\xi}\left(s\right)\right\| \\ \leq\ &\frac{1}{16}\left[\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|+\left\|\xi\left(s\right)-\widehat{\xi}\left(s\right)\right\|\right]. \end{aligned}$$

Analogously,

$$\begin{aligned}
& \left\|\Re_2\left(s,\psi,\phi,\xi\right)-\Re_2\left(s,\widehat{\psi},\widehat{\phi},\widehat{\xi}\right)\right\| \\
\leq\ & \frac{1}{18}\left[\left\|\psi\left(s\right)-\widehat{\psi}\left(s\right)\right\|+\left\|\phi\left(s\right)-\widehat{\phi}\left(s\right)\right\|+\left\|\xi\left(s\right)-\widehat{\xi}\left(s\right)\right\|\right]
\end{aligned}$$

where $\xi=\xi\left(\psi,\phi\right)$ and $\widehat{\xi}=\widehat{\xi}\left(\widehat{\psi},\widehat{\phi}\right)$. Thus, $N_{\Re_1}=\frac{1}{16}$ and $N_{\Re_2}=\frac{1}{18}$. Furthermore,

$$\begin{aligned}
\left\|\Re_1\left(s,\psi,\phi,\xi\right)\right\| &= \left\|\frac{se^{-\sqrt{s}}}{\ln 10}+\frac{\tan^{-1}\psi\left(s\right)}{18(1+s^2)}+\frac{s}{15+s^3}\ln\left(1+\phi\left(s\right)\right)\right. \\
&\quad \left.+\frac{1}{20(1+\sqrt{s})}{}^{ABC}D_s^{\frac{1}{2}}\left[\psi\left(s\right)-\frac{s^4}{20+20s^4}\left(\sin\psi\left(s\right)+\cos\phi(s)\right)\right]\right\| \\
&\leq \frac{1}{\ln\left(10\right)}+\frac{1}{36}\left\|\psi\right\|+\frac{1}{16}\left\|\phi\right\|+\frac{1}{40}\left\|\xi\right\|,
\end{aligned}$$

and

$$\begin{aligned}
\left\|\Re_2\left(s,\psi,\phi,\xi\right)\right\| &= \left\|\frac{s}{\sqrt{2}+2s}+\frac{\cos\sqrt{s}}{25(1+s)}\psi\left(s\right)+\frac{1-s}{18}\phi\left(s\right)\right. \\
&\quad \left.+\frac{\sin\left(s\right)}{30}{}^{ABC}D_s^{\frac{1}{2}}\left[\phi\left(s\right)-\frac{\sqrt{s}}{8+7\sqrt{s}}\left(\psi\left(s\right)+\sin\left(\phi\left(1-s\right)\right)\right)\right]\right\| \\
&\leq \frac{1}{4}+\frac{1}{50}\left\|\psi\right\|+\frac{1}{18}\left\|\phi\right\|+\frac{1}{30}\left\|\xi\right\|.
\end{aligned}$$

where $\xi=\xi\left(\psi,\phi\right)$. Thus, $N_{\Re_{1,1}}=\frac{1}{\ln(10)}$, $N_{\Re_{2,1}}=\frac{1}{4}$, $N_{\Re_{1,2}}=\frac{1}{36}$, $N_{\Re_{2,2}}=\frac{1}{50}$, $N_{\Re_{1,3}}=\frac{1}{16}$, $N_{\Re_{2,3}}=\frac{1}{18}$, $N_{\Re_{1,4}}=\frac{1}{40}$, and $N_{\Re_{2,4}}=\frac{1}{30}$.

It follows from (A_4) that $N_1^*\approx 0.704051$ and $N_2^*\approx 0.1707534$. Choose $\Lambda(\rho)=1$, then $r\geq 1.031137>0$. Based on the values above, we can find $\Psi_1\approx 0.073378$ and $\Psi_2\approx 0.135903$.Thus, $\Psi_i<1$ for $i=1,2$. Therefore all assertions of Theorem 4.1 are satisfied. Hence, the system (6.2) has a unique solution.

In addition to, since $\Psi_i<1$ for $i=1,2$, the solution to the system (6.1) is UH, generalized UH, UHR, and generalized UHR stability, according to Theorems 5.8 and 5.10.

7 Conclusion and open problems

Implicit coupled neutral FDEs are crucial in nonlinear analysis because they provide a powerful framework for modeling complex systems with nonlocal behavior, memory effects, and hereditary properties. These equations are particularly effective in capturing intricate dynamics in fields like physics and engineering, where the state of a system at a given moment depends not just on its current conditions but also on its entire past history. The "fractional" component allows for derivatives of non-integer order, which more accurately describe real-world phenomena like anomalous diffusion and viscoelasticity than traditional integer-order models. The "coupled" nature signifies that the equations involve multiple interconnected variables, making them suitable for analyzing systems where different components influence each other. The "implicit" and "neutral" aspects mean that the equations can involve the derivative of the unknown function both at the current and delayed times, enabling the study of more general and complex types of nonlinearities and delays. This combination makes them an essential tool for understanding and predicting the behavior of a wide range of nonlinear systems.

So, in this paper, we investigated the existence, uniqueness, and stability of a class of nonlinear implicit coupled neutral FDEs with the ABC derivative. We established the existence of solutions using Krasnoselskii's FP theorem and proved both existence and uniqueness with the Banach contraction principle. The stability analysis covered UH, generalized UH, UHR, and generalized UHR results. Finally, a practical example demonstrated the existence, uniqueness, and stability of the solutions.

While significant progress has been made in the theory of implicit coupled neutral FDEs, several open problems remain. One major challenge is developing effective and robust numerical schemes for solving these complex equations, as their implicit and nonlocal nature makes traditional methods computationally expensive or unstable. Another key area is the investigation of qualitative properties beyond existence, uniqueness, and stability, such as the analysis of chaos, bifurcation, and blow-up phenomena, which are often difficult to study due to the memory effects inherent in fractional derivatives. Furthermore, extending the theory to more complex settings, including those with stochastic terms or non-instantaneous impulses, is a crucial research direction to model real-world systems more accurately. Moreover, exploring novel applications of these equations in emerging fields like finance, epidemiology, and materials science, and validating the theoretical results with experimental data, represents a significant gap between theoretical advancements and practical utility.

Data Availability

No data is associated with this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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References

[1] I. Podlubny, Fractional Differential Equations, Academic Press, (1999).

[2] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, Imperial College Press, (2010).

[3] B. Ross, A brief history and exposition of the fundamental theory of fractional calculus, In Fractional Calculus and its Applications, Springer, (1974).

[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, (2006).

[5] M. Caputo, F. Mainardi, Linear models of dissipation in anelastic solids, La Rivista del Nuovo Cimento, 1, 161–198, (1971).

[6] J. Sabatier, O.P. Agrawal, J.T. Machado, Advances in Fractional Calculus: Theoretical and Numerical Developments and Applications, Springer, (2007).

[7] V.E. Tarasov, Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media, Springer, (2010).

[8] S.S. Redhwan, M. Han, M. A. Almalahi , M.A. Alyami, M. Alsulami, N. Alghamdi, Piecewise implicit coupled system under ABC fractional differential equations with variable order, AIMS Mathematics, 9(7), 1622–1638.

[9] Asma, S. Shabbir, K. Shah, T. Abdeljawad, Stability analysis for a class of implicit fractional differential equations involving Atangana–Baleanu fractional derivative, Adv. Differ. Equ., 2021, 395 (2021).

[10] K. Trachoo, I. Chaiya, S. Phakmee, D. Prathumwan, Dynamics of bone remodeling by using mathematical model under ABC time-fractional derivative, Symmetry, 17, 905, (2025).

[11] M.A. Almalahi, S.K. Panchal, F. Jarad, Results on implicit fractional pantograph equations with Mittag-Leffler kernel and nonlocal condition, J. Math., 2022, 2, (2022).

[12] B. Ghanbari, A. Atangana, A new application of fractional Atangana–Baleanu derivatives: Designing ABC-fractional masks in image processing, Phy. A: Stat. Mech. Appli., 123516, (2020).

[13] H.A. Hammad, M.G. Alshehri, Application of the Mittag-Leffler kernel in stochastic differential systems for approximating the controllability of nonlocal fractional derivatives, Chaos, Solitons & Fractals, 182, 114775, (2024).

[14] D. Vivek, K. Kanagarajan, E.M. Elsayed, Existence and stability of fractional implicit differential equations with complex order, Fixed Point Theory Appl., 2018, 1–10, (2018).

[15] H. Khan, A. Khan, F. Jarad, A. Shah, Existence and data dependence theorems for solutions of an ABC-fractional order impulsive system, Chaos, Solitons & Fractals, 131, 109477, (2020).

[16] D. Aimene, D. Baleanu, D. Seba, Controllability of semilinear impulsive Atangana-Baleanu fractional differential equations with delay, Chaos, Solitons & Fractals, 28, 51–57, (2019).

[17] L. Mahto, S. Abbas, A. Favini, Analysis of caputo impulsive fractional order differential equations with applications, Int. J. Differ. Equ., 2013, 1–11, (2013).

[18] T.L. Guo, W. Jiang, Impulsive fractional functional differential equations, Comput. Appl. Math., 64, 3414–3424, (2012).

[19] N. K. Sooppy, K. Logeswari, C. Ravichandran, S. Sabarinathan, New frame of fractional neutral ABC-derivative with IBC and mixed delay, Chaos Solitons & Fractals, 2(174), 114050, (2023)

[20] K.D. Kucche, J.J. Nieto, V. Venktesh, Theory of nonlinear implicit fractional differential equations, Differ. Equ. Dyn. Syst., 28, (2020), 1–17.

[21] H.A. Hammad, R.A. Rashwan, A. Nafea, M.E. Samei, M. De la Sen, Stability and existence of solutions for a tripled problem of fractional hybrid delay differential equations, Symmetry, 14, 2579, (2022).

[22] K.D. Kucche, S.T. Sutar, Stability via successive approximations for nonlinear implicit fractional differential equations, Moroc J Pure Appl. Anal., 3, 36–54, (2017).

[23] S.T. Sutar, K.D. Kucche, Global existence and uniqueness for implicit differential equation of arbitrary order, Fract. Differ. Calc., 5, 199-208, (2015).

[24] S.S. Asma, K. Shah, T. Abdeljawad, Stability analysis for a class of implicit fractional differential equations involving Atangana-Baleanu fractional derivative, Adv. Differ. Equ., 2021, 395, (2021).

[25] V. Sowbakiy, R. Nirmalkumar, K. Loganathan, C. Selvamani, Study on existence and stability analysis for implicit neutral fractional differential equations of ABC derivative, Partial Diff. Equ. Appl. Math., 15, 101276, (2025).

[26] S.T.M. Thabet, I. Kedim, M.E. Samei, T. Abdeljawad, Analysis study of hybrid Caputo-Atangana-Baleanu fractional pantograph system under integral boundary conditions, Math. Modelling Anal., 30(2), 386-404, (2025).

[27] S.T.M. Thabet, I. Kedim, T. Abdeljawad, Exploring the solutions of Hilfer delayed Duffing problem on the positive real line, Bound Value Probl., 2024, 95 (2024).

[28] G. Mani, A. Lakshmi S. R, S.T.M. Thabet, I. Kedim, M. Vivas-Cortez, New results on coupled Caputo-Hadamard fractional neutral differential equations supplemented by unbounded delays, J. Math. Computer Sci., 39(3), 361-375, (2025).

[29] M.A.A. Ali, S.T.M. Thabet, T. Abdeljawad, I. Kedim. New results on an impulsive differential equation involving a $q-$analogue of the $\Psi-$Caputo fractional derivative, Bound. Value Probl., 2025, 139 (2025).

[30] H.A. Hammad, H. Aydi, M. Zayed. Involvement of the topological degree theory for solving a tripled system of multi-point boundary value problems, AIMS Mathematics, 8(1), 2257-2271, (2023).

[31] H.A. Hammad, M. Qasymeh, M. Abdel-Aty, Existence and stability results for a Langevin system with Caputo-Hadamard fractional operators, Int. J. Geometric Meth. Modern Phy., 21(13), 2450218, (2024).

[32] E. Bassiouny, Mathematical model for hyperbolic two temperature fractional-order thermoelastic materials subjected to thermal loading, Appl. Math. Inf. Sci., 15(1), 23-29, (2021).

[33] M. Senol, H.D. Kasmaei, Perturbation-iteration algorithm for systems of fractional differential equations and convergence analysis, Prog. Fract. Diff. Appl., 03(4), 271-279, (2017).

[34] K.D Kucche, S.T Sutar, N.K. Sooppy, Analysis of nonlinear implicit fractional differential equations with the Atangana-Baleanu derivative via measure of non-compactness, AIMS Math., 9(10), 27058–27079, (2024).

[35] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and nonsingular kernel: Theory and application to heat transfer model, Therm Sci., 20(2), 763–769, (2016).

[36] T. Abdeljawad, D. Baleanu, Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels, Adv. Differ. Equ., 2016(1), 1–18, (2016).

[37] U. Cakan, I. Ozdemir, An application of krasnoselskii fixed point theorem to some nonlinear functional integral equations, Nevsehir Bilim Ve Teknol Derg, 3(2), 66–73, (2014).

[38] A. Grans, J. Dugundji, Fixed point theory, New york: Springer-verlag, (2003).

[39] R.P. Agarwal, Y. Zhou, Y He, Existence of fractional neutral functional differential equations, Comput. Appl. Math., 59(3), 1095–1100, (2012).

[40] S.S. Chang, Y.J. Cho, N.J. Huang, Coupled fixed point theorems with applications, J. Korean Math. Soc., 33(3), 575–585, (2005).

[41] A. Khan, M. Alzabut,T. Abdeljawad, Ulam-Hyers stability results for nonlinear Caputo fractional differential equations, Alex. Eng. J., 61(4), 3125–3134, (2022).

[42] H.M. Srivastava, K. Shah, R.A. Khan, Finite-time stability analysis for impulsive fractional differential equations involving Hilfer derivative, Fractal and Fract., 7(2), 155, (2023).

[43] M.E.M. Abdalla, Z. Liu, H.A. Hammad, Existence and stability analysis of impulsive Caputo fractional delay systems, Bound. Value Probl., 2026, 32, (2026).

[44] M. De la Sen, A. Atangana, D. Baleanu, Generalized Ulam stability for Atangana–Baleanu fractional evolution equations, Chaos, Solitons & Fractals, 178, 114356, (2024).