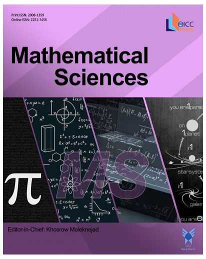

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Research Article

An Efficient Numerical Approach for Solving the Mittag-Leffler Fractional Differential Equations

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Abstract:

In this paper, we introduce a fractional integral operator related to the recently proposed Mittag-Leffler-Caputo-Fabrizio (MLCF) fractional derivative, which has a non-singular Mittag-Leffler kernel. We provide sufficient conditions for the existence of unique solutions for a certain class of nonlinear fractional differential equations by fixed point methods. Our analysis yields an explicit inequality involving fractional orders, the Lipschitz constant, and the finite time, thus guaranteeing the existence of a unique solution. Furthermore, we introduce a novel numerical scheme, the Euler MLCF method, for approximating the solutions to these equations. We prove that this scheme is convergent with first-order accuracy. The new scheme is rigorously validated through numerous numerical examples. The results, in terms of absolute error and empirical convergence order, are consistent with the theoretical prediction. This paper presents theoretical and practical advancements in solving fractional differential equations using the MLCF operator.

Keywords: MLCF fractional derivative; MLCF fractional integral; Existence and uniqueness; Fixed-point theorem; Fractional differential equations; Numerical approach

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1. Introduction

Fractional calculus (FC) is the branch of mathematics that generalizes differentiation and integration to non-integer ordered operators. Besides this, the concepts are extended to the real and complex domains, and the applications are numerous. For instance, Tenreiro Machado [1] used entropy-based analysis of dynamical systems to provide a firm ground for the understanding of the behavior of fractional-order systems. Atanackovic et al. [2] pumped up the area with their work on the establishment of mechanical models for vibrations and diffusion processes involving fractional calculus. Carpinteri and Mainardi [3] crossed the border of the theory by using fractional calculus in continuum mechanics, specifically when fractals were involved. Baleanu et al. [4] came up with complete numerical methods for fractional differential equations that are very helpful in the practical side of the work. Abdelkawy et al. [5] employed fractional calculus in a very effective way to simulate advection-dispersion models thereby proving the huge potential of FC in the environmental and engineering sectors. Magin [6] looked into FC in bioengineering and came up with the groundbreaking idea of using it to model complex biological systems. Sadek and Jarad [7] came up with new fractional operators that helped in broadening the theoretical framework of FC. Sadek et al. [8]

and Sadek [9] also made significant enhancements to the field by studying, respectively, the aspects of controllability in fractional dynamical systems and stability of the same systems. The contributions made by these researchers are a good example of how FC is an interdisciplinary field and its potential applications are wide-ranging.

Most of the concepts can be directly linked to Liouville's groundbreaking 1832 work [10], where the first fractional operators were defined, operators which later were understood to be the Riemann-Liouville and Caputo derivatives in particular. Liouville did not only develop the theory but also envisaged its applications, which made fractional calculus an inherently interdisciplinary field. Present-day fractional calculus studies are still very dynamic and can be found in almost all scientific and engineering fields, and the fractional calculus is said to have unique abilities in explaining the complex phenomena. Just to mention a few: Anomalous dielectric relaxation phenomena can be very well described by fractional equations whose solutions are Mittag-Leffler functions [11]; Using local fractional calculus concepts, fractal models are developed for LC-electric circuits [12]; The Havriliak-Negami type fractional relaxation equations describe dielectric response functions [13]. The list of changes is extended with novel analytical methods [14, 15, 16], time-domain studies of HN models [17], and advanced fractional derivatives for better characterization of nonlocal systems [18, 19].

The Mittag-Leffler (ML) function is the main point of fractional calculus (FC), and its significance has been repeated and confirmed by numerous studies during the last several decades. Mainardi and Gorenflo [20] were the first to apply ML functions to fractional differential equations, especially for describing viscoelastic materials and anomalous diffusion. Srivastava and Tomovski [21] took these ideas one step further by developing generalized ML functions. This led to new analytical solutions for fractional-order problems. Next, Hilfer's research [22] took these ideas one step further by establishing connections between ML functions and probabilistic aspects of fractional operators. The foundation of ML functions in FC was drastically altered by research done on integral transforms. Samko [23] was the first to introduce groundbreaking research on spontaneous ML functions in fractional integral operators. Kilbas et al. [24] took these ideas and developed them further. Haubold and Mathai [25] continued these ideas by exploring deeper connections between ML functions and special functions of mathematical physics.

The recent innovations have mostly focused on developing useful fractional operators that involve ML kernels. Jafari et al. [26] proposed a new ML-based operator for fuzzy fractional systems, thus exhibiting the better ability of the system in handling uncertain parameters. In a similar vein, Boudaoui et al. [27] proposed a new ML-kernel operator that was tailored for epidemiological modeling; the operator was more accurate in explaining the dynamics of disease outbreaks. Improvements to ML-based convolution integrals [20, 21] have also created room for a revolution in the theory of fractional derivatives. Baleanu and Atangana [28] pioneered the development of a non-singular ML-kernel derivative that revolutionized the entire concept of modeling heat transfer phenomena. Ahokposi et al. [29] have also recorded great success in using these concepts for modeling fractal flow in porous media. Tateishi et al. [30] went on to adjust these operators to anomalous diffusion processes more precisely, and Baleanu and Fernández [31] set up the basic mathematical properties of these operators.

One of the major advantages of modern ML-kernel operators is their less complex structure as compared to traditional Riemann-Liouville types. In the past, ML generalizations were more complicated as they needed several indices [20, 21], but the current versions [31] are not only mathematically sound but also more user-friendly since they have fewer parameters. The recent research topics are: Thorough study of the ML-calculus features with the main focus on nonlocality and memory effects [31]; The invention of fast numerical techniques for ML-kernel operators [32]; The use of variational calculus and Euler-Lagrange equations [33]; The establishment of discrete models that are computationally efficient [34, 35].

Research for these improvements has been conducted, and the results have been confirmed through various applications. Firstly, the ML-kernel method has been extremely successful in thermodynamics [28], hydrology [29], and biological systems modeling [30, 31] fields. In the latest development, Alqhtani et al., have introduced a novel ML-based derivative [36] that not only extends the potential of this framework but also keeps the calculation simple.

One of the major issues in fractional calculus (FC) is the need for the development of efficient numerical methods to solve fractional differential equations (FDEs). Moreover, fractional methods can't be simply extended to the fractional case due to the existence of memory effects, which leads to essentially different computational approaches. This article is devoted to fractional derivatives with ML kernels, which have different asymptotic behavior than other FC operators. We claim that this feature is essential for the correct "capture" of complicated nature-like phenomena. On the other hand, FDEs with nonsingular ML kernels are still a "deep ocean" to explore, especially with regard to numerical solution methods, before their advantages can be demonstrated in real applications. Driven by such considerations, we use fixed point theory to locate conditions under which uniqueness and existence of nonlinear ML-kernel FDEs can be guaranteed. We also create a corresponding numerical scheme and confirm our theoretical results by computational experiments.

The Caputo-Fabrizio derivative may have a non-singular kernel, but the exponential kernel of it is not flexible enough to represent more diverse memory ef-

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fects. Generalizations by means of the Mittag-Leffler function like the one in [36] achieve this flexibility with the extra parameter β. Nevertheless, the integral operator associated with this and a detailed numerical analysis of equations with such a derivative are still not completely figured out.

This paper presents a comprehensive theoretical and computational framework for FDEs involving the MLCF fractional integral. The work completes the calculus begun in our earlier work [36], where the MLCF derivative was first proposed. Specifically, this paper:

- Introduces the fractional integral operator of MLCF.
- Proves theorems regarding the existence and uniqueness of solutions of nonlinear FDEs.
- Constructs a computational method that is both effective and efficient and proves its convergence and stability.
- Supports the theoretical findings through numerical experiments.

In recent times, new discoveries have also been made in the area of fractional calculus and their applications. In a research done by Kumar [37], generalized Mittag-Leffler functions and fractional integrals were studied. This was an important theoretical foundation. Reynolds [38] also derived and evaluated double and single integrals of the Mittag-Leffler function. This was also an important theoretical foundation. Atalan [39] also introduced new iterative processes based on almost contraction operators. This was also an important contribution to the area of numerical analysis. Sawar et al. [40] also applied the concept of fractional variable order memristive hyper-chaotic systems in secure image transmission. This was also an important contribution. Ayalew [41] also presented a numerical solution for time fractional Klein-Gordon equations using the Yang-Abdel-Cattani fractional derivative operator. Madani et al. [42] also introduced a new model for HIV/AIDS dynamics using the Caputo-Fabrizio derivative. This was also an important contribution, showing the power of using fractional derivative in epidemiology.

The structure of this paper is as follows. Section 2 is devoted to the main concepts of fractional derivatives. Section 3 is about the MLCF fractional integral and its main features. In Section 4, we discuss the closeness and existence of FDE solutions. Section 5 is about the MLCF Euler method for solving FDEs. Examples of solving FDEs using this method are presented in Section 6.

2. Preliminaries in fractional operators

This subsection introduces the key mathematical concepts that form the basis of our investigation.

Definition 2.1 *For $\alpha \in (0,1)$, the Caputo fractional derivative of a continuous function $y : [0,\varpi] \to \mathbb{R}$ is defined as*

$$^{C}D^{\alpha}_{\varpi} y(\varpi) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\varpi} \frac{y'(s)}{(\varpi - s)^{\alpha}} ds,$$

where $\Gamma(\cdot)$ represents the Euler gamma function and $\varpi > 0$.

Definition 2.2 *For a fractional order $\alpha \in (0,1)$, the Caputo-Fabrizio derivative is given by*

$$^{CF}D^{\alpha}_{\varpi} y(\varpi) = \frac{M(\alpha)}{1-\alpha} \int_0^{\varpi} y'(\tau) e^{-\frac{\alpha}{1-\alpha}(\varpi-\tau)} d\tau,$$

with the normalization function $M(\alpha)$ defined as

$$M(\alpha) = (1-\alpha) + \frac{\alpha}{\Gamma(\alpha)},$$

where the exponential kernel involves the fractional order parameter α. Note that $M(\alpha)$ is positive and bounded for $\alpha \in (0,1)$, with $M(\alpha) \in (0,2)$.

Definition 2.3 *[36] Let $0 < \alpha < 1$ and $0 < \beta \le 1$. The MLCF fractional derivative is defined as follows*

$$^{MLCF}\mathcal{D}^{\alpha,\beta}_{\varpi} y(\varpi) = \frac{M(\alpha)}{1-\alpha} \int_0^{\varpi} \frac{\mathbb{E}_{\beta}(\mu_{\alpha} s^{\beta})}{\mathbb{E}_{\beta}(\mu_{\alpha}\varpi^{\beta})} y'(s) ds, \tag{1}$$

where $\mu_{\alpha} = \frac{\alpha}{1-\alpha}$ and the one parameter ML function is

$$\mathbb{E}_{\beta}(\varpi) = \sum_{j=0}^{\infty} \frac{\varpi^{j}}{\Gamma(\beta j + 1)}.$$

The MLCF operator offers several advantages over existing fractional derivatives:

- Enhanced flexibility: The additional parameter β in the Mittag-Leffler kernel allows for more precise modeling of complex memory effects.
- Mathematical consistency: The operator maintains the non-singular property while extending the Caputo-Fabrizio framework.
- Computational tractability: Despite its generality, the operator admits efficient numerical implementation as demonstrated in this work.
- Physical interpretability: The Mittag-Leffler kernel naturally arises in various physical processes with power-law memory.

These features make the MLCF operator a valuable addition to the family of fractional calculus tools.

Remark 2.4 *The following operator extends the Caputo-Fabrizio derivative by introducing an additional parameter β. Notably, when $\beta = 1$, it reduces to the standard CF operator given in Definition 2.2.*

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Definition 2.5 *For parameters $\alpha \in (0,1)$ and $\beta \in (0,1]$, the Riemann-Liouville variant of the Mittag-Leffler-Caputo-Fabrizio derivative for a function $y(\varpi)$ is given by*

$$^{MLCFR}\mathcal{D}_{\varpi}^{\alpha,\beta} y(\varpi) = \frac{M(\alpha)}{1-\alpha}\frac{d}{d\varpi}\int_0^{\varpi}\frac{\mathbb{E}_\beta(\mu_\alpha s^\beta)}{\mathbb{E}_\beta(\mu_\alpha \varpi^\beta)} y(s)ds, \tag{2}$$

where $\mu_\alpha = \frac{\alpha}{1-\alpha}$.

3. The Mittag-Leffler-Caputo-Fabrizio integral

In this section, we present the MLCF fractional integral associated with the MLCF fractional derivative. Consider the FDE with the MLCF fractional derivative

$$MLCFD\varpi\alpha\beta g(\varpi) = y(\varpi), \quad g(0) = 0, \tag{3}$$

where the MLCF derivative is defined as

$$^{MLCF}\mathcal{D}_{\varpi}^{\alpha,\beta} g(\varpi) = \tag{4}$$
$$\frac{M(\alpha)\mathbb{E}_\beta\left(\mu_\alpha\varpi^\beta\right)^{-1}}{1-\alpha}\int_0^{\varpi}\mathbb{E}_\beta\left(\mu_\alpha s^\beta\right) g'(s)\,ds.$$

Let

$$I(\varpi) = \int_0^{\varpi}\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right) g'(s)\,ds. \tag{5}$$

Then the equation becomes

$$\frac{M(\alpha)}{1-\alpha}\mathbb{E}_\beta\left(\mu_\alpha\varpi^\beta\right)^{-1} I(\varpi) = y(\varpi), \tag{6}$$

so

$$I(\varpi) = \frac{1-\alpha}{M(\alpha)}\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}\varpi^\beta\right) y(\varpi). \tag{7}$$

From (5), we have

$$\frac{d}{d\varpi}I(\varpi) = \mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}\varpi^\beta\right) g'(\varpi). \tag{8}$$

From Eqs. (7) and (8), we have

$$\frac{d}{d\varpi}\left(\frac{1-\alpha}{M(\alpha)}\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}\varpi^\beta\right) y(\varpi)\right) \tag{9}$$
$$= \mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}\varpi^\beta\right) g'(\varpi).$$

Then

$$g'(\varpi) = \frac{1-\alpha}{M(\alpha)}\left(y'(\varpi) + y(\varpi)\frac{d}{d\varpi}\ln\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}\varpi^\beta\right)\right), \tag{10}$$

therefore

$$g(\varpi) = \frac{1-\alpha}{M(\alpha)}\int_0^{\varpi}\left(y'(s) + y(s)\frac{d}{ds}\ln\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right)\right) ds. \tag{11}$$

The general solution is

$$g(\varpi) = \frac{1-\alpha}{M(\alpha)}\int_0^{\varpi}\left(y'(s) + y(s)\frac{d}{ds}\ln\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right)\right) ds. \tag{12}$$

Remark 3.1 *Special Case: $\beta = 1$ (Caputo-Fabrizio-Type). For $\beta = 1$, $\mathbb{E}_1(z) = e^z$, and*

$$\frac{d}{d\varpi}\ln\mathbb{E}_1\left(\frac{\alpha}{1-\alpha}\varpi\right) = \frac{\alpha}{1-\alpha}. \tag{13}$$

Thus:

$$g'(\varpi) = \frac{1-\alpha}{M(\alpha)}\left(y'(\varpi) + \mu_\alpha y(\varpi)\right), \tag{14}$$

which integrates to

$$g(\varpi) = \frac{1-\alpha}{M(\alpha)}y(\varpi) + \frac{\alpha}{M(\alpha)}\int_0^{\varpi} y(s)\,ds. \tag{15}$$

Definition 3.2 *Let $0 < \alpha < 1$ and $0 < \beta \leq 1$. The MLCF integral is defined as follows*

$$^{MLCF}\mathcal{I}_{\varpi}^{\alpha,\beta} y(\varpi) = \frac{1-\alpha}{M(\alpha)}y(\varpi)$$
$$+ \frac{1-\alpha}{M(\alpha)}\int_0^{\varpi} y(s)\frac{d}{ds}\ln\mathbb{E}_\beta\left(\mu_\alpha s^\beta\right) ds. \tag{16}$$

Remark 3.3 *Special Case: $\beta = 1$ (Caputo-Fabrizio-Integral-Type). We have*

$$^{CF}\mathcal{I}_{\varpi}^{\alpha} y(\varpi) = \frac{1-\alpha}{M(\alpha)}y(\varpi) + \frac{\alpha}{M(\alpha)}\int_0^{\varpi} y(s)\,ds. \tag{17}$$

Corollary 3.4 *The MLCF fractional derivative and integral satisfy the fundamental composition relations*

$$^{MLCF}\mathcal{D}_{\varpi}^{\alpha,\beta}\left(^{MLCF}\mathcal{I}_{\varpi}^{\alpha,\beta} y\right)(\varpi) = y(\varpi).$$

Theorem 3.5 *The MLCF fractional derivative and integral satisfy the fundamental composition relations*

$$^{MLCF}\mathcal{I}_{\varpi}^{\alpha,\beta}\left(^{MLCF}\mathcal{D}_{\varpi}^{\alpha,\beta} y\right)(\varpi) = y(\varpi) - y(0). \tag{18}$$

Proof. Let $0 < \alpha < 1$ and $0 < \beta \leq 1$. Begin with the MLCF derivative from Definition 2.3, we have

$$^{MLCF}\mathcal{D}_{\varpi}^{\alpha,\beta} y(\varpi) = \tag{19}$$
$$\frac{M(\alpha)}{1-\alpha}\mathbb{E}_\beta\left(\tfrac{\alpha}{1-\alpha}\varpi^\beta\right)^{-1}\int_0^{\varpi}\mathbb{E}_\beta\left(\tfrac{\alpha}{1-\alpha}s^\beta\right) y'(s)ds.$$

Apply the MLCF integral operator from Definition 3.2, we have

$$^{MLCF}\mathcal{I}_{\varpi}^{\alpha,\beta}\left(^{MLCF}\mathcal{D}_{\varpi}^{\alpha,\beta} y\right)(\varpi)$$
$$= \frac{1-\alpha}{M(\alpha)}\left(^{MLCF}\mathcal{D}_{\varpi}^{\alpha,\beta} y(\varpi)\right)$$
$$+ \frac{1-\alpha}{M(\alpha)}\int_0^{\varpi}\left(^{MLCF}\mathcal{D}_{s}^{\alpha,\beta} y(s)\right)\frac{d}{ds}\ln\mathbb{E}_\beta\left(\tfrac{\alpha}{1-\alpha}s^\beta\right) ds$$
$$= \underbrace{\mathbb{E}_\beta\left(\tfrac{\alpha}{1-\alpha}\varpi^\beta\right)^{-1}\int_0^{\varpi}\mathbb{E}_\beta\left(\tfrac{\alpha}{1-\alpha}s^\beta\right) y'(s)ds}_{\text{Term I}}$$
$$+ \underbrace{\int_0^{\varpi}\left[\mathbb{E}_\beta\left(\tfrac{\alpha}{1-\alpha}s^\beta\right)^{-1}\int_0^{s}\mathbb{E}_\beta\left(\tfrac{\alpha}{1-\alpha}\tau^\beta\right) y'(\tau)d\tau\right] \times\frac{d}{ds}\ln\mathbb{E}_\beta\left(\tfrac{\alpha}{1-\alpha}s^\beta\right) ds}_{\text{Term II}}.$$

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Simplifying Term II: Let $u(s) = \int_0^s \mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}\tau^\beta\right) y'(\tau)d\tau$, then

$$\begin{aligned}\text{Term II} &= \int_0^{\varpi} \frac{u(s)}{\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right)} \frac{\frac{d}{ds}\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right)}{\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right)} ds \\ &= \int_0^{\varpi} -u(s)\frac{d}{ds}\left(\frac{1}{\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right)}\right) ds \\ &= \left.\frac{-u(s)}{\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right)}\right|_0^{\varpi} + \int_0^{\varpi} \frac{y'(s)\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right)}{\mathbb{E}_\beta\left(\frac{\alpha}{1-\alpha}s^\beta\right)} ds \\ &= -\text{Term I} + \int_0^{\varpi} y'(s)ds.\end{aligned}$$

Thus, combining both terms

$$\begin{aligned}&{}^{MLCF}\mathcal{I}_{\varpi}^{\alpha,\beta}\left(MLCFD\varpi\alpha\beta y\right)(\varpi) = \text{Term I} \\ &+(-\text{Term I} + \int_0^{\varpi} y'(s)ds) = y(\varpi) - y(0).\end{aligned}$$

Now let the nonlinear FDE involving MLCF nonsingular kernel

$$\begin{cases} {}^{MLCF}\mathcal{D}_{\varpi}^{\alpha,\beta} y(\varpi) = f(\varpi, y(\varpi)), \quad \varpi \in [0,T], \\ y(0) = d. \end{cases} \tag{20}$$

4. Existence and uniqueness

Next, we perform an analysis on the existence and uniqueness of the newly proposed FDE Eq. (20).

Lemma 4.1 *For fixed parameters $\alpha \in (0,1)$ and $\beta \in (0,1]$, the function $\varphi(\varpi) = \ln \mathbb{E}_\beta(\mu_\alpha \varpi^\beta)$ is continuous on the interval $[0,T]$ as the composition of continuous functions. Consequently, it attains a maximum on $[0,T]$, i.e.,*

$$\max_{\varpi \in [0,T]} \ln \mathbb{E}_\beta(\mu_\alpha \varpi^\beta) = \ln \mathbb{E}_\beta(\mu_\alpha T^\beta) < \infty.$$

Theorem 4.2 *Let $0 < \alpha < 1$ and $0 < \beta \le 1$. Let $f(\varpi, y)$ be a real-valued continuous function defined on a domain $G \subset \mathbb{R}^2$ in the (ϖ, y)-plane. Assume that f satisfies a Lipschitz condition in G with respect to y; that is,*

$$|f(\varpi, y_1(\varpi)) - f(\varpi, y_2(\varpi))| \le L_f |y_1(\varpi) - y_2(\varpi)|. \tag{21}$$

Under this condition, the fractional initial value problem described by Eq. (20) has a unique solution $y(\varpi)$, provided the following inequality is fulfilled

$$\frac{1-\alpha}{M(\alpha)}L_f + \frac{1-\alpha}{M(\alpha)}L_f \ln \mathbb{E}_\beta\left(\mu_\alpha T^\beta\right) < 1. \tag{22}$$

Proof. We begin by applying the MLCF fractional integral of order α, β, as defined in Eq. (16), to both sides of Eq. (20). This yields

$$\begin{aligned}&{}^{MLCF}\mathcal{I}_{\varpi}^{\alpha,\beta}\left[{}^{MLCF}\mathcal{D}_{\varpi}^{\alpha,\beta} y(\varpi)\right] \\ &= {}^{MLCF}\mathcal{I}_{\varpi}^{\alpha,\beta}[f(\varpi, y(\varpi))].\end{aligned} \tag{23}$$

By employing Eqs. (16) and (18) to evaluate the left and right-hand sides of Eq. (23), we have

$$\begin{aligned}y(\varpi) &= d + \frac{1-\alpha}{M(\alpha)} f(\varpi, y(\varpi)) \\ &+ \frac{1-\alpha}{M(\alpha)} \int_0^{\varpi} f(s, y(s)) \frac{d}{ds} \ln \mathbb{E}_\beta\left(\mu_\alpha s^\beta\right) ds.\end{aligned} \tag{24}$$

Clearly, any function $y(\varpi)$ that solves Eq. (20) must also satisfy Eq. (24). Conversely, applying the MLCF derivative defined in Eq. (1) to Eq. (24) recovers the original fractional differential equation (20).

Let $\Sigma = (0,T)$, and define an operator $O : C(\Sigma, \mathbb{R}) \to C(\Sigma, \mathbb{R})$ by

$$\begin{aligned}O[y(\varpi)] &= d + \frac{1-\alpha}{M(\alpha)} f(\varpi, y(\varpi)) \\ &+ \frac{1-\alpha}{M(\alpha)} \int_0^{\varpi} f(s, y(s)) \frac{d}{ds} \ln \mathbb{E}_\beta\left(\mu_\alpha s^\beta\right) ds.\end{aligned} \tag{25}$$

With this operator, Eq. (24) can be equivalently rewritten as the fixed-point equation

$$y(\varpi) = O[y(\varpi)]. \tag{26}$$

By Lemma 4.1, the function $\ln \mathbb{E}_\beta(\mu_\alpha s^\beta)$ is continuous and bounded on $[0,T]$, attaining its maximum at $s = T$. This boundedness is essential for the contraction estimate that follows.

Let $\|\cdot\|_\Sigma$ denote the supremum norm on the interval Σ, defined by

$$\|y(\varpi)\|_\Sigma = \sup_{\varpi \in \Sigma} |y(\varpi)|, \quad \text{for } y(\varpi) \in C(\Sigma, \mathbb{R}). \tag{27}$$

We now estimate the difference between two evaluations of the operator O:

$$\begin{aligned}&\|O[y_1(\varpi)] - O[y_2(\varpi)]\|_\Sigma \\ &\quad = \sup_{\varpi \in \Sigma} |O[y_1(\varpi)] - O[y_2(\varpi)]|.\end{aligned} \tag{28}$$

From the definition of O in Eq. (25) along with the Lipschitz condition from Eq. (21), we obtain

$$\begin{aligned}&\|O[y_1] - O[y_2]\|_\Sigma \le \\ &\quad \frac{1-\alpha}{M(\alpha)} \|f(\varpi, y_1(\varpi)) - f(\varpi, y_2(\varpi))\|_\Sigma \\ &\quad + \frac{1-\alpha}{M(\alpha)} \|f(\varpi, y_1(\varpi)) - f(\varpi, y_2(\varpi))\|_\Sigma \\ &\quad \times \sup_{s \in (0,T)} \left(\ln \mathbb{E}_\beta\left(\mu_\alpha s^\beta\right)\right) \\ &\le \left(\frac{1-\alpha}{M(\alpha)}L_f + \frac{1-\alpha}{M(\alpha)}L_f \ln \mathbb{E}_\beta\left(\mu_\alpha T^\beta\right)\right) \|y_1 - y_2\|_\Sigma.\end{aligned} \tag{29}$$

Hence, we obtain

$$\|O[y_1(\varpi)] - O[y_2(\varpi)]\|_\Sigma \le L \|y_1(\varpi) - y_2(\varpi)\|_\Sigma, \tag{30}$$

where the constant L is given by

$$L = \frac{1-\alpha}{M(\alpha)}L_f + \frac{1-\alpha}{M(\alpha)}L_f \ln \mathbb{E}_\beta\left(\mu_\alpha T^\beta\right).$$

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If the condition stated in Eq. (22) is satisfied, then $L < 1$, and the operator O becomes a contraction on $C(\Sigma, \mathbb{R})$. Therefore, by Banach's fixed-point theorem, the equation (20) admits a unique solution in $C(\Sigma, \mathbb{R})$. As concluded from Theorem 4.2, the inequality in Eq. (22) provides a sufficient criterion for ensuring the uniqueness and existence of the solution to the FDE (20). This condition is governed by four key parameters: the Lipschitz constant L_f, the fractional orders α and β, the normalization function $M(\alpha)$, and the terminal time T.

5. The MLCF Euler method

This section presents the MLCF Euler method to solve the FDE Eq. (20). We convert the above to

$$y(\varpi) = y(0) + \frac{1-\alpha}{M(\alpha)} f(\varpi, y(\varpi)) + \frac{1-\alpha}{M(\alpha)} \int_0^{\varpi} f(\xi, y(\xi)) \frac{d}{d\xi} \ln \mathbb{E}_\beta \left(\mu_\alpha \xi^\beta\right) d\xi,$$

so

$$y_{n+1} = y_0 + \frac{1-\alpha}{M(\alpha)} f(\varpi_{n+1}, y_{n+1}) + \frac{1-\alpha}{M(\alpha)} \sum_{j=0}^{n} \int_{\varpi_j}^{\varpi_{j+1}} f(\varpi, y(\varpi)) \frac{d}{dt} \ln \mathbb{E}_\beta \left(\mu_\alpha t^\beta\right) dt.$$

The integral is then replaced by

$$\int_{\varpi_j}^{\varpi_{j+1}} f(z) g(z) \mathrm{d}z \approx g(\varpi_j) \int_{\varpi_j}^{\varpi_{j+1}} f(z) dz,$$

therefore

$$y_{n+1} = y_0 + \frac{1-\alpha}{M(\alpha)} f(\varpi_{n+1}, y_{n+1}) + \frac{1-\alpha}{M(\alpha)} \sum_{j=0}^{n} f(\varpi_j, y(\varpi_j)) \int_{\varpi_j}^{\varpi_{j+1}} \frac{d}{dt} \ln \mathbb{E}_\beta \left(\mu_\alpha t^\beta\right) dt,$$

then

$$y_{n+1} = y_0 + \frac{1-\alpha}{M(\alpha)} f(\varpi_{n+1}, y_{n+1}) + \frac{1-\alpha}{M(\alpha)} \sum_{j=0}^{n} f(\varpi_j, y_j) \times \left[\ln \mathbb{E}_\beta \left(\mu_\alpha \varpi_{j+1}^\beta\right) - \ln \mathbb{E}_\beta \left(\mu_\alpha \varpi_j^\beta\right) \right]. \tag{31}$$

Note that equation (31) is implicit in y_{n+1} due to the term $f(\varpi_{n+1}, y_{n+1})$. In practice, we solve for y_{n+1} using a fixed-point iteration, e.g., $y_{n+1}^{(k+1)} = \Phi(y_{n+1}^{(k)})$, where Φ is the right-hand side of (31). Under the conditions of Theorem 1, the mapping Φ is contractive, guaranteeing convergence of the iteration. The computational cost per time step is $O(\text{Iterations} \cdot n)$, where Iterations is the number of fixed-point iterations needed to converge to a specified tolerance. In Algorithm 1, we present the MLCF Euler method to solve Eq. (20).

Algorithm 1 MLCF Euler method for solving Eq.(20).

Input: $n \in \mathbb{N}$, $\beta, \alpha \in (0, 1)$, tolerance ϵ.
1. Set $h = \frac{T}{n}$.
2. Set $\varpi_k = kh$ for $k = 0, 1, \ldots, n$.
3. Set $y_0 = d$.
4. Solve $y_1 = y_0 + \frac{1-\alpha}{M(\alpha)} f(\varpi_1, y_1) + \frac{1-\alpha}{M(\alpha)} f(\varpi_0, y_0) \ln \mathbb{E}_\beta \left(\mu_\alpha \varpi_1^\beta\right)$ using fixed-point iteration.
5. for $k = 1, \ldots, n$
6. Solve $y_{k+1} = y_0 + \frac{1-\alpha}{M(\alpha)} f(\varpi_{k+1}, y_{k+1}) + \frac{1-\alpha}{M(\alpha)} f(\varpi_0, y_0) \ln \mathbb{E}_\beta \left(\mu_\alpha \varpi_1^\beta\right) + \frac{1-\alpha}{M(\alpha)} \sum_{j=1}^{k} f(\varpi_j, y_j) \left[\ln \mathbb{E}_\beta \left(\mu_\alpha \varpi_{j+1}^\beta\right) - \ln \mathbb{E}_\beta \left(\mu_\alpha \varpi_j^\beta\right) \right]$ using fixed-point iteration.

Output: The approximate solution y_k for $k = 0, 1, \ldots, n$.

Theorem 5.1 (Convergence of MLCF Euler Method) *Under the Lipschitz condition on $f(\varpi, y)$ with constant L_f, and assuming the exact solution $y(\varpi) \in C^2[0, T]$, the MLCF Euler method defined by Eq. (31) converges with order 1, i.e.,*

$$\max_{0 \le k \le n} |y(\varpi_k) - y_k| = O(h).$$

Proof. The proof proceeds in several steps, adapting standard numerical analysis techniques to the specific structure of the MLCF operator.

Step 1: The equivalent Volterra integral equation to the FDE (20) is

$$y(\varpi) = y(0) + \frac{1-\alpha}{M(\alpha)} f(\varpi, y(\varpi)) + \frac{1-\alpha}{M(\alpha)} \int_0^{\varpi} f(s, y(s)) \frac{d}{ds} \ln \mathbb{E}_\beta(\mu_\alpha s^\beta)\, ds.$$

Let $\varpi_k = kh$ for $k = 0, 1, \ldots, n$, with $h = T/n$. The exact solution at ϖ_{n+1} satisfies:

$$y(\varpi_{n+1}) = y_0 + \frac{1-\alpha}{M(\alpha)} f(\varpi_{n+1}, y(\varpi_{n+1})) + \frac{1-\alpha}{M(\alpha)} \int_0^{\varpi_{n+1}} f(s, y(s)) \frac{d}{ds} \ln \mathbb{E}_\beta(\mu_\alpha s^\beta)\, ds.$$

The numerical scheme (31) is

$$y_{n+1} = y_0 + \frac{1-\alpha}{M(\alpha)} f(\varpi_{n+1}, y_{n+1}) + \frac{1-\alpha}{M(\alpha)} \sum_{j=0}^{n} f(\varpi_j, y_j) \int_{\varpi_j}^{\varpi_{j+1}} \frac{d}{dt} \ln \mathbb{E}_\beta(\mu_\alpha t^\beta) dt.$$

Step 2: Let $e_k = y(\varpi_k) - y_k$ be the global error at time ϖ_k. Subtracting (31) from the exact equation at

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ϖ_{n+1}, we have

$$e_{n+1} = \frac{1-\alpha}{M(\alpha)}[f(\varpi_{n+1}, y(\varpi_{n+1})) - f(\varpi_{n+1}, y_{n+1})] + \frac{1-\alpha}{M(\alpha)}\left\{\int_0^{\varpi_{n+1}} f(s, y(s))g'(s)\,ds - \sum_{j=0}^{n} f(\varpi_j, y_j)\int_{\varpi_j}^{\varpi_{j+1}} g'(t)\,dt\right\},$$

where $g(s) = \ln \mathbb{E}_\beta(\mu_\alpha s^\beta)$, so $g'(s) = \frac{d}{ds}\ln \mathbb{E}_\beta(\mu_\alpha s^\beta)$. Define the quadrature error

$$\tau_{n+1} = \int_0^{\varpi_{n+1}} F(s)ds - \sum_{j=0}^{n} F(\varpi_j)\int_{\varpi_j}^{\varpi_{j+1}} g'(t)dt,$$

where $F(s) = f(s, y(s))g'(s)$.

Step 3: Estimate the Quadrature Error τ_{n+1}. Since $y(\varpi) \in C^2[0,T]$ and $g'(s)$ is continuous on $[0,T]$, then $F(s) \in C^1[0,T]$. On each subinterval $[\varpi_j, \varpi_{j+1}]$, the local error is

$$\delta_j = \int_{\varpi_j}^{\varpi_{j+1}} [F(s) - F(\varpi_j)]\,ds.$$

By the Mean Value Theorem

$$|F(s) - F(\varpi_j)| \le \|F'\|_\infty (s - \varpi_j) \le \|F'\|_\infty h.$$

Thus

$$|\delta_j| \le \int_{\varpi_j}^{\varpi_{j+1}} \|F'\|_\infty h\,ds = \|F'\|_\infty h^2.$$

The global quadrature error is bounded by

$$|\tau_{n+1}| = \left|\sum_{j=0}^{n}\delta_j\right| \le \sum_{j=0}^{n}\|F'\|_\infty h^2 = (n+1)\|F'\|_\infty h^2.$$

Since $\varpi_{n+1} = (n+1)h \le T$, we have $(n+1) \le T/h$, hence

$$|\tau_{n+1}| \le \frac{T}{h}\cdot\|F'\|_\infty h^2 = C_1 h,$$

where $C_1 = T\|F'\|_\infty$ is constant.

Step 4: Rewriting the error equation:

$$e_{n+1} = \frac{1-\alpha}{M(\alpha)}[f(\varpi_{n+1}, y(\varpi_{n+1})) - f(\varpi_{n+1}, y_{n+1})] + \frac{1-\alpha}{M(\alpha)}\tau_{n+1} + \frac{1-\alpha}{M(\alpha)} \times \sum_{j=0}^{n}[f(\varpi_j, y(\varpi_j)) - f(\varpi_j, y_j)]\int_{\varpi_j}^{\varpi_{j+1}} g'(t)dt.$$

Let $K = \frac{1-\alpha}{M(\alpha)}$. Using the Lipschitz condition and taking absolute values, we get

$$|e_{n+1}| \le KL_f|e_{n+1}| + K|\tau_{n+1}| + KL_f\sum_{j=0}^{n}|e_j|\int_{\varpi_j}^{\varpi_{j+1}}|g'(t)|\,dt.$$

Let $G = \max_{t\in[0,T]}|g'(t)|$. Then $\int_{\varpi_j}^{\varpi_{j+1}}|g'(t)|dt \le Gh$, and we obtain

$$|e_{n+1}| \le KL_f|e_{n+1}| + KC_1h + KL_fGh\sum_{j=0}^{n}|e_j|.$$

Step 5: Apply a Discrete Gronwall Inequality. Rearranging

$$(1 - KL_f)|e_{n+1}| \le KC_1h + KL_fGh\sum_{j=0}^{n}|e_j|.$$

Assuming $KL_f < 1$, we have

$$|e_{n+1}| \le \frac{KC_1}{1-KL_f}h + \frac{KL_fG}{1-KL_f}h\sum_{j=0}^{n}|e_j|.$$

Let $A = \frac{KC_1}{1-KL_f}$ and $B = \frac{KL_fG}{1-KL_f}$. Then

$$|e_{n+1}| \le Ah + Bh\sum_{j=0}^{n}|e_j|.$$

By the discrete Gronwall inequality, we get

$$\max_{0\le k\le n}|e_k| \le Ah\cdot e^{BT}.$$

Step 6: Since A and B are constants independent of h, we have

$$\max_{0\le k\le n}|y(\varpi_k) - y_k| \le Ch,$$

where $C = Ae^{BT}$. This proves

$$\max_{0\le k\le n}|y(\varpi_k) - y_k| = O(h).$$

6. Examples

In this section, we illustrate the effectiveness of the proposed approach through several examples. All numerical computations were carried out using MATLAB (R2023a) on a laptop with 64 GB of memory. To evaluate the performance of the methods, we calculate the Experimental Order of Convergence (EOC) using the formula

$$\text{EOC} = \log_2\left(\frac{E_n}{E_{2n}}\right),$$

where E_n is the absolute error computed using a step size $h = T/n$, and E_{2n} is the error computed using a refined step size $h/2 = T/(2n)$. The Mittag-Leffler function was evaluated using the built-in 'ml' function (by Roberto Garrappa) available on the MATLAB File Exchange, MATLAB Central File Exchange, which provides high accuracy.

Example 6.1 Let fractional problem

$$\begin{aligned} {}^{MLCF}\mathcal{D}_\varpi^{\alpha,\beta}y(\varpi) &= \varpi f(\varpi),\ \varpi \in]0,1], \\ y(0) &= 0, \end{aligned} \tag{32}$$

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where

$$f(\varpi) = \frac{M(\alpha)\mathbb{E}_{\beta}(\frac{\alpha}{1-\alpha}\varpi^{\beta})^{-1}}{1-\alpha}\mathbb{E}_{\beta,2}\left(\frac{\alpha}{1-\alpha}\varpi^{\beta}\right).$$

The exact solution is $y(\varpi) = \varpi$. Figure 1, Figure 2, Figure 3, Figure 4 illustrate the exact and numerical solutions as well as the absolute errors for different α and β values. The absolute errors, CPU time, and experimental orders of convergence (EOC) for the fractional Euler method are given in Table 1, Table 2, Table 3 for different parameter settings. It is seen that the EOC for method (31) is always close to 1, thus indicating first-order convergence. The figures demonstrate that the numerical solutions are in a very good agreement with the exact ones in the whole domain.

In Figure 5, Log-log plot of the absolute error vs. step size for Example 6.1 with $\alpha = 0.5$, demonstrating first-order convergence. The dashed line shows a slope of 1.

To complement the sensitivity analysis in Example 6.1, we also investigate the case where α is very small ($\alpha = 0.01$) with various β values. This represents another critical regime where the fractional order approaches zero. Table 4 presents the convergence results.

The results demonstrate that the MLCF Euler method maintains stable convergence even for extremely small α values (Figure 6), with EOC consistently close to 1. The method exhibits particularly robust behavior for $\beta = 0.01$ and $\beta = 0.99$, while for $\beta = 0.7$ the EOC gradually approaches 1 as the mesh refines.

Example 6.2 Let the fractional problem:

$$^{MLCF}\mathcal{D}_{\varpi}^{\alpha,0.998}y(\varpi) = u(\varpi)\Big(g(\varpi)+1\Big) - \varpi^{0.998},$$
$$\varpi \in]0,1], \quad u(0) = 0, \qquad (33)$$

where

$$g(\varpi) = \frac{M(\alpha)0.998}{1-\alpha}\mathbb{E}_{0.998}\left(\frac{\alpha}{1-\alpha}\varpi^{0.998}\right)^{-1} \times\mathbb{E}_{0.998,1.998}\left(\frac{\alpha}{1-\alpha}\varpi^{0.998}\right).$$

The exact solution is $y(\varpi) = \varpi^{0.998}$.

Figure 7 and Figure 8 show the exact solution, numerical solution, and absolute error for different values of α and β. Table 5 shows the absolute error, EOC, and CPU time for the fractional Euler method with different parameter values. It is evident that the method given in Eq. (31) has an EOC of almost 1, which verifies that the method is of order 1. In addition, the numerical solution is found to match the exact solution very well over the entire domain.

Example 6.3 Let fractional problem:

$$^{MLCF}\mathcal{D}_{\varpi}^{\alpha,\beta}y(\varpi) = \cos(y(\varpi)) - \cos\left(\frac{1-\alpha}{M(\alpha)}(\varpi+1)\right) + \varpi\frac{\mathbb{E}_{\beta,2}(\mu_{\alpha}\varpi^{\beta})}{\mathbb{E}_{\beta}(\mu_{\alpha}\varpi^{\beta})}, \qquad (34)$$
$$\varpi \in]0,2], \quad y(0) = \frac{1-\alpha}{M(\alpha)}.$$

The exact solution is $y(\varpi) = \frac{1-\alpha}{M(\alpha)}(\varpi+1)$.

Figure 9, Figure 10, Figure 11 show the exact and numerical solutions along with the absolute error for different values of α and β. The numerical solutions obtained by taking different values of α and β, along with the absolute error and CPU time, are shown in Table 6, Table 7, Table 8. From the above examples, it can be observed that the EOC of the scheme given by Eq. (31) is always equal to 1. Hence, the proposed method is a first-order accurate method. Figure 9, Figure 10, Figure 11 show the exact and numerical solutions obtained by the proposed method along with the absolute error for the given examples. It can be observed from the figures that the numerical solutions obtained by the proposed method match well with the exact solutions over the entire interval.

7. Conclusion

This work has paved the way for a detailed framework for the MLCF. We went on to define the associated MLCF fractional integral, thus, fully completing the fundamental calculus for this new derivative. By means of fixed-point theory, we established quite strong measures under which a solution exists and is unique to a system of nonlinear fractional differential equations involving the MLCF derivative. In order to make the theory transferable into practice, we invented a computational scheme, the MLCF Euler method, for equation-solving operations in this field. We have also provided a rigorous convergence analysis, proving that the method is first-order accurate. The efficiency of the introduced technique was confirmed by conducting several representative examples. All the numerical results such as absolute error, experimental order of convergence (EOC), and computational time were in line with the first-order accuracy of the scheme. The numerical solutions coincide very well with the exact ones for all tested problems, thus, confirming the soundness of our method. Briefly, this work is a step forward in terms of both theory and computations when dealing with FDEs having Mittag-Leffler nonsingular kernels. The next projects will revolve around employing these MLCF fractional operators to simulate complicated natural phenomena, for instance, tracing the prevalence pattern of HIV/AIDS with special attention to the undiagnosed individuals. Furthermore, extending this method to fractional partial differential equations (FPDEs) presents a promising avenue for future research. The primary challenges would involve coupling the temporal MLCF discretization with a suitable spatial discretization method (e.g., finite differences or spectral methods), which would lead to a system of algebraic equations to be solved at each time step.

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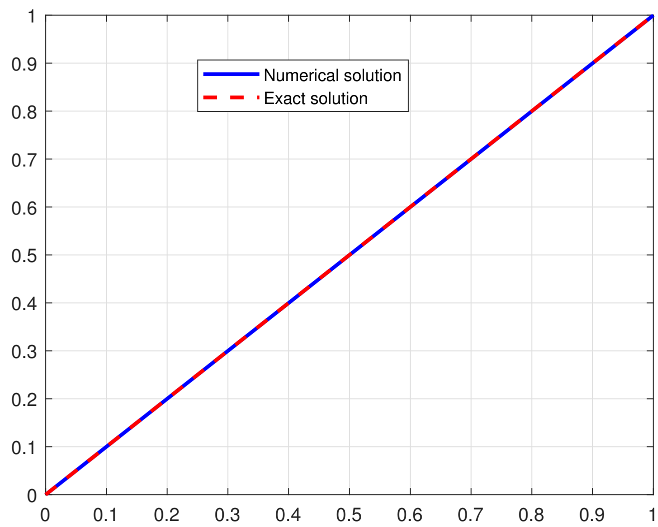

Figure 1. The numerical solution and the exact solution with $\beta = 0.9$, $\alpha = 0.5$, $n = 640$ for Example 6.1

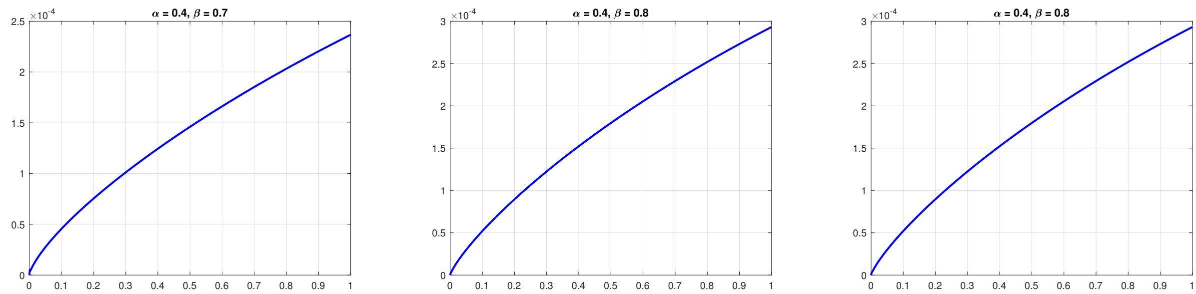

Figure 2. The absolute error with $\alpha = 0.4$, $n = 640$ for different β

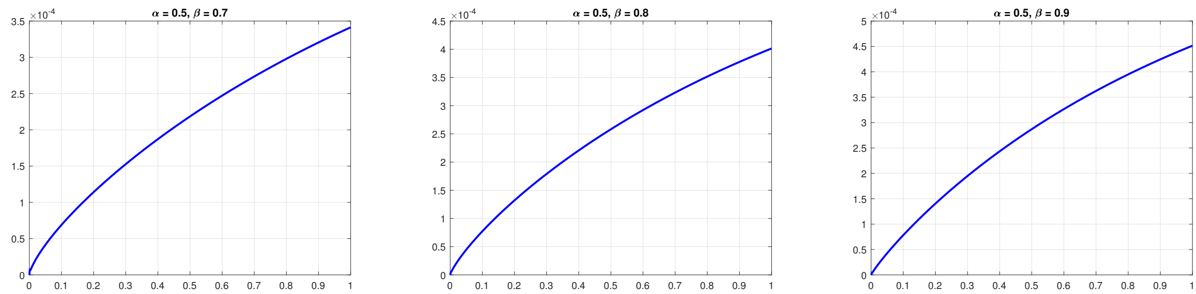

Figure 3. The absolute error with $\alpha = 0.5$, $n = 640$ for different β

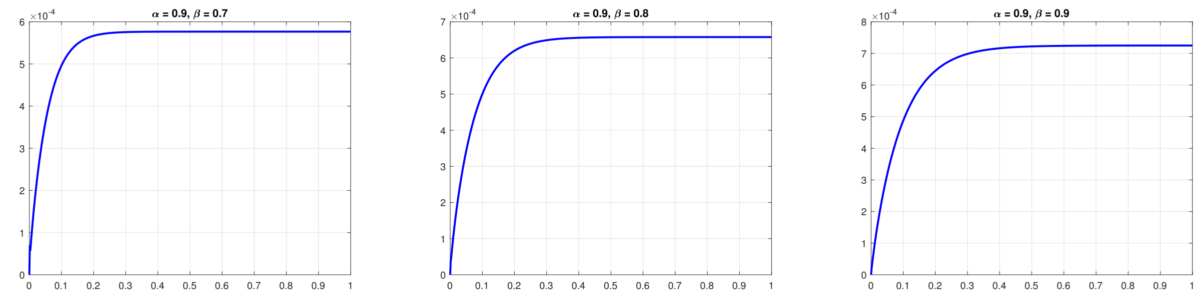

Figure 4. The absolute error with $\alpha = 0.9$, $n = 640$ for different β

Table 1. The absolute error (AE), EOC, and CPU time in seconds (CTs) of $\alpha = 0.4$ for Example 6.1

n	$\beta = 0.7$			$\beta = 0.8$			$\beta = 0.9$		
	AE	EOC	CTs	AE	EOC	CTs	AE	EOC	CTs
40	0.0037	–	0.09	0.0047	–	0.10	0.0054	–	0.08
80	0.0019	0.98	0.14	0.0023	0.99	0.15	0.0027	1.00	0.13
160	9.39e-04	0.99	0.27	0.0012	0.99	0.25	0.0014	1.00	0.25
320	4.71e-04	0.99	0.71	5.8574e-04	0.99	0.63	6.8059e-04	0.99	0.59
640	2.36e-04	0.99	2.05	2.9315e-04	0.99	1.85	3.4033e-04	0.99	1.83

Table 2. The AE, EOC and CTs of $\alpha = 0.5$ for Example 6.1

n	$\beta = 0.7$			$\beta = 0.8$			$\beta = 0.9$		
	AE	EOC	CTs	AE	EOC	CTs	AE	EOC	CTs
40	0.0053	–	0.08	0.0064	–	0.08	0.0072	–	0.10
80	0.0027	0.98	0.18	0.0032	0.99	0.16	0.0036	1.00	0.14
160	0.0014	0.99	0.28	0.0016	0.99	0.28	0.0018	1.00	0.26
320	6.8042e-04	0.99	0.80	8.0189e-04	0.99	0.80	9.0275e-04	1.00	0.64
640	3.4133e-04	0.99	2.66	4.0134e-04	0.99	2.11	4.5141e-04	0.99	2.00

Table 3. The AE, EOC and CTs of $\alpha = 0.9$ for Example 6.1

n	$\beta = 0.7$			$\beta = 0.8$			$\beta = 0.9$		
	AE	EOC	CTs	AE	EOC	CTs	AE	EOC	CTs
40	0.0100	–	0.11	0.0110	–	0.12	0.0121	–	0.09
80	0.0047	1.09	0.26	0.0053	0.99	0.20	0.0059	1.03	0.17
160	0.0023	1.03	0.82	0.0026	1.01	0.52	0.0029	1.01	0.47
320	0.0011	1.00	2.27	0.0013	1.00	1.82	0.0015	1.00	1.30
640	5.7663e-04	0.99	7.88	6.5767e-04	1.00	5.86	7.2489e-04	1.00	5.43

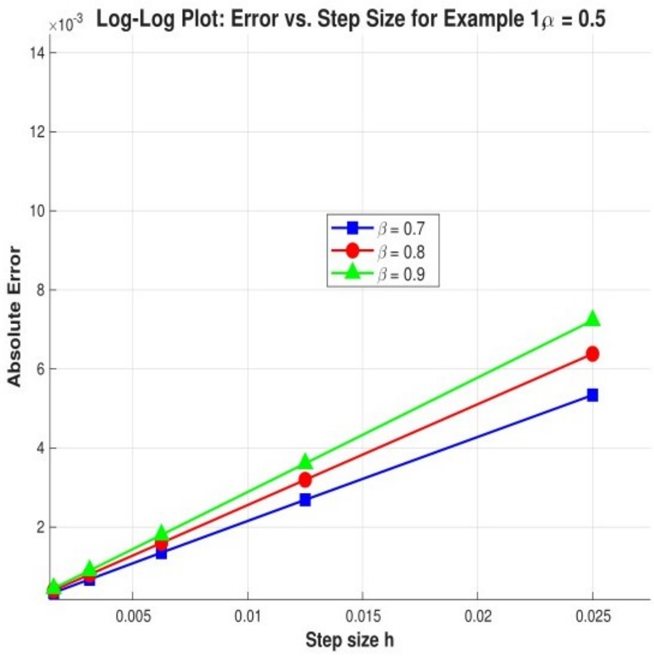

Figure 5. Log-log plot of the absolute error vs. step size for Example 6.1 with $\alpha = 0.5$, demonstrating first-order convergence. The dashed line shows a slope of 1

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Table 4. Convergence analysis for $\alpha = 0.01$ with different β values

n	$\beta = 0.01$		$\beta = 0.7$		$\beta = 0.99$	
	AE	EOC	AE	EOC	AE	EOC
40	2.46e-04	—	5.38e-05	—	1.24e-04	—
80	1.22e-04	1.005	2.72e-05	0.984	6.18e-05	1.000
160	6.10e-05	1.005	1.37e-05	0.988	3.09e-05	1.000
320	3.04e-05	1.005	6.89e-06	0.992	1.55e-05	1.000
640	1.52e-05	1.005	3.46e-06	0.995	7.73e-06	1.000

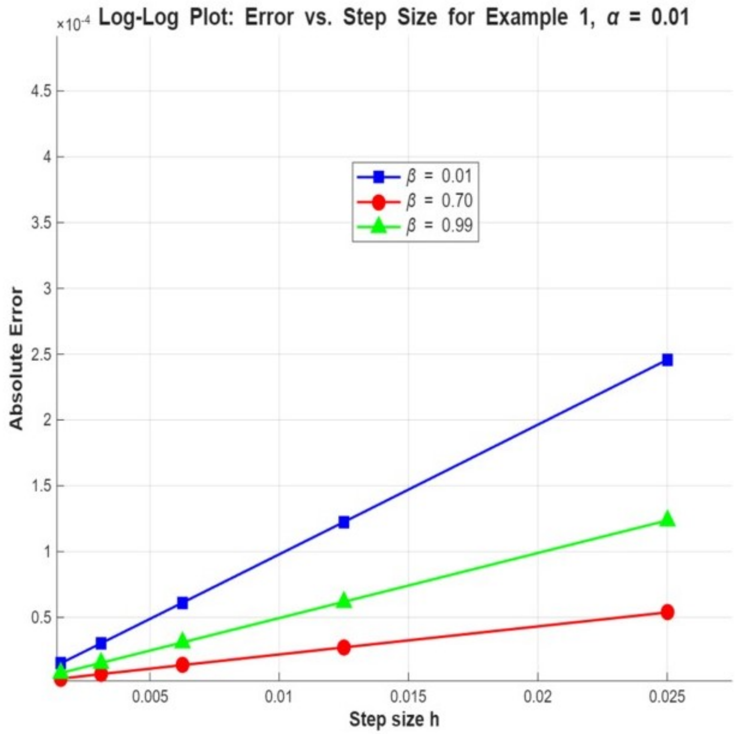

Figure 6. Log-log plot of absolute error vs. step size for $\alpha = 0.01$ and different β values. The dashed line represents first-order convergence (slope = 1)

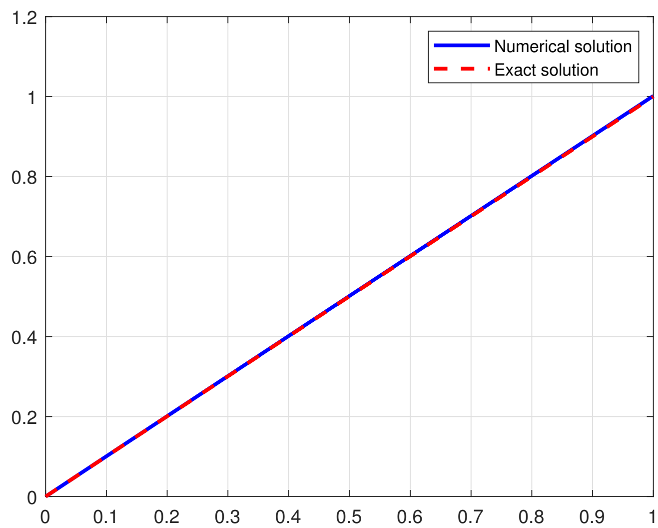

Figure 7. The exact solution and numerical solution for Example 6.2.

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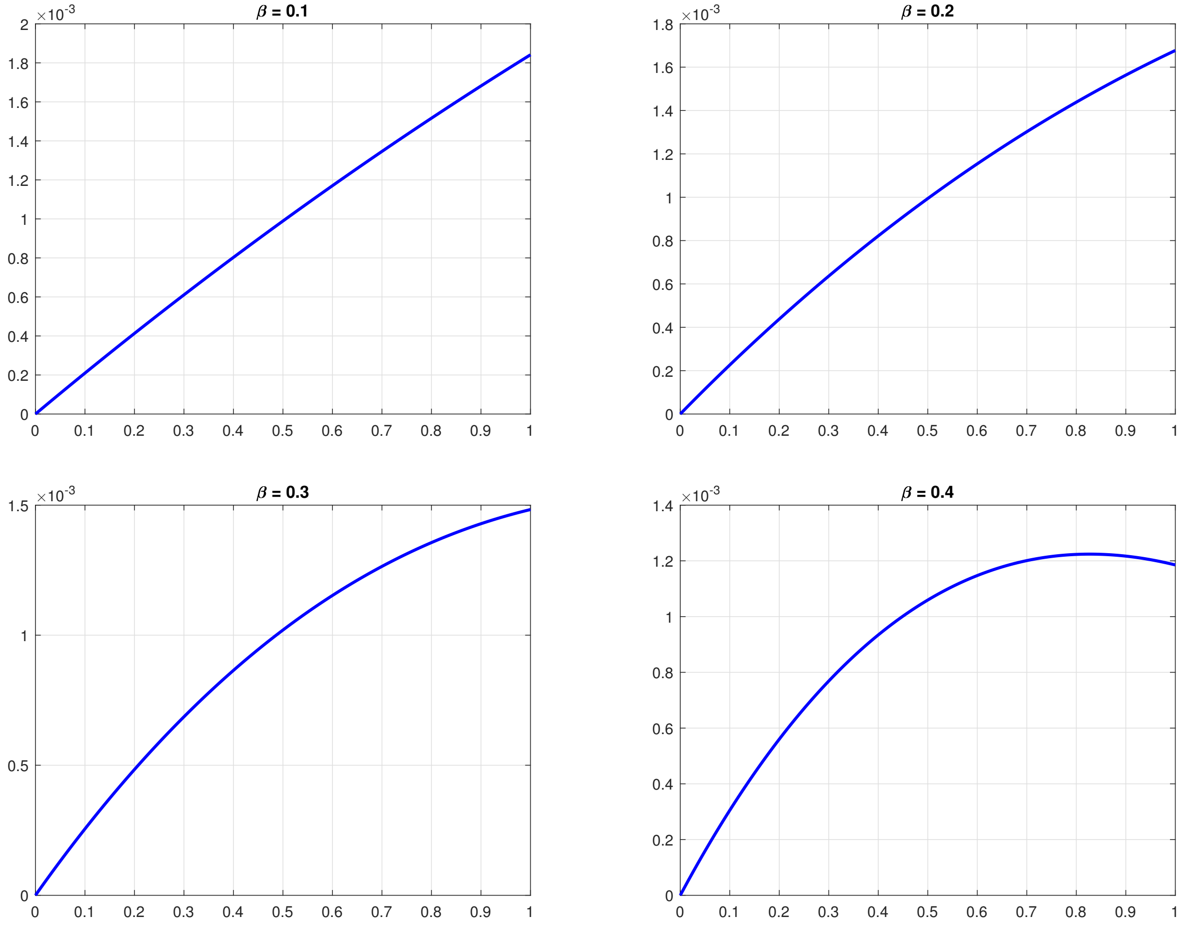

Figure 8. The absolute error with $\alpha = 0.9$, $n = 640$ for different β

Table 5. The AE and CTs of different α for Example 6.2

n	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
	AE	CTs	AE	CTs	AE	CTs	AE	CTs
40	0.0030	0.31	0.0041	0.35	0.0051	0.31	0.0059	0.33
80	0.0024	0.13	0.0028	0.15	0.0032	0.13	0.0034	0.15
160	0.0021	0.26	0.0022	0.30	0.0022	0.26	0.0021	0.28
320	0.0019	0.50	0.0018	0.55	0.0017	0.60	0.0015	0.59
640	0.0018	1.40	0.0017	1.52	0.0015	1.66	0.0012	1.87

Table 6. The AE, EOC and CTs of $\alpha = 0.4$ for Example 6.3

n	$\beta = 0.7$			$\beta = 0.8$			$\beta = 0.9$		
	AE	EOC	CTs	AE	EOC	CTs	AE	EOC	CTs
40	0.0041	–	0.32	0.0050	–	0.32	0.0059	–	0.34
80	0.0021	0.98	0.34	0.0025	0.99	0.18	0.0029	1.00	0.17
160	0.0010	0.99	0.36	0.0013	0.99	0.54	0.0015	1.00	0.32
320	5.2206e-04	0.99	0.88	6.3153e-04	0.99	0.76	7.2875e-04	1.00	0.77
640	2.6177e-04	0.99	2.60	3.1601e-04	0.99	2.55	3.6433e-04	1.00	2.30

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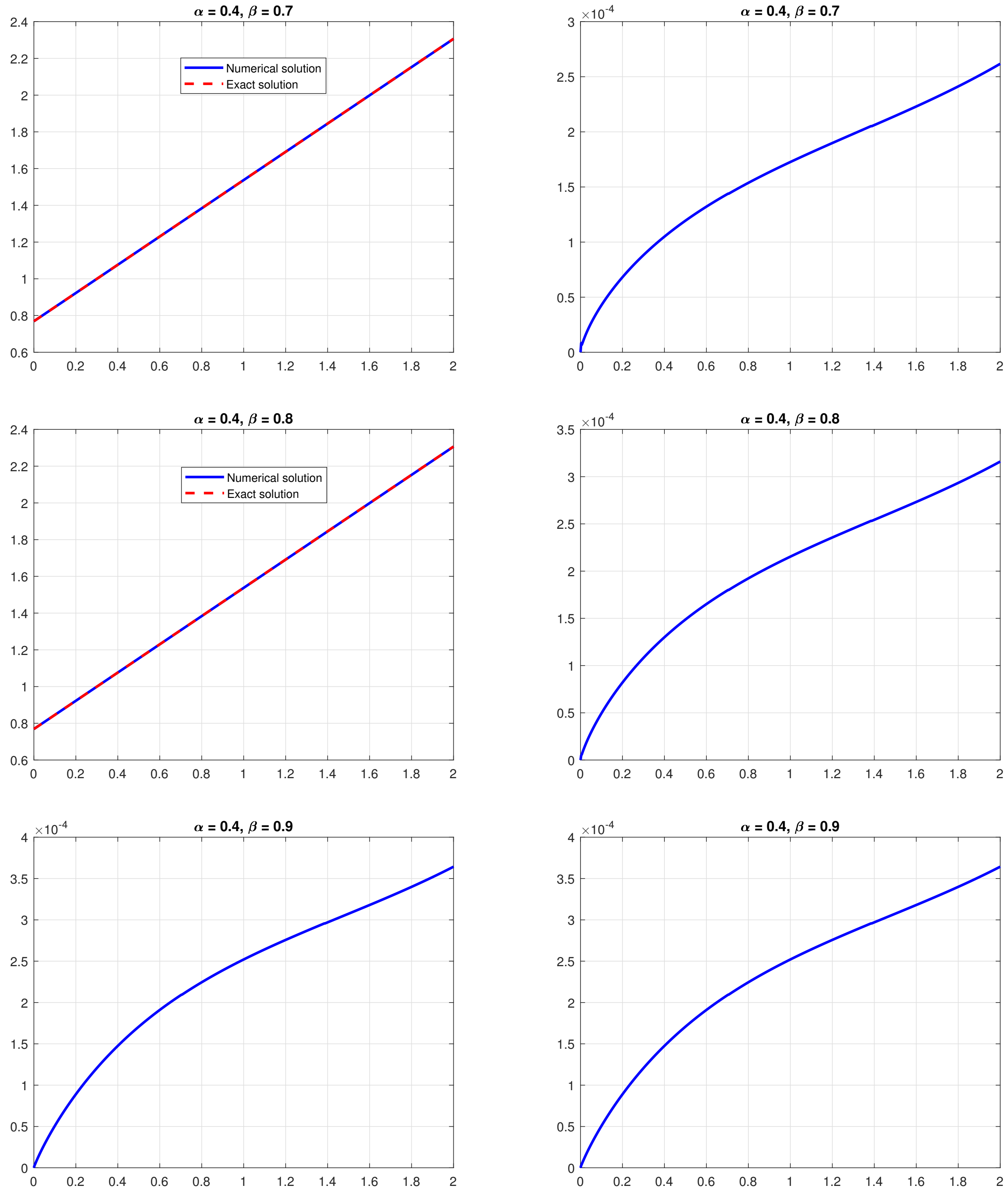

Figure 9. The numerical solution, the exact solution and the absolute error with $\alpha = 0.4$, $n = 640$ for different β

Table 7. The AE, EOC and CTs of $\alpha = 0.5$ for Example 6.3

n	$\beta = 0.7$			$\beta = 0.8$			$\beta = 0.9$		
	AE	EOC	CTs	AE	EOC	CTs	AE	EOC	CTs
40	0.0037	–	0.19	0.0043	–	0.20	0.0049	–	0.18
80	0.0018	0.99	0.10	0.0022	1.00	0.09	0.0025	1.00	0.09
160	9.2600e-04	0.99	0.19	0.0011	0.99	0.19	0.0012	1.00	0.18
320	4.6476e-04	0.99	0.56	5.4026e-04	0.99	0.51	6.1274e-04	1.00	0.47
640	2.3329e-04	0.99	1.64	2.7050e-04	0.99	1.48	3.0619e-04	1.00	1.43

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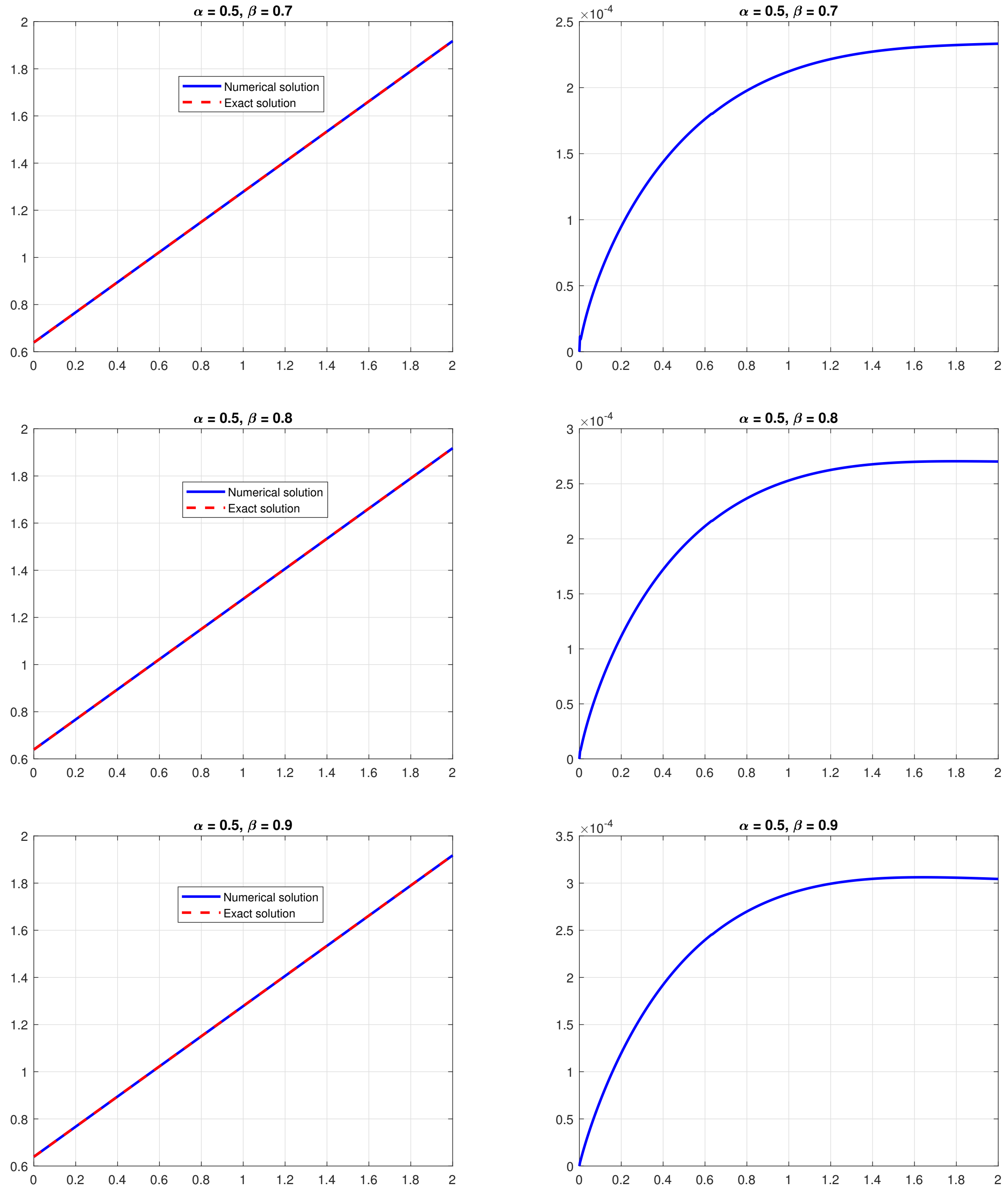

Figure 10. The numerical solution, the exact solution and the absolute error with $\alpha = 0.5$, $n = 640$ for different β

Table 8. The AE, EOC and CTs of $\alpha = 0.9$ for Example 6.3

n	$\beta = 0.7$			$\beta = 0.8$			$\beta = 0.9$		
	AE	EOC	CTs	AE	EOC	CTs	AE	EOC	CTs
40	0.0023	–	0.22	0.0024	–	0.31	0.0025	–	0.19
80	9.8846e-04	1.21	0.20	0.0011	1.11	0.16	0.0012	1.07	0.12
160	4.6397e-04	1.09	0.56	5.2588e-04	1.04	0.39	5.7970e-04	1.03	0.32
320	2.2698e-04	1.03	2.20	2.5969e-04	1.01	1.32	2.8685e-04	1.01	1.09
640	1.1313e-04	1.00	7.05	1.2940e-04	1.00	4.81	1.4278e-04	1.00	3.74

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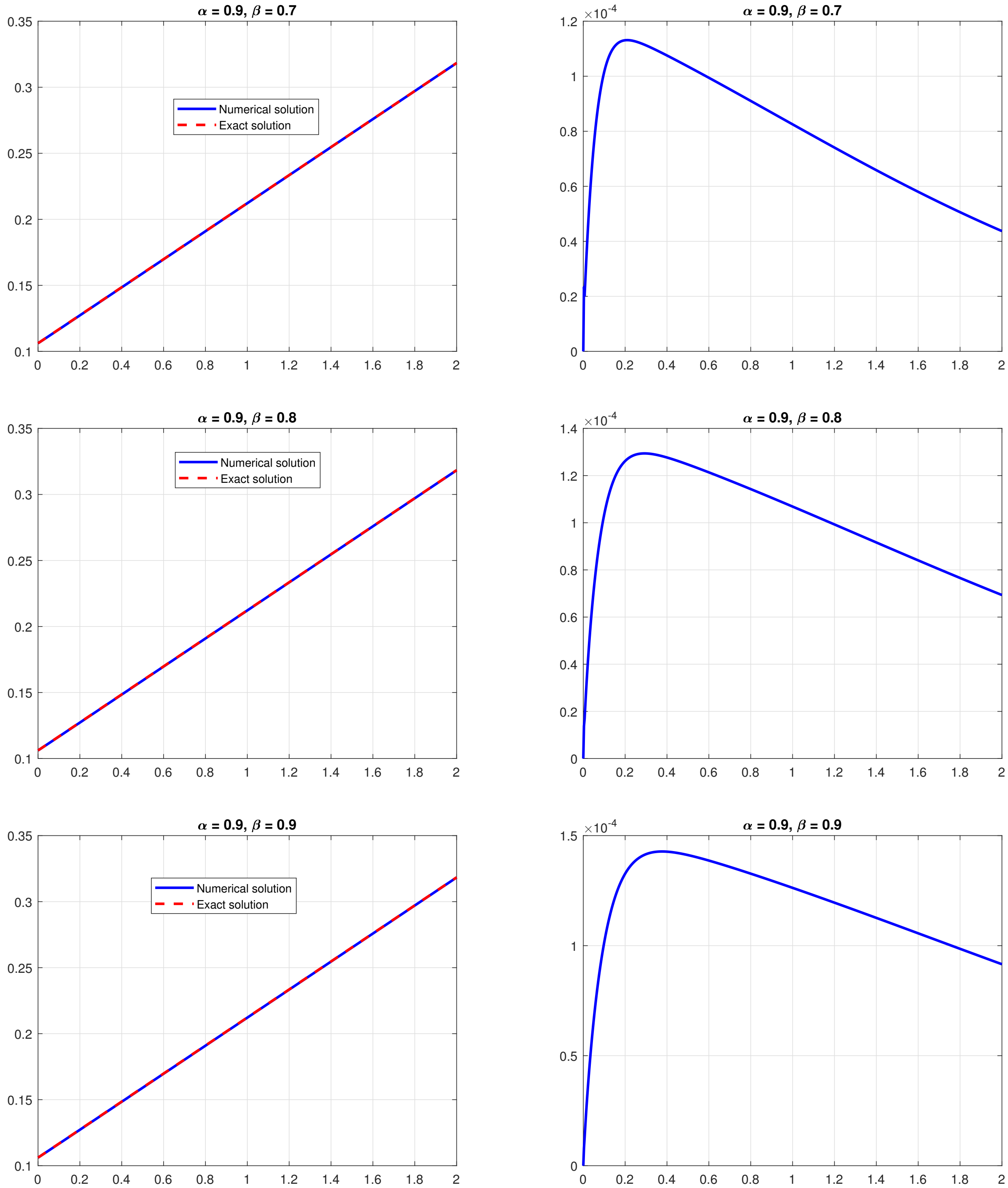

Figure 11. The numerical solution, the exact solution and the absolute error with $\alpha = 0.9$, $n = 640$ for different β

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Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.
Conflict of interests
All authors declare that they have no conflict of interest.

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