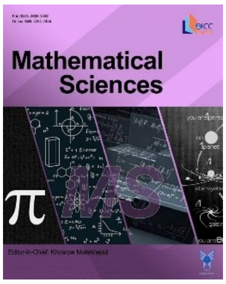

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Research Article

Power Unit Haq Distribution: A Flexible Probability Model for Lifetime Data Analysis

Mohammed Ahmed Alomair[1], Muhammad Ahsan-ul-Haq[2,3,*]

[1]*Department of Quantitative Methods, School of Business, King Faisal University, 31982, Al-Ahsa, Saudi Arabia*
[2]*College of Statistical Sciences, University of the Punjab, Lahore Pakistan*
[3]*Quality Enhancement Cell, National College of Arts, Lahore Pakistan*

***Corresponding author:** ahsanshani36@gmail.com

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Abstract

In this study, a new unit interval distribution defined on the unit interval is developed through a power transformation approach and termed the *Power Unit Haq (PUH)* distribution. Several key statistical properties of the new distribution are derived, including incomplete moments, moments, and associated measures, moment generating function, hazard function, mean residual life function, and Rényi entropy. The parameters of the proposed distribution are estimated using five estimation approaches, and their performance is evaluated through extensive Monte Carlo simulations. The flexibility and practical relevance of the new distribution are further demonstrated by utilizing three real datasets-one involving radiation, reactor pump failures, and kidney dialysis patients. The proposed distribution exhibits superior fitting performance compared to established competing unit interval distributions. Additionally, Bayesian estimation of the model parameter is carried out, enhancing the distribution's applicability for real-world scenarios.

Keywords: Unit interval model; Moments; Reliability analysis; Inference, lifetime data, Bayesian analysis

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1. Introduction

Compositional data analysis plays a crucial role in various scientific and applied disciplines, including radiology, hydrology, economics, sociology, environmental sciences, psychology, epidemiology, and insurance risk management. Such types of datasets often appear in the form of proportions, fractions, or percentages that are naturally constrained to the unit interval support [0,1] [1]. Since these measurements lie between [0,1]. The classical probability distributions defined on the real line may fail to adequately represent the inherent underlying properties. Consequently, the derivation of statistical distributions that operate exclusively within the unit interval has become increasingly interval. For example, in psychology, proportions and percentages are important in assessing probabilities, such as the fraction of mental activity associated with a specific region. In the field of economics, variables such as capital structure, market share, and the proportion of income spent on non-permanent expenses are frequently measured within unit intervals. The probability models defined on the bounded interval are noted for their failure rate (hazard function) shapes, resembling increasing and bathtub curves. The development of current probability distributions using appropriate methodologies has fueled the derivation of extended models. Various modified models have been extensively derived over the last decades for the analysis of real-world data sets in different areas. The extended or generalized distributions can be derived by adding one or more parameters to the baseline models, using mixture approaches, or other approaches that provide good flexibility in analyzing datasets in the field. The common generalization strategies are generalization are Azzalini's approach [2], Beta-G family [3], transmuted-G family [4], Kumaraswamy-G family [5], Fréchet-G family [6], Weibull-G family [7], and Lindley power series [8], inverted Nadarajah–Haghighi power series [9], and weighted distributions [10].

The concept of variable power transformation has sparked widespread attention in probability models and statistical analysis. Power transformation has been widely utilized to propose a variety of adaptable probability distributions that improve the analysis capacity of real-world data across various disciplines. Several important models have been introduced using this approach, for example, the power half-logistic distribution [11] expands on the half-logistic distribution by including a power component, allowing for more versatility in modeling skewed data. Further extensions include the power Lomax distribution [12], power Raleigh distribution [13], power Erlang distribution [14], power inverted Topp–Leone [15], power modified Kies-exponential distribution [16], power inverted Nadarajah–Haghighi distribution [17], and power Quasi-Xgamma distribution [18].

The Unit Haq (UH) distribution was recently proposed by [19] using log transformation. The cumulative distribution function (CDF) and probability density function (PDF) of the UH distribution are presented below, respectively.

$$F(z;\theta)=\left(\frac{(1+\theta)^2-\theta\ln z+\frac{\theta^2(\ln z)^2}{2}}{(1+\theta)^2}\right)z^{\theta},$$

$$\theta>0,\ \ 0<z<1$$

and

$$f(z;\theta)=\frac{\theta^2}{(1+\theta)^2}\left(2+\theta+\frac{\theta(\ln z)^2}{2}\right)z^{\theta-1}.$$

Motivated by the above-discussed considerations, a new generalization of the UH distribution is proposed using a power transformation of the random variable, and the resultant model is named the Power Unit Haq (PUH) distribution. The main objectives of this study are as follows:

- The prime aim of this was to introduce a power unit Haq distribution that would enhance the flexibility of the unit-Haq distribution. We derived and explored its various statistical properties.
- To derive its main reliability characteristics, including survival, hazard function, mean residual life, and entropy.
- To estimate the distribution parameters using five different estimation methods and perform an extensive simulation study to examine the behavior of these derived estimators.
- The practical relevance and flexibility of the new model are demonstrated using three datasets related to radiation, engineering, and kidney patients.
- Another key objective was to perform a Bayesian analysis.

The rest of the study is planned as follows. Section 2 is based on the derivation of a new distribution and analysis of its density behavior. Theoretical properties are derived and explored in Section 3. In Section 4, the model parameters are estimated using five different estimation methods. An extensive Monte Carlo simulation study is conducted with different parameter choices and sample sizes in the same section. The flexibility of the new distribution is explored using three datasets in Section 5. Section 6 is devoted to Bayesian analysis. The key findings, along with concluding remarks, are presented in Section 7.

2. Derivation of New Distribution

The Power Unit Haq distribution is introduced in this section. The CDF of the PUH distribution is given by

$$F(z;\theta,\alpha)=\left(1-\frac{\theta\alpha\ln z}{(1+\theta)^2}+\frac{\theta^2(\alpha\ln z)^2}{2(1+\theta)^2}\right)z^{\theta\alpha}. \tag{1}$$

Some CDF curves for different choices of parameters are presented in Figure 2.

The PDF corresponding to equation (1) is

$$f(z;\theta,\alpha)=\frac{\alpha\theta^2}{(1+\theta)^2}\left(2+\theta+\frac{\theta\alpha^2(\ln z)^2}{2}\right)z^{\alpha\theta-1}. \tag{2}$$

Now we explore the behavior function of the density function for different choices of parameters as presented in Figure 1. It is observed that the density function of PUH distribution shows decreasing and increasing trends.

3. Statistical Characteristics

In this section, we derived and explored various statistical properties of the PUH distribution. The key properties are survival function, hazard function, moments, reversed hazard function, cumulative hazard

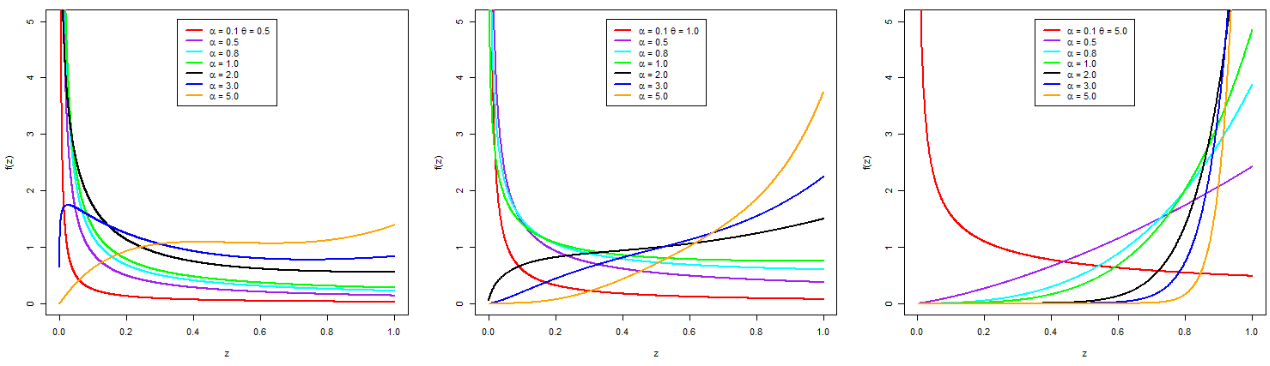

Figure 1. PDF curves of PUH distribution based on different choices of parameters

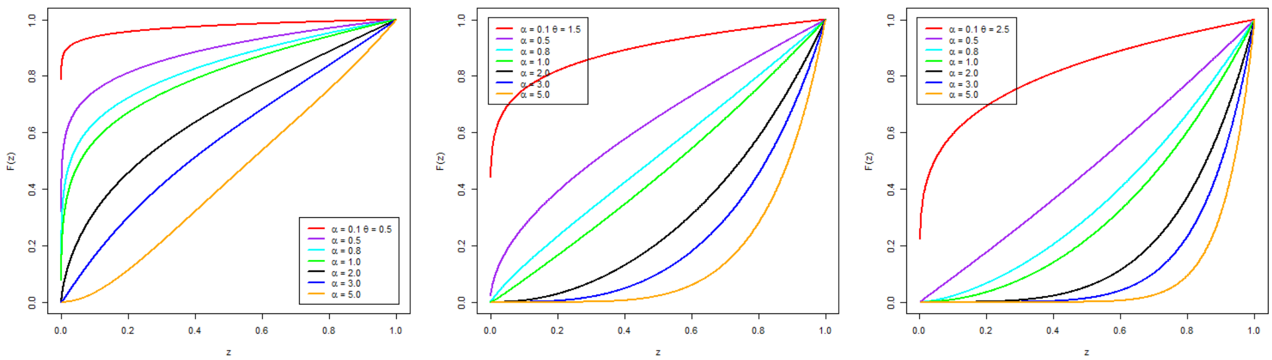

Figure 2. CDF curves of PUH distribution based on different choices of parameters

function, moment generating function, Rényi entropy, and mean residual life function.

3.1. Survival Function

The survival function for PUH distribution is defined as

$$S(z;\theta,\alpha) = 1 - \left(1 - \frac{\theta\alpha\ln z}{(1+\theta)^2} + \frac{\theta^2(\alpha\ln z)^2}{2(1+\theta)^2}\right) z^{\theta\alpha}.$$

Survival function plots based on some parameter values are presented in Figure 3.

3.2. Hazard Function

The hazard function is also known as “the failure of an event”. The hazard function of the PUH distribution is

$$h(z;\theta,\alpha) = \frac{\frac{\alpha\theta^2}{(1+\theta)^2}\left(2+\theta+\frac{\theta\alpha^2(\ln z)^2}{2}\right) z^{\alpha\theta-1}}{1-\left(1-\frac{\theta\alpha\ln z}{(1+\theta)^2}+\frac{\theta^2(\alpha\ln z)^2}{2(1+\theta)^2}\right) z^{\theta\alpha}}.$$

Hazard function plots based on some parameter values are presented in Figure 4.

3.3. Reverse Hazard Function

The reverse hazard function for the PUH distribution is given below.

$$r(z;\theta,\alpha) = \frac{\frac{\alpha\theta^2}{(1+\theta)^2}\left(2+\theta+\frac{\theta\alpha^2(\ln z)^2}{2}\right)}{\left(1-\frac{\theta\alpha\ln z}{(1+\theta)^2}+\frac{\theta^2(\alpha\ln z)^2}{2(1+\theta)^2}\right) z}.$$

3.4. Cumulative Hazard Function

The cumulative hazard function is defined as

$$H(z;\theta,\alpha) = \log\left\{1-\left(1-\frac{\theta\alpha\ln z}{(1+\theta)^2}+\frac{\theta^2(\alpha\ln z)^2}{2(1+\theta)^2}\right) z^{\theta\alpha}\right\}.$$

3.5. Incomplete moments

In this section, we calculate incomplete moments for the PUH distribution. The incomplete moments are important tools for analyzing different metrics, such as Pietra and Gini indices, and Lorenz curves, especially for income distributions. The Incomplete moments may be defined as

$$M_r(p) = \int_0^p z^r\, g(z;\theta,\alpha)dz,$$

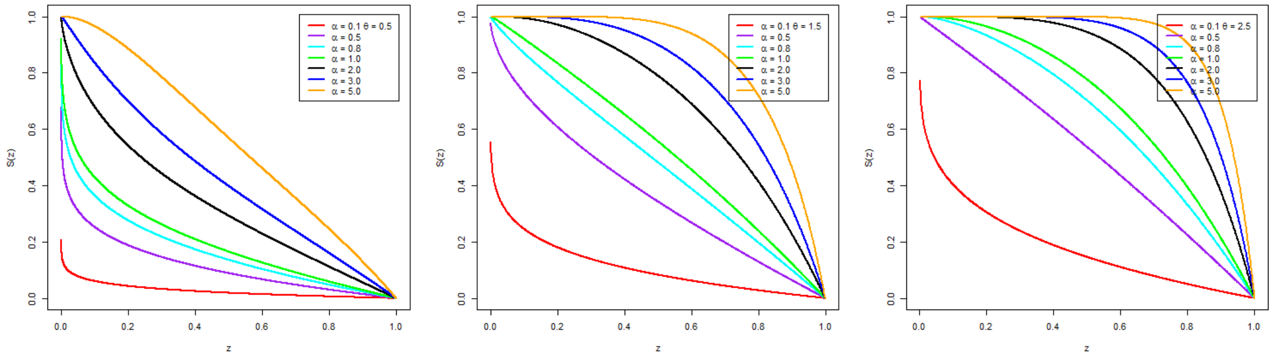

Figure 3. Survival function curves of PUH distribution based on different choices of parameters

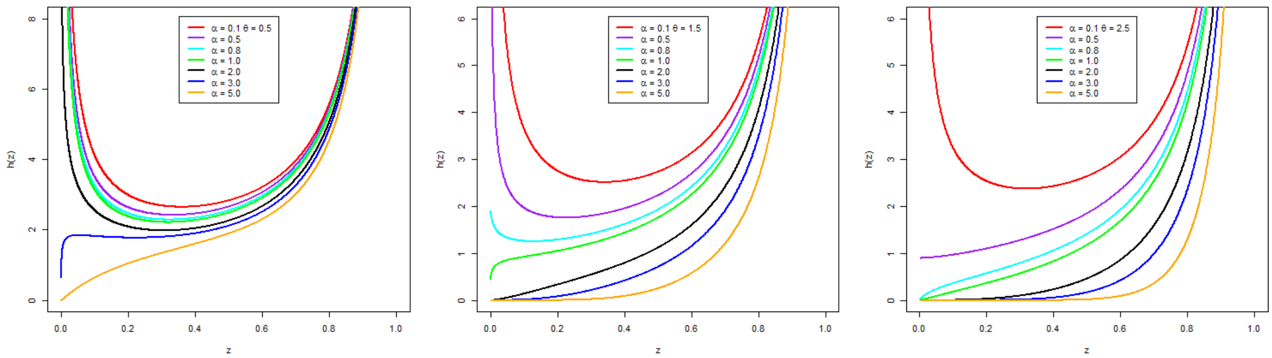

Figure 4. Hazard function curves of PUH distribution based on different choices of parameters

Now using the density function (2). Thus, the r-th incomplete moments are

$$M_r(p) = \frac{\alpha\theta^2}{(1+\theta)^2}\int_0^p z^{r+\alpha\theta-1}\left(2+\theta+\frac{\theta\alpha^2(\ln z)^2}{2}\right)dz,$$

$$= \frac{\alpha\theta^2}{(1+\theta)^2}\left[(2+\theta)\int_0^p z^{r+\alpha\theta-1}\,dz + \frac{\theta\alpha^2}{2}\int_0^p (\ln z)^2 z^{r+\alpha\theta-1}\,dz\right],$$

$$= \frac{\alpha\theta^2}{(1+\theta)^2}\left[\frac{(2+\theta)p^{r+\alpha\theta}}{r+\alpha\theta} + \frac{\theta\alpha^2 p^{r+\alpha\theta}}{2(r+\alpha\theta)^3}\big(2+((r+\alpha\theta)^2(\ln p)^2 - 2(r+\alpha\theta)\ln p)\big)\right].$$

3.6. Moments

The r-th moment of PUH distribution is defined as:

$$E(Z^r) = \int_0^1 z^r\, g(z;\theta,\alpha)dz,$$

Using the density function (2)

$$E(Z^r) = \frac{\alpha\theta^2}{(1+\theta)^2}$$

$$\times\int_0^1 z^{r+\alpha\theta-1}\left(2+\theta+\frac{\theta\alpha^2(\ln z)^2}{2}\right)dz,$$

$$= \frac{\alpha\theta^2(2+\theta)}{(1+\theta)^2}\int_0^1 z^{r+\alpha\theta-1}\,dz \tag{4}$$

$$+\frac{\theta^3\alpha^3}{2(1+\theta)^2}\int_0^1 (\ln z)^2 z^{r+\alpha\theta-1}\,dz,$$

$$E(Z^r) = \frac{\alpha\theta^2(2+\theta)}{(1+\theta)^2(r+\alpha\theta)} + \frac{\alpha^2\theta^3}{(1+\theta)^2(r+\alpha\theta)^3}.$$

The expression of the first four ordinary moments of PUH distribution is obtained from equation (4).

$$E(Z) = \frac{\alpha\theta^2}{(1+\theta)^2}\left[\frac{2+\theta}{1+\alpha\theta} + \frac{\alpha^2\theta}{(1+\alpha\theta)^3}\right],$$

$$E(Z^2) = \frac{\alpha\theta^2}{(1+\theta)^2}\left[\frac{2+\theta}{2+\alpha\theta} + \frac{\alpha^2\theta}{(2+\alpha\theta)^3}\right],$$

$$E(Z^3) = \frac{\alpha\theta^2}{(1+\theta)^2}\left[\frac{2+\theta}{3+\alpha\theta} + \frac{\alpha^2\theta}{(3+\alpha\theta)^3}\right],$$

$$E(Z^4) = \frac{\alpha\theta^2}{(1+\theta)^2}\left[\frac{2+\theta}{4+\alpha\theta} + \frac{\alpha^2\theta}{(4+\alpha\theta)^3}\right].$$

The variance of the PUH distribution is

$$Var(Z) = \frac{\alpha\theta^2}{(1+\theta)^2}\left[\frac{2+\theta}{2+\alpha\theta} + \frac{\alpha^2\theta}{(2+\alpha\theta)^3}\right] - \left\{\frac{\alpha\theta^2}{(1+\theta)^2}\left[\frac{2+\theta}{1+\alpha\theta} + \frac{\alpha^2\theta}{(1+\alpha\theta)^3}\right]\right\}^2.$$

The coefficients of skewness (CS) and kurtosis (CK) can be obtained using the following expressions:

$$CS(Z) = \frac{E(Z^3) - 3E(Z^2)E(Z) + 2\left(E(Z)\right)^3}{\left(E(Z^2) - \left(E(Z)\right)^2\right)^{\frac{3}{2}}},$$

and

$$CK(Z) = \frac{E(Z^4) - 4E(Z^3)E(Z) + 6E(Z^2)\left(E(Z)\right)^2 - 3\left(E(Z)\right)^3}{\left(E(Z^2) - \left(E(Z)\right)^2\right)^2}.$$

Table 1 shows the mean, variance, CV, skewness, and kurtosis for the PUH distribution at different θ and α values. We also plotted these measures via heatmaps and presented them in Figure 5.
As the parameters θ increase, an interesting pattern emerges: the mean and variance of the distribution consistently rise, reflecting a shift towards higher expected values. At the same time, the skewness and kurtosis exhibit contrasting behavior, showing a decreasing trend.

3.7. Rényi Entropy

The Rényi entropy for the random variable Z is defined as

$$H_\delta(Z) = \frac{1}{1-\delta}\log\left[\int_0^1 f(z;\alpha,\theta)^\delta\, dz\right],$$

$$H_\delta(Z) = \tfrac{1}{1-\delta} \times \log\left[\int_0^1 \left\{\frac{\alpha\theta^2}{(1+\theta)^2}\left(2+\theta+\frac{\theta\alpha^2(\ln z)^2}{2}\right)z^{\alpha\theta-1}\right\}^\delta dz\right], \quad (6)$$

Consider

$$I = \int_0^1 \left\{\frac{\alpha\theta^2}{(1+\theta)^2}\left(2+\theta+\frac{\theta\alpha^2(\ln z)^2}{2}\right)z^{\alpha\theta-1}\right\}^\delta dz,$$

$$I = \left(\frac{\alpha\theta^2(2+\theta)}{(1+\theta)^2}\right)^\delta \int_0^1 \left(1+\frac{\theta\alpha^2(\ln z)^2}{2(2+\theta)}\right)^\delta z^{\delta(\alpha\theta-1)}\, dz,$$

Now using the following binomial expansion $(1+x)^\eta = \sum_{k=0}^{\infty}\binom{\eta}{i}x^k$

$$I = \left(\frac{\alpha\theta^2(2+\theta)}{(1+\theta)^2}\right)^\delta \times \sum_{k=0}^{\infty}\binom{\delta}{k}\left(\frac{\theta\alpha^2}{2(2+\theta)}\right)^k \int_0^1 z^{\delta(\alpha\theta-1)}(\ln z)^{2k}\, dz,$$

$$I = \left(\frac{\alpha\theta^2(2+\theta)}{(1+\theta)^2}\right)^\delta \times \sum_{k=0}^{\infty}\binom{\delta}{k}\left(\frac{\theta\alpha^2}{2(2+\theta)}\right)^k \frac{e^{2ki\pi}\Gamma(1+2i)}{\left(1+\delta(\alpha\theta-1)\right)^{2k+1}}.$$

The final expression is

$$H_\delta(Z) = \frac{1}{1-\delta}\log\left[\left(\frac{\alpha\theta^2(2+\theta)}{(1+\theta)^2}\right)^\delta \sum_{k=0}^{\infty}\binom{\delta}{k}\left(\frac{\theta\alpha^2}{2(2+\theta)}\right)^k \frac{e^{2ki\pi}\Gamma(1+2i)}{\left(1+\delta(\alpha\theta-1)\right)^{2k+1}}\right]$$

3.8. Mean Residual Life

The mean residual life (MRL) represents the expected lifetime given that the random variable Z has already survived beyond time t. The MRL for PUH distribution is defined as

$$M(t) = \frac{1}{S(t)}\int_t^1 zf(z)\, dz - t \quad for\ t > 0. \quad (7)$$

We have

$$\int_t^1 zf(z)\, dz = \frac{\alpha\theta^2}{(1+\theta)^2} \times \int_0^z \left(2+\theta+\frac{\theta(\ln z^\alpha)^2}{2}\right)z^{r+\alpha\theta-1}\, dz,$$

$$\int_t^1 zf(z)\,dz = \frac{\alpha\theta^2(2+\theta)}{(1+\theta)^2}\int_0^z z^{r+\alpha\theta-1}\,dz + \frac{\theta^3\alpha^3}{2(1+\theta)^2}\int_0^z (\ln z)^2 z^{r+\alpha\theta-1}\,dz,$$

$$\int_t^1 zf(z)\,dz = \frac{\alpha^3\theta^3\left(2 - t^{r+\alpha\theta}\left(2+(r+\alpha\theta)\ln t\left((r+\alpha\theta)\ln t - 2\right)\right)\right)}{2(1+\theta)^2(r+\alpha\theta)^3} - \frac{\left(t^{r+\alpha\theta}-1\right)\alpha\theta^2(2+\theta)}{(1+\theta)^2(r+\alpha\theta)}.$$

Table 1. Moments and associated measures for the PUH distribution based on some choices of parameters

Parameters		Measures							
α	θ	E(Z)	$E(Z^2)$	$E(Z^3)$	$E(Z^4)$	Var(Z)	CV(Z)	CS(Z)	CK(Z)
0.1	0.1	0.00172	0.00086	0.00058	0.00043	0.00086	17.0642	22.6637	579.056
	0.5	0.02650	0.01356	0.00911	0.00686	0.01285	4.27786	5.53659	36.0074
	1.0	0.06837	0.03574	0.02420	0.01830	0.03107	2.57802	3.19776	13.0699
	2.0	0.14866	0.08089	0.05558	0.04234	0.05879	1.63100	1.82933	5.36659
	3.0	0.21711	0.12242	0.08527	0.06543	0.07528	1.26375	1.25896	3.41062
	5.0	0.32510	0.19467	0.13897	0.10806	0.08897	0.91751	0.67186	2.18285
0.5	0.1	0.00835	0.00425	0.00285	0.00214	0.00418	7.73501	10.1694	117.640
	0.5	0.11467	0.06234	0.04294	0.03277	0.04919	1.93419	2.24649	7.22243
	1.0	0.25926	0.15200	0.10787	0.08368	0.08478	1.12312	0.99249	2.72048
	2.0	0.45833	0.30041	0.22396	0.17867	0.09034	0.65579	0.12728	1.75608
	3.0	0.57600	0.40671	0.31481	0.25695	0.07493	0.47523	-0.28114	1.95945
	5.0	0.70457	0.54489	0.44453	0.37551	0.04847	0.31248	-0.72002	2.71030
1.0	0.1	0.01640	0.00835	0.00563	0.00425	0.00808	5.48314	7.18635	59.5011
	0.5	0.20165	0.11467	0.08066	0.06234	0.07401	1.34909	1.37554	3.70529
	1.0	0.40625	0.25926	0.19141	0.15200	0.09422	0.75558	0.32946	1.79991
	2.0	0.62551	0.45833	0.36267	0.30041	0.06707	0.41401	-0.45646	2.15971
	3.0	0.72949	0.57600	0.47656	0.40671	0.04384	0.28703	-0.82668	2.95150
	5.0	0.82626	0.70457	0.61442	0.54489	0.02186	0.17895	-1.19484	4.21335
1.5	0.1	0.02447	0.01239	0.00835	0.00631	0.01179	4.43712	5.83769	39.8376
	0.5	0.27308	0.16053	0.11467	0.08947	0.08596	1.07362	0.94763	2.62021
	1.0	0.50400	0.34111	0.25926	0.20962	0.08709	0.58554	-0.01742	1.74805
	2.0	0.71354	0.55733	0.45833	0.38970	0.04819	0.30766	-0.76760	2.76996
	3.0	0.80128	0.66978	0.57600	0.50560	0.02773	0.20783	-1.10839	3.83727
	5.0	0.87693	0.78121	0.70457	0.64176	0.01221	0.12603	-1.42157	5.23072
2.0	0.1	0.03275	0.01640	0.01105	0.00835	0.01533	3.77988	5.01127	29.8376
	0.5	0.33333	0.20165	0.14583	0.11467	0.09053	0.90267	0.67036	2.14918
	1.0	0.57407	0.40625	0.31600	0.25926	0.07669	0.48239	-0.24805	1.88792
	2.0	0.76800	0.62551	0.52867	0.45833	0.03569	0.24599	-0.97098	3.33391
	3.0	0.84293	0.72949	0.64352	0.57600	0.01896	0.16336	-1.28620	4.54389
	5.0	0.90471	0.82626	0.76051	0.70457	0.00776	0.09739	-1.55663	5.94055
2.5	0.1	0.04132	0.02042	0.01373	0.01038	0.01871	3.31017	4.42968	23.7355
	0.5	0.38485	0.23896	0.17471	0.13827	0.09085	0.78321	0.46788	1.92349
	1.0	0.62682	0.45953	0.36439	0.30269	0.06663	0.41180	-0.41781	2.08507
	2.0	0.80504	0.67541	0.58268	0.51288	0.02732	0.20532	-1.11673	3.82470
	3.0	0.87014	0.77089	0.69242	0.62875	0.01374	0.13471	-1.40986	5.10804
	5.0	0.92226	0.85592	0.79862	0.74861	0.00536	0.07939	-1.64666	6.45995
3.0	0.1	0.05021	0.02447	0.01640	0.01239	0.02195	2.95091	3.98681	19.6061
	0.5	0.42933	0.27308	0.20165	0.16053	0.08875	0.69390	0.30989	1.81853
	1.0	0.66797	0.50400	0.40625	0.34111	0.05782	0.35998	-0.55006	2.29681
	2.0	0.83188	0.71354	0.62551	0.55733	0.02152	0.17636	-1.22711	4.24734
	3.0	0.88931	0.80128	0.72949	0.66978	0.01040	0.11468	-1.50120	5.56524
	5.0	0.93435	0.87693	0.82626	0.78121	0.00392	0.06702	-1.71108	6.85535

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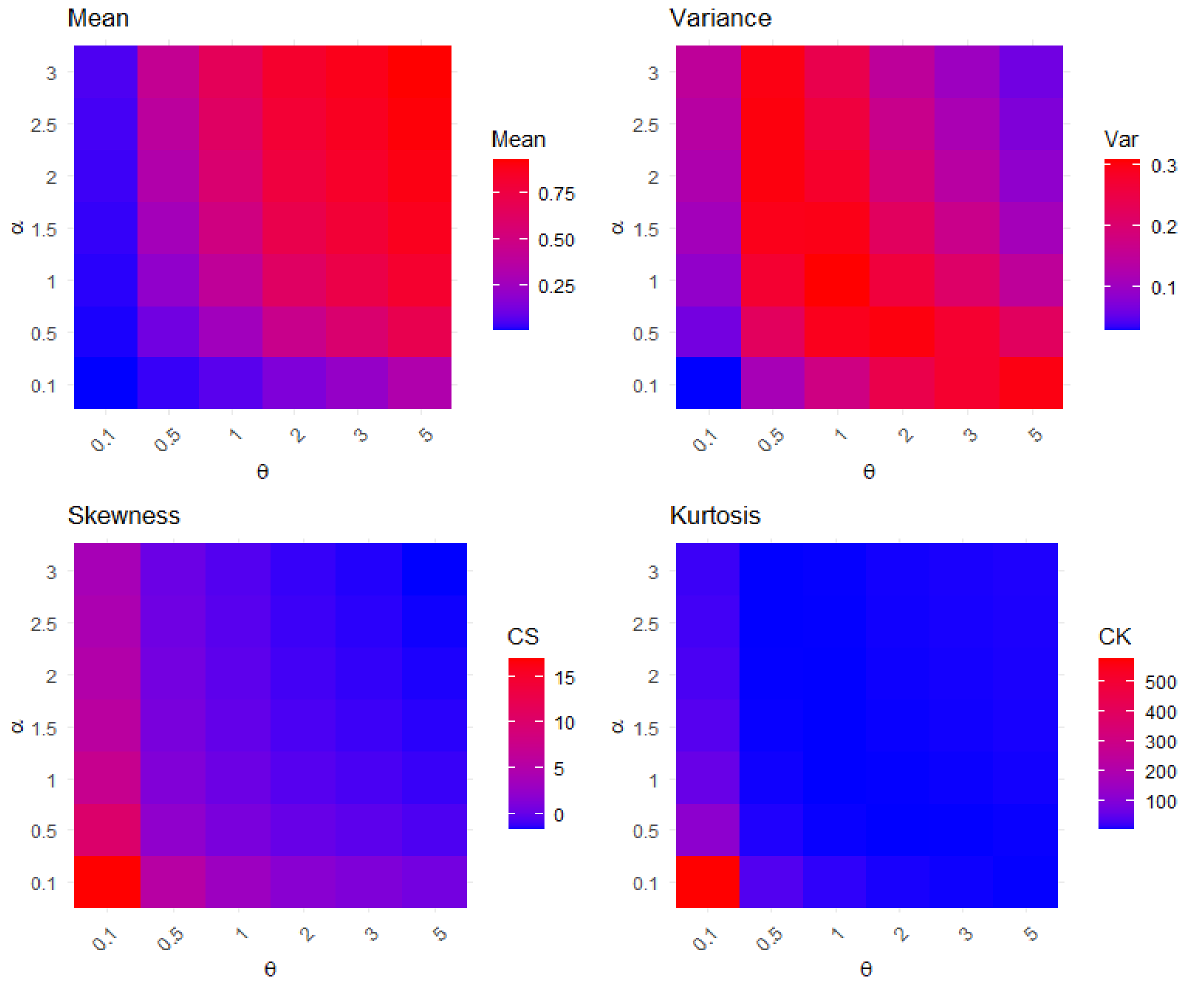

Figure 5. Heatmap plots of mean, Var, skewness, and Kurtosis

Put in equation (7)

$$M(t)=\frac{1}{S(t)}\left\{\frac{\alpha^3\theta^3\left(2-t^{r+\alpha\theta}\left(2+(r+\alpha\theta)\ln t\left((r+\alpha\theta)\ln t-2\right)\right)\right)}{2(1+\theta)^2(r+\alpha\theta)^3}-\frac{(t^{r+\alpha\theta}-1)\alpha\theta^2(2+\theta)}{(1+\theta)^2(r+\alpha\theta)}\right\}-t.$$

4. Estimation of Parameters

In this section, we estimate the model parameters of the PUH distribution using five different approaches. The considered estimation approaches are maximum likelihood (ML), Anderson Darling (AD), Cramer von Mises (CVM), Ordinary Least Squares (OLS), and Weighted Least Squares (WLS). An extensive simulation study is utilized to illustrate behavior and identify the efficient estimation method. The A statistic is defined as

$$\mathrm{A}=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left[F\left(z_{(i)};\widehat{\Theta}\right)\right]+\log\left[1-F\left(z_{(n+1-i)};\widehat{\Theta}\right)\right]\right],$$

4.1. Maximum Likelihood Estimation

Let $z_1, z_2, \ldots, z_n$ be a random sample of size n taken from the PUH distribution. The likelihood function is given by

$$L(\theta,\alpha)=\prod_{i=1}^{n}\frac{\alpha\theta^2}{(1+\theta)^2}\left(2+\theta+\frac{\theta\left(\ln z_i^{\alpha}\right)^2}{2}\right)z_i^{\alpha\theta-1}.$$

Thus, the log-likelihood function for the sample is

$$l(\theta,\alpha)=n\log(\alpha)+2n\log(\theta)-2n\log(1+\theta)+\sum_{i=1}^{n}\log\left(2+\theta+\frac{\theta\alpha^2(\ln z_i)^2}{2}\right)$$

$$+(\alpha\theta - 1)\sum_{i=1}^{n}\log(z_i).$$

To find the MLE for θ and α, we differentiate the $l(\theta,\alpha)$ with respect to θ and α.

$$\frac{\partial l(\theta,\alpha)}{\partial\theta} = \frac{2n}{\theta} - \frac{2n}{1+\theta} + \sum_{i=1}^{n}\frac{2+\alpha^2(\ln z_i)^2}{4+2\theta+\theta\alpha^2(\ln z_i)^2}$$

$$+\alpha\sum_{i=1}^{n}\log(z_i),$$

and

$$\frac{\partial l(\theta,\alpha)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{2\alpha(\ln z_i)^2}{4+2\theta+\theta\alpha^2(\ln z_i)^2}$$

$$+\theta\sum_{i=1}^{n}\log(z_i).$$

The parameter estimates based on the ML approach are attained by solving the above equations simultaneously. As the exact solution of the above equations is not possible, we will utilize an iterative procedure using ***the R language***.

4.2. Anderson Darling Estimation

Consider the random sample $(z_1, z_2, \dots, z_n)$ is drawn from the PUH distribution and the $Z_{(1)}, Z_{(2)}, \dots, Z_{(n)}$ is the ordered sample. The Aderson Darling estimates (ADEs) for the parameters θ and α are acquired by minimizing the following distance.

$$ADE(\theta,\alpha) = -n - \sum_{i=1}^{n}\frac{2i-1}{n} \times \left[\log\left[F\left(z_{(i)};\theta,\alpha\right)\right] + \log\left[1-F\left(z_{(n+1-i)};\theta,\alpha\right)\right]\right]$$

4.3. Cramer-von Mises Estimation

The Cramer-von Mises estimates (CVME) for the parameters θ and α can be gained by minimizing the following expression for the parameters.

$$CVME(\theta,\alpha) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(z_{(i)}\right) - \frac{2i-1}{2n}\right)^2,$$

and

$$CVM(\theta,\alpha) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\left(1 - \frac{\theta\alpha\ln z_{(i)}}{(1+\theta)^2} + \frac{\theta^2\left(\alpha\ln z_{(i)}\right)^2}{2(1+\theta)^2}\right)z_{(i)}^{\theta\alpha} - \frac{2i-1}{2n}\right]^2.$$

4.4. Ordinary Least Squares Estimation

Let $Z_{(1)}, Z_{(2)}, \dots, Z_{(n)}$ is the ordered sample of size n drawn from the PUH distribution. The Ordinary Least-Squares estimates (OLSE) for the PUH distribution parameters are attained by minimizing the following difference between the empirical and theoretical cumulative distribution functions.

$$OLSE(\theta,\alpha) = \sum_{i=1}^{n}\left[\left(1 - \frac{\theta\alpha\ln z_{(i)}}{(1+\theta)^2} + \frac{\theta^2\left(\alpha\ln z_{(i)}\right)^2}{2(1+\theta)^2}\right)z_{(i)}^{\theta\alpha} - \frac{i}{n+1}\right]^2.$$

4.5. Weighted Ordinary Least Squares Estimation

A random sample is taken $(z_1, z_2, \dots, z_n)$ from the proposed distribution. The Weighted Least Squares estimates (WLSE) for the distribution's parameters can be obtained by minimizing the distance.

$$WLSE(\theta,\alpha) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left(1 - \frac{\theta\alpha\ln z_{(i)}}{(1+\theta)^2} + \frac{\theta^2\left(\alpha\ln z_{(i)}\right)^2}{2(1+\theta)^2}\right)z_{(i)}^{\theta\alpha} - \frac{i}{n+1}\right]^2.$$

4.6. Monte Carlo Simulation

In this subsection, a Monte Carlo simulation study is presented for a comprehensive analysis of the performance of five estimation methods for the PUH distribution. The random samples are generated from PUH distribution for various choices of parameter values with sample sizes (n = 20, 50, 100, 200, and 300). The simulation is repeated for N =10,000 times for all sample sizes with a combination of parameter choices. The results from these simulations will be compared based on the following criteria:

Absolute Bias (AB): The AB measures the average absolute distance of the estimator from the true parameter values.

$$AB(\hat{\varphi}) = \frac{1}{N}\sum_{i=1}^{N}|\hat{\varphi}_i - \varphi|.$$

Mean Squared Error (MSE): The MSE measures the average squared deviation of the estimator from the true parameter value.

$$MSE(\hat{\varphi}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\varphi}_i - \varphi)^2.$$

Mean Relative Error (MRE): The MRE deals with the average relative distance of the estimator from the true parameter.

$$MRE(\hat{\varphi}) = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{\hat{\varphi}_i - \varphi}{\varphi}\right|.$$

Tables 2-10 exhibit simulation results for AB, MRE, and MSE using various sample sizes and θ and α values. Further, we also plot the heatmaps based on MSE values for each set of parameters presented in Figures 6 and 7.

It is interesting to note that:

- The ADE and MLE approaches are the most effective estimators based on MSE and AB criteria. These methods deliver the most accurate parameter estimations with the fewest errors.
- The CVME approach performs moderately, but with somewhat higher AB and MSE, making it less trustworthy than the ADE and MLE.
- The OLS and WLS estimation methods have higher AB and MSE for small values of sample size but improve with an increase in sample size.
- For high sample sizes such as n>100, all the estimation approaches improve, but the MLE and ADE remain the best methods.

5. Application

The current section evaluates PUH distribution's applicability and flexibility by analyzing its performance on three real-life datasets. These considered datasets are related to kidney dialysis patients, the duration between secondary reactor pump failures, and radiation. We aim to determine if the PUH distribution can appropriately analyze these real-world datasets. We also fit and evaluate different renowned unit interval distributions such as Beta distribution, Kumaraswamy distribution, Unit-Haq (UH) distribution, Unit-Xgamma (UXg) distribution, and generalized exponential uniform distribution (GEUD) [20]. These probability models are commonly utilized in the literature to describe unit interval datasets, each with varying degrees of skewness, tail behavior, and form flexibility. Their probability density functions are summarized below.

- **Beta distribution:** A random variable Z follows a Beta distribution with parameters $\alpha > 0$ and $\beta > 0$ if its PDF is

$$f(z;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} z^{\alpha-1}(1-z)^{\beta-1},$$
$$0 < z < 1.$$

- **Kumaraswamy distribution:** Kumaraswamy distribution with parameters $\alpha > 0$ and $\beta > 0$ PDF

$$f(z;\alpha,\beta) = \alpha\beta z^{\alpha-1}(1-z^{\alpha})^{\beta-1},$$
$$0 < z < 1.$$

- **GEU distribution:** The density function of the GEU distribution is

$$f(z;\alpha,\beta) = \frac{\alpha\beta z^{\beta-1}}{(1-z^{\beta})^2}\exp\left\{-\alpha\left(\frac{z^{\beta}}{1-z^{\beta}}\right)\right\},$$
$$0 < z < 1.$$

- **UH distribution:** The density function of the unit Haq distribution is

$$f(z;\alpha) = \frac{\alpha^2}{(1+\alpha)^2}\left(2+\alpha+\frac{\alpha(\ln(z))^2}{2}\right)z^{\alpha-1},$$
$$0 < z < 1.$$

In the context of the comparative analysis, we analyze the ability of the PUH distribution to capture the data pattern and whether it serves better based on statistical values such as the log-likelihood value, AIC, BIC, Anderson Darling Statistic (A), Kolmogorov Smirnov Statistic (KS), and Cramer von Mises Statistic (W). The AIC and BIC are defined as

$$\text{AIC} = -2l(\hat{\Theta}) + 2k,$$

$$BIC = -2l(\hat{\Theta}) + k\log(n),$$

where $l(\hat{\Theta})$ is the maximized log-likelihood, k is the number of parameters, and n is the sample size.

The KS statistic is given by

$$\text{KS} = \sup_{z}|F_n(z) - F(z;\hat{\Theta})|,$$

where $F_n(z)$ denotes the empirical distribution function and $F(z;\hat{\Theta})$ is the fitted CDF.

Table 2. PUH distribution parameter estimates for the parameters $\theta = 0.5$ and $\alpha = 0.5$

n	Para.	Est.	MLE	ADE	CVME	OLSE	WLSE
20	θ	AE	0.48685	0.50962	0.49978	0.51611	0.51172
		AB	0.01315	0.00962	0.00022	0.01611	0.01172
		MRE	0.02630	0.01925	0.00044	0.03223	0.02344
		MSE	0.01133	0.01382	0.01449	0.01424	0.01365
	α	AE	0.52407	0.50585	0.52057	0.49728	0.50122
		AB	0.02407	0.00585	0.02057	0.00272	0.00122
		MRE	0.04813	0.01171	0.04114	0.00544	0.00244
		MSE	0.00500	0.00505	0.00827	0.00750	0.00622
50	θ	AE	0.49302	0.50291	0.49783	0.50424	0.50639
		AB	0.00698	0.00291	0.00217	0.00424	0.00639
		MRE	0.01395	0.00582	0.00434	0.00848	0.01277
		MSE	0.00570	0.00643	0.00679	0.00692	0.00646
	α	AE	0.51162	0.50408	0.51075	0.49979	0.49960
		AB	0.01162	0.00408	0.01075	0.00021	0.00040
		MRE	0.02323	0.00816	0.02150	0.00042	0.00079
		MSE	0.00225	0.00258	0.00343	0.00340	0.00272
75	θ	AE	0.49336	0.50404	0.49890	0.50580	0.50554
		AB	0.00664	0.00404	0.00110	0.00580	0.00554
		MRE	0.01328	0.00809	0.00220	0.01160	0.01108
		MSE	0.00358	0.00410	0.00468	0.00468	0.00444
	α	AE	0.50685	0.50086	0.50753	0.49952	0.50033
		AB	0.00685	0.00086	0.00753	0.00048	0.00033
		MRE	0.01370	0.00173	0.01506	0.00097	0.00066
		MSE	0.00126	0.00156	0.00226	0.00218	0.00178
100	θ	AE	0.49543	0.49991	0.49843	0.50621	0.50475
		AB	0.00457	0.00009	0.00157	0.00621	0.00475
		MRE	0.00915	0.00018	0.00314	0.01243	0.00950
		MSE	0.00246	0.00288	0.00336	0.00356	0.00359
	α	AE	0.50563	0.50255	0.50657	0.49780	0.50058
		AB	0.00563	0.00255	0.00657	0.00220	0.00058
		MRE	0.01125	0.00510	0.01314	0.00439	0.00116
		MSE	0.00091	0.00111	0.00160	0.00165	0.00133
200	θ	AE	0.49787	0.50006	0.49791	0.50276	0.49921
		AB	0.00213	0.00006	0.00209	0.00276	0.00079
		MRE	0.00427	0.00011	0.00418	0.00553	0.00157
		MSE	0.00109	0.00111	0.00174	0.00179	0.00152
	α	AE	0.50315	0.50071	0.50392	0.49884	0.50147
		AB	0.00315	0.00071	0.00392	0.00116	0.00147
		MRE	0.00631	0.00142	0.00785	0.00233	0.00294
		MSE	0.00037	0.00042	0.00085	0.00075	0.00064
300	θ	AE	0.49697	0.49895	0.49974	0.50130	0.50028
		AB	0.00303	0.00105	0.00026	0.00130	0.00028
		MRE	0.00607	0.00211	0.00052	0.00259	0.00055
		MSE	0.00064	0.00065	0.00115	0.00129	0.00104
	α	AE	0.50291	0.50135	0.50165	0.49930	0.50147
		AB	0.00291	0.00135	0.00165	0.00070	0.00147
		MRE	0.00583	0.00269	0.00329	0.00141	0.00293
		MSE	0.00024	0.00026	0.00051	0.00060	0.00042

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Table 3. PUH distribution parameter estimates for the parameters $\theta = 0.5$ and $\alpha = 1.0$

n	Para.	Est.	MLE	ADE	CVME	OLSE	WLSE
20	θ	AE	0.49122	0.51055	0.49716	0.51503	0.50969
		AB	0.00878	0.01055	0.00284	0.01503	0.00969
		MRE	0.01756	0.02109	0.00568	0.03006	0.01938
		MSE	0.01345	0.01292	0.01425	0.01457	0.01263
	α	AE	1.04478	1.00736	1.04763	0.99564	0.99886
		AB	0.04478	0.00736	0.04763	0.00436	0.00114
		MRE	0.04478	0.00736	0.04763	0.00436	0.00114
		MSE	0.02180	0.02050	0.03270	0.03005	0.02363
50	θ	AE	0.49649	0.50809	0.49892	0.50774	0.50240
		AB	0.00351	0.00809	0.00108	0.00774	0.00240
		MRE	0.00701	0.01618	0.00216	0.01548	0.00480
		MSE	0.00644	0.00676	0.00748	0.00733	0.00712
	α	AE	1.02660	1.00293	1.01526	0.99402	1.00642
		AB	0.02660	0.00293	0.01526	0.00598	0.00642
		MRE	0.02660	0.00293	0.01526	0.00598	0.00642
		MSE	0.00989	0.00991	0.01430	0.01418	0.01137
75	θ	AE	0.49832	0.50152	0.49957	0.50426	0.49948
		AB	0.00168	0.00152	0.00043	0.00426	0.00052
		MRE	0.00337	0.00304	0.00086	0.00852	0.00103
		MSE	0.00453	0.00418	0.00439	0.00475	0.00424
	α	AE	1.01536	1.00353	1.01520	0.99833	1.00695
		AB	0.01536	0.00353	0.01520	0.00167	0.00695
		MRE	0.01536	0.00353	0.01520	0.00167	0.00695
		MSE	0.00673	0.00673	0.00851	0.00884	0.00789
100	θ	AE	0.49785	0.50272	0.49822	0.50798	0.49974
		AB	0.00215	0.00272	0.00178	0.00798	0.00026
		MRE	0.00430	0.00544	0.00356	0.01597	0.00053
		MSE	0.00307	0.00326	0.00369	0.00374	0.00342
	α	AE	1.00832	1.00284	1.01430	0.99248	1.00370
		AB	0.00832	0.00284	0.01430	0.00752	0.00370
		MRE	0.00832	0.00284	0.01430	0.00752	0.00370
		MSE	0.00421	0.00519	0.00696	0.00681	0.00523
200	θ	AE	0.49711	0.49900	0.49831	0.50435	0.49935
		AB	0.00289	0.00100	0.00169	0.00435	0.00065
		MRE	0.00577	0.00200	0.00338	0.00870	0.00129
		MSE	0.00150	0.00175	0.00176	0.00200	0.00157
	α	AE	1.00751	1.00443	1.00601	0.99693	1.00236
		AB	0.00751	0.00443	0.00601	0.00307	0.00236
		MRE	0.00751	0.00443	0.00601	0.00307	0.00236
		MSE	0.00210	0.00256	0.00328	0.00334	0.00251
300	θ	AE	0.50012	0.50053	0.49925	0.50069	0.50279
		AB	0.00012	0.00053	0.00075	0.00069	0.00279
		MRE	0.00023	0.00105	0.00151	0.00139	0.00558
		MSE	0.00106	0.00103	0.00115	0.00119	0.00105
	α	AE	1.00178	0.99999	1.00296	1.00022	1.00038
		AB	0.00178	0.00001	0.00296	0.00022	0.00038
		MRE	0.00178	0.00001	0.00296	0.00022	0.00038
		MSE	0.00139	0.00152	0.00208	0.00196	0.00160

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Table 4. PUH distribution parameter estimates for the parameters $\theta = 0.5$ and $\alpha = 1.5$

n	Para.	Est.	MLE	ADE	CVME	OLSE	WLSE
20	θ	AE	0.48687	0.50809	0.49816	0.51299	0.51630
		AB	0.01313	0.00809	0.00184	0.01299	0.01630
		MRE	0.02625	0.01619	0.00368	0.02598	0.03259
		MSE	0.01213	0.01250	0.01419	0.01453	0.01382
	α	AE	1.57377	1.50891	1.55695	1.49713	1.49403
		AB	0.07377	0.00891	0.05695	0.00287	0.00597
		MRE	0.04918	0.00594	0.03797	0.00191	0.00398
		MSE	0.04913	0.04666	0.07221	0.06194	0.05821
50	θ	AE	0.49449	0.50127	0.50085	0.51124	0.50750
		AB	0.00551	0.00127	0.00085	0.01124	0.00750
		MRE	0.01103	0.00254	0.00171	0.02248	0.01499
		MSE	0.00573	0.00690	0.00739	0.00693	0.00707
	α	AE	1.53185	1.51269	1.52179	1.48185	1.49910
		AB	0.03185	0.01269	0.02179	0.01815	0.00090
		MRE	0.02124	0.00846	0.01452	0.01210	0.00060
		MSE	0.01911	0.02416	0.02971	0.02752	0.02382
75	θ	AE	0.49586	0.50100	0.50080	0.50629	0.49959
		AB	0.00414	0.00100	0.00080	0.00629	0.00041
		MRE	0.00829	0.00201	0.00159	0.01258	0.00082
		MSE	0.00414	0.00446	0.00450	0.00465	0.00402
	α	AE	1.52698	1.50931	1.51734	1.49313	1.50304
		AB	0.02698	0.00931	0.01734	0.00687	0.00304
		MRE	0.01799	0.00620	0.01156	0.00458	0.00202
		MSE	0.01345	0.01499	0.01906	0.02011	0.01392
100	θ	AE	0.50016	0.50264	0.49887	0.50468	0.49914
		AB	0.00016	0.00264	0.00113	0.00468	0.00086
		MRE	0.00033	0.00529	0.00226	0.00936	0.00171
		MSE	0.00336	0.00352	0.00390	0.00373	0.00321
	α	AE	1.51527	1.50471	1.51456	1.49509	1.50612
		AB	0.01527	0.00471	0.01456	0.00491	0.00612
		MRE	0.01018	0.00314	0.00971	0.00328	0.00408
		MSE	0.01008	0.01172	0.01562	0.01452	0.01117
200	θ	AE	0.50038	0.50182	0.50136	0.50366	0.50050
		AB	0.00038	0.00182	0.00136	0.00366	0.00050
		MRE	0.00076	0.00364	0.00273	0.00731	0.00099
		MSE	0.00144	0.00177	0.00165	0.00176	0.00163
	α	AE	1.50698	1.49869	1.50666	1.49325	1.50186
		AB	0.00698	0.00131	0.00666	0.00675	0.00186
		MRE	0.00465	0.00088	0.00444	0.00450	0.00124
		MSE	0.00442	0.00585	0.00715	0.00726	0.00547
300	θ	AE	0.50180	0.50186	0.49794	0.49972	0.49977
		AB	0.00180	0.00186	0.00206	0.00028	0.00023
		MRE	0.00359	0.00371	0.00411	0.00056	0.00047
		MSE	0.00102	0.00104	0.00109	0.00126	0.00106
	α	AE	1.50273	1.49930	1.50702	1.50113	1.50385
		AB	0.00273	0.00070	0.00702	0.00113	0.00385
		MRE	0.00182	0.00047	0.00468	0.00075	0.00257
		MSE	0.00318	0.00354	0.00452	0.00502	0.00390

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Table 5. PUH distribution parameter estimates for the parameters $\theta = 1.0$ and $\alpha = 0.5$

n	Para.	Est.	MLE	ADE	CVME	OLSE	WLSE
20	θ	AE	0.99916	1.00578	1.01397	1.01363	1.01708
		AB	0.00084	0.00578	0.01397	0.01363	0.01708
		MRE	0.00084	0.00578	0.01397	0.01363	0.01708
		MSE	0.03274	0.03128	0.04053	0.03448	0.03550
	α	AE	0.52895	0.51117	0.53048	0.49601	0.50851
		AB	0.02895	0.01117	0.03048	0.00399	0.00851
		MRE	0.05790	0.02235	0.06095	0.00799	0.01702
		MSE	0.00676	0.00635	0.01418	0.00989	0.00956
50	θ	AE	0.99263	1.00023	1.00602	1.01277	1.00417
		AB	0.00737	0.00023	0.00602	0.01277	0.00417
		MRE	0.00737	0.00023	0.00602	0.01277	0.00417
		MSE	0.01400	0.01342	0.01900	0.01794	0.01661
	α	AE	0.51693	0.50467	0.51351	0.49849	0.50148
		AB	0.01693	0.00467	0.01351	0.00151	0.00148
		MRE	0.03387	0.00935	0.02703	0.00302	0.00296
		MSE	0.00279	0.00279	0.00521	0.00415	0.00356
75	θ	AE	0.98971	0.99788	1.00280	1.00401	1.00341
		AB	0.01029	0.00212	0.00280	0.00401	0.00341
		MRE	0.01029	0.00212	0.00280	0.00401	0.00341
		MSE	0.00781	0.00876	0.01277	0.01130	0.01139
	α	AE	0.51165	0.50540	0.50970	0.49859	0.50218
		AB	0.01165	0.00540	0.00970	0.00141	0.00218
		MRE	0.02329	0.01080	0.01939	0.00282	0.00437
		MSE	0.00166	0.00174	0.00321	0.00296	0.00235
100	θ	AE	0.99282	0.99846	1.00288	1.00446	1.00148
		AB	0.00718	0.00154	0.00288	0.00446	0.00148
		MRE	0.00718	0.00154	0.00288	0.00446	0.00148
		MSE	0.00588	0.00539	0.00883	0.00803	0.00856
	α	AE	0.50984	0.50551	0.50373	0.50038	0.50266
		AB	0.00984	0.00551	0.00373	0.00038	0.00266
		MRE	0.01967	0.01102	0.00746	0.00075	0.00533
		MSE	0.00112	0.00116	0.00202	0.00234	0.00191
200	θ	AE	0.99645	0.99588	1.00267	1.00378	1.00003
		AB	0.00355	0.00412	0.00267	0.00378	0.00003
		MRE	0.00355	0.00412	0.00267	0.00378	0.00003
		MSE	0.00195	0.00248	0.00412	0.00411	0.00401
	α	AE	0.50413	0.50346	0.50337	0.49924	0.50262
		AB	0.00413	0.00346	0.00337	0.00076	0.00262
		MRE	0.00825	0.00692	0.00674	0.00153	0.00525
		MSE	0.00037	0.00045	0.00106	0.00102	0.00089
300	θ	AE	0.99642	0.99816	1.00251	1.00364	1.00134
		AB	0.00358	0.00184	0.00251	0.00364	0.00134
		MRE	0.00358	0.00184	0.00251	0.00364	0.00134
		MSE	0.00107	0.00104	0.00265	0.00283	0.00256
	α	AE	0.50301	0.50183	0.50134	0.49871	0.50011
		AB	0.00301	0.00183	0.00134	0.00129	0.00011
		MRE	0.00601	0.00366	0.00267	0.00258	0.00021
		MSE	0.00018	0.00022	0.00069	0.00071	0.00057

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Table 6. PUH distribution parameter estimates for the parameters $\theta = 1.0$ and $\alpha = 1.0$

n	Para.	Est.	MLE	ADE	CVME	OLSE	WLSE
20	θ	AE	1.00079	1.01297	1.01041	1.01693	1.01766
		AB	0.00079	0.01297	0.01041	0.01693	0.01766
		MRE	0.00079	0.01297	0.01041	0.01693	0.01766
		MSE	0.03930	0.03283	0.03611	0.03662	0.03439
	α	AE	1.05655	1.02019	1.05498	0.99615	1.00605
		AB	0.05655	0.02019	0.05498	0.00385	0.00605
		MRE	0.05655	0.02019	0.05498	0.00385	0.00605
		MSE	0.03043	0.03131	0.05125	0.03679	0.03731
50	θ	AE	1.00377	1.00191	1.00731	1.01066	1.00186
		AB	0.00377	0.00191	0.00731	0.01066	0.00186
		MRE	0.00377	0.00191	0.00731	0.01066	0.00186
		MSE	0.01780	0.01719	0.01946	0.01770	0.01661
	α	AE	1.02416	1.00745	1.03099	0.99499	1.00958
		AB	0.02416	0.00745	0.03099	0.00501	0.00958
		MRE	0.02416	0.00745	0.03099	0.00501	0.00958
		MSE	0.01310	0.01393	0.01975	0.01822	0.01369
75	θ	AE	0.99997	1.00861	1.00100	1.00540	1.00016
		AB	0.00003	0.00861	0.00100	0.00540	0.00016
		MRE	0.00003	0.00861	0.00100	0.00540	0.00016
		MSE	0.01111	0.01109	0.01167	0.01170	0.01104
	α	AE	1.02158	1.00487	1.01972	0.99478	1.00998
		AB	0.02158	0.00487	0.01972	0.00522	0.00998
		MRE	0.02158	0.00487	0.01972	0.00522	0.00998
		MSE	0.00849	0.00815	0.01284	0.01202	0.01025
100	θ	AE	1.00362	1.00753	1.00487	1.00281	1.00139
		AB	0.00362	0.00753	0.00487	0.00281	0.00139
		MRE	0.00362	0.00753	0.00487	0.00281	0.00139
		MSE	0.00830	0.00846	0.00897	0.00800	0.00859
	α	AE	1.01131	1.00372	1.01531	0.99593	1.00978
		AB	0.01131	0.00372	0.01531	0.00407	0.00978
		MRE	0.01131	0.00372	0.01531	0.00407	0.00978
		MSE	0.00596	0.00674	0.00897	0.00825	0.00757
200	θ	AE	1.00193	0.99933	1.00297	1.00424	1.00171
		AB	0.00193	0.00067	0.00297	0.00424	0.00171
		MRE	0.00193	0.00067	0.00297	0.00424	0.00171
		MSE	0.00419	0.00394	0.00445	0.00409	0.00441
	α	AE	1.00678	1.00258	1.00578	0.99863	1.00276
		AB	0.00678	0.00258	0.00578	0.00137	0.00276
		MRE	0.00678	0.00258	0.00578	0.00137	0.00276
		MSE	0.00265	0.00308	0.00435	0.00411	0.00354
300	θ	AE	0.99806	1.00390	1.00264	1.00064	1.00117
		AB	0.00194	0.00390	0.00264	0.00064	0.00117
		MRE	0.00194	0.00390	0.00264	0.00064	0.00117
		MSE	0.00263	0.00257	0.00291	0.00288	0.00259
	α	AE	1.00492	0.99879	0.99893	0.99744	1.00218
		AB	0.00492	0.00121	0.00107	0.00256	0.00218
		MRE	0.00492	0.00121	0.00107	0.00256	0.00218
		MSE	0.00166	0.00223	0.00276	0.00256	0.00204

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Table 7. PUH distribution parameter estimates for the parameters $\theta = 1.0$ and $\alpha = 1.5$

n	Para.	Est.	MLE	ADE	CVME	OLSE	WLSE
20	θ	AE	1.00728	1.00435	1.01229	1.00782	1.01938
		AB	0.00728	0.00435	0.01229	0.00782	0.01938
		MRE	0.00728	0.00435	0.01229	0.00782	0.01938
		MSE	0.03513	0.03108	0.03928	0.03383	0.03670
	α	AE	1.59327	1.51473	1.58543	1.49782	1.50517
		AB	0.09327	0.01473	0.08543	0.00218	0.00517
		MRE	0.06218	0.00982	0.05695	0.00146	0.00345
		MSE	0.06895	0.06185	0.10644	0.08958	0.08180
50	θ	AE	1.00024	1.00469	1.00666	1.00353	1.00955
		AB	0.00024	0.00469	0.00666	0.00353	0.00955
		MRE	0.00024	0.00469	0.00666	0.00353	0.00955
		MSE	0.01671	0.01700	0.01822	0.01724	0.01671
	α	AE	1.54165	1.51065	1.54566	1.49851	1.50912
		AB	0.04165	0.01065	0.04566	0.00149	0.00912
		MRE	0.02776	0.00710	0.03044	0.00099	0.00608
		MSE	0.02971	0.02822	0.04683	0.03968	0.03617
75	θ	AE	1.00082	1.00602	1.00164	1.00481	1.00904
		AB	0.00082	0.00602	0.00164	0.00481	0.00904
		MRE	0.00082	0.00602	0.00164	0.00481	0.00904
		MSE	0.01090	0.01063	0.01192	0.01138	0.01036
	α	AE	1.52939	1.50609	1.53258	1.50117	1.50632
		AB	0.02939	0.00609	0.03258	0.00117	0.00632
		MRE	0.01959	0.00406	0.02172	0.00078	0.00421
		MSE	0.01835	0.01962	0.02963	0.02330	0.02075
100	θ	AE	0.99994	1.00086	0.99854	1.00687	1.00291
		AB	0.00006	0.00086	0.00146	0.00687	0.00291
		MRE	0.00006	0.00086	0.00146	0.00687	0.00291
		MSE	0.00731	0.00771	0.00847	0.00823	0.00766
	α	AE	1.52094	1.50674	1.52522	1.49845	1.50718
		AB	0.02094	0.00674	0.02522	0.00155	0.00718
		MRE	0.01396	0.00450	0.01681	0.00103	0.00479
		MSE	0.01252	0.01570	0.01966	0.02086	0.01584
200	θ	AE	1.00179	1.00456	1.00107	0.99918	1.00113
		AB	0.00179	0.00456	0.00107	0.00082	0.00113
		MRE	0.00179	0.00456	0.00107	0.00082	0.00113
		MSE	0.00407	0.00398	0.00426	0.00402	0.00402
	α	AE	1.51267	1.50466	1.51154	1.49753	1.50392
		AB	0.01267	0.00466	0.01154	0.00247	0.00392
		MRE	0.00845	0.00311	0.00769	0.00165	0.00261
		MSE	0.00669	0.00701	0.01108	0.00931	0.00807
300	θ	AE	1.00023	0.99961	1.00419	1.00382	0.99901
		AB	0.00023	0.00039	0.00419	0.00382	0.00099
		MRE	0.00023	0.00039	0.00419	0.00382	0.00099
		MSE	0.00239	0.00289	0.00289	0.00295	0.00241
	α	AE	1.50647	1.50042	1.50767	1.49656	1.50118
		AB	0.00647	0.00042	0.00767	0.00344	0.00118
		MRE	0.00431	0.00028	0.00512	0.00229	0.00079
		MSE	0.00393	0.00483	0.00679	0.00708	0.00500

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Table 8. PUH distribution parameter estimates for the parameters $\theta = 1.5$ and $\alpha = 0.5$

n	Para.	Est.	MLE	ADE	CVME	OLSE	WLSE
20	θ	AE	1.53410	1.53020	1.55047	1.52040	1.52840
		AB	0.03410	0.03020	0.05047	0.02040	0.02840
		MRE	0.02274	0.02013	0.03365	0.01360	0.01893
		MSE	0.06743	0.07151	0.10300	0.09489	0.08426
	α	AE	0.53208	0.51124	0.53343	0.50004	0.50776
		AB	0.03208	0.01124	0.03343	0.00004	0.00776
		MRE	0.06416	0.02247	0.06686	0.00009	0.01553
		MSE	0.00765	0.00670	0.01311	0.01027	0.00897
50	θ	AE	1.49760	1.50836	1.52989	1.50687	1.49832
		AB	0.00240	0.00836	0.02989	0.00687	0.00168
		MRE	0.00160	0.00558	0.01992	0.00458	0.00112
		MSE	0.02740	0.02778	0.04196	0.03771	0.03507
	α	AE	0.51620	0.50914	0.51426	0.49788	0.50547
		AB	0.01620	0.00914	0.01426	0.00212	0.00547
		MRE	0.03240	0.01828	0.02852	0.00424	0.01094
		MSE	0.00264	0.00310	0.00538	0.00457	0.00366
75	θ	AE	1.50018	1.50278	1.51816	1.49993	1.51229
		AB	0.00018	0.00278	0.01816	0.00007	0.01229
		MRE	0.00012	0.00185	0.01211	0.00005	0.00820
		MSE	0.01680	0.01505	0.02525	0.02591	0.02522
	α	AE	0.51342	0.50648	0.50896	0.49955	0.50364
		AB	0.01342	0.00648	0.00896	0.00045	0.00364
		MRE	0.02685	0.01295	0.01791	0.00090	0.00727
		MSE	0.00159	0.00167	0.00318	0.00296	0.00268
100	θ	AE	1.49845	1.50287	1.50652	1.50254	1.50722
		AB	0.00155	0.00287	0.00652	0.00254	0.00722
		MRE	0.00103	0.00191	0.00435	0.00169	0.00481
		MSE	0.00933	0.01068	0.01951	0.01819	0.01831
	α	AE	0.50813	0.50591	0.50466	0.50008	0.50304
		AB	0.00813	0.00591	0.00466	0.00008	0.00304
		MRE	0.01626	0.01182	0.00933	0.00016	0.00607
		MSE	0.00092	0.00113	0.00211	0.00232	0.00182
200	θ	AE	1.49511	1.49897	1.50347	1.50052	1.50414
		AB	0.00489	0.00103	0.00347	0.00052	0.00414
		MRE	0.00326	0.00069	0.00231	0.00035	0.00276
		MSE	0.00282	0.00320	0.00875	0.00906	0.00794
	α	AE	0.50368	0.50308	0.50488	0.50091	0.50115
		AB	0.00368	0.00308	0.00488	0.00091	0.00115
		MRE	0.00735	0.00616	0.00977	0.00182	0.00229
		MSE	0.00026	0.00034	0.00116	0.00110	0.00086
300	θ	AE	1.49566	1.49877	1.50672	1.50150	1.50324
		AB	0.00434	0.00123	0.00672	0.00150	0.00324
		MRE	0.00289	0.00082	0.00448	0.00100	0.00216
		MSE	0.00134	0.00134	0.00637	0.00572	0.00550
	α	AE	0.50301	0.50122	0.50285	0.49926	0.50050
		AB	0.00301	0.00122	0.00285	0.00074	0.00050
		MRE	0.00602	0.00244	0.00570	0.00148	0.00099
		MSE	0.00016	0.00013	0.00079	0.00069	0.00061

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Table 9. PUH distribution parameter estimates for the parameters $\theta = 1.5$ and $\alpha = 1.0$

n	Para.	Est.	MLE	ADE	CVME	OLSE	WLSE
20	θ	AE	1.56017	1.52995	1.57689	1.53460	1.52286
		AB	0.06017	0.02995	0.07689	0.03460	0.02286
		MRE	0.04011	0.01997	0.05126	0.02307	0.01524
		MSE	0.08744	0.07310	0.11769	0.08932	0.07497
	α	AE	1.06376	1.01357	1.06597	0.99707	1.00701
		AB	0.06376	0.01357	0.06597	0.00293	0.00701
		MRE	0.06376	0.01357	0.06597	0.00293	0.00701
		MSE	0.03179	0.02762	0.05044	0.04117	0.03322
50	θ	AE	1.51541	1.51880	1.51767	1.50958	1.50353
		AB	0.01541	0.01880	0.01767	0.00958	0.00353
		MRE	0.01027	0.01254	0.01178	0.00639	0.00235
		MSE	0.03573	0.03461	0.03983	0.03182	0.03488
	α	AE	1.03219	1.01227	1.02033	1.00288	1.00725
		AB	0.03219	0.01227	0.02033	0.00288	0.00725
		MRE	0.03219	0.01227	0.02033	0.00288	0.00725
		MSE	0.01353	0.01428	0.01991	0.01921	0.01615
75	θ	AE	1.51233	1.51132	1.51184	1.50938	1.51315
		AB	0.01233	0.01132	0.01184	0.00938	0.01315
		MRE	0.00822	0.00755	0.00789	0.00625	0.00877
		MSE	0.02152	0.02079	0.02539	0.02338	0.02234
	α	AE	1.02068	1.00281	1.02236	0.99673	1.00874
		AB	0.02068	0.00281	0.02236	0.00327	0.00874
		MRE	0.02068	0.00281	0.02236	0.00327	0.00874
		MSE	0.00916	0.00925	0.01370	0.01162	0.01017
100	θ	AE	1.50753	1.50451	1.51077	1.49990	1.50558
		AB	0.00753	0.00451	0.01077	0.00010	0.00558
		MRE	0.00502	0.00301	0.00718	0.00007	0.00372
		MSE	0.01620	0.01515	0.01787	0.01699	0.01708
	α	AE	1.01352	0.99984	1.01574	0.99517	1.00406
		AB	0.01352	0.00016	0.01574	0.00483	0.00406
		MRE	0.01352	0.00016	0.01574	0.00483	0.00406
		MSE	0.00647	0.00653	0.00957	0.00905	0.00742
200	θ	AE	1.50483	1.50325	1.50530	1.50293	1.50652
		AB	0.00483	0.00325	0.00530	0.00293	0.00652
		MRE	0.00322	0.00217	0.00353	0.00195	0.00435
		MSE	0.00833	0.00781	0.00896	0.00926	0.00807
	α	AE	1.00747	1.00620	1.00673	0.99920	1.00373
		AB	0.00747	0.00620	0.00673	0.00080	0.00373
		MRE	0.00747	0.00620	0.00673	0.00080	0.00373
		MSE	0.00300	0.00352	0.00460	0.00458	0.00384
300	θ	AE	1.49888	1.50114	1.50418	1.50440	1.50138
		AB	0.00112	0.00114	0.00418	0.00440	0.00138
		MRE	0.00074	0.00076	0.00279	0.00293	0.00092
		MSE	0.00505	0.00487	0.00649	0.00544	0.00537
	α	AE	1.00378	0.99775	1.00683	1.00061	1.00348
		AB	0.00378	0.00225	0.00683	0.00061	0.00348
		MRE	0.00378	0.00225	0.00683	0.00061	0.00348
		MSE	0.00193	0.00199	0.00311	0.00304	0.00235

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Table 10. PUH distribution parameter estimates for the parameters $\theta = 1.5$ and $\alpha = 1.5$

n	Para.	Est.	MLE	ADE	CVME	OLSE	WLSE
20	θ	AE	1.55225	1.52767	1.58509	1.50471	1.51439
		AB	0.05225	0.02767	0.08509	0.00471	0.01439
		MRE	0.03484	0.01844	0.05672	0.00314	0.00959
		MSE	0.09004	0.07474	0.10986	0.07534	0.07687
	α	AE	1.58963	1.50765	1.58427	1.49193	1.49525
		AB	0.08963	0.00765	0.08427	0.00807	0.00475
		MRE	0.05975	0.00510	0.05618	0.00538	0.00317
		MSE	0.07353	0.06014	0.11907	0.08540	0.08462
50	θ	AE	1.52075	1.51523	1.51308	1.50908	1.50748
		AB	0.02075	0.01523	0.01308	0.00908	0.00748
		MRE	0.01383	0.01016	0.00872	0.00606	0.00499
		MSE	0.03607	0.03509	0.03709	0.03770	0.03440
	α	AE	1.54092	1.50538	1.55375	1.49963	1.52189
		AB	0.04092	0.00538	0.05375	0.00037	0.02189
		MRE	0.02728	0.00359	0.03583	0.00024	0.01459
		MSE	0.03124	0.03062	0.04925	0.04339	0.03824
75	θ	AE	1.50981	1.50781	1.52052	1.51083	1.51300
		AB	0.00981	0.00781	0.02052	0.01083	0.01300
		MRE	0.00654	0.00521	0.01368	0.00722	0.00867
		MSE	0.02131	0.02188	0.02573	0.02551	0.02165
	α	AE	1.53394	1.50917	1.53661	1.49759	1.50932
		AB	0.03394	0.00917	0.03661	0.00241	0.00932
		MRE	0.02262	0.00611	0.02441	0.00160	0.00621
		MSE	0.01968	0.02116	0.03318	0.02640	0.02109
100	θ	AE	1.50785	1.50601	1.51193	1.50596	1.50999
		AB	0.00785	0.00601	0.01193	0.00596	0.00999
		MRE	0.00523	0.00401	0.00795	0.00397	0.00666
		MSE	0.01557	0.01704	0.01848	0.01769	0.01726
	α	AE	1.51587	1.50773	1.51627	1.49738	1.49419
		AB	0.01587	0.00773	0.01627	0.00262	0.00581
		MRE	0.01058	0.00515	0.01085	0.00174	0.00387
		MSE	0.01457	0.01602	0.02245	0.01917	0.01630
200	θ	AE	1.50292	1.50214	1.51028	1.49965	1.50544
		AB	0.00292	0.00214	0.01028	0.00035	0.00544
		MRE	0.00195	0.00143	0.00685	0.00023	0.00363
		MSE	0.00831	0.00804	0.00871	0.00858	0.00846
	α	AE	1.51292	1.50340	1.50634	1.50723	1.49875
		AB	0.01292	0.00340	0.00634	0.00723	0.00125
		MRE	0.00861	0.00227	0.00422	0.00482	0.00083
		MSE	0.00679	0.00773	0.00971	0.01036	0.00736
300	θ	AE	1.50337	1.49928	1.50200	1.50351	1.50650
		AB	0.00337	0.00072	0.00200	0.00351	0.00650
		MRE	0.00224	0.00048	0.00133	0.00234	0.00434
		MSE	0.00514	0.00533	0.00595	0.00582	0.00545
	α	AE	1.50590	1.50386	1.50819	1.49964	1.50426
		AB	0.00590	0.00386	0.00819	0.00036	0.00426
		MRE	0.00393	0.00257	0.00546	0.00024	0.00284
		MSE	0.00414	0.00509	0.00684	0.00634	0.00531

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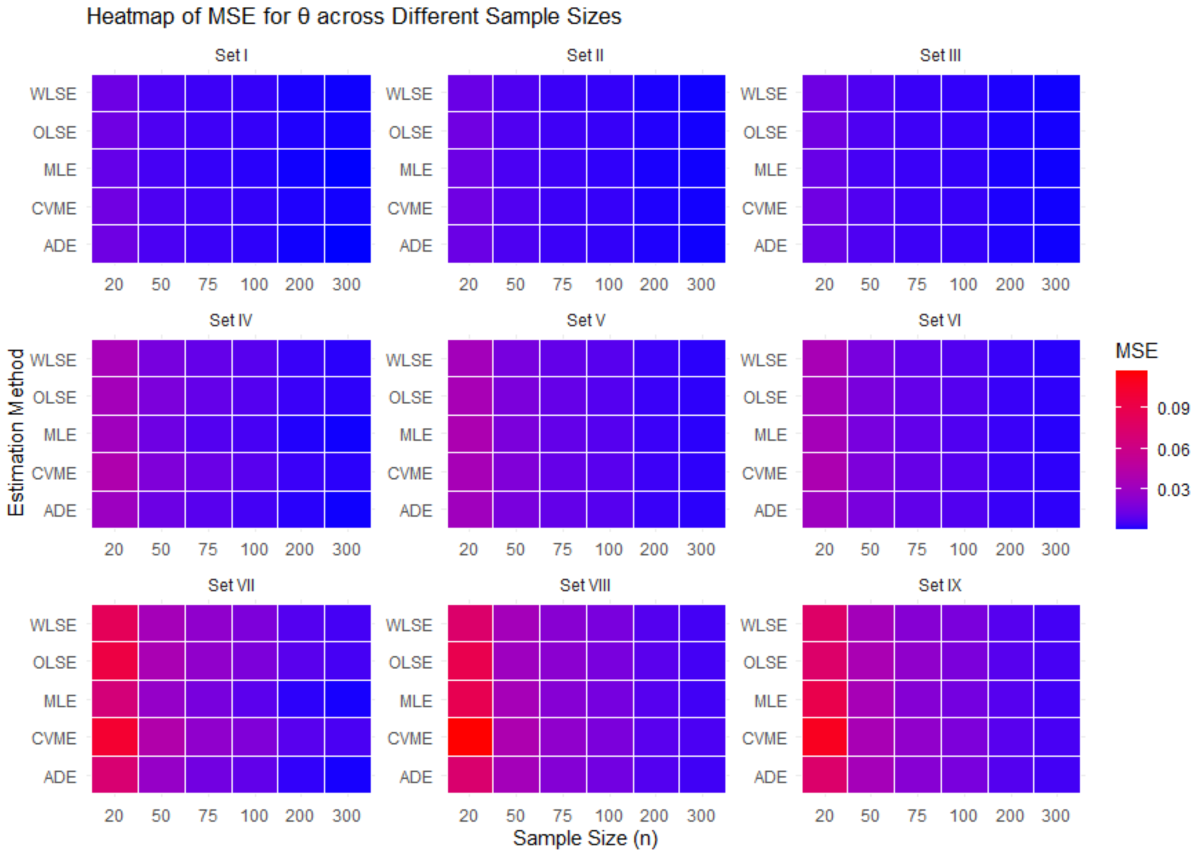

Figure 6. Heatmap of MSEs for θ based on different estimation methods and sample sizes

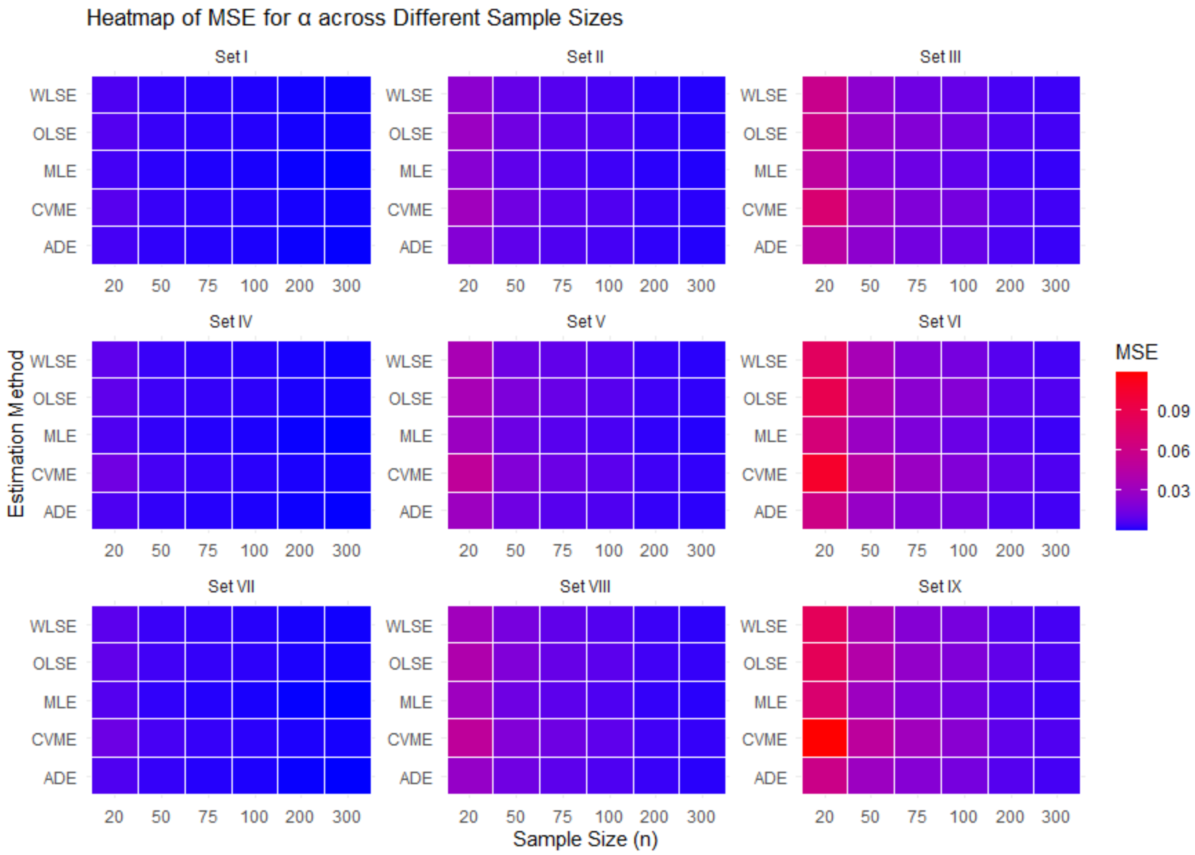

Figure 7. Heatmap of MSEs for α based on different estimation methods and sample sizes

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The A statistic is defined as

$$A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left[F\left(z_{(i)};\hat{\Theta}\right)\right] + \log\left[1 - F\left(z_{(n+1-i)};\hat{\Theta}\right)\right]\right],$$

where $z_{(i)}$ are the order observations.

The W statistic is given by

$$W = \frac{1}{12n} + \sum_{i=1}^{n}\left[F\left(z_{(i)};\hat{\Theta}\right) - \frac{2i-1}{2n}\right]^2.$$

We also illustrate the flexibility of the PUH distribution using graphical representations of the density function, cumulative distribution function, hazard function, profile log-likelihood, and contour plots through this comparative analysis. The findings of this assessment will shed light on the PUH distribution's possible benefits over current probability models and applicability for real-world uses.

Data Set I (Kidney Dialysis Patients):

The first dataset is about the times of kidney dialysis patients studied by [21]. The data observations are

0.083	0.116	0.083	0.116	0.483	0.116	0.150
33333	66667	33333	66667	33333	66667	00000
0.416	0.250	0.183	0.216	0.216	0.250	0.916
66667	00000	33333	66667	66667	00000	66667
0.250	0.283	0.316	0.350	0.383	0.416	0.450
00000	33333	66667	00000	33333	66667	00000
0.483	0.716	0.716	0.750	0.250	0.750	0.850
33333	66667	66667	00000	00000	00000	00000

Some key descriptive measures are derived from the first dataset and presented in Table 11. These provide insight into its central tendency, dispersion index, and general distribution. Then again, some non-parametric plots like the TTT plot, box plot, violin plot, and Q-Q plot in Figure 8 elaborate further on the characteristics of the kidney dialysis patient's dataset.

The TTT plot shows an upside-down bathtub shape, which indicates a certain pattern in the data's failure rate or hazard function. The box plot, violin plot, and finally, the Q-Q plot indicate that there is a right skewness in this dataset, meaning that there is a heavier tail towards the right, possibly indicating a departure from normality. Table 12 displays the estimates of the parameters and the model selection criteria for the kidney dialysis patient's dataset. Figure 9 visually shows the estimated PDF, CDF, and hazard function which provide insights into the dataset's distributional features. Additionally, it includes the profile log-likelihood and contour plots which illustrate the stability and reliability of the parameter estimates obtained through kidney dialysis patients' data.

Model comparison reveals the best fit provided by the PUH distribution, which provides the minimum AIC and BIC values, non-significant test results for KS, A, and W tests, and an adequate fit to the data. The other models, such as Kumaraswamy and Beta distributions, also present acceptable fits, though the PUH model outperforms these models when refereed by the criteria of information. Contrasting this, the models of GEUD and UXg performed rather poorly with larger information criteria and significant goodness-of-fit test results, whereas UH only reached borderline adequacy.

Table 11. Some descriptive statistics for the kidney dialysis patients' dataset

n	Min.	Max.	Mean	Median	Variance	Skewness	Kurtosis
22	0.08333	0.91667	0.37738	0.30000	0.06107	0.76505	2.42188

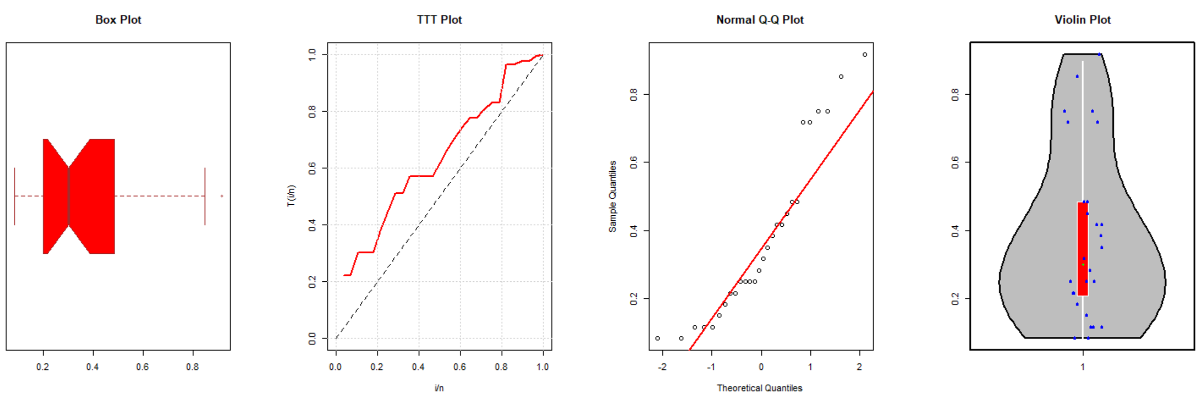

Figure 8. Box, TTT, Q-Q, and Violin plots for the kidney dialysis patient's dataset

Table 12. The parameter estimates and model selection measures for the kidney dialysis patient's dataset

Model	Estimate (S.E.)		*l*	AIC	BIC	KS (p-value)	A (p-value)	W (p-value)
PUH	0.0959(0.0823)	23.194(21.590)	4.6205	-5.2411	-2.5767	0.1200 (0.8145)	0.4156 (0.8318)	0.0664 (0.7777)
Kum	1.2651(0.2544)	2.0798(0.5714)	3.6625	-3.3249	-0.6605	0.1377 (0.6629)	0.7083 (0.5505)	0.1203 (0.4967)
Beta	1.3566(0.3332)	2.1057(0.5496)	3.7776	-3.5552	-0.8908	0.1411 (0.6321)	0.7162 (0.5440)	0.1216 (0.4913)
GEUD	0.5994(0.3065)	0.8216(0.3233)	-1.7434	7.4869	10.151	0.5275 (0.0331)	2.5764 (0.0456)	0.2660 (0.0380)
UH	1.1906(0.1734)	-	-0.2870	2.5740	3.9062	0.4009 (0.0711)	2.0900 (0.0824)	0.2561 (0.0507)
UXg	1.3062(0.1806)	-	-5.1461	12.292	13.624	0.8572 (0.0049)	4.2426 (0.0067)	0.3318 (0.0041)

It follows, therefore, that PUH distribution is the most suitable model for the kidney dialysis patient's dataset as compared to these considered competitive models.

Data Set II (Times between Failures of Secondary Reactor Pumps):

The second dataset is about the duration between secondary reactor pump failures [22]. The data observations are

0.2160	0.0150	0.4082	0.0746	0.0358	0.0199	0.0402	0.0101	0.0605	0.0954	0.1359	0.0273
0.0491	0.3465	0.0070	0.6560	0.1060	0.0062	0.4992	0.0614	0.5320	0.0347	0.1921	

Table 13 summarizes some key descriptive characteristics obtained from the second dataset. Figure 10 presents the TTT plots, Violin, box plot, and Q-Q plot generated using the time between failures of secondary reactor pump data. The box, violin, and Q-Q plots show favorably skewed data distribution. Furthermore, the TTT plot shows a bathtub-shaped failure rate in the time between failures of the secondary reactor pump dataset. Table 14 shows the parameter estimates and model selection measures for the reactor pump dataset.

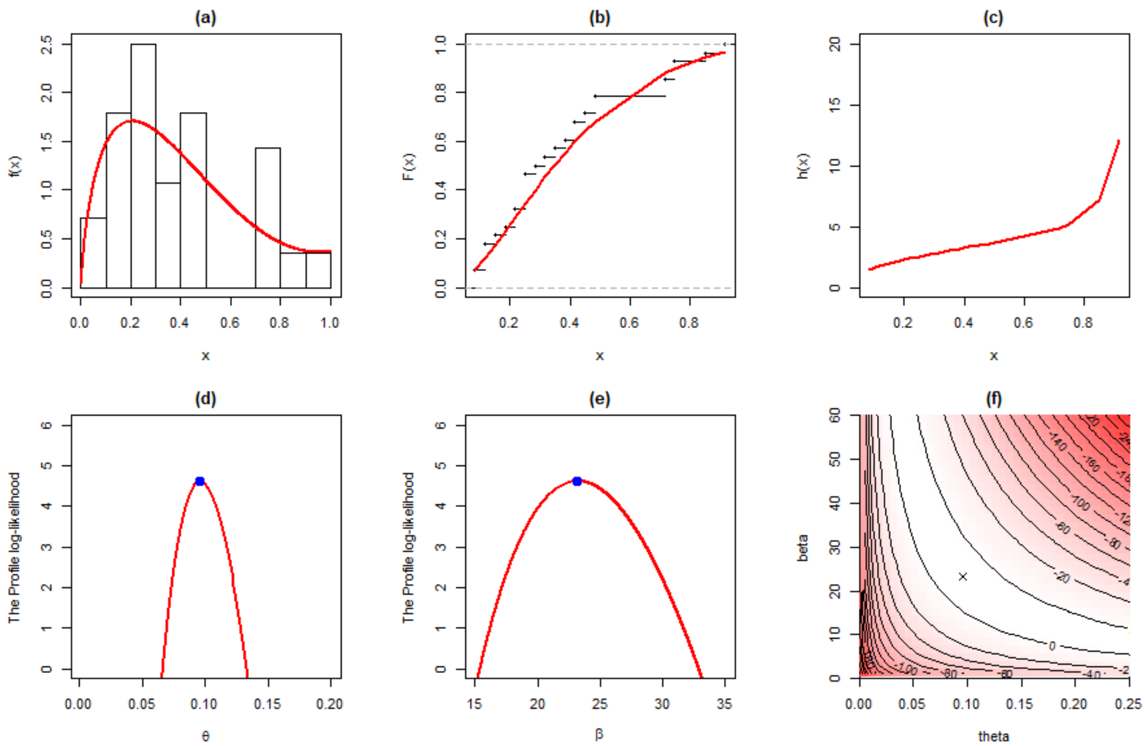

Figure 9. Visualization of the fitted density function, CDF, hazard function, log-likelihood, and contour plots for the kidney dialysis patient's dataset

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Figure 11 depicts the estimated PDF, CDF, and hazard function, which offers information about the dataset's distributional characteristics. It also provides profile log-likelihood graphs and contour plots, which demonstrate the stability and reproducibility of parameter estimations derived from the time between failures of secondary reactor pump data. These graphical representations are essential for assessing the model's fitness and parameter sensitivity. Table 14 presents the parameter estimates, log-likelihood, information criteria, and results of the goodness-of-fit tests for six competing distributions fitted to the reactor pumps dataset. Based on the highest for its log-likelihood and lowest values of AIC and BIC, the proposed PUH distribution provides a better fit as compared considered competitive distributions. Therefore, the PUH distribution is the most suitable and reliable model for the reactor pumps dataset, while Kumaraswamy and Beta can be acceptable alternatives.

Data Set III (Radiation):

The third dataset examined the F1 adults of *Stegobium paniceum* produced from parents fed on peppermint obtained from both unirradiated and irradiated packets, utilizing a non-choice, non-packet test. The peppermint samples were exposed gamma radiation (6,8,10 KGy) or microwaves (1, 2, 3 min), resulting in the following adult counts: 148, 145, 152, 64, 64, 64, 52, 59, 57, 33, 36, 31, 87, 85, 87, 69, 65, 67, 48, 42, 46 [23]. Later, [24] normalized this dataset to the bounded using the transformation $X = (X_i/\max(X_i) + 1)$ before applying the unit interval distributions

Table 13. Some descriptive statistics for the reactor pump dataset

n	Min.	Max.	Mean	Median	Variance	Skewness	Kurtosis
23	0.0062	0.6560	0.1578	0.0614	0.0372	1.3643	3.5445

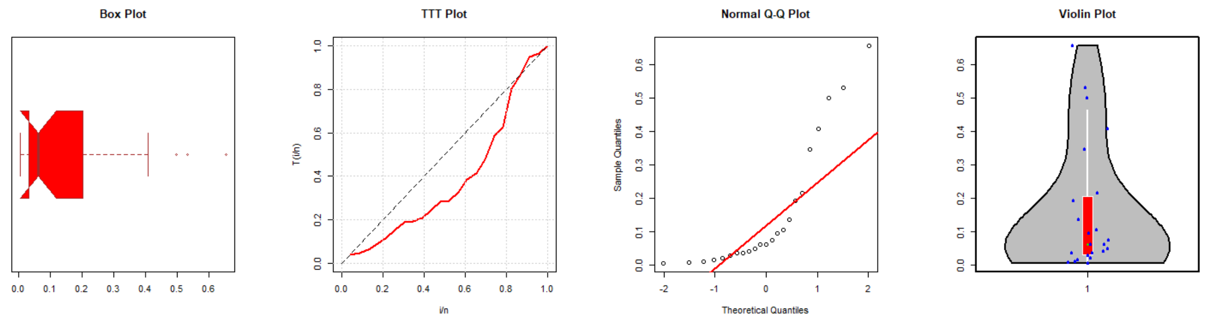

Figure 10. TTT, box, violin, and Q-Q plots for the reactor pump dataset

Table 14. The parameter estimates and model selection measures for the reactor pumps dataset

Model	Estimate (S.E.)		l	AIC	BIC	KS (p-value)	A (p-value)	W (p-value)
PUH	0.0077(0.0046)	146.08(91.829)	20.7112	-37.422	-35.151	0.0812 (0.6892)	0.5032 (0.7417)	0.1308 (0.7788)
Kum	0.6766(0.1406)	2.9360(0.9557)	20.329	-36.659	-34.388	0.0988 (0.5945)	0.5754 (0.6696)	0.1393 (0.7123)
Beta	0.6307(0.1575)	3.2317(1.0647)	20.028	-36.057	-33.786	0.1264 (0.4730)	0.6886 (0.5667)	0.1541 (0.5918)
GEUD	1.3434(0.6634)	0.5298(0.1707)	18.575	-33.150	-30.879	0.2440 (0.1965)	1.2484 (0.2495)	0.2075 (0.2396)
UH	0.6683(0.0976)	-	15.699	-29.399	-28.263	0.3899 (0.0761)	2.1236 (0.0791)	0.2237 (0.1713)
UXg	4.6480(0.8122)	-	16.093	-30.186	-29.050	0.7021 (0.0118)	3.9935 (0.0089)	0.3068 (0.0202)

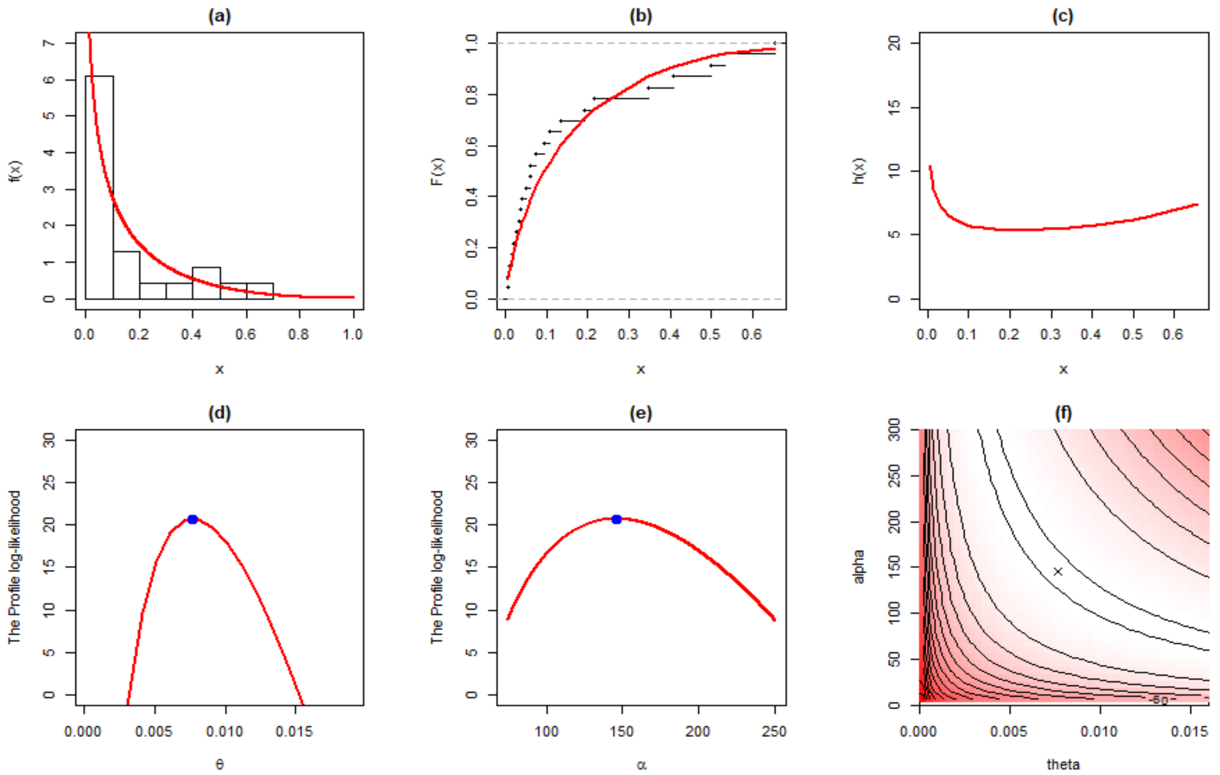

Figure 11. Visualization of the fitted density function, CDF, hazard function, log-likelihood, and contour plots for the reactor pumps dataset

Table 15. Some descriptive statistics for the radiation dataset

n	Min.	Max.	Mean	Median	Variance	Skewness	Kurtosis
21	0.2026	0.9935	0.4672	0.4183	0.0551	1.2586	3.5820

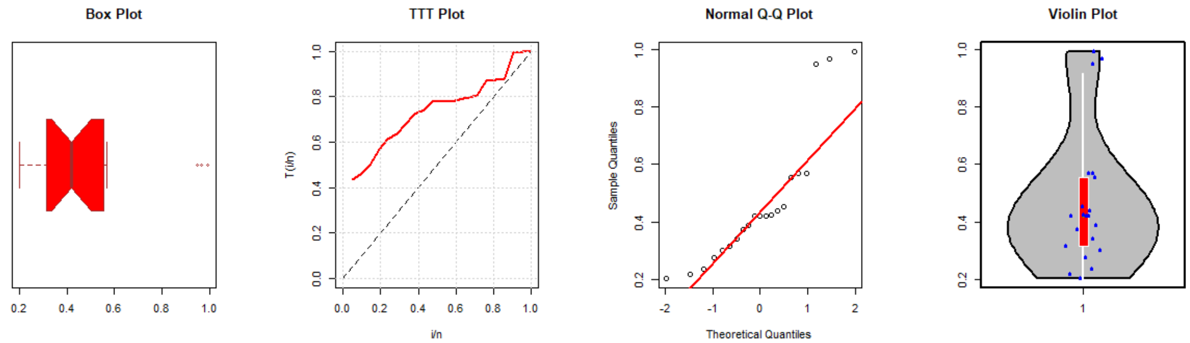

Figure 12. Box, TTT, Q-Q, and violin plots for the radiation dataset

The transformed data observations are

0.967320	0.947712	0.993464	0.418301	0.418301	0.339869	0.385621	0.372549
0.235294	0.202614	0.568627	0.555556	0.568627	0.450980	0.424837	0.437908
0.313725	0.274509	0.300654	0.215686	0.418301			

Table 15 presents descriptive measures for the radiation dataset. We also plot some non-parametric plots, such as TTT, violin box, and Q-Q to illustrate the nature of this dataset, and list them in Figure 12. The descriptive measures and plots indicate a positively skewed

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distribution. Moreover, the TTT plot exhibits an increasing failure rate.

Table 16: Parameter estimation and model selection criteria for radiation data. Figure 13 depicts the estimated PDF, CDF, and hazard function, which offers information about the dataset's distributional characteristics. It also provides profile log-likelihood graphs and contour plots, which demonstrate the stability and reproducibility of parameter estimations derived from radiation data. These graphical representations are essential for assessing the model's fitness and parameter sensitivity.

From Table 16, it is observed that the PUH, Kumaraswamy, and Beta models have adequate data fitting, and their KS, Anderson–Darling, Cramér–von Mises test and p-values. But based on measures like AIC and BIC values, it is observed that the PUH model performs reasonably well compared to the other two. Although GEUD and UXg have extreme values of information criterion statistics, they are not suitable models because of highly significant values of the KS test statistics. The performance of UH is not satisfactory compared to the PUH distribution model. Hence, it can be concluded that the proposed generalization is the best model for radiation data.

Table 16. The parameter estimates and model selection measures for the radiation dataset

Model	Estimate (S.E.)		l	AIC	BIC	KS (p-value)	A (p-value)	W (p-value)
PUH	0.1138(0.0749)	26.549(18.688)	4.7397	-5.4795	-3.3905	0.2388 (0.1819)	1.3683 (0.2112)	0.2281 (0.2196)
Kum	1.3115(0.3364)	1.2377(0.3491)	0.4848	3.0302	5.1193	0.3055 (0.0396)	2.3434 (0.0605)	0.4540 (0.0512)
Beta	1.3654(0.3968)	1.2691(0.3643)	0.5344	2.9311	5.0201	0.3037 (0.0415)	2.3489 (0.0601)	0.4516 (0.0519)
GEUD	0.0271(0.0473)	0.2815(0.4831)	-27.830	59.660	61.749	0.7115 (0.0000)	18.012 (0.0000)	2.9541 (0.0000)
UH	1.5201(0.2667)	-	-0.5203	3.0407	4.0853	0.3513 (0.0112)	2.7251 (0.0384)	0.5821 (0.0238)
UXg	0.1999(0.0300)	-	-56.271	114.54	115.58	0.8164 (0.0000)	38.767 (0.0000)	4.0242 (0.0000)

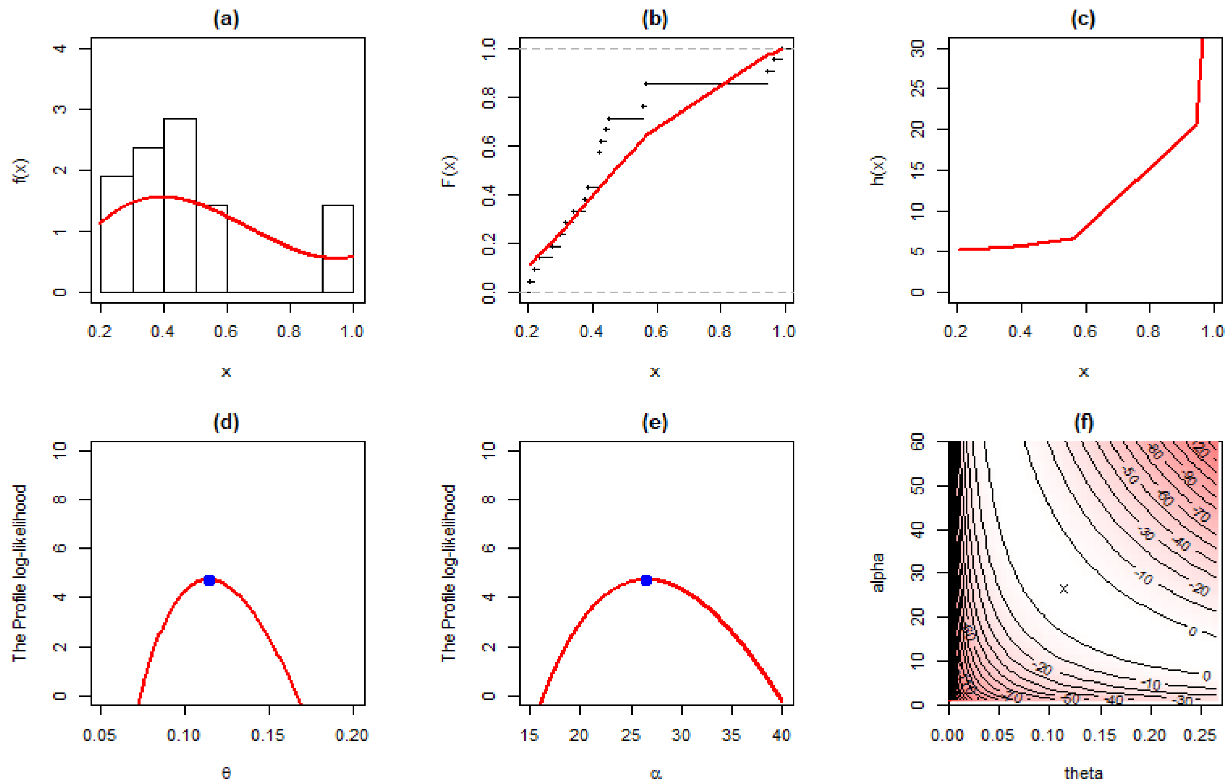

Figure 13. Visualization of the fitted density function, CDF, hazard function, log-likelihood, and contour plots for the radiation dataset

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4.4. Bayesian Analysis

In this section, the Bayesian paradigm is used to estimate the PUH distribution parameters and alternative classical estimation methods. We utilized the Gamma distribution for both θ and α parameters. Consequently, $\theta \sim Gamma(a_1, b_1)$ and $\alpha \sim Gamma(a_2, b_2)$, where a_1, b_1, a_2, b_2 are the positive hyperparameters. The prior distributions for the parameters are

$$p_1(\theta) = \frac{{b_1}^{a_1}}{\Gamma(a_1)} \theta^{a_1 - 1} \exp(-\theta b_1),$$

and

$$p_2(\alpha) = \frac{{b_2}^{a_2}}{\Gamma(a_2)} \alpha^{a_2 - 1} \exp(-\alpha b_2).$$

Therefore, the joint prior density function is

$$p(\boldsymbol{\theta}) \propto \theta^{a_1 - 1} \alpha^{a_2 - 1} \exp(-\theta b_1 - \alpha b_2).$$

The posterior density function is obtained as

$$p(\theta, \alpha | x) \propto \theta^{a_1 - 1} \alpha^{a_2 - 1} \exp(-\theta b_1 - \alpha b_2)$$

$$\times \frac{\alpha \theta^2}{(1+\theta)^2} \prod_{i=1}^{n} \left(2 + \theta + \frac{\theta (\ln z_i^{\alpha})^2}{2}\right) z_i^{\alpha\theta - 1}.$$

The above posterior density function does not have a closed-form formulation. We will generate samples from the posterior density using the Markov Chain Monte Carlo (MCMC) approach that was developed with R software. A total of 1,005,000 samples were generated, with the first 5,000 removed as a burn-in phase to minimize the effect of initial values. The convergence of the simulated samples was assessed through trace plots and Geweke diagnostic tests.

Table 17 shows the Bayesian estimates as well as other relevant variables such as standard deviation, standard error, Geweke's Z-score, and the 95% HPD range for both datasets.Further Trace plots, autocorrelation plots, and posterior density plots are all used to further analyze Markov chain convergence.

Figures 14–16 show no discernible trends in the trace plots, indicating that the Markov chains have achieved convergence.

Table 17. Posterior summary statistics for both datasets using the PUH distribution

	Kidney dialysis patients' data	Reactor pumps data	Radiation data
Bayes Estimates	0.1111	0.0998	0.1970
	20.545	10.790	13.961
Standard Deviation	0.0256	0.0268	0.0429
	4.6316	2.8849	3.1742
HPD	(0.0650, 0.16209)	(0.0553, 0.1552)	(0.1189, 0.2817)
	(11.368, 29.246)	(5.5416, 16.455)	(8.4937, 20.664)
Z-score	0.2958	-0.7728	0.4720
	-0.1624	0.7179	0.0408

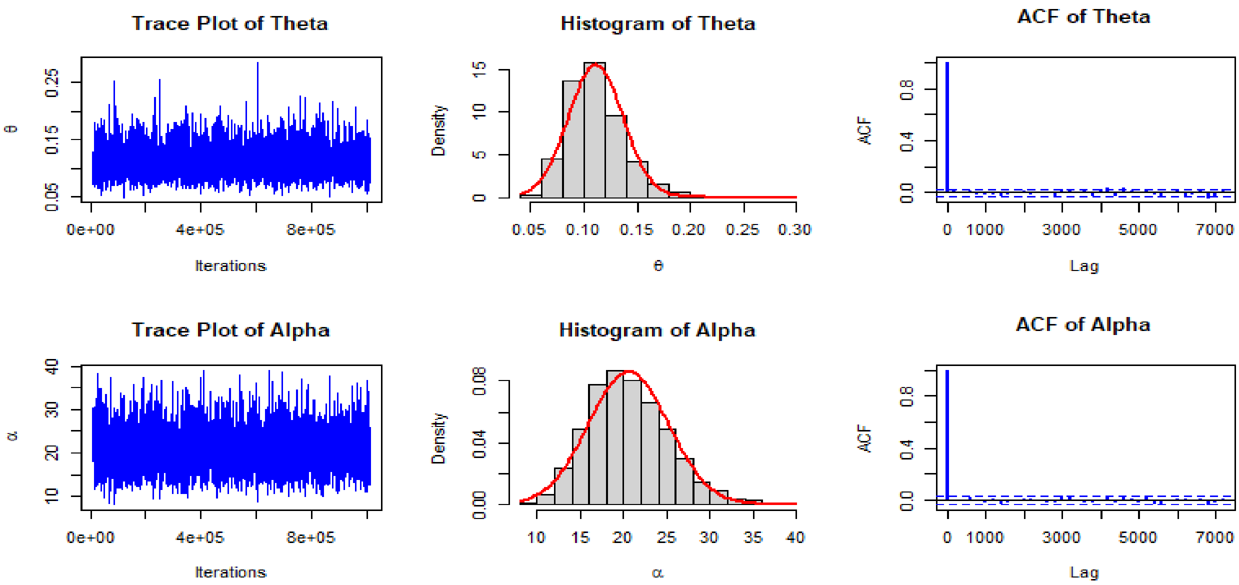

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Figure 14. The MCMC convergence diagnostics plots for the kidney dialysis patient's dataset

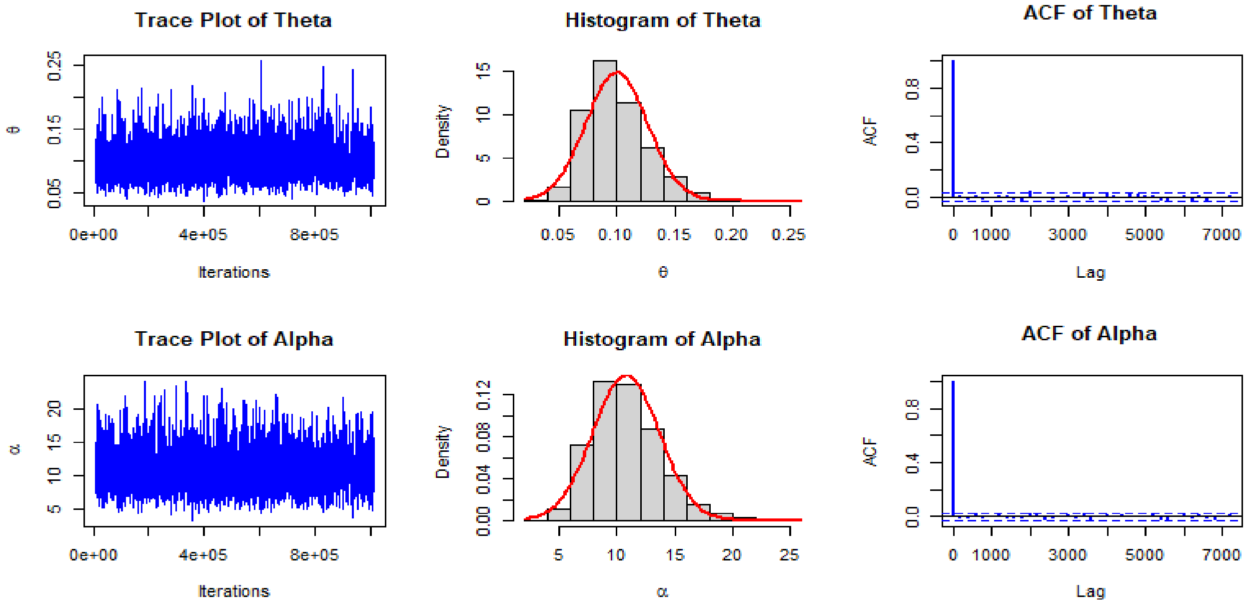

Figure 15. The MCMC convergence diagnostics plots for the reactor pumps dataset

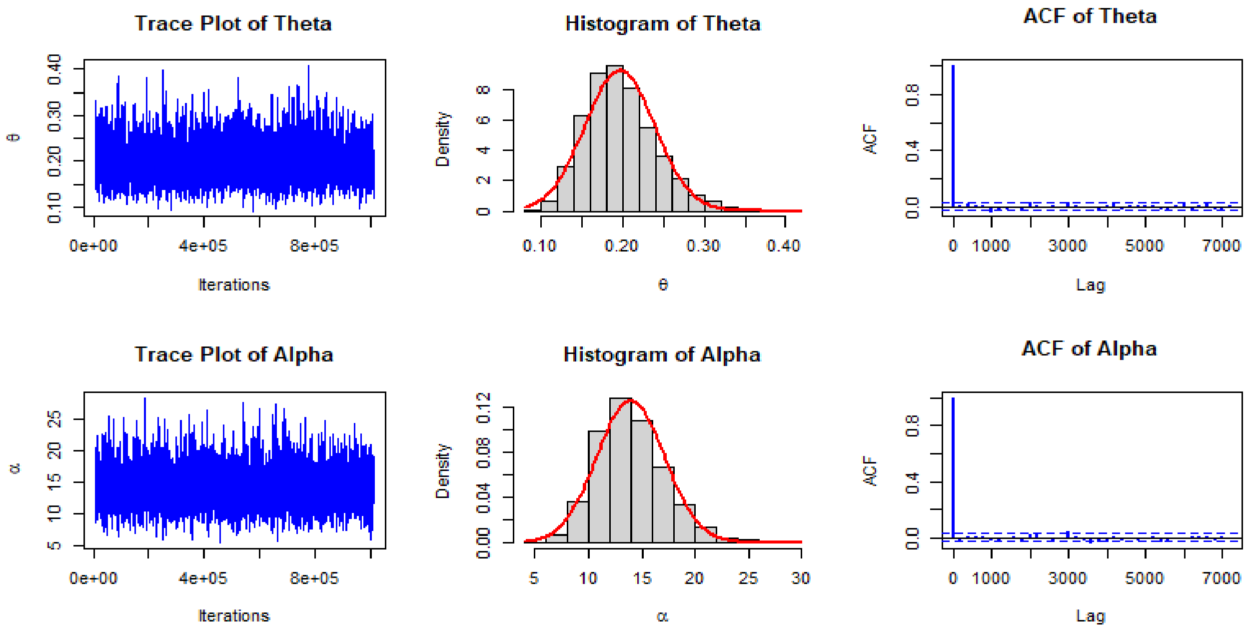

Figure 16. The MCMC convergence diagnostics plots for the radiation dataset

7. Conclusion

In this paper, a new flexible probability model for data supported on the unit interval has been proposed, called the Power Unit Haq distribution, developed through the power transformation approach. The new distribution in this paper can be applied because it provides extra flexibility for the modeling of unit interval distributions when facing real data with different patterns. A comprehensive set of statistical properties of the PUH distribution was obtained, including all its moments, incomplete moments, and related descriptive measures. Besides these, the moment generating function, hazard rate function, mean residual life function, and Rényi entropy were derived. Such theoretical results ensure more profound insight into the structural behavior of the model and emphasize its appropriateness for reliability analysis and modeling lifetime data on the unit interval.

Parameter estimation was addressed using five different estimation techniques, including the maximum likelihood method and minimum distance-based methods. A comprehensive Monte Carlo simulation study was conducted to compare the small-sample properties of the proposed estimators. From the simulation study results, it was observed that the proposed estimators were consistent and efficient, especially the maximum likelihood estimator. The usefulness and flexibility that can be realized using the PUH distribution are highlighted through the example applications carried out on three different datasets, namely the measurement data concerning radionuclides, the data on failures in the reactor pumps, and the patients on kidney dialysis. Also, a Bayesian approach has been developed and applied in the study, which increases the usefulness of the distribution when prior information or the estimation of uncertainty becomes necessary. In conclusion, it is found that PUH distribution is an advantageous addition to the list of unit interval distributions. It is one of the most tractable models with desirable flexibility in shape parameters, superior estimation capability, and outstanding empirical fitting performance, making it an attractive and useful tool in reliability studies, medical, and other applied sciences. Future research may explore multivariate extensions, regression developments to broaden its scope of applications.

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Authors Contribution
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The author states that there is no conflict of interest.

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