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ORIGINAL RESEARCH

Survival Analysis Based on the AFT Approach with a Semi-Continuous Response under Zero-Inflation:An Application in Life Insurance

Nafiseh Khojasteh Bakht[1†], Shirin Shoaee[2*†], Ehsan Bahrami Samani[3†]

[1]Department of Actuarial Science, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran.
[2]Department of Actuarial Science, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran..
[3]Department of Statistics, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran.

*Corresponding author(s). E-mail(s): Sh_Shoaee@sbu.ac.ir;
Contributing authors: nkhojastebakht@gmail.com; ehsansamani@yahoo.com;
†These authors contributed equally to this work.

Abstract

Mortality models in actuarial science are important tools used to develop models from historical data to predict future trends and mortality rates, particularly in life insurance and premium assessments. These models need to be flexible enough to accommodate effects such as age and product type. Censoring also features as a unique element providing a basis for survival analysis, as it is not very often that the entire duration values are present as non-negative or asymmetric for various censored cases. Standard regression methods are unsuitable for such data, and specialized techniques such as Cox Proportional Hazards and Accelerated Failure Time (AFT) models are required. AFT models allow for flexible distributional assumptions while providing interpretable results in complex settings.
This study extends the AFT framework by introducing a Zero-Inflation component to accommodate semi-continuous lifetime data characterized by excess zeros, thereby transforming a purely continuous distribution into a semi-continuous one. By integrating zero inflation with both right- and left-censoring mechanisms, the proposed models expand the applicability of survival analysis in actuarial contexts.

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The empirical analysis is based on a real dataset obtained from a medium-sized pension fund, including individual-level information on future lifetimes and relevant covariates. The results demonstrate that the zero-inflated Gompertz AFT model provides superior performance compared to competing specifications, particularly in capturing tail behavior and age-dependent mortality dynamics. Furthermore, the inclusion of zero inflation significantly improves model flexibility and estimation accuracy in the presence of excess-zero observations. From an actuarial perspective, the proposed modelling framework offers improved estimation of survival patterns and annuity factors, supporting more accurate longevity risk assessment and better-informed decision-making in life insurance and pension applications.

Keywords: Future LifeTime Variable, Zero-Inflation, Accelerated Failure Time (AFT), Analysis, Annuity Factor.

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1 Introduction

Mortality modelling constitutes one of the earliest quantitative tools in actuarial science and plays a central role in premium calculation, risk assessment, and longevity forecasting. Traditional life tables summarize historical mortality experience; however, they often require smoothing, extrapolation, and extensive tabular structures. Parametric and semi-parametric statistical models provide a more parsimonious and analytically tractable alternative, enabling actuaries to incorporate covariate information such as age, product characteristics, and demographic factors when estimating mortality rates and future lifetimes. ([1],[2],[3],[4],[5],[6],[7],[8],[9].). A fundamental type of data in actuarial and biomedical studies is time-to-event (TTE) data, defined as the duration from a well-specified starting point until the occurrence of a particular event. Such data are non-negative, often right-skewed, and frequently subject to censoring when the event of interest is not observed during the study period ([10],[11],[12])). Censoring may arise due to loss to follow-up, administrative limits, or early termination of observation and can occur in several forms, including right-, left-, and interval-censoring ([13]). These characteristics make classical linear or generalized linear regression methods unsuitable and motivate the use of specialized survival analysis techniques. Most survival analytic procedures are implemented in standard statistical software, and a common assumption is that the censoring mechanism is non-informative, meaning that censoring does not provide additional prognostic information about future survival. Although right-censoring is most frequently encountered, left-censoring may also arise in actuarial applications and should be incorporated within a unified modelling framework. Recent developments further emphasize the role of frailty-based survival models for capturing unobserved heterogeneity through latent random effects ([14],[15]).

Two principal modelling paradigms dominate survival analysis: the Cox Proportional Hazards (PH) model and the Accelerated Failure Time (AFT) model. The Cox PH model ([16]) is a semi-parametric approach that specifies covariate effects through hazard ratios without fully specifying the baseline hazard function. In contrast, the

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AFT model ([17]) directly models the logarithm of survival time as a function of covariates, yielding an explicit regression structure. A key advantage of the AFT formulation is that regression coefficients admit a direct interpretation in terms of time acceleration or deceleration factors, which is particularly appealing in actuarial applications where lifetime scaling is of primary interest.

Parametric AFT models require specifying an appropriate error distribution; common choices include the log-normal, log-logistic, generalized gamma, and Weibull distributions ([18]). While the proportional hazards assumption may be restrictive in heterogeneous populations, AFT models often provide a more interpretable and practically relevant alternative when an appropriate distribution can be identified ([19, 20]). Extensive developments of AFT methodology have been reported in the literature, with pplications in reliability theory, industrial experiments, and survivorship data ([21],[22],[23],[24],[25]).Comprehensive discussions are also available in standard textbooks ([13],[26],[27]).

Before defining the AFT model, consider a type of survival data known as future lifetime. The future lifetime T of an individual currently aged x is a random variable, and we denote by $f_0(x)$, $F_0(x)$, and $S_0(x) = 1 - F_0(x)$ the density function, the distribution function, and the corresponding survival function of this random variable. As usual, we denote by μ_x the force of mortality at age x, which is given by $\mu_x = \frac{f_0(x)}{1-F_0(x)}$. We then obtain the following two well-known results (see, for example, [28]):

$$S_{x_i}(t) = \exp\left(-\int_0^t \mu_{x_i+s}\, ds\right), \tag{1}$$

$$f_{x_i}(t) = S_{x_i}(t)\mu_{x_i+t}. \tag{2}$$

where $S_{x_i}(t)$ and $f_{x_i}(t)$ denote, respectively, the conditional survival function and the associated probability density function of the future lifetime for an individual aged x_i at duration t. Let T denote the future lifetime and Z a $p \times 1$ vector of explanatory covariates. Within the AFT framework, the relationship between covariates and survival time is typically expressed as $\log T = Z'\theta + \sigma\epsilon$, where $\sigma > 0$ is a scale parameter and ϵ denotes the error term.Throughout this paper, log denotes the natural logarithm.

In this study, we consider two parametric specifications within the AFT framework. The first is the Weibull AFT model, obtained when the error term follows a standard Extreme Value Distribution, allowing increasing, decreasing, or constant hazard rates. The second specification is derived from the classical Gompertz mortality model, which is widely used in actuarial and demographic analysis. Embedding the Gompertz structure within the AFT formulation provides an alternative parametric representation that preserves actuarial interpretability while offering additional flexibility in modelling mortality dynamics. Recent research in survival analysis has increasingly focused on modelling complex lifetime data structures characterized by excess zeros and heterogeneous risk populations. Zero-inflated and semi-continuous survival formulations have been proposed to accommodate outcomes with a non-negligible probability mass at zero, extending classical continuous survival models ([29],[30]). In

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parallel, mixture cure-rate models provide an alternative framework by allowing a subgroup of individuals to be effectively immune to the event of interest, thereby capturing long-term survival behavior ([31],[32]). Moreover, frailty-based approaches introduce latent random effects to account for unobserved heterogeneity and dependence structures in survival data ([14],[15]). Although these developments have substantially advanced survival modelling, their joint integration within a parametric AFT framework for semi-continuous future lifetime data remains limited, particularly in actuarial applications. Despite extensive developments in AFT methodology, actuarial lifetime data may exhibit semi-continuous behavior, where a non-negligible proportion of observations accumulate at zero.

In actuarial contexts, such excess zeros may arise when individuals exit immediately after policy entry, when observation begins at the time of enrollment in pension or insurance schemes, or due to administrative recording structures in which very short durations are recorded as zero. Such Zero-Inflated structures may arise due to immediate exit after entry, administrative recording practices, or specific insurance product designs.[33],[29],[34]) These features motivate the need for modelling approaches that can explicitly accommodate a discrete mass at zero alongside a continuous positive lifetime component. Standard continuous AFT models are not designed to explicitly accommodate excess zeros, particularly in the presence of censoring and unobserved heterogeneity.

To address these limitations, we extend the AFT framework by introducing a Zero-Inflation component that produces a semi-continuous distribution consisting of a degenerate mass at zero and a continuous positive component. The proposed framework accommodates both right- and left-censoring and incorporates random effects to capture unobserved heterogeneity. The remainder of this paper is organized as follows. Section 2 presents the proposed modelling methodology, including the Zero-Inflated semi-continuous AFT framework and the Gompertz-based AFT specification. Section 3 provides numerical experiments, including simulation studies and a real data analysis based on pension fund data. Section 4 discusses actuarial implications through annuity factor calculations. Finally, Section 5 concludes the paper.

2 Modelling Framework

Recent developments in survival analysis have focused on enhancing the flexibility of Accelerated Failure Time (AFT) models for time-to-event data. Various extensions have been proposed, including flexible error distributions and alternative random effect specifications ([35],[36],[37],[38]). These contributions highlight the importance of distributional flexibility and robust treatment of unobserved heterogeneity in AFT modelling. Motivated by these developments, we investigate the structure of the distribution associated with the future lifetime variable and adopt an AFT framework for its analysis. The presence of zero inflation renders the response semi-continuous, while unobserved heterogeneity motivates the inclusion of random effects. We consider Weibull and Gompertz specifications within the AFT structure and introduce a modified error formulation for the Gompertz case. Simulation is performed using the

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inverse transform method. Overall, the proposed framework aims to improve flexibility and reliability in actuarial survival modelling.

2.1 AFT Modelling Framework without Zero Inflation

Let $T > 0$ be a random variable representing the observed future lifetime, and let t denote a point within its support.For the i-th subject, it is denoted as:

$$T_i = \begin{cases} \min(T_i^*, C_i) & \text{right-censored,} \\ \max(T_i^*, C_i) & \text{left-censored.} \end{cases}$$

where T_i^* is the true (or real) event time, and C_i denotes the potential censoring time.

A censoring indicator δ_i is defined as 0 for censored observations and 1 for fully observed events:

$$\delta_i = \begin{cases} I(T_i^* \leq C_i), & \text{for right-censored,} \\ I(T_i^* \geq C_i), & \text{for left-censored.} \end{cases}$$

Here, $I(\cdot)$ denotes the indicator function. If the event of interest (death) occurs at time t_i, the i-th subject experiences the event at that time and no further measurements are recorded afterward. Therefore, the observation schedule for subject i satisfies $t_i \leq T_i$.

Under this setting, we assume a mixed-effects AFT model of the following form:

$$\log(T_i|\gamma; \vartheta_i, \sigma) = \mathbf{Z}_i'\gamma + \mathbf{W}_i'\vartheta_i + \sigma\epsilon_i.$$

Here, $\mathbf{Z} = (1, Z_1, \ldots, Z_p)'$ is the vector of basic risk factors, and $\mathbf{W}$ is a $q \times 1$ sub-vector of $\mathbf{Z}$ ($q < p$). Also, $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_p)'$ and $\vartheta_i = (\vartheta_{i1}, \vartheta_{i2}, \ldots, \vartheta_{iq})'$ are vectors of fixed effects parameters for the risk factors and unobserved random effects parameters, respectively. We assume that the error term ϵ_i and the random effects ϑ_i are independent.

2.1.1 Future Lifetime Under Weibull AFT Regression Model (WAFT Model)

If ϵ follows the standard extreme value distribution (EVD) with density $f(\epsilon) = \exp\{\epsilon - e^{\epsilon}\}$, where $\epsilon \in \mathbb{R}$, the resulting sub-model corresponds to a WAFT regression model. In this case, T_0, representing the future lifetime at birth, follows a Weibull distribution with scale parameter

$$\log(\lambda^*) = -\sigma^{-1}\{Z'\gamma + W'\vartheta\},$$

and shape parameter σ^{-1}, which determines the shape of the hazard function. Here, the parameter λ^* denotes a covariate-dependent component induced by the AFT linear predictor and governs the effective scale of the survival distribution under the chosen parametric specification. Note that λ^* depends on the covariates, whereas σ^{-1} is assumed constant across individuals. In this case, we will write:

$$f_0(t_i|\vartheta_i; \gamma, \sigma) = \sigma^{-1}\lambda_i^* t_i^{\sigma^{-1}-1} \exp(-\lambda_i^* t_i^{\sigma^{-1}}).$$

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The logarithm of the survival function, incorporating covariate effects through λ_i^*, is given by

$$\log S_0(t_i|\vartheta_i;\gamma,\sigma) = -\lambda_i^* t_i^{\sigma^{-1}} = -t_i^{\sigma^{-1}} \exp\big(-\sigma^{-1}\{Z_i'\gamma + W_i'\vartheta_i\}\big).$$

Based on equations (1) and (2), for T_{x_i} we will have:

$$\log S_{x_i}(t_i|\vartheta_i;\gamma,\sigma) = -\lambda_i^*\big((x_i+t_i)^{\sigma^{-1}} - x_i^{\sigma^{-1}}\big) = \big((x_i+t_i)^{\sigma^{-1}} - x_i^{\sigma^{-1}}\big)\exp\big(-\sigma^{-1}\{Z_i'\gamma + W_i'\vartheta_i\}\big),$$

Moreover,

$$f_{x_i}(t_i|\vartheta_i;\gamma,\sigma) = S_{x_i}(t_i|\vartheta_i;\gamma,\sigma)\mu_{x_i+t_i} = \exp\left(-\lambda_i^*\left((x_i+t_i)^{\sigma^{-1}} - x_i^{\sigma^{-1}}\right)\right)\sigma^{-1}\lambda_i^*(x_i+t_i)^{\sigma^{-1}-1}.$$

While the WAFT model is flexible and widely used, alternative distributions such as the Gompertz may better capture age-dependent mortality dynamics; therefore, we consider a Gompertz-based AFT specification as an alternative.

2.1.2 Future Lifetime Under Gompertz AFT Regression Model (GAFT Model)

In developing the Gompertz-based AFT specification, three main considerations are taken into account. First, the Gompertz distribution is adopted because of its well-established role in actuarial and demographic modelling, particularly its ability to capture the exponential increase in hazard rates at older ages and thus provide realistic mortality behavior. Second, within the AFT framework defined by $\log(T_x) = \lambda^* + \sigma\epsilon$, the corresponding error distribution is obtained through a monotone transformation so that the induced lifetime distribution remains Gompertz; hence, the error density arises naturally from the model construction rather than being imposed ad hoc (see Appendix A). Third, the resulting formulation belongs to the log-location-scale family underlying parametric AFT models and satisfies standard regularity conditions for likelihood-based inference; in particular, identifiability follows from the one-to-one mapping between (λ^*, σ) and the induced Gompertz distribution, while numerical stability is supported by a negative definite observed Hessian matrix with eigenvalues sufficiently bounded away from zero. In this case, we will write:

$$f_0(t_i|\vartheta_i;\gamma,\sigma) = \sigma\lambda_i^* B C^{\sigma t_i} \exp\left(\frac{-B\lambda_i^*}{\log C}[C^{\sigma t_i} - 1]\right),$$

Here, $f_0(\cdot)$ denotes the baseline conditional density of T^* given $T^* > 0$, corresponding to the WAFT component.

Under this construction, T_0 follows a Gompertz distribution with

$$\lambda^* = Z'\gamma + W'\vartheta,$$

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and scale and shape parameters given by $\sigma \log C$ and $\frac{B\lambda^*}{\log C}$, respectively, where $B > 0$ and $C > 1$ are standard Gompertz constants controlling the baseline level and exponential growth rate of mortality.

The logarithmic survival function, incorporating covariate effects through λ_i^*, is expressed as

$$S_0(t_i|\vartheta_i;\gamma,\sigma) = \exp\left(\frac{-B\lambda_i^*}{\log C}\left[C^{\sigma t_i} - 1\right]\right).$$

Upon substitution of Equations (1) and (2), the corresponding form for T_{x_i} is derived as follows:

$$S_{x_i}(t_i|\vartheta_i;\gamma,\sigma) = \exp\left(\frac{-B\lambda_i^*}{\log C}C^{\sigma x_i}\left[C^{\sigma t_i} - 1\right]\right).$$

Moreover, the associated probability density function is characterized by the expression:

$$f_{x_i}(t_i|\vartheta_i;\gamma,\sigma) = \sigma\lambda_i^* BC^{\sigma t_i} \exp\left(\frac{-B\lambda_i^*}{\log C}C^{\sigma x_i}\left[C^{\sigma t_i} - 1\right]\right).$$

The (AFT) modelling based on the Weibull and Gompertz distributions has been completed. As previously noted, in mortality analyses a substantial proportion of observations are censored. Accordingly, the subsequent section investigates the performance of these model specifications under censoring conditions. To derive the joint density of (T_i, δ_i), we assume that, conditional on $\mathbf{Z}_i$ and $\mathbf{W}_i$, the pairs (T_i^*, C_i) are independent for $i = 1, \ldots, n$. Under non-informative censoring and given the random effects, the joint density for right-censored observations is

$$f_{x_i}(t_i, \delta_i|\vartheta_i;\gamma,\sigma) = [f_{x_i}(t_i|\vartheta_i;\gamma,\sigma)]^{\delta_i}[S_{x_i}(t_i|\vartheta_i;\gamma.\sigma)]^{1-\delta_i}. \quad (3)$$

Similarly, the joint density function in the left-censored case is equal to:

$$f_{x_i}(t_i, \delta_i|\vartheta_i;\gamma,\sigma) = [f_{x_i}(t_i|\vartheta_i;\gamma,\sigma)]^{\delta_i}[F_{x_i}(t_i|\vartheta_i;\gamma,\sigma)]^{1-\delta_i}. \quad (4)$$

The random effects ϑ_i are assumed to follow an independent normal distribution with mean zero and variance σ_ϑ^2, i.e., $\vartheta_i \sim N(0, \sigma_\vartheta^2)$.

Here, σ_ϑ^2 is the variance component in the survival parts. Let the unknown parameter vector in the proposed model be $\Psi = (\gamma, \sigma_\vartheta^2, \sigma)$.

The full likelihood function is defined as

$$L(\Psi) = \prod_{i=1}^{n} L(\Psi; t_i, \delta_i) = \prod_{i=1}^{n} \int f_{x_i}(t_i, \delta_i|\vartheta_i;\gamma,\sigma) f(\vartheta_i) d\vartheta_i,$$

f_x is fully specified as (3) or (4). Finally, the full log-likelihood can be expressed as follows:

$$l(\Psi) = \sum_{i=1}^{n} \log\left\{\int f_{x_i}(t_i, \delta_i|\vartheta_i;\gamma,\sigma) f(\vartheta_i) d\vartheta_i\right\}, \quad (5)$$

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Equation(5) is used to obtain the maximum likelihood estimates (MLEs). Numerical optimization can be implemented using the nlminb function in the R software, while the observed Hessian matrix (e.g., via fdHess) provides standard error estimates. Specifically, the inverse of the observed Hessian matrix evaluated at the MLEs provides an estimate of the variance–covariance matrix of the parameter estimates. These standard errors are subsequently used for statistical inference and for assessing estimator performance, including the calculation of measures such as bias and mean squared error (MSE) reported in the application section.

2.2 Zero-Inflated AFT Modelling Framework

Building upon the non-inflated AFT framework introduced in the previous subsection, we extend the model to accommodate zero inflation, which allows the framework to capture semi-continuous lifetime data with an excess probability mass at zero. In this subsection, we incorporate a Zero-Inflated structure into the survival component. Specifically, we assume that the conditional distribution of T_i^* follows a mixture formulation given by

$$F_x(t_i|\vartheta_{1i}, \vartheta_{2i}; \gamma_1, \gamma_2) = \pi_i + (1 - \pi_i)F_{x\{T^*>0\}}(t_i|\vartheta_{1i}; \gamma_1, \sigma),$$

where

$$F_{x\{T^*>0\}}(t|\vartheta_1; \gamma_1, \sigma) = \Pr(T^* \leq t|T^* > 0, \vartheta_1; \gamma_1, \sigma).$$

is a continuous conditional distribution for the non-zero component of T^*, and $\pi = P(T^* = 0|\vartheta_1; \gamma_1)$ denotes the conditional probability associated with the zero mass. We refer to these two models as the Zero-Inflated Gompertz AFT model (ZGAFT) and the Zero-Inflated Weibull AFT model (ZWAFT), respectively. Similar to Subsection (2.1), for the conditional distribution $F_{x,\{T^*>0\}}$, we consider

$$\log(\lambda^*) = -\sigma^{-1}\{Z_1'\gamma_1 + W_1'\vartheta_1\},$$

and

$$\lambda^* = Z_1'\gamma_1 + W_1'\vartheta_1.$$

for ZWAFT and ZGAFT regression models, respectively.

The Zero-Inflation probability π is modeled using a logistic regression specification

$$\text{logit}(\pi) = Z_2'\gamma_2 + W_2'\vartheta_2,$$

where Z_1' and Z_2' are vectors of covariates. Additionally, $\gamma = (\gamma_1, \gamma_2)$ and $\vartheta = (\vartheta_1, \vartheta_2)$ are vectors of fixed and random effects parameters, respectively.

For right-censored observations, the likelihood contribution is expressed as

$$f_x(t_i, \delta_i|\vartheta_i; \gamma, \sigma) = [f_x(t_i|\vartheta_i; \gamma, \sigma)]^{\delta_i} [S_x(t_i|\vartheta_i; \gamma, \sigma)]^{1-\delta_i}.$$

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which simplifies to:

$$\left[\frac{d}{dt}F_x(t_i|\vartheta_i;\gamma,\sigma)\right]^{\delta_i}\left[1-F_x(t_i|\vartheta_i;\gamma,\sigma)\right]^{1-\delta_i}.$$

This expression can equivalently be written as

$$(1-\pi_i)\left[f_{x\{T^*>0\}}(t_i|\vartheta_{1i};\gamma_1,\sigma)\right]^{\delta_i}\left[S_{x\{T^*>0\}}(t_i|\vartheta_{1i};\gamma_1,\sigma)\right]^{1-\delta_i}. \tag{6}$$

Similarly, for left-censored observations

$$f_x(t_i,\delta_i|\vartheta_i;\gamma,\sigma)=\left[f_x(t_i|\vartheta_i;\gamma,\sigma)\right]^{\delta_i}\left[F_x(t_i|\vartheta_i;\gamma,\sigma)\right]^{1-\delta_i}.$$

written as:

$$(1-\pi_i)\left[f_{x\{T^*>0\}}(t_i|\vartheta_{1i};\gamma_1,\sigma)\right]^{\delta_i}\left[1-(1-\pi_i)S_{x\{T^*>0\}}(t_i|\vartheta_{1i};\gamma_1,\sigma)\right]^{1-\delta_i}. \tag{7}$$

Here, Σ_ϑ denotes the covariance matrix of the parameter vector ϑ, given by

$$\Sigma_\vartheta=\begin{bmatrix}\sigma^2_{\vartheta_1} & \sigma_{\vartheta_1,\vartheta_2}\\ \sigma_{\vartheta_1,\vartheta_2} & \sigma^2_{\vartheta_2}\end{bmatrix},$$

where $\sigma^2_{\vartheta_1}$ and $\sigma^2_{\vartheta_2}$ denote the variance components. The likelihood under the inflated formulation follows Equation (5), with corresponding adjustments in parameter dimensionality.

3 Numerical Experiments

In this section, simulation studies are performed to evaluate the proposed models. Also, a real data set is used to investigate the purposes of the models.

3.1 Simulation Study

In this section, we assess the performance of our proposed models: the GAFT and WAFT regression models for future lifetime variables, with or without inflation, as defined in section 2. To demonstrate the performance of the proposed models, the nlminb function in R software was used to ensure that the same numerical algorithms were employed for likelihood maximization, and to assess the accuracy of the models. we consider different sets of simulations, each with a sample size $n = 50, 100$, and 500. Future lifetime data, t_i^*, were generated from the AFT model:

$$\log(t_i^*)=2+x_i-Z_{1i}+\sigma_{b_i}b_i+0.5\times\epsilon_i,$$

and set $t_i=\min(t_i^*,C_i)$ or $t_i=\max(t_i^*,C_i)$ in the right or left-censored cases, respectively. Z_{1i} is generated as a binary variable taking a value of 1 with probability 0.4 and 0 otherwise. x_i is the age, and the values of the coefficients were $\gamma=(\gamma_0,\gamma_1,\gamma_2)'=(2,1,-1)'$.

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The random cluster effects, b_i, were independently drawn from a standard normal distribution, and we set $\sigma_{b_i} = 0.5$. Moreover, for the WAFT model, the error term ϵ follows a standard Extreme Value Distribution (EVD), and the scale parameter σ was set to 0.5.

Because the error distribution in the GAFT model is not a standard statistical distribution (as detailed in the Appendix A), we used the inverse transform method along with the distribution function to generate the error term, which will be explained later. Assume that $T_x \sim \text{Gompertz}\left(\frac{B\lambda^*}{\log C}, \sigma \log C\right)$, where the Gompertz constants are treated as fixed actuarial parameters, with values $C = 1.07$ and $B = 0.0003$ as reported in [28]. $Y = \log T_x = \lambda^* + \sigma\epsilon$ [1]. Consider that $F_Y(y) = U$, $U \sim U(0, 1)$. Therefore,

$$P(Y \leqslant y) = P(\log T_x \leqslant y) = P(T_x \leqslant e^y) = F(e^y) = U$$

Thus, we obtain:

$$\begin{aligned}
1 - e^{-\frac{B\lambda^*}{\log C} C^{\sigma x}\left(C^{\sigma e^y} - 1\right)} &= U \\
e^{-\frac{B\lambda^*}{\log C} C^{\sigma x}\left(C^{\sigma e^y} - 1\right)} &= 1 - U \\
-\frac{B\lambda^*}{\log C} C^{\sigma x}\left(C^{\sigma e^y} - 1\right) &= \log(1 - U) \\
\left(C^{\sigma e^y} - 1\right) &= -\frac{\log(1 - U)\log C}{B\lambda^* C^{\sigma x}} \\
\left[\frac{\log(1 - U)\log C}{B\lambda^* C^{\sigma x}} - 1\right] &= C^{\sigma e^y} \\
\log_C \left[\frac{\log(1 - U)\log C}{B\lambda^* C^{\sigma x}} - 1\right] &= \sigma e^y \\
\log\left[\frac{1}{\sigma} \log_C \left[\frac{\log(1 - U)\log C}{B\lambda^* C^{\sigma x}} - 1\right]\right] &= y
\end{aligned}$$

Using this method, the error expression under the proposed model was obtained as $\epsilon = \frac{y - \lambda^*}{\sigma}$. In the second set of simulations, inflation was added. Let $U \sim U(0, 1)$. In the right-censored case, the survival data t_i was obtained by assigning $t_i = 0$ if $U_i \leq \pi_i = \frac{\exp(\gamma_{20} + \gamma_{21} x_i + \gamma_{22} Z_{2i})}{1 + \exp(\gamma_{20} + \gamma_{21} x_i + \gamma_{22} Z_{2i})}$ and $t_i = \min(t_i^*, C_i)$ otherwise, where $t_i^* = \exp(2 + x_i - Z_{1i} + \sigma_{b_i} b_i + 0.5 \times \epsilon_i)$ Similarly, in the left-censored case, $t_i = \max(t_i^*, C_i)$. We set $\gamma_1 = (\gamma_{10}, \gamma_{11}, \gamma_{12})' = (2, 1, -1)'$ and $\gamma_2 = (\gamma_{20}, \gamma_{21}, \gamma_{22})' = (1, 1, 1)'$. To avoid ambiguity in parameter notation, we adopt a consistent naming convention throughout the manuscript. Parameters $\gamma_0, \gamma_1, \gamma_2$ denote the regression coefficients of the baseline AFT component, γ_{1j} (e.g., $\gamma_{10}, \gamma_{11}, \gamma_{12}$) represent the regression parameters associated with the survival component in the zero-inflated AFT models, and γ_{2j} (e.g., $\gamma_{20}, \gamma_{21}, \gamma_{22}$) correspond to the coefficients of the logistic component that governs the probability of zero inflation. The censoring time, C_i, was generated from a uniform distribution over $(5, 10)$ and $(0, 5)$ for right and left censoring, respectively.These choices

[1] λ^* is defined in Section 2.1.2

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produce censoring proportions of approximately 30%–39% across the simulation scenarios, as reported in Tables 1–4. Accordingly, we computed the MLE, Standard Error (SE), bias (B), and MSE, all of which are displayed in Tables1–4. These results show that, a more detailed examination of Tables 1–4 reveals several systematic patterns regarding estimation accuracy and model behavior. First, across all simulation scenarios, increasing the sample size from $n = 50$ to $n = 500$ leads to a consistent reduction in SE and MSE, indicating improved estimator stability and convergence toward the true parameter values. This pattern reflects the expected asymptotic behavior of maximum likelihood estimators under correctly specified parametric AFT models. Second, bias generally decreases as the sample size increases, although the magnitude of reduction varies across parameters and model specifications. In particular, parameters associated with the baseline survival component tend to stabilize more rapidly than those related to random effects or Zero-Inflation components, which require larger sample sizes to achieve comparable precision. Third, comparing the Weibull and Gompertz specifications indicates that both models provide competitive estimation performance, albeit with distinct characteristics. The WAFT model generally exhibits more stable estimation behavior, particularly in smaller samples. In contrast, the proposed GAF formulation offers greater flexibility in capturing mortality dynamics; however, its performance in terms of MSE is not uniformly superior across all parameters. In larger samples, the MSE values for GAFT become comparable to those of the Weibull model for some parameters, although discrepancies remain for others. Furthermore, the introduction of zero inflation affects estimation variability in small samples, as reflected by higher bias and dispersion in some parameters when $n = 50$. However, as the sample size increases, the Zero-Inflated models demonstrate improved stability, suggesting that the additional mixture component is identifiable and estimable under moderate-to-large sample regimes. This behavior highlights the importance of sufficient sample size when modelling semi-continuous lifetime data with excess zeros. Overall, the simulation results indicate that the proposed modelling framework remains statistically stable under both censoring mechanisms and across different sample sizes. The decreasing trends in SE and MSE, together with improved bias properties for larger samples, provide empirical support for the robustness and practical applicability of the Zero-Inflated AFT models. From a methodological perspective, the simulation findings also provide insight into the identifiability and robustness of the proposed Zero-Inflated AFT framework under censoring. The consistent reduction in estimation variability across both right- and left-censoring scenarios suggests that the likelihood formulation remains stable despite the added complexity of zero inflation and random effects. Moreover, the comparable performance between Weibull and Gompertz specifications indicates that the proposed error formulation for the GAFT model does not introduce numerical instability, supporting its practical feasibility. These results collectively demonstrate that incorporating zero inflation within the AFT framework can enhance modelling flexibility while preserving reliable inferential properties under realistic actuarial data structures.

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Table 1 Simulation results for the WAFT model under right- and left-censoring mechanisms without zero inflation. The table reports maximum likelihood estimates (MLE), standard errors (SE), bias (B), and mean squared errors (MSE). A consistent reduction in SE and MSE as sample size increases illustrates improved estimator stability and convergence.

Parameter	True	Right-Censored											
		$n = 50$				$n = 100$				$n = 500$			
		MLE	SE	B	MSE	MLE	SE	B	MSE	MLE	SE	B	MSE
γ_0	2.000	2.764	0.244	-0.764	0.642	2.284	0.084	-0.284	0.087	2.132	0.185	-0.132	0.052
γ_1	1.000	1.441	0.184	-0.441	0.227	1.300	0.195	-0.300	0.129	1.410	0.200	-0.410	0.208
γ_2	-1.000	-1.533	0.467	0.533	0.500	-1.470	0.152	0.470	0.247	-1.166	0.222	0.166	0.077
σ_b	0.500	0.841	0.270	-0.341	0.188	0.924	0.247	-0.424	0.241	0.853	0.187	-0.353	0.159
σ	0.500	0.500	0.374	0.000	0.140	0.500	0.270	0.000	0.073	0.500	0.205	0.000	0.042
Parameter	True	Left-Censored											
		$n = 50$				$n = 100$				$n = 500$			
		MLE	SE	B	MSE	MLE	SE	B	MSE	MLE	SE	B	MSE
γ_0	2.000	2.088	0.222	-0.088	0.057	2.016	0.161	-0.016	0.026	2.150	0.059	-0.150	0.027
γ_1	1.000	0.875	0.228	0.125	0.067	1.325	0.170	-0.325	0.134	1.280	0.214	-0.280	0.129
γ_2	-1.000	-0.700	0.361	-0.300	0.220	-1.180	0.237	0.180	0.090	-1.085	0.089	0.085	0.015
σ_b	0.500	0.840	0.270	-0.340	0.188	0.924	0.247	-0.424	0.241	0.853	0.187	-0.353	0.159
σ	0.500	0.500	0.374	0.000	0.140	0.500	0.273	0.000	0.075	0.500	0.200	0.000	0.040

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Table 2 Simulation results for the GAFT model under right- and left-censoring without zero inflation. The table reports maximum likelihood estimates (MLE), standard errors (SE), bias (B), and mean squared errors (MSE). Compared with the Weibull specification, larger bias values are observed for some parameters in small samples, while estimation accuracy improves as sample size increases.

Parameter	True	Right-Censored											
		$n = 50$				$n = 100$				$n = 500$			
		MLE	SE	B	MSE	MLE	SE	B	MSE	MLE	SE	B	MSE
γ_0	2.000	0.658	0.100	1.342	1.806	0.761	0.144	1.239	1.556	0.825	0.206	1.175	1.444
γ_1	1.000	0.052	0.097	0.948	0.907	0.092	0.097	0.908	0.835	0.223	0.093	0.777	0.613
γ_2	-1.000	-0.054	0.204	-0.946	0.935	0.232	1.023	-1.232	2.547	0.580	0.340	-1.580	2.607
σ_b	0.500	0.500	0.139	0.000	0.019	0.500	0.124	0.000	0.015	0.500	0.129	0.000	0.017
σ	0.500	0.500	0.013	0.000	0.000	0.500	0.010	0.000	0.000	0.500	0.011	0.000	0.000

Parameter	True	Left-Censored											
		$n = 50$				$n = 100$				$n = 500$			
		MLE	SE	B	MSE	MLE	SE	B	MSE	MLE	SE	B	MSE
γ_0	2.000	1.970	0.520	0.030	0.271	2.070	0.120	-0.070	0.019	1.080	0.560	0.920	1.162
γ_1	1.000	1.310	0.140	-0.310	0.116	1.000	0.490	0.000	0.240	0.992	0.280	0.008	0.078
γ_2	-1.000	-0.130	0.640	-0.870	1.167	-1.300	0.750	0.300	0.653	-1.180	0.190	0.180	0.068
σ_b	0.500	0.500	0.410	0.000	0.168	0.500	0.500	0.000	0.250	0.500	0.640	0.000	0.410
σ	0.500	0.500	0.200	0.000	0.040	0.500	0.340	0.000	0.116	0.500	0.150	0.000	0.023

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Table 3 Simulation results for the ZWAFT model under right- and left-censoring mechanisms. The table reports maximum likelihood estimates (MLE), standard errors (SE), bias (B), and mean squared errors (MSE). Compared with the non-inflated case, additional variability is observed for zero-inflation parameters in smaller samples, while estimation accuracy improves substantially as sample size increases.

		Right-Censored											
Parameter	True	$n = 50$				$n = 100$				$n = 500$			
		MLE	SE	B	MSE	MLE	SE	B	MSE	MLE	SE	B	MSE
γ_{10}	2.000	1.301	0.400	0.699	0.649	1.331	0.433	0.669	3.016	2.690	0.149	-0.690	0.505
γ_{11}	1.000	0.607	0.211	0.393	0.200	1.170	0.530	-0.170	0.308	1.181	0.211	-0.181	0.076
γ_{12}	-1.000	-0.780	0.220	-0.220	0.097	-0.880	0.374	-0.120	0.155	-1.530	0.212	0.530	0.326
γ_{20}	1.000	0.830	0.378	0.170	0.171	-0.530	0.256	1.530	2.411	1.125	0.133	-0.125	0.033
γ_{21}	1.000	1.383	0.210	-0.383	0.191	1.200	0.082	-0.200	0.047	1.240	0.178	-0.240	0.088
γ_{22}	1.000	1.193	0.870	-0.193	0.796	1.050	0.470	-0.050	0.223	0.980	0.268	0.020	0.072
σ_b	0.500	0.650	0.370	-0.150	0.162	0.500	0.271	0.000	0.073	0.500	0.291	0.000	0.085
σ	0.500	0.500	0.552	0.000	0.305	0.644	0.179	-0.144	0.053	0.780	0.250	-0.280	0.141

		Left-Censored											
Parameter	True	$n = 50$				$n = 100$				$n = 500$			
		MLE	SE	B	MSE	MLE	SE	B	MSE	MLE	SE	B	MSE
γ_{10}	2.000	2.390	0.280	-0.390	0.230	2.108	0.118	-0.108	0.025	2.235	0.125	-0.235	0.070
γ_{11}	1.000	1.280	0.464	-0.280	0.295	1.014	0.093	-0.014	0.009	0.610	0.316	0.390	0.254
γ_{12}	-1.000	-0.265	0.686	-0.735	1.013	-1.727	0.153	0.727	0.551	-1.613	0.556	0.613	0.688
γ_{20}	1.000	1.332	0.068	-0.332	0.115	1.164	0.399	-0.164	0.186	1.814	0.257	-0.814	0.726
γ_{21}	1.000	1.166	0.450	-0.166	0.230	1.480	0.106	-0.480	0.241	1.416	0.730	-0.416	0.707
γ_{22}	1.000	0.330	0.445	0.670	0.647	0.540	0.830	0.460	0.901	0.932	0.520	0.068	0.275
σ_b	0.500	0.500	0.216	0.000	0.047	0.500	0.317	0.000	0.100	0.500	0.135	0.000	0.018
σ	0.500	0.500	0.170	0.000	0.029	0.500	0.250	0.000	0.063	0.730	0.104	-0.230	0.064

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Table 4 Simulation results for the ZGAFT model under right- and left-censoring mechanisms. The table reports maximum likelihood estimates (MLE), standard errors (SE), bias (B), and mean squared errors (MSE). The results indicate increased variability for zero-inflation parameters in small samples, while estimation stability improves substantially as sample size increases.

		Right-Censored											
Parameter	True	$n = 50$				$n = 100$				$n = 500$			
		MLE	SE	B	MSE	MLE	SE	B	MSE	MLE	SE	B	MSE
γ_{10}	2.000	1.650	0.460	0.350	0.333	0.942	0.523	1.058	1.393	1.171	1.017	0.829	1.721
γ_{11}	1.000	1.139	0.400	-0.139	0.179	1.650	0.460	-0.650	0.663	1.170	1.017	-0.170	1.063
γ_{12}	-1.000	-0.341	0.982	-0.659	1.401	-0.532	0.307	-0.468	0.314	0.326	0.719	-1.326	2.253
γ_{20}	1.000	1.139	0.406	-0.139	0.184	1.990	0.376	-0.990	1.142	1.370	0.330	-0.370	0.246
γ_{21}	1.000	1.056	0.410	-0.056	0.171	0.874	1.120	0.126	1.270	0.610	0.750	0.390	0.715
γ_{22}	1.000	1.260	0.660	-0.260	0.503	1.040	0.713	-0.040	0.510	1.330	0.780	-0.330	0.713
σ_b	0.500	0.500	0.122	0.000	0.015	0.500	0.023	0.000	0.001	0.500	0.013	0.000	0.000
σ	0.500	0.500	0.122	0.000	0.015	0.500	0.281	0.000	0.079	0.500	0.188	0.000	0.035
		Left-Censored											
Parameter	True	$n = 50$				$n = 100$				$n = 500$			
		MLE	SE	B	MSE	MLE	SE	B	MSE	MLE	SE	B	MSE
γ_{10}	2.000	0.874	1.120	1.126	2.519	0.940	0.523	1.060	1.401	1.170	0.620	0.830	1.073
γ_{11}	1.000	0.850	0.320	0.150	0.125	1.056	0.410	-0.056	0.171	1.080	0.191	-0.080	0.043
γ_{12}	-1.000	-1.588	0.473	0.588	0.569	-0.530	0.307	-0.470	0.315	-1.336	0.400	0.336	0.292
γ_{20}	1.000	1.640	0.495	-0.640	0.653	1.510	0.460	-0.510	0.471	1.136	0.450	-0.136	0.220
γ_{21}	1.000	1.110	0.520	-0.110	0.293	1.040	0.090	-0.040	0.010	1.210	0.810	-0.210	0.705
γ_{22}	1.000	0.025	0.825	0.975	1.636	0.940	0.116	0.060	0.017	1.060	0.099	-0.060	0.014
σ_b	0.500	0.500	0.840	0.000	0.706	0.500	0.280	0.000	0.078	0.500	0.700	0.000	0.490
σ	0.500	0.500	0.097	0.000	0.009	0.500	0.023	0.000	0.001	0.505	0.068	-0.005	0.005

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3.2 Real Data Analysis

Society problems are always involved in mortality modelling. Mortality models based on frailty were investigated using the generalization of the Gompertz model by [39]. Another new idea in the study of survival is random mortality, in which mortality is considered a random process, and the first research in this field was conducted by [40]. But the Markov chain approach in mortality modelling so that the time distribution is specific and has a type 3 fuzzy distribution has been proposed by [41]. [42] modeled mortality based on this idea. In a continuation of the research conducted in this field, [43] used a mortality model with several risk factors. They addressed the issue that the introduced mortality models are mainly focused on the age and gender of people; however, other factors, including the socioeconomic status of people, such as the level of education, income, place of residence, job, marriage, and behavioral factors, can affect the mortality rate and future lifetime will be affected. The empirical analysis in this study is based on observations of members of a medium-sized pension fund. In particular, we observe the values of the two covariates for every member of the fund. We refer to this as the complete dataset.

3.2.1 Description of Dataset

The complete dataset had the following characteristics:

- 18,741 records of pensions in payment.

- 172,601.4 person-years of time exposed to risks.
- 4,956 observed deaths.
- Period of observation: 10th November 1992-31st December 2009.
- The annual benefit amount is observed for each individual.
- The annual benefit amount is observed for each individual Z_i.
  The geo-demographic profile is observed at individual level on the basis of MOSAIC profiler, widely used in the UK for pricing longevity swaps and bulk buy-outs.

The last two items comprise a covariate vector. Although the benefit amount is observed in pounds and can be treated as continuous, we treat it as categorical for two reasons. First, [9] showed that its direct use within the model did not improve fit. Second, treating a continuous variable as categorical is more convenient in terms of the parsimony of the modelling approach. Furthermore, [44] observed that a low pension amount can be a misleading indicator of individual affluence, as it could be observed because an employee was low-paid with long service or highly paid with short service.

For these reasons, demographic and geographic characteristics are converted into categorical variables and represented by C. This classification is determined using the postal code; therefore, it depends on the geographic region of each member's postal address. The demographic-geographic characteristics are based on Based on the 18 MOSAIC codes presented in Table 5

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Table 5 Composition of demographic and geographic characteristics of the available data.

Group	Count	Group	Count	Group	Count
A	206	G	285	M	2,049
B	1,236	H	129	N	1,164
C	218	I	2,128	O	413
D	1,523	J	3,105	90	3
E	1,025	K	512	91	1
F	3,272	L	1,466	92	6

Initially, individuals with codes 90, 91, and 92 are grouped together and uniquely represented as $9X$ owing to the small number of individuals with these codes. Additionally, individuals with missing demographic and geographic characteristics, as well as those with codes 98 and 99, were excluded. Based on these initial considerations, grouping was performed in three stages, using the minimum distance criterion. Consequently, the dendrogram in Fig 1 is produced, which graphically represents hierarchical clustering (this diagram is derived from the method proposed by [45]).

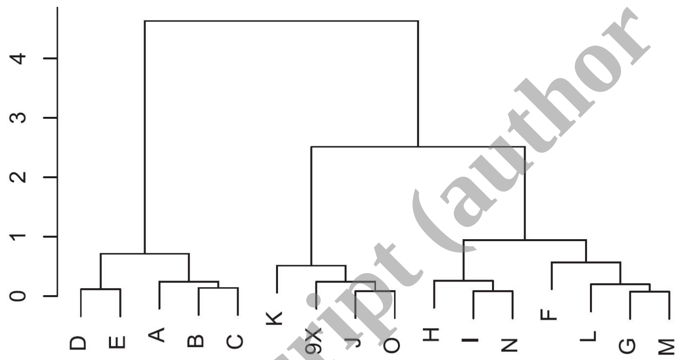

Fig. 1 Dendrogram of geo-demographic clustering obtained using Ward's hierarchical clustering method. The dendrogram displays the hierarchical merging of groups based on the minimum-distance criterion, providing a graphical representation of the clustering structure used to identify the final geo-demographic groups.

Various indices are available to determine the optimal number of clusters, although the choice depends on the selected clustering method. Therefore, for simplicity, the following three groups were chosen:

- Level 0 includes: $9X, O, K, J$;
- Level 1 includes: N, M, L, I, H, G, F;
- Level 2 includes E, D, C, B, A.

Through cluster classification, three levels of C were established: 0, 1, and 2. Each level corresponds to a set of demographic-geographic characteristics, where level 0 represents the most deprived regions of England and level 2 represents the least deprived areas.

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Similarly, the benefit amount is divided into a categorical variable with two levels, represented by B. This random variable can take high (high benefits) and low (low benefits) values. The same method was applied to the auxiliary benefit variable. All individuals were classified into four subpopulations based on the four benefit amounts, and by obtaining the dendrogram shown in Fig 2, they were divided into four subgroups , Additionally, descriptive statistics for the two covariates, demographic-geographic characteristics (C) and benefit level (B), are calculated in Table 7.

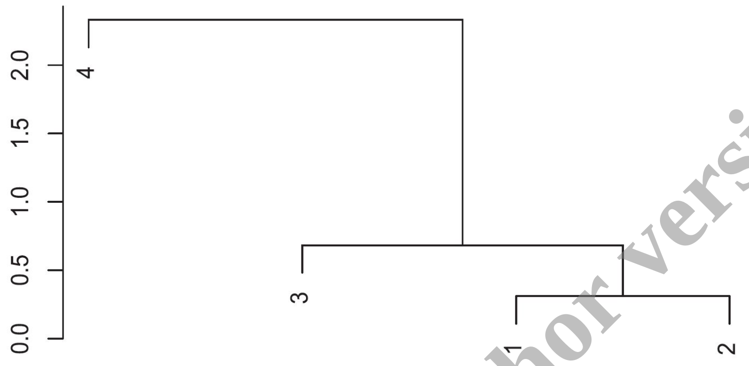

Fig. 2 Dendrogram diagram of clustering based on benefit quartiles.

Table 6 The benefit amount range based on the dendrogram clustering diagram.

Quartile	Benefit Range
First	$0 - 2,376.65$
Second	$2,376.66 - 4,811.97$
Third	$4,811.98 - 8,553.09$
Fourth	$8,553.10 - 116,584.44$

Therefore, considering the proposed dendrogram division that separates individuals in the higher quartile from the rest, a threshold of 8,500$ was chosen. This means that $14,003$ individuals (74.62% of the pension plan population) will be classified as retirees with low benefits, whereas the remaining $4,738$ individuals (25.38% of the pension plan population) will be classified as retirees with high benefits.

3.2.2 Empirical Application

In this section, the real-data modelling based on the proposed frameworks is presented, separately for the cases with and without inflation. Subsequently, the empirical results obtained from these models are analyzed in detail.

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Table 7 Relative frequency of benefit levels and demographic-geographic characteristics for a retirement plan with 18,741 members.

		Demographic-Geographic Characteristics (C)			
		0	1	2	
Benefit Level (B)	Low	0.19	0.43	0.13	0.75
	High	0.03	0.13	0.09	0.25
		0.22	0.56	0.22	1

3.2.2.1 Analysis Without Zero Inflation

In this section, we analyze model empirical data based on the proposed model. Without considering the inflation of the future lifetime process, the model is given by:

$$\log(T_{x_i}|b_i;\gamma,\sigma) = \gamma_0 + \gamma_1 x_i + \gamma_2 1_{[b_i=High]} + \gamma_3 1_{[c_i=1]} + \gamma_4 1_{[c_i=2]} + \sigma_{b_i} b_i + \sigma\epsilon_i.$$

So that

$$T_{x_i}|b_i;\gamma,\sigma \sim WAFT(\sigma^{-1},\lambda^*),$$
$$\log(\lambda^*) = \sigma^{-1}\left[\gamma_0 + \gamma_1 x_i + \gamma_2 1_{[b_i=High]} + \gamma_3 1_{[c_i=1]} + \gamma_4 1_{[c_i=2]} + \sigma_{b_i} b_i\right].$$

and

$$T_{x_i}|b_i;\gamma,\sigma \sim GAFT(\sigma\log C, \frac{B\lambda^*}{\log C}),$$
$$\log(\lambda^*) = \left[\gamma_0 + \gamma_1 x_i + \gamma_2 1_{[b_i=High]} + \gamma_3 1_{[c_i=1]} + \gamma_4 1_{[c_i=2]} + \sigma_{b_i} b_i\right].$$

The same parameter notation is used in the empirical application to maintain consistency with the simulation framework.

In this case, the shared random effects, b_i, follow a $N(0,1)$ distribution, while ϵ_i is assumed to follow a standard Extreme Value Distribution (EVD) in the Weibull model. However, in the Gompertz model, the distribution of ϵ_i remains statistically undetermined (see Appendix A).

3.2.2.2 Analysis With Zero Inflation

Considering the inflation for the future lifetime variable, we will have:

$$T_{x_i}|b_i;\gamma,\sigma \sim ZWAFT(\sigma^{-1},\lambda^*),$$
$$\log(\lambda^*) = \sigma^{-1}\left[\gamma_{10} + \gamma_{11} x_i + \gamma_{12} 1_{[b_i=High]} + \gamma_{13} 1_{[c_i=1]} + \gamma_{14} 1_{[c_i=2]} + \sigma_{b_i} b_i\right].$$

$$\text{logit}(\pi) = \gamma_{20} + \gamma_{21} x_i + \gamma_{22} 1_{[b_i=High]} + \gamma_{23} 1_{[c_i=1]} + \gamma_{24} 1_{[c_i=2]}.$$

and

$$T_{x_i}|b_i;\gamma,\sigma \sim ZGAFT(\sigma\log C, \frac{B\lambda^*}{\log C}),$$
$$\log(\lambda^*) = \left[\gamma_{10} + \gamma_{11} x_i + \gamma_{12} 1_{[b_i=High]} + \gamma_{13} 1_{[c_i=1]} + \gamma_{14} 1_{[c_i=2]} + \sigma_{b_i} b_i\right].$$

$$\text{logit}(\pi) = \gamma_{20} + \gamma_{21} x_i + \gamma_{22} 1_{[b_i=High]} + \gamma_{23} 1_{[c_i=1]} + \gamma_{24} 1_{[c_i=2]}.$$

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Discussion of Empirical Results:

Summaries of the parameter estimates (Est), the standard errors (S.E.), and the 95% equal-tail confidence intervals for the parameters from the no-inflation models and those from the inflation models will be given in Table 8 also include the Akaike Information Criterion (AIC) that was used to compare these joint models. An important aspect is that in addition to age, the data pertaining to the current retirement plan also had two variables (demographic-geographic characteristics and benefit level), which are key factors affecting further longevity (response variable). This was because in an endeavor to develop a model that accommodated all factors without collinearity of variables, the desired variables were then added stepwise into the model. (In addition to this method, in Appendix B, a chart of mortality rates as a function of future lifetime for individuals, based on the two suggested auxiliary variables, is provided. The results indicate a significant relationship between them). This was important in improving the AIC of the model, and the model improved with all proposed variables. Besides, the inclusion of random effects as a factor for assessing unobservable variables heightened the model. Within the AFT framework, the Weibull and Gompertz regression models were fitted on real data to see if the inflation component should be included. The fitted models turned out significantly accurate, especially for the inflated observations. Even in the instances of weak influence by the inflation component, including it into the model aided to better the fit, as measured by the Akaike Information Criterion (AIC). In addition to the results presented in Table 8, which ensure that the issue of accumulation should be considered, the descriptive statistics obtained for future lifetime in Appendix B also support this point. In contrast to the Weibull model was the Gompertz model, a new approach within the AFT framework that performed better, particularly because the Gompertz model posed no assumption about known error distribution. However, given the complexity of taking into account all factors affecting the dependent variable, this model includes. To enhance the interpretative depth of the empirical analysis, we provide additional discussion of the estimated coefficients within the AFT framework. In AFT models, regression parameters can be interpreted as time-acceleration factors; therefore, negative coefficient estimates indicate shorter expected future lifetimes, while positive values imply prolonged survival duration. The results in Table 8 and suggest that age plays a significant role in accelerating failure time, particularly under the GAFT specification, where the age coefficient is consistently negative and statistically significant under both censoring mechanisms. This finding aligns with actuarial expectations that increasing age reduces the remaining lifetime horizon. Similarly, the demographic–geographic classification (levels of C) demonstrates measurable heterogeneity in longevity, indicating that socio-demographic characteristics materially influence survival outcomes. The benefit level variable also shows meaningful effects, suggesting that pension-related financial indicators may capture underlying health or socioeconomic differences among retirees. From an actuarial perspective, these findings highlight the importance of incorporating heterogeneous covariate structures when modelling future lifetime distributions, as ignoring such factors may lead to biased longevity estimates and inaccurate annuity pricing. In particular, the improved fit observed for the Gompertz-based AFT model suggests that mortality dynamics exhibiting exponential hazard growth may

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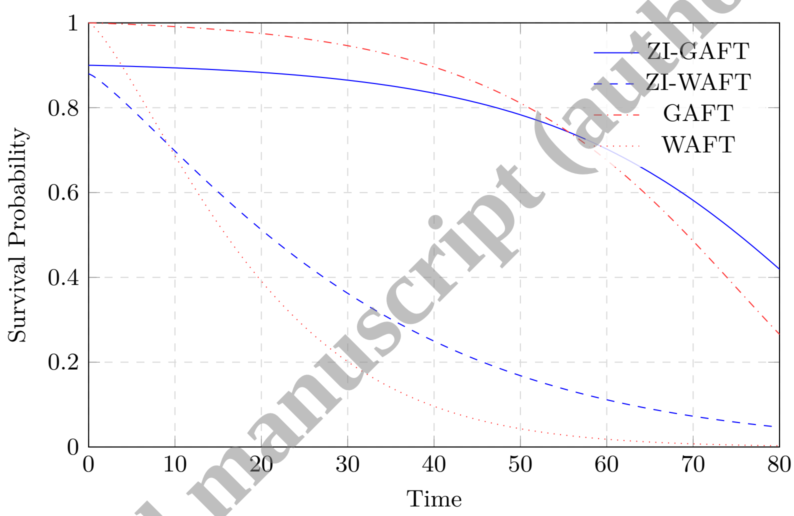
be more suitable for pension populations. In addition to AIC comparisons, graphical assessment based on predicted survival functions was performed to evaluate model adequacy across different covariate profiles. The comparison of fitted survival curves under the Weibull and Gompertz specifications provides complementary insight into the tail behavior and overall goodness-of-fit of the models beyond purely numerical criteria. To complement the numerical results, Fig 3.2.2 provides a graphical comparison of the fitted survival patterns under the four competing model specifications. The zero-inflated models exhibit an initial reduction in survival probability, reflecting the excess-zero component in the data. Among all specifications, the ZGAFT model shows the most plausible and stable survival pattern, followed by the ZWAFT model, whereas the non-inflated GAFT and WAFT models provide comparatively less adequate representations. Overall, this graphical evidence is consistent with the AIC-based comparison and further supports the superior performance of the ZGAFT model.

The comparative analysis in this study focuses on Weibull and GAFT specifications to provide a clear evaluation of the proposed inflation-based future lifetime framework under structurally comparable parametric models. This focused comparison allows a direct assessment of how incorporating inflation influences model behavior, estimation stability, and empirical fit within the AFT setting.



Estimated survival curves under the four fitted models. The zero-inflated Gompertz specification shows the best overall fit, followed by the zero-inflated Weibull, Gompertz, and Weibull models.

4 Annuity Factors under Zero-Inflated AFT Models

Let Y_x be the random variable representing the present value of the cash flows a pension fund member aged x receives during his remaining lifetime. If cash flows are

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Table 8 Results of fitting future lifetime under two censoring mechanisms without and with inflation.

Model / Parameters	Right Censoring			Left Censoring		
	Est.	S.E.	95% CI	Est.	S.E.	95% CI
Panel A: WAFT Model						
γ_0	-8.31	3.47	(-14.21, 7.02)	-11.03	5.59	(-21.90, 11.43)
γ_1	-1.23	0.79	(-1.36, 7.14)	0.97	0.54	(-0.43, 4.08)
γ_2	-1.40	0.62	(-0.20, 0.90)	-2.30	1.01	(-4.02, 0.04)
γ_3	-0.91	0.04	(-1.36, -0.37)	-1.20	0.05	(-2.26, 1.92)
γ_4	-0.45	0.06	(-1.70, 0.23)	-0.67	0.04	(-0.97, 0.37)
σ_b	0.67	0.05	(0.42, 0.73)	0.52	0.05	(0.31, 0.71)
σ	0.56	0.08	(-0.12, 0.59)	0.61	0.09	(-0.03, 0.92)
AIC	2485.157			2540.026		
Panel B: GAFT Model						
γ_0	-12.47	4.68	(-22.56, -2.28)	-15.13	6.21	(-27.52, -2.74)
γ_1	-2.74	1.12	(-4.85, -0.63)	-3.10	1.30	(-5.56, -0.64)
γ_2	-0.88	0.45	(-1.75, -0.01)	-0.65	0.36	(-1.36, 0.06)
γ_3	-1.53	0.52	(-2.54, -0.52)	-1.90	0.60	(-3.09, -0.71)
γ_4	-0.96	0.50	(-1.96, 0.04)	-1.10	0.46	(-2.02, -0.18)
σ_b	0.72	0.07	(0.59, 0.85)	0.66	0.07	(0.52, 0.80)
σ	0.80	0.10	(0.61, 0.99)	0.85	0.09	(0.67, 1.03)
AIC	2224.40			2279.27		
Panel C: ZWAFT Model						
γ_{10}	-5.83	2.74	(-11.15, -0.51)	-7.94	3.22	(-14.18, -1.71)
γ_{11}	-0.32	0.62	(-1.55, 0.91)	1.21	0.47	(0.29, 2.13)
γ_{12}	-0.76	0.59	(-1.90, 0.38)	-1.59	0.78	(-3.11, -0.07)
γ_{13}	-1.19	0.25	(-1.67, -0.71)	-1.42	0.29	(-2.00, -0.83)
γ_{14}	-0.55	0.03	(-0.61, -0.49)	-0.49	0.05	(-0.60, -0.37)
γ_{20}	0.88	0.05	(0.80, 0.96)	0.70	0.04	(0.63, 0.77)
γ_{21}	0.34	0.06	(0.22, 0.46)	0.51	0.07	(0.38, 0.64)
γ_{22}	0.22	0.08	(0.06, 0.38)	0.43	0.09	(0.28, 0.58)
γ_{23}	0.36	0.07	(0.23, 0.49)	0.41	0.08	(0.26, 0.56)
γ_{24}	0.11	0.03	(0.05, 0.17)	0.26	0.06	(0.16, 0.36)
σ_b	0.52	0.03	(0.47, 0.57)	0.47	0.03	(0.41, 0.53)
σ	0.41	0.06	(0.29, 0.53)	0.52	0.07	(0.39, 0.65)
AIC	2922.08			2976.95		
Panel D: ZGAFT Model						
γ_{10}	-10.72	3.85	(-18.23, -3.21)	-13.04	4.88	(-22.59, -3.49)
γ_{11}	-2.34	1.02	(-4.35, -0.33)	-2.94	1.13	(-5.18, -0.70)
γ_{12}	-0.92	0.48	(-1.85, 0.01)	-0.78	0.36	(-1.48, -0.08)
γ_{13}	-1.34	0.43	(-2.19, -0.49)	-1.68	0.51	(-2.67, -0.68)
γ_{14}	-0.70	0.39	(-1.48, 0.08)	-0.98	0.46	(-1.88, -0.08)
γ_{20}	0.63	0.05	(0.53, 0.73)	0.57	0.04	(0.49, 0.65)
γ_{21}	0.37	0.06	(0.27, 0.47)	0.42	0.08	(0.28, 0.56)
γ_{22}	0.29	0.07	(0.16, 0.42)	0.38	0.08	(0.21, 0.55)
γ_{23}	0.33	0.07	(0.20, 0.46)	0.39	0.09	(0.21, 0.57)
γ_{24}	0.26	0.04	(0.18, 0.34)	0.35	0.05	(0.24, 0.46)
σ_b	0.59	0.05	(0.48, 0.70)	0.54	0.04	(0.47, 0.61)
σ	0.71	0.08	(0.56, 0.86)	0.75	0.07	(0.61, 0.89)
AIC	2678.64			2667.95		

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assumed to be 1$ per year paid continuously, the expected value of Y_x, called the annuity factor, is denoted by a_x and calculated as follows ([28]):

$$\bar{a}_x = E(Y_x) = \int_0^{120-x} e^{-rt} S_x(t)\, dt,$$

where r is the interest rate (with continuous compounding), 120 is the assumed maximum age, and $S_x(t)$ is the survival function.

In the present study, the survival function $S_x(t)$ is obtained directly from the fitted parametric AFT models described in Section 1. Specifically, for each model specificationWAFT, GAFT, and their Zero-Inflated extensions, the estimated model parameters determine the corresponding survival distribution, which is then used to compute the annuity factor through numerical integration. Therefore, the annuity calculations represent a direct actuarial application of the proposed survival modelling framework rather than a separate post hoc analysis. The annuity calculations rely on several standard assumptions, including continuous benefit payments, a constant interest rate, and independence between financial discounting and mortality dynamics. In the Zero-Inflated models, the survival function incorporates the additional probability mass at zero through the mixture formulation, which affects expected lifetime and consequently modifies annuity values. From a sensitivity perspective, differences in annuity factors across models arise from the distinct hazard structures implied by the Weibull and Gompertz AFT specifications, as well as the presence or absence of zero inflation.

In addition, the annuity factors were computed under two alternative interest-rate assumptions (1% and 3%) in order to examine the sensitivity of the results to the discount rate. Consistent with standard actuarial theory, annuity factors calculated at the lower interest rate are larger than those obtained at the higher rate, reflecting the inverse relationship between interest rates and the present value of annuity payments. In particular, the inclusion of zero inflation tends to reduce expected remaining lifetimes, leading to smaller annuity values compared with non-inflated models. Similarly, distributional assumptions influence the tail behavior of the survival function, which directly impacts discounted lifetime expectations. The annuity factor for different mortality rates under the proposed models at ages 65 and 73 years for right censoring was calculated, and Fig 4 is shown.

5 Conclusions

In this study, a modelling approach was used to consider future lifetime data inflated at the zero point. Unlike most survival studies, parametric models tend to have higher efficiency in comparison with Cox proportional hazards models. Thus, a parametric AFT regression model based on Weibull and Gompertz distributions was employed for the modelling of future lifetime incorporating a vector of random effects. In addition, inflation at the zero point was taken as the response, whereas two types of censoring mechanisms-right and left- were also incorporated. In order to evaluate the performance of the models suggested, a simulation study was carried out and then came the parameter estimation. The simulation results suggested that standard errors were reduced and estimates were in closer proximity to the true values, as the sample

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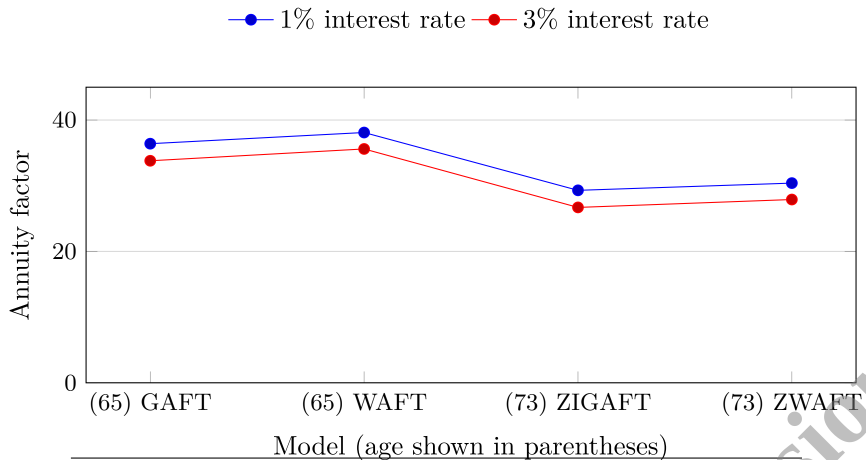

	(65) GAFT	(65) WAFT	(73) ZIGAFT	(73) ZWAFT
1%	36.4	38.1	29.3	30.4
3%	33.8	35.6	26.7	27.9

Annuity factors computed under the fitted AFT models at ages 65 and 73 for two alternative interest-rate assumptions (1% and 3%). The legend is placed outside the plotting area and a summary table is included below the graph to improve readability.

size increased. In the application section, the proposed models were used to analyze real datasets on retirement plans. Moreover, with the goodness of fit established, the models found further utility in the field of insurance calculations. Overall, the results indicate that incorporating zero inflation has a meaningful effect on future lifetime modelling, particularly in improving model fit and actuarial quantities such as annuity factors. The comparative analysis suggests that the Gompertz AFT model provides a flexible and competitive alternative to the Weibull AFT model in semi-continuous lifetime settings. These findings underline the practical relevance of the proposed framework for actuarial applications involving longevity and annuity calculations.

A Derivation of error distribution

To obtain the error distribution in the regression model, we consider Theorem 1[46].
Theorem 1: If T has a continuous distribution $f_T(t)$ and $g(t) = \epsilon$ is a strictly monotonic and differentiable function, then the probability density function of $\mathcal{E} = g(T)$ is given by:

$$f_{\mathcal{E}}(\epsilon) = f_T\left(g^{-1}(\epsilon)\right)\left|\frac{dg^{-1}(\epsilon)}{d\epsilon}\right|, \quad \frac{dt}{d\epsilon} \neq 0$$

Using Theorem 1, we assume that T_x follows the AFT model:

$$T_x \sim \text{Gompertz}\left(\frac{B\lambda^*}{\log C}, \sigma \log C\right), \quad \log T_x = \lambda^* + \sigma\epsilon.$$

Thus:

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$$f_{\mathcal{E}}(\epsilon) = f_{T_x}\left(g^{-1}(\epsilon)\right)\left|\frac{dt}{d\epsilon}\right| = f_{T_x}\left(e^{\lambda^*+\sigma\epsilon}\right)\left|\sigma e^{\lambda^*+\sigma\epsilon}\right|.$$

Based on this, the density function will be as follows,

$$f_{\mathcal{E}}(\epsilon) = \sigma^2\lambda^* BC^{\sigma e^{\lambda^*+\sigma\epsilon}} e^{\lambda^*+\sigma\epsilon} \exp\left(\frac{-B\lambda^*}{\log C} C^{\sigma\epsilon}\left[C^{\sigma e^{\lambda^*+\sigma\epsilon}} - 1\right]\right).$$

The following remarks explained the theoretical and statistical properties of the derived error distribution:

- Model construction and actuarial interpretation. This construction is adopted to preserve the actuarial interpretability of the Gompertz law within a log-location-scale AFT framework. Since the transformation from T_x to ϵ is strictly monotone and differentiable, the induced error distribution is uniquely determined by the target Gompertz distribution of T_x together with the AFT relation $\log T_x = \lambda^* + \sigma\epsilon$. Therefore, the error density is not introduced in an ad hoc manner but arises naturally from the specified survival-time model.
- Statistical framework and tail behavior. From a statistical perspective, this formulation places the model within the general log-location-scale family underlying parametric AFT models, where the choice of the baseline lifetime distribution determines the implied error structure. The resulting specification inherits the characteristic hazard behavior of the Gompertz law, namely an exponentially increasing hazard, which provides appropriate tail behavior for actuarial lifetime modelling.
- Identifiability and regularity conditions. Identifiability follows from the one-to-one mapping between (λ^*, σ) and the induced Gompertz parameterization under the assumed mixed-effects structure. In practice, local identifiability and numerical stability are supported by the observed Hessian matrix evaluated at the maximum of the log-likelihood; for a valid interior maximum, the Hessian is negative definite with eigenvalues sufficiently bounded away from zero, indicating a well-defined curvature of the likelihood surface and stable parameter estimation.

B Descriptive Analysis of Mortality Rates and Future Lifetime Distribution

The crude death rate at a given single year of age, Dx/Ex, is given in Fig 3. The death rates were plotted for both benefit levels (left plot) and three groups according to the geo-demographic profile (right plot). Here, Dx denotes the number of deaths occurring between exact ages x and x+1, and Ex is the total period during which all individuals were observed and alive between those ages (i.e., exposed-to-risk). There were three key observations drawn from the Fig 3: 1. there is a bit of a log-linear relationship with age and death rates, even at advanced ages, where sampling errors are larger because of lower exposures; 2. lower mortality was associated with higher levels of benefits; and 3. lower mortality was associated with geo-demographic status concerning deprivation.

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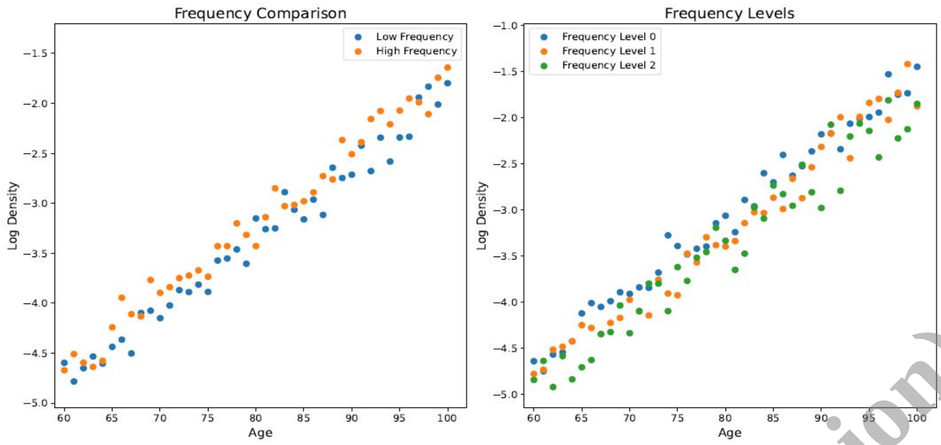

Fig. 3 Crude death rates Dx/Ex by age for individuals with high and low benefit levels on a log scale (LHS) and crude death rates by age for individuals with different geo-demographic profiles on a log scale (RHS).

Fig 4 shows a histogram of the future lifetimes of individuals. For the first time period, the number of people whose future lifetime is zero or close to zero is higher than that in other periods, except for the last period. the issue of accumulation for future lifetime responses cannot be ignored. The mixed distribution for the future lifetime response, in the form of a probability function that includes a continuous distribution for positive values and a degenerate distribution for the zero point, transforms it into a semi-continuous variable, which could be a new concept in survival analysis.

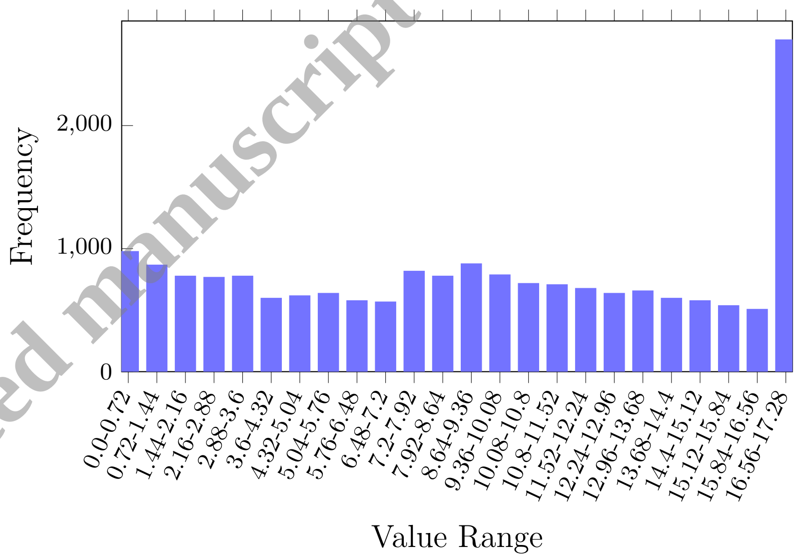

Fig. 4 Histogram plot of the response variable (future lifetime of individuals).

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References

[1] Gompertz, B.: On the nature of the function expressive of the law of human mortality, and on a new method of determining the value of life contingencies. Philosophical Transactions of the Royal Society of London **115**, 513–585 (1825) https://doi.org/10.1098/rstl.1825.0026

[2] Makeham, W.M.: On the further mathematical theory of the mortality of the human body. Journal of the Institute of Actuaries **25**(3), 191–212 (1885)

[3] Heligman, L., Pollard, J.H.: The age pattern of mortality. Journal of the Institute of Actuaries **107**(1), 49–80 (1980) https://doi.org/10.1017/S0020268100040257

[4] Lee, R.D., Carter, L.R.: Modelling and forecasting us mortality. Journal of the American Statistical Association **87**(419), 659–671 (1992) https://doi.org/10.1080/01621459.1992.10475265

[5] Pedroza, C.: A bayesian forecasting model: predicting us male mortality. Biostatistics **7**(4), 530–550 (2006) https://doi.org/10.1093/biostatistics/kxj024

[6] Vaupel, J.W.: Biodemography of human ageing. Nature **464**(7288), 536–542 (2010) https://doi.org/10.1038/nature08984

[7] Melnikov, A., Romaniuk, Y.: Evaluating the performance of gompertz, makeham and lee–carter mortality models for risk management with unit-linked contracts. Insurance: Mathematics and Economics **39**(3), 310–329 (2006) https://doi.org/10.1016/j.insmatheco.2005.09.004

[8] Basellini, U., Camarda, C.G., Booth, H.: Thirty years on: A review of the lee–carter method. International Journal of Forecasting **39**(3), 1033–1049 (2023) https://doi.org/10.1016/j.ijforecast.2022.11.002

[9] Macdonald, A.S., Richards, S.J., Currie, I.D.: Modelling Mortality with Actuarial Applications. Cambridge University Press, Cambridge (2018)

[10] Dobson, A.J., Barnett, A.G.: An Introduction to Generalized Linear Models. Chapman and Hall/CRC, Boca Raton (2018)

[11] Hosmer, D.W., Lemeshow, S., May, S.: Applied Survival Analysis: Regression Modelling of Time-to-Event Data. John Wiley & Sons, Hoboken, NJ (2008)

[12] Zhao, Y., et al.: A tutorial on modern survival analysis methods. Journal of Biomedical Statistics (2024)

[13] Kleinbaum, D.G., Klein, M.: Introduction to Survival Analysis. Springer, Atlanta (2012)

[14] Balan, T.A., Putter, H.: A tutorial on frailty models in survival analysis. Statistics

Accepted manuscript (author version)

Accepted manuscript (author version)

in Medicine **39**(20), 3424–3454 (2020) https://doi.org/10.1002/sim.8627

[15] Gorfine, M., Hsu, L., *et al.*: Shared frailty models for survival data: A review. Annual Review of Statistics and Its Application **10**, 1–25 (2023)

[16] Cox, D.R.: Regression models and life-tables. JRSS Series B **34**(2), 187–202 (1972) https://doi.org/10.1111/j.2517-6161.1972.tb00899.x

[17] Wei, L.-J.: The accelerated failure time model: a useful alternative to the cox regression model in survival analysis. Statistics in Medicine **11**(14-15), 1871–1879 (1992) https://doi.org/10.1002/sim.4780111409

[18] Bradburn, M.J., Clark, T.G., Love, S.B., Altman, D.G.: Survival analysis part ii: multivariate data analysis. British Journal of Cancer **89**(3), 431–436 (2003) https://doi.org/10.1038/sj.bjc.6601119

[19] Bakhshi, E., Khoei, R.A.A., Azarkeivan, A., Kooshesh, M., Biglarian, A.: Survival analysis of thalassemia major patients. Medical Journal of the Islamic Republic of Iran **31**, 97 (2017) https://doi.org/10.14196/mjiri.31.97

[20] Rizopoulos, D.: Latent Variable Models for Survival Data. Chapman and Hall/CRC, Boca Raton, FL (2010)

[21] Buckley, J., James, I.: Linear regression with censored data. Biometrika **66**(3), 429–436 (1979) https://doi.org/10.1093/biomet/66.3.429

[22] Tsiatis, A.: A nonidentifiability aspect of competing risks. PNAS **72**(1), 20–22 (1975) https://doi.org/10.1073/pnas.72.1.20

[23] Tseng, Y.-K., Hsieh, F., Wang, J.-L.: Joint modelling of accelerated failure time and longitudinal data. Biometrika **92**(3), 587–603 (2005) https://doi.org/10.1093/biomet/92.3.587

[24] Saikia, R., Barman, M.P.: A review on accelerated failure time models. Int J Stat Syst **12**(2), 311–322 (2017)

[25] Qi, J.: Comparison of proportional hazards and accelerated failure time models. PhD thesis, University of Saskatchewan (2009)

[26] Kalbfleisch, J.D., Prentice, R.L.: The Statistical Analysis of Failure Time Data. Wiley, London (2002)

[27] Lee, E.T., Wang, J.W.: Statistical Methods for Survival Data Analysis. John Wiley & Sons, Hoboken, NJ (2003)

[28] Dickson, D.C.M., Hardy, M.R., Waters, H.R.: Actuarial Mathematics for Life Contingent Risks. Cambridge University Press, London (2020)

Accepted manuscript (author version)

Accepted manuscript (author version)

[29] Liu, L.: Statistical analysis of zero-inflated nonnegative continuous data: A review. Statistical Science **34**(2), 253–279 (2019) https://doi.org/10.1214/18-STS669

[30] Farewell, V.T., Long, D.L., Tom, B.D.M.: Two-part and related regression models for semicontinuous data. Annual Review of Statistics and Its Application **4**, 283–315 (2017) https://doi.org/10.1146/annurev-statistics-031017-100323

[31] Maller, R.A., Zhou, X.: Mixture cure models in survival analysis: Recent developments. Statistics Surveys **18**, 1–40 (2024) https://doi.org/10.1214/24-SS156

[32] Latimer, N.R.: Mixture and non-mixture cure models: A practical tutorial. PharmacoEconomics **42**, 1–15 (2024) https://doi.org/10.1007/s40273-023-01329-9

[33] Lambert, D.: Zero-inflated poisson regression. Technometrics **34**(1), 1–14 (1992) https://doi.org/10.1080/00401706.1992.10485228

[34] Gould, A.L., Boye, M.E., Crowther, M.J.: Joint modelling of semicontinuous longitudinal data and survival outcomes. Biostatistics **15**(3), 465–478 (2014) https://doi.org/10.1093/biostatistics/kxt051

[35] Oh, S., Lee, H., Kang, S., Seo, B.: Adaptive accelerated failure time modelling with a semiparametric skewed error distribution. arXiv:2402.02128 (2024)

[36] Rubio, F.J., Hong, Y.: Survival and lifetime data analysis with a flexible class of distributions. Journal of Applied Statistics **43**(10), 1794–1813 (2016) https://doi.org/10.1080/02664763.2015.1110030

[37] Seo, B., Ha, I.D.: Semiparametric accelerated failure time models. Computational Statistics & Data Analysis **195**, 107958 (2024) https://doi.org/10.1016/j.csda.2024.107958

[38] Parvej, M., Khan, A.A.: Bayesian extension of the weibull aft shared frailty model. Journal of Applied Statistics, 1–29 (2024) https://doi.org/10.1080/02664763.2024.2350988

[39] Shoaee, S., Keshmarzi, M.M.: Generalization of stochastic mortality models. Journal of Decisions and Operations Research **8**(2), 352–369 (2023)

[40] Yashin, A.I., Manton, K.G., Vaupel, J.W.: Mortality and aging in a heterogeneous population. Theoretical Population Biology **27**(2), 154–175 (1985) https://doi.org/10.1016/0040-5809(85)90008-6

[41] Mittal, K.e.a.: A comprehensive review on type 2 fuzzy logic applications. Engineering Applications of AI **95**, 103916 (2020) https://doi.org/10.1016/j.engappai.2020.103916

[42] Melin, P.e.a.: Design of type-3 fuzzy systems and ensemble neural networks.

Accepted manuscript (author version)

Accepted manuscript (author version)

Axioms **11**(8), 410 (2022) https://doi.org/10.3390/axioms11080410

[43] Balia, S., Jones, A.M.: Mortality, lifestyle and socio-economic status. Journal of Health Economics **27**(1), 1–26 (2008) https://doi.org/10.1016/j.jhealeco.2007.03.001

[44] Madrigal, A.M.e.a.: What longevity predictors should be allowed for. British Actuarial Journal **16**(1), 1–38 (2011) https://doi.org/10.1017/S1357321711000048

[45] Ward Jr, J.H.: Hierarchical grouping to optimize an objective function. JASA **58**(301), 236–244 (1963) https://doi.org/10.1080/01621459.1963.10500845

[46] Ross, S.M.: Introduction to Probability Models. Academic Press, Los Angeles (2014)

Accepted manuscript (author version)