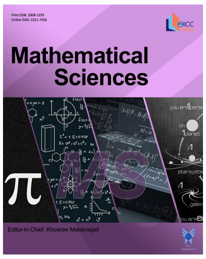

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Research Article

Existence, Uniqueness, and Stability Analysis of Coupled Fractional Differential Equations with Hilfer and Weighted Caputo Derivatives

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Abstract:

In this paper, we first establish the necessary conditions for the existence, uniqueness, and stability of solutions to system (1) by applying an alternative fixed-point theorem. We then provide specific examples and utilize the Banach fixed-point numerical method to demonstrate the applicability and reliability of our approach; notably, one of these examples is solved numerically in full, and the results from this numerical approximation further confirm the effectiveness and accuracy of the proposed method. These results verify the feasibility of the proposed approach for approximating solutions to complex fractional differential systems.

Keywords: Fractional derivatives; Fixed-point theorems; Mittag-Leffer functions; Control/observation systems; Stability of solutions; Numerical integration and quadrature

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1. Introduction

Fractional calculus, an extension of classical calculus, has seen significant advancements due to its ability to generalize differentiation and integration to non-integer orders [1]. The concept, initially suggested by mathematicians such as Joseph Fourier, has become instrumental in solving complex differential equations, particularly in fields where classical methods are insufficient to capture certain dynamical behaviors [2]. The Caputo derivative, a notable improvement on the traditional Riemann-Liouville derivative, introduced integer-order initial conditions, providing a more practical tool for modeling fractional-order systems [3]. The development of various definitions of fractional derivatives, including Grunwald-Letnikov, Riemann-Liouville, Hilfer, and Caputo derivatives, has enabled more accurate representations of physical, engineering, and biological phenomena [4, 5, 6, 7]. These fractional operators are particularly useful in describing systems with memory and hereditary properties, common in disciplines such as control theory, viscoelasticity, and signal processing [8, 9, 10, 11]. The versatility of fractional calculus has made it increasingly relevant in the qualitative analysis of solutions to fractional differential equations (FDEs), leading to extensive research on the existence, uniqueness, and stability of solutions for these equations [12, 13, 14, 5, 15, 16]. Ulam raised the stability issue of solutions to functional equations related to group homomorphisms in 1940. Subsequently, Hyers provided a positive solution to the Ulam question in Banach spaces, marking a significant advancement. Since then, numerous articles have explored various generalizations of the Ulam problem. Additionally, researchers have investigated the stability of Hyers-Ulam (HU) and Hyers-Ulam-Rassias (HUR) and other generalizations using fractional derivatives [17, 18, 19].

Motivated by the work in [20, 4], this paper focuses on coupled fractional differential equations, which frequently arise in applications involving interconnected dynamic systems. Such coupled systems often exhibit complex behavior that can only be accurately described through fractional derivatives. Here, we consider a system defined by Hilfer and weighted Caputo derivatives, aiming to establish sufficient conditions for the existence, uniqueness, and stability of its solutions. By employing an alternative fixed-point theorem, we derive criteria that ensure these qualitative properties for the system under study.

To validate our theoretical findings, we provide illustrative examples and apply the Banach fixed-point method as a numerical approach to approximate solutions. In this context, one of the examples has been solved numerically, demonstrating the convergence of iterative computations toward the desired solution.This numerical method, based on complete metric spaces and the Banach fixed-point theorem, enables the analysis of existence, uniqueness, and stability of solutions, in line with similar strategies employed in references such as [21, 22, 23]. The examples, particularly the one solved numerically, clearly illustrate the effectiveness and accuracy of the proposed approach in handling systems characterized by memory effects governed by fractional dynamics. This study contributes to a deeper understanding of fractional differential systems by offering a structured framework for analyzing coupled systems involving Hilfer and weighted Caputo derivatives, supporting future applications in applied mathematics and engineering.

$$\begin{cases} {}^{\mathscr{H}}\mathbb{D}_{0^+}^{i,\Upsilon;\alpha}x(t) &= \mu_1(t,x(t),y(t)), & t\in(0,\Psi], \\ {}_{m}^{c}\mathbb{D}_{0^+}^{i,\alpha}y(t) &= \mu_2(t,x(t),y(t)), & t\in(0,\Psi], \\ \mathbb{I}_{0^+}^{1-\beta;\alpha}x(0^+) &= x_0, & x_0\in\mathbb{R}, \\ y(0) &= y_0, & y_0\in\mathbb{R}, \end{cases} \tag{1}$$

where $\mu_1,\mu_2:(0,\Psi]\times\mathbb{R}^2\to\mathbb{R}$ are the given functions. The weighted function $m(t)$ for $t\in(0,\Psi]$ is nonzero, and $\alpha\colon (0,\Psi]\to\mathbb{R}^+$ is strictly increasing such that $\alpha\in C^1(0,\Psi]$ with $\alpha'(t)\neq 0$ for all $t\in(0,\Psi]$. Therefore, ${}^{\mathscr{H}}\mathbb{D}_{0^+}^{i,\Upsilon;\alpha}(.)$ is the α-Hilfer fractional derivative of order $0<i\leq 1$ and type $0\leq\Upsilon\leq 1$, further ${}_{m}^{c}\mathbb{D}_{0^+}^{i,\alpha}(.)$ is the weighted generalized Caputo fractional derivative (WCFD) of order $0<i\leq 1$ concerning the function α, and $\mathbb{I}_{0^+}^{1-\beta;\alpha}(.)$ is the Riemann-Liouville fractional integral of order $1-\beta$, in which $\beta=i+\Upsilon(1-i)$. Let $C^n[\alpha,\theta]$ be the space of n differentiable functions with each of derivatives is continuous.

2. Preliminaries

This section contains the important theorems, lemmas, and definitions that are pertinent to main results.

Theorem 2.1 ([24]) [Banach fixed point] Suppose contractive mapping $\Phi: h\to h$ with constant $0\leq q<1$ so that nonempty closed set h be in a Banach space Ξ. Thus:

1. There is the unique $j^*\in h$ such that $j^*=\Phi(j^*)$.
2. For any $j_0\in h$, the sequence $\{j_n\}\subset h$ is defined by $j_{n+1}=\Phi(j_n)$, $n\in\mathbb{N}$, converges to j^*:

$$\|j_n-j^*\|_\Xi\to 0, \qquad \text{as} \quad n\to\infty.$$

Theorem 2.2 ([25]) [Alternative fixed point] Consider generalized complete metric space (Ξ,Θ), and Lipschitz mapping $\Phi\colon \Xi\longrightarrow\Xi$ with constant $q<1$. Then, for some $i\in\Xi$ either,

$$\Theta(\Phi^m(i),\Phi^{m+1}(i))=+\infty \quad (m\geq 0),$$

or we can find a natural number m_0 such that:

$$\Theta(\Phi^m(i),\Phi^{m+1}(i))<+\infty \quad (\forall m\geq m_0),$$

then the followings are true:

1. The fixed point j of Φ is the convergence point of the sequence $\{\Phi^n(i)\}$;
2. In the set $\Lambda=\{j^*\in\Xi\colon \Theta(\Phi^m(i),j^*)<+\infty\}$, j is the unique fixed point of Φ;
3. For all $j^*\in\Lambda$, $\Theta(j^*,j)\leq\frac{1}{1-q}\Theta(j^*,\Phi(j^*))$.

Definition 2.3 ([5]) *Assume the gamma function Γ. A generalization of the exponential function is the Mittag-Leffler function given by*

$$E_i(\mu):=\sum_{\tau=0}^{\infty}\frac{\mu^\tau}{\Gamma(\tau i+1)}, \qquad i\in C,\ \mathbf{Re}(i)>0. \tag{2}$$

Definition 2.4 ([4, 7]) *Suppose (θ,i) on the real line $\mathbb{R}$, so that $i>0$. Furthermore, assume positive and non-decreasing monotone function $\alpha(t)$ on $(\theta,i]$ which has a continuous derivative $\alpha'(t)$ on (θ,i). The α-Hilfer and weighted fractional integrals of a function x, concerning α, on $[\theta,i]$ are defined by*

$$\mathbb{I}_{\alpha^+}^{i;\alpha}x(t)=\frac{1}{\Gamma(i)}\int_\theta^t \alpha'(s)\left(\alpha(t)-\alpha(s)\right)^{i-1}x(s)ds, \tag{3}$$

and

$${}_{m}\mathbb{I}_{0^+}^{i,\alpha}y(t)=\frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t)-\alpha(s))^{i-1}y(s)ds, \tag{4}$$

respectively.

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Definition 2.5 (**[4, 7]**) *Suppose non-decreasing* $\alpha \in C^q[\theta, i]$ *with* $\alpha'(t) \neq 0$ *for* $t \in [\theta, i]$, *and* $x \in C^q[\theta, i]$. *Further let* $i \in (q-1, q)$, *and* $q \in \mathbb{N}$. *The fractional derivatives of a function x of type* $0 \leq \Upsilon \leq 1$ *and order i is defined by*

$$^{\mathscr{H}}\mathbb{D}_{t_0^+}^{i,\Upsilon;\alpha}x(t) = \mathbb{I}_{t_0^+}^{\Upsilon(q-i);\alpha}\left(\frac{1}{\alpha'(t)}\frac{d}{dt}\right)^q \mathbb{I}_{t_0^+}^{(1-\Upsilon)(q-i);\alpha}x(t),$$

and

$$^{c}_{m}\mathbb{D}_{0^+}^{i,\alpha}y(t) = {}_m\mathbb{I}_{0^+}^{q-i,\alpha}m(t)\left(\frac{1}{\alpha'(t)}\frac{d}{dt}\right)^q y(t).$$

Theorem 2.6 (**[4, 7]**) *Let* $x, y \in C^1[0, i], i \in (0,1)$ *and* $\Upsilon \in [0,1]$, *then*

$$\begin{aligned}
&^{\mathscr{H}}\mathbb{D}_{0^+}^{i,\Upsilon;\alpha}\mathbb{I}_{0^+}^{i;\alpha}x(t) = x(t),\\
&^{c}_{m}\mathbb{D}_{0^+}^{i;\alpha}{}_m\mathbb{I}_{0^+}^{i;\alpha}y(t) = y(t),\\
&\mathbb{I}_{0^+}^{i;\alpha\,\mathscr{H}}\mathbb{D}_{0^+}^{i,\Upsilon;\alpha}x(t) = x(t)\\
&\qquad -\frac{(\alpha(t)-\alpha(0))^{\beta-1}}{\Gamma(\beta)}\mathbb{I}_{0^+}^{(1-\Upsilon)(1-i);\alpha}x_0,\\
&{}_m\mathbb{I}_{0^+}^{i;\alpha\,c}{}_m\mathbb{D}_{0^+}^{i;\alpha}y(t) = y(t) - \frac{m(0)y_0}{m(t)}.
\end{aligned}$$

Lemma 2.7 (**[18]**) *Considering the continuous functions* $\mu_i: (0,\Psi]\times\mathbb{R}^2 \to \mathbb{R}$ *for* $i = 1, 2$. *Then, the system* (1) *is equivalent to*

$$\begin{aligned}
(x(t), y(t)) = \Bigg(&\frac{(\alpha(t)-\alpha(0))^{\beta-1}}{\Gamma(\beta)}x_0\\
&+\frac{1}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t)-\alpha(s))^{i-1}\mu_1(s, x(s), y(s))ds,\\
&\frac{m(0)y_0}{m(t)} + \frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t)\\
&-\alpha(s))^{i-1}\mu_2(s, x(s), y(s))ds\Bigg).
\end{aligned}$$

Definition 2.8 *Suppose* $\varphi \in C((0,\Psi],\mathbb{R})$, *if for each* $(x, y) \in C((0,\Psi],\mathbb{R}) \times C((0,\Psi],\mathbb{R})$ *the following condition is verified:*

$$\begin{cases}
\Big|\frac{(\alpha(t)-\alpha(0))^{\beta-1}}{\Gamma(\beta)}x_0 + \frac{1}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t)-\alpha(s))^{i-1}\\
\quad\times\mu_1(s, x(s), y(s))ds - x(t)\Big| \leq \varphi(t),\\
\Big|\frac{m(0)y_0}{m(t)} + \frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t)-\alpha(s))^{i-1}\\
\quad\times\mu_2(s, x(s), y(s))ds - y(t)\Big| \leq \frac{\varphi(t)}{m(t)}
\end{cases} \tag{5}$$

then the system (1) *is HUR stable and there exist positive constants* σ_1, σ_2 *and a solution* $(h, z) \in C((0,\Psi],\mathbb{R}) \times C((0,\Psi],\mathbb{R})$ *of* (1) *such that*

$$\begin{cases}
|h(t) - x(t)| \leq \sigma_1\varphi(t), & t \in (0,\Psi),\\
|z(t) - y(t)| \leq \sigma_2\frac{\varphi(t)}{m(t)}, & t \in (0,\Psi).
\end{cases} \tag{6}$$

To simplify, we define $\Omega := C((0,\Psi],\mathbb{R}) \times C((0,\Psi],\mathbb{R})$.

Remark 2.9 *In particular, when* $\varphi(t) = E_i(-|(\alpha(t) - \alpha(0))^i|)$, *the system is said to be Hyers-Ulam-Mittag-Leffler (HUML) stable.*

3. HUR stability

In this section, we prove the existence of a unique solution for the system (1). Further, the stability result of HUR for the system (1), according to the alternative fixed point theorem is investigated. For more details, refer to [20].

Lemma 3.1 *Let* $\varphi \in C((0,\Psi],\mathbb{R})$, *and* $(x_1(t), y_1(t)), (x_2(t), y_2(t)) \in \Omega$ *for* $t \in (0,\Psi]$, *if we define a mapping* $\phi: \Omega\times\Omega \to [0, +\infty]$, *by*

$$\begin{aligned}
&\phi\left((x_1(t), y_1(t)), ((x_2(t), y_2(t))\right) :=\\
&\inf\left\{c_1 \geq 0 : |x_1(t) - x_2(t)| \leq c_1\varphi(t)\right\}\\
&+\inf\left\{c_2 \geq 0 : |y_1(t) - y_2(t)| \leq c_2\frac{\varphi(t)}{m(t)}\right\}.
\end{aligned} \tag{7}$$

Then we demonstrate that (Ω, ϕ) *constitutes a generalized metric space.*

Proof. For every $t \in (0,\Psi]$ we show that $\phi\left((x_1(t), y_1(t)), (x_2(t), y_2(t))\right) = 0$ if and only if $(x_1(t), y_1(t)) = (x_2(t), y_2(t))$. Assume $\phi\left((x_1(t), y_1(t)), (x_2(t), y_2(t))\right) = 0$, then for all $(x_1(t), y_1(t)), (x_2(t), y_2(t)) \in \Omega$, and $t \in (0,\Psi]$, we have:

$$\begin{aligned}
&\inf\left\{c_1 \geq 0 : |x_1(t) - x_2(t)| \leq c_1\varphi(t)\right\}\\
&+\inf\left\{c_2 \geq 0 : |(y_1(t) - y_2(t))| \leq c_2\frac{\varphi(t)}{m\,(t)}\right\} = 0,
\end{aligned}$$

so for every $t \in (0,\Psi]$ we have $(x_1(t), y_1(t)) = (x_2(t), y_2(t))$, and conversely.
This can be easily demonstrated

$$\begin{aligned}
&\phi\left((x_1(t), y_1(t)), (x_2(t), y_2(t))\right)\\
&= \phi\left((x_2(t), y_2(t)), (x_1(t), y_1(t))\right),
\end{aligned}$$

For every $(x_1(t), y_1(t)), (x_2(t), y_2(t)) \in \Omega$, and $t \in (0,\Psi]$, we have

$$\begin{aligned}
&\phi\left((x_1(t), y_1(t)), (x_2(t), y_2(t))\right)\\
&= \inf\left\{c_1 \geq 0 : |x_1(t) - x_2(t)| \leq c_1\varphi(t)\right\}\\
&+\inf\left\{c_2 \geq 0 : |(y_1(t) - y_2(t))| \leq c_2\frac{\varphi(t)}{m\,(t)}\right\}\\
&= \inf\left\{c_1 \geq 0 : |x_2(t) - x_1(t)| \leq c_1\varphi(t)\right\}\\
&+\inf\left\{c_2 \geq 0 : |(y_2(t) - y_1(t))| \leq c_2\frac{\varphi(t)}{m\,(t)}\right\}\\
&= \phi\left((x_2(t), y_2(t)), (x_1(t), y_1(t))\right).
\end{aligned}$$

Now, consider for $i = 1, \ldots, 4$; there exist constants g_i such that $\phi\left((x_1(t), y_1(t)), (x_2(t), y_2(t))\right) = g_1 + g_2$ and $\phi\left((x_2(t), y_2(t)), (x_3(t), y_3(t))\right) = g_3 + g_4$. Then, we have

$$\begin{cases} |x_1(t)-x_2(t)| \leq g_1\varphi(t), \\ |y_1(t)-y_2(t)| \leq g_2\dfrac{\varphi(t)}{m\,(t)}, \\ |x_2(t)-x_3(t)| \leq g_3\varphi(t), \\ |y_2(t)-y_3(t)| \leq g_4\dfrac{\varphi(t)}{m\,(t)}, \end{cases} \tag{8}$$

according to the triangular property in absolute magnitudes, we can define $k_1 = g_1 + g_3$ and $k_2 = g_2 + g_4$ such that:

$$\begin{cases} |x_1(t)-x_3(t)| \leq |x_1(t)-x_2(t)| \\ \qquad + |x_2(t)-x_3(t)| \leq k_1\varphi(t), \\ \\ |y_1(t)-y_3(t)| \leq |y_1(t)-y_2(t)| \\ \qquad + |y_2(t)-y_3(t)| \leq k_2\dfrac{\varphi(t)}{m(t)}. \end{cases} \tag{9}$$

If we consider the above relationships for k_1, k_2 and the sum of their infimum values, we obtain:

$$\begin{aligned} &\phi\left((x_1(t), y_1(t)), (x_3(t), y_3(t))\right) \\ &= \inf\left\{k_1 \geq 0 : |x_1(t)-x_3(t)| \leq k_1\varphi(t)\right\} \\ &\quad + \inf\left\{k_2 \geq 0 : |y_1(t)-y_3(t)| \leq k_2\frac{\varphi(t)}{m\,(t)}\right\} \\ &\leq k_1 + k_2 \\ &= (g_1+g_2) + (g_3+g_4) \\ &= \phi\left((x_1(t), y_1(t)), (x_2(t), y_2(t))\right) \\ &+ \phi\left((x_2(t), y_2(t)), (x_3(t), y_3(t))\right). \end{aligned}$$

Now, we demonstrate the completeness of (Ω, ϕ). Consider a Cauchy sequence $\{(x_\upsilon(t), y_\upsilon(t))\}_\upsilon$. This means that for any positive number i, there exists a natural N_i such that for all $\upsilon, k \geqslant \mathsf{N}_i$, we have:

$$\phi((x_\upsilon(t), y_\upsilon(t)), (x_k(t), y_k(t))) < i.$$

With definition (7), for all $t \in (0, \Psi]$ there exists i_1 and i_2 such that $i = i_1 + i_2$, and then we can express:

$$\begin{cases} |x_\upsilon(t)-x_k(t)| \leqslant i_1\varphi(t), \\ |y_\upsilon(t)-y_k(t)| \leqslant i_2\dfrac{\varphi(t)}{m\,(t)}. \end{cases} \tag{10}$$

Let $t \in (0, \Psi]$ is fixed, so the sequence $\{(x_\upsilon(t), y_\upsilon(t))\}_\upsilon$ is a Cauchy sequence in $\mathbb{R}^2$ and, due to the completeness of $\mathbb{R}^2$, converges for each $t \in (0, \Psi]$. Consequently, we can define a function $(x(t), y(t))\colon (0, \Psi] \times (0, \Psi] \to \mathbb{R}^2$ by

$$(x(t), y(t)) = \lim_{\upsilon\to\infty} (x_\upsilon(t), y_\upsilon(t)).$$

Given that $\varphi(t)$ is continuous and bounded over the compact interval $[0, \Psi]$. Thus, (10) implies that $\{(x_\upsilon(t), y_\upsilon(t))\}_\upsilon$ converges uniformly to $(x(t), y(t))$ in the usual topology of $\mathbb{R}^2$. Hence, $(x, y) \in \Omega$. If we suppose k approaches to infinity, for $i > 0$ a natural N_i exists such that for $\upsilon \geqslant \mathsf{N}_i$ it follows from (10) that

$$\begin{cases} |x_\upsilon(t)-x(t)| \leqslant i_1\varphi(t), \\ |y_\upsilon(t)-y(t)| \leqslant i_2\dfrac{\varphi(t)}{m\,(t)}, \end{cases} \tag{11}$$

for all posetive i there exists natural N_i such that for all $\upsilon > \mathsf{N}_i$ considering equation (7), we obtain

$$\phi((x(t), y(t)), (x_\upsilon(t), y_\upsilon(t))) \leqslant i.$$

This indicates that the Cauchy sequence$\{(x_\upsilon, y_\upsilon)\}$ converges to (x, y) in (Ω, ϕ), demonstrating the completeness of (Ω, ϕ).

Theorem 3.2 *Suppose $(x, y) \in \Omega$ satisfies (5) and $\mathbb{I}_{0^+}^{1-\beta;\alpha} x\,(0^+) = x_0 \in \mathbb{R}$. Considering $\varphi \in C\left((0, \Psi], \mathbb{R}\right)$, and conditions $(M_1), (M_2)$ hold:*

(M_1) *Exists $w \in (0, 1)$ that*

$$\begin{aligned} \mathbb{I}_{0^+}^{i;\alpha}\varphi(t) &= \frac{1}{\Gamma(i)}\int_0^t \alpha'(t_0)(\alpha(t)-\alpha(t_0))^{i-1}\varphi(t_0)dt_0 \\ &\leq w\varphi(t), \end{aligned} \tag{12}$$

$$\begin{aligned} &{}_m\mathbb{I}_{0^+}^{i;\alpha}\frac{\varphi(t)}{m(t)} \\ &= \frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(t_0)\alpha'(t_0)(\alpha(t)-\alpha(t_0))^{i-1}\frac{\varphi(t_0)}{m(t_0)}dt_0 \\ &\leq w\frac{\varphi(t)}{m(t)}. \end{aligned} \tag{13}$$

(M_2) *If $\mu_1, \mu_2 \in C\left((0, \Psi] \times \mathbb{R}^2, \mathbb{R}\right)$, then positive constants δ and ϵ exist so*

$$\begin{cases} |\mu_1(t, q_1, q_2) - \mu_1(t, \lambda_1, \lambda_2)| \\ \quad \leq \delta\left(|q_1-\lambda_1| + m(t)\,|q_2-\lambda_2|\right), \ q_t, \lambda_t \in \mathbb{R}, \\ \\ |\mu_2(t, q_1, q_2) - \mu_2(t, \lambda_1, \lambda_2)| \\ \quad \leq \epsilon\left(m^{-1}(t)\,|q_1-\lambda_1| + |q_2-\lambda_2|\right), \ q_t, \lambda_t \in \mathbb{R}, \\ \\ (\delta+\epsilon)w < 1. \end{cases} \tag{14}$$

Thus the unique solution of the system (1) is $(h, z) \in \Omega$ such that

$$\begin{aligned} (h(t), z(t)) = \Bigg(&\frac{(\alpha(t)-\alpha(0))^{\beta-1}}{\Gamma(\beta)}h_0 + \\ &\frac{1}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t)-\alpha(s))^{i-1}\mu_1(s, h(s), z(s))ds, \\ &\frac{m(0)z_0}{m(t)} + \frac{m^{-1}(t)}{\Gamma(i)} \times \\ &\int_0^t m(s)\alpha'(s)(\alpha(t)-\alpha(s))^{i-1}\mu_2(s, h(s), z(s))ds\Bigg), \end{aligned}$$

in which, we have $\mathbb{I}_{0^+}^{1-\beta;\alpha} h\,(0^+) := x_0 \in \mathbb{R}$, and $\lambda := (\delta+\epsilon)w$

$$\begin{cases} |x(t)-h(t)| \leq \left(\dfrac{c_1}{1-\lambda}\right)\varphi(t), \\ |y(t)-z(t)| \leq \left(\dfrac{c_2}{1-\lambda}\right)\dfrac{\varphi(t)}{m\,(t)}. \end{cases} \tag{15}$$

Proof. Assume that $(x_1(t), y_1(t)), (x_2(t), y_2(t)) \in \Omega$ and $t \in (0, \Psi]$ define a mapping $\phi : \Omega \times \Omega \to [0, +\infty]$

by

$$\phi\left((x_1(t), y_1(t)), ((x_2(t), y_2(t))\right) = \inf\left\{c_1 \geq 0 : |x_1(t) - x_2(t)| \leq c_1\varphi(t)\right\} + \inf\left\{c_2 \geq 0 : |y_1(t) - y_2(t)| \leq c_2\frac{\varphi(t)}{m(t)}\right\}. \quad (16)$$

According to Lemma 3.1 (Ω, ϕ) is a generalized metric space. Step 1. In this step, we prove the contractionary of the operator $H\colon \Omega \to \Omega$ that is defined by

$$H(x(t), y(t)) = \Bigg(\frac{(\alpha(t) - \alpha(0))^{\beta-1}}{\Gamma(\beta)}x_0 + \frac{1}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t) - \alpha(s))^{i-1}\mu_1(s, x(s), y(s))\,ds, \quad (17)$$
$$\frac{m(0)\,y_0}{m(t)} + \frac{m^{-1}(t)}{\Gamma(i)} \times \int_0^t m(s)\alpha'(s)(\alpha(t) - \alpha(s))^{i-1}\mu_2(s, x(s), y(s))ds\Bigg).$$

Based on the fundamental theorem of calculus, we can conclude that $H(x(t), y(t))$ is continuously differentiable on the interval $[0, \Psi]$, since $(x(t), y(t))$ is a continuous function, which implies that $H(x(t), y(t)) \in \Omega$. We claim that H is contractive on Ω. Let $(x(t), y(t)), (h(t), z(t)) \in \Omega$, and $\phi\left((x(t), y(t)), (h(t), z(t))\right) = b$ so that $b \in [0, \infty]$ be a constant. With definition of metric (7), for all $t \in (0, \Psi]$ there are b_1, b_2 so $b = b_1 + b_2$. Furthermore, we can express:

$$\begin{cases} |x(t) - h(t)| \leq b_1\varphi(t), \\ |y(t) - z(t)| \leq b_2\dfrac{\varphi(t)}{m(t)}. \end{cases} \quad (18)$$

From (7) and (17), we can deduce that

$$\Bigg|\frac{1}{\Gamma(i)}\int_0^t \alpha'(s)\big(\alpha(t) - \alpha(s)\big)^{i-1}\mu_1\big(s, x(s), y(s)\big)\,ds - \frac{1}{\Gamma(i)}\int_0^t \alpha'(s)\big(\alpha(t)-\alpha(s)\big)^{i-1}\mu_1\big(s, h(s), z(s)\big)\,ds\Bigg|$$
$$= \Bigg|\frac{1}{\Gamma(i)}\int_0^t \alpha'(s)\big(\alpha(t) - \alpha(s)\big)^{i-1} \times \Big(\mu_1\big(s, x(s), y(s)\big) - \mu_1\big(s, h(s), z(s)\big)\Big)\,ds\Bigg|$$
$$\leq \frac{1}{\Gamma(i)}\int_0^t \alpha'(s)\big(\alpha(t) - \alpha(s)\big)^{i-1} \times \Big|\mu_1\big(s, x(s), y(s)\big) - \mu_1\big(s, h(s), z(s)\big)\Big|\,ds,$$

using (14) we have:

$$\leq \frac{\delta}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t) - \alpha(s))^{i-1}|x(s) - h(s)|\,ds + \frac{\delta}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t)-\alpha(s))^{i-1}|y(s) - z(s)|\,ds,$$

by utilizing (18) and (M_1), we get:

$$\leq \frac{\delta}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t) - \alpha(s))^{i-1}b_1\varphi(s)ds + \frac{\delta}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t) - \alpha(s))^{i-1}b_2\varphi(s)ds$$
$$\leq \frac{\delta(b_1 + b_2)}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t) - \alpha(s))^{i-1}\varphi(s)ds$$
$$\leq \delta(b_1' + b_2')w\varphi(t). \quad (19)$$

Similarly, with the help of relations (7) and (17), we can deduce that

$$\Bigg|\frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t) - \alpha(s))^{i-1}\mu_2(s, x(s), y(s))ds - \frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t) - \alpha(s))^{i-1}\mu_2(s, h(s), z(s))ds\Bigg|$$
$$= \Bigg|\frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t) - \alpha(s))^{i-1}(\mu_2(s, x(s), y(s)) - \mu_2(s, h(s), z(s)))ds\Bigg|$$
$$\leq \frac{m^{-1}(t)}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t) - \alpha(s))^{i-1}m(s)|\mu_2(s, x(s), y(s)) - \mu_2(s, h(s), z(s))|ds,$$

using (14) we have:

$$\leq \frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t) - \alpha(s))^{i-1} \times \epsilon\Big(m^{-1}(s)\,|x(s) - h(s)| + |y(s) - z(s)|\Big)\,ds$$
$$= \frac{\epsilon m^{-1}(t)}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t) - \alpha(s))^{i-1}\,|x(s) - h(s)|\,ds + \frac{\epsilon m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t) - \alpha(s))^{i-1}|y(s) - z(s)|ds,$$

by utilizing (18) and (M_1), we get:

$$\leq \frac{\epsilon m^{-1}(t)}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t) - \alpha(s))^{i-1}b_1\varphi(s)ds + \frac{\epsilon m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t) - \alpha(s))^{i-1}b_2\frac{\varphi(s)}{m(s)}ds$$
$$\leq m^{-1}(t)\frac{\epsilon(b_1 + b_2)}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t) - \alpha(s))^{i-1}\varphi(s)ds$$
$$\leq \epsilon(b_1' + b_2')w\frac{\varphi(t)}{m(t)}. \quad (20)$$

So due to (19)-(20), and $\phi\left((x(t), y(t)), (h(t), z(t))\right) = b$, we can derive:

$$\phi\big(H(x(t), y(t)), H(h(t), z(t))\big) \leq (\epsilon + \delta)w\phi\big((x(t), y(t)), (h(t), z(t))\big). \quad (21)$$

Due to (M_2), we get H is contractionary.

Step 2. Let $(x(t), y(t)) \in \Omega$, then we try to illustrate that $\phi\big(\,(x(t), y(t))\,, H\,(x(t), y(t))\,\big) < \infty$. According to

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(7), and (17), we have

$$\phi\big((x(t),y(t)) ,H(x(t),y(t))\big) = \phi\Bigg((x(t),y(t)),\Bigg(\frac{(\alpha(t)-\alpha(0))^{\beta-1}}{\Gamma(\beta)}x_0 \frac{1}{\Gamma(i)}\int_0^t \alpha^{'}(s)\left(\alpha(t)-\alpha(s)\right)^{i-1}\mu_1\left(s,x(s),y(s)\right)ds, \frac{m(0)y_0}{m(t)}+\frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha^{'}(s)\left(\alpha(t)-\alpha(s)\right)^{i-1} \times\mu_2\left(s,x\left(s\right),y\left(s\right)\right)ds\Bigg)\Bigg)$$

$$= \inf\Bigg\{c_1>0:\Bigg|x\left(t\right)-\frac{\left(\alpha\left(t\right)-\alpha\left(0\right)\right)^{\beta-1}}{\Gamma\left(\beta\right)}x_0 +\frac{1}{\Gamma\left(i\right)}\int_0^t\alpha^{'}\left(s\right)\left(\alpha\left(t\right)-\alpha\left(s\right)\right)^{i-1} \times\mu_1(s,x\left(s\right),y\left(s\right))ds\Bigg|<c_1\varphi(t)\Bigg\}$$

$$+\inf\Bigg\{c_2>0:\Bigg|y\left(t\right)-\frac{m\left(0\right)y_0}{m\left(t\right)} +\frac{m^{-1}\left(t\right)}{\Gamma\left(i\right)}\int_0^t m\left(s\right)\alpha^{'}\left(s\right)\left(\alpha\left(t\right)-\alpha\left(s\right)\right)^{i-1} \times\mu_2\left(s,x(s),y(s)\right)ds\Bigg|<c_2\frac{\varphi(t)}{m\left(t\right)}\Bigg\}<c_1+c_2, \tag{22}$$

from (5), we have $c_1=c_2=1$; therefore, the inequality $c_1+c_2\leq 2<\infty$ holds. Now, according to the Theorem 2.2, we can find the element (h,z) in Ω that satisfies the following conditions:

1. The sequence $\{H^m\left(x(t),y(t)\right)\}$ converges to the fixed point $(h(t),z(t))\in H$;
2. $(h(t),z(t))\in\Omega^*$ is the unique fixed point of H, such that,

$$H(h(t),z(t)) = \Bigg(\frac{(\alpha(t)-\alpha(0))^{\beta-1}}{\Gamma(\beta)}h_0 +\frac{1}{\Gamma(i)}\int_0^t\alpha^{'}(s)(\alpha(t)-\alpha(s))^{i-1}\mu_1(s,h(s),z(s))ds, \frac{m(0)z_0}{m(t)}+\frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha^{'}(s)(\alpha(t)-\alpha(s))^{i-1} \times\mu_2(s,h(s),z(s))ds\Bigg),$$

   which $\Omega^*=\{(x(t),y(t))\in\Omega:\phi(H\left((x_\circ(t),y_\circ(t)\right),(x(t),y(t))))<\infty\}$.
3. If $(x(t),y(t))\in\Omega^*$, and $\lambda:=(\epsilon+\delta)w$, then we have

$$\phi\left(\left(x(t),y(t)\right),\left(h(t),z(t)\right)\right) \leq\frac{1}{1-\lambda}\phi\left(H\left(x(t),y(t)\right),\left(x(t),y(t)\right)\right).$$

From (22) we get

$$\phi\left(\left(x(t),y(t)\right),\left(h(t),z(t)\right)\right)\leq\frac{1}{1-\lambda}(c_1+c_2).$$

Then

$$\begin{cases} |x(t)-h(t)|\leq\left(\frac{c_1}{1-\lambda}\right)\varphi(t),\\ |y(t)-z(t)|\leq\left(\frac{c_2}{1-\lambda}\right)\frac{\varphi(t)}{m\left(t\right)}. \end{cases} \tag{23}$$

4. HUML stability

This section demonstrates that the system (1) has a solution and the solution of (1) is unique. Further, the stability of HUML for (1) according to the alternative fixed point theorem is investigated. For more details, refer to [20].

Lemma 4.1 *If* $\alpha\in C^1((0,\Psi],[0,1])$*, and there exists* $w^{'}\in(0,1)$ *such that*

$$\mathbb{I}_{0^+}^{i;\alpha}E_i(-|(\alpha(t)-\alpha(0))^i|) = \frac{1}{\Gamma(i)}\int_0^t\alpha'\left(t_0\right)\left(\alpha(t)-\alpha\left(t_0\right)\right)^{i-1} \times E_i(-|(\alpha(t_0)-\alpha(0))^i|)dt_0 \leq w^{'}E_i(-|(\alpha(t)-\alpha(0))^i|), \tag{24}$$

$$_m\mathbb{I}_{0^+}^{i;\alpha}\frac{E_i(-|(\alpha(t)-\alpha(0))^i|)}{m(t)} = \frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(t_0)\alpha'\left(t_0\right)\left(\alpha(t)-\alpha\left(t_0\right)\right)^{i-1} \times\frac{E_i(-|(\alpha(t_0)-\alpha(0))^i|)}{m(t_0)}dt_0 \leq w^{'}\frac{E_i(-|(\alpha(t)-\alpha(0))^i|)}{m(t)}. \tag{25}$$

Proof. We prove relation (24).

$$\frac{1}{\Gamma(i)}\int_0^t\alpha'(t_0)(\alpha(t)-\alpha(t_0))^{i-1}E_i(-|(\alpha(t_0)-\alpha(0))^i|)dt_0 = \frac{1}{\Gamma(i)}\int_0^t\alpha'\left(t_0\right)\left(\alpha(t)-\alpha\left(t_0\right)\right)^{i-1} \times\sum_{k=0}^{\infty}\frac{(-1)^{ki}\left(\alpha\left(t_0\right)-\alpha(0)\right)^{ki}}{\Gamma(ki+1)}dt_0 = \frac{1}{\Gamma(i)}\sum_{k=0}^{\infty}\frac{(-1)^{ki}}{\Gamma(ki+1)} \times\int_0^t\alpha'\left(t_0\right)\left(\alpha(t)-\alpha\left(t_0\right)\right)^{i-1}\left(\alpha\left(t_0\right)-\alpha(0)\right)^{ki}dt_0,$$

if we choose $s=(\alpha\left(t_0\right)-\alpha(0))$, then:

$$= \frac{1}{\Gamma(i)}\sum_{k=0}^{\infty}\frac{(-1)^{ki}}{\Gamma(ki+1)}\int_0^{\alpha(t)-\alpha(0)}(\alpha(t)-\alpha(0)-s)^{i-1}s^{ki}ds = \frac{1}{\Gamma(i)}\sum_{k=0}^{\infty}\frac{(-1)^{ki}(\alpha(t)-\alpha(0))^{i-1}}{\Gamma(ki+1)} \times\int_0^{\alpha(t)-\alpha(0)}\left(1-\frac{s}{\alpha(t)-\alpha(0)}\right)^{i-1}s^{ki}ds,$$

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let $z = \dfrac{s}{\alpha(t)-\alpha(0)}$,

$$= \frac{1}{\Gamma(i)} \sum_{k=0}^{\infty} \frac{(-1)^{ki}(\alpha(t)-\alpha(0))^{(k+1)i-1}}{\Gamma(ki+1)} \times \int_0^1 (1-z)^{i-1} z^{ki} dz$$
$$= \frac{1}{\Gamma(i)} \sum_{k=0}^{\infty} \frac{(-1)^{ki}(\alpha(t)-\alpha(0))^{(k+1)i-1}\Gamma(ki+1)\Gamma(i)}{\Gamma(ki+1)\Gamma((k+1)i+1)}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^{ki}(\alpha(t)-\alpha(0))^{(k+1)i-1}}{\Gamma((k+1)i+1)}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^{ki}(\alpha(t)-\alpha(0))^{ki}(\alpha(t)-\alpha(0))^{i-1}}{(k+1)i\Gamma((k+1)i)},$$

Similarly, using the properties of the gamma function, we have

$$\frac{1}{\Gamma(i)}\int_0^t \alpha'(t_0)(\alpha(t)-\alpha(t_0))^{i-1}E_i(-|(\alpha(t_0)-\alpha(0))^i|)dt_0$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^{ki}(\alpha(t)-\alpha(0))^{ki}(\alpha(t)-\alpha(0))^{i-1}}{(k+1)i(ki+i-1)\cdots(ki+1)\Gamma(ki+1)}$$
$$= \sum_{k=0}^{\infty} \frac{(\alpha(t)-\alpha(0))^{i-1}}{(k+1)i(ki+i-1)\cdots(ki+1)} \times E_i(-|(\alpha(t_0)-\alpha(0))^i|),$$

where α is bounded, because $\alpha \in C^1((0,\Psi],[0,1])$, so there exists a positive costant w' such that we can write:

$$\frac{1}{\Gamma(i)} \int_0^t \alpha'(t_0)(\alpha(t)-\alpha(t_0))^{i-1} \times E_i(-|(\alpha(t_0)-\alpha(0))^i|)dt_0 \le w' E_i(-|(\alpha(t)-\alpha(0))^i|),$$

therefore

$$\mathbb{I}_{0^+}^{i;\alpha} E_i(-|(\alpha(t)-\alpha(0))^i|) \le w' E_i(-|(\alpha(t)-\alpha(0))^i|).$$

In the same way (25) is also proved.

Corollary 4.2 *(Ω,ϕ) is a generalized metric space if a mapping $\phi : \Omega\times\Omega \longrightarrow [0,+\infty]$ is defind by*

$$\phi\left((x_1(t),y_1(t)),((x_2(t),y_2(t))\right) \tag{26}$$
$$= \inf\left\{c_1 \ge 0 : |x_1(t)-x_2(t)| \le c_1 E_i(-|(\alpha(t)-\alpha(0))^i|)\right\}$$
$$+\inf\left\{c_2 \ge 0 : |y_1(t)-y_2(t)| \le c_2 \frac{E_i(-|(\alpha(t)-\alpha(0))^i|)}{m(t)}\right\},$$

when $(x_1(t),y_1(t)),(x_2(t),y_2(t)) \in \Omega$ and $t \in (0,\Psi]$.

Proof. Similarly to Lemma 3.1, we can prove this corollary.

Corollary 4.3 *Suppose $(x,y) \in \Omega$ satisfies (5) and $\mathbb{I}_{0^+}^{1-C;\alpha}x(0^+) = x_0 \in \mathbb{R}$. Let the following condition hold:*

(M_1') *If $\mu_1,\mu_2 \in C\left((0,\Psi]\times\mathbb{R}^2,\mathbb{R}\right)$, then positive constants δ and ϵ exist so*

$$\begin{cases} |\mu_1(t,q_1,q_2)-\mu_1(t,\lambda_1,\lambda_2)| \\ \le \delta\left(|q_1-\lambda_1| + m(t)|q_2-\lambda_2|\right), \quad q_t,\lambda_t \in \mathbb{R}, \\ |\mu_2(t,q_1,q_2)-\mu_2(t,\lambda_1,\lambda_2)| \\ \le \epsilon\left(m^{-1}(t)|q_1-\lambda_1| + |q_2-\lambda_2|\right), \quad q_t,\lambda_t \in \mathbb{R}, \\ (\delta+\epsilon)w' < 1, \end{cases} \tag{27}$$

where w' is introduced in Lemma 4.1.

Thus the unique solution of the system (1) is $(h,z) \in \Omega$ such that

$$(h(t),z(t)) = \Bigg(\frac{(\alpha(t)-\alpha(0))^{\beta-1}}{\Gamma(\beta)}h_0 + \frac{1}{\Gamma(i)}\int_0^t \alpha'(s)(\alpha(t)-\alpha(s))^{i-1}\mu_1(s,h(s),z(s))ds,$$
$$\frac{m(0)z_0}{m(t)} + \frac{m^{-1}(t)}{\Gamma(i)}\int_0^t m(s)\alpha'(s)(\alpha(t)-\alpha(s))^{i-1} \times \mu_2(s,h(s),z(s))ds\Bigg),$$

in which $\mathbb{I}_{0^+}^{1-c;\alpha}h(0^+) = x_0 \in \mathbb{R}$,

$$\begin{cases} |x(t)-h(t)| \le \left(\dfrac{c_1}{1-\lambda'}\right) E_i\left(-|(\alpha(t)-\alpha(0))^i|\right), \\ |y(t)-z(t)| \le \left(\dfrac{c_2}{1-\lambda'}\right) \dfrac{E_i\left(-|(\alpha(t)-\alpha(0))^i|\right)}{m(t)}. \end{cases} \tag{28}$$

Proof. Similarly to Theorem 3.2, we can prove this corollary.

5. Application

This section applies the main result to solve the following examples.

Example 5.1 *The fractional differential system*

$$\begin{cases} {}^{\mathscr{H}}\mathbb{D}_{0^+}^{\frac{1}{2},\Upsilon;\sin(t)}x(t) = \\ \dfrac{tanh(t+x(t)+y(t))(1+sech^2(t))}{20}, \quad t \in \left(0,\frac{\pi}{2}\right], \\ {}^{c}_{1+sech^2(t)}\mathbb{D}_{0^+}^{\frac{1}{2},\sin(t)}y(t) = \\ \dfrac{\tan^{-1}(t+x(t)+y(t))}{20(1+sech^2(t))}, \quad t \in \left(0,\frac{\pi}{2}\right], \\ \mathbb{I}_{0^+}^{1-\beta;\sin(t)}x(0^+) = x_0, \quad x_0 \in \mathbb{R}, \\ y(0) = y_0, \quad y_0 \in \mathbb{R}, \end{cases} \tag{29}$$

is an example of the system (1). $\mathbb{I}_{0^+}^{1-\beta;\sin(t)}(.)$ is the Riemann-Liouville fractional integral according to the function $\sin(t)$ and the order $1-\beta$, such that $\beta = \frac{1}{2}+\frac{1}{2}\Upsilon$. Furthermore, ${}^{\mathscr{H}}\mathbb{D}_{0^+}^{\frac{1}{2},\Upsilon;\sin(t)}(.)$ is the $\sin(t)$-Hilfer fractional derivative of type $0 \le \Upsilon \le 1$ with order $\frac{1}{2}$

and ${}^{c}_{1+sech^2(t)}\mathbb{D}_{0^+}^{\frac{1}{2},\sin(t)}(.)$ *is the weighted generalized Caputo fractional derivative (WCFD) of order* $\frac{1}{2}$.

First, we choose

$$(x(t), y(t)) := \left(\frac{(\sin(t) - \sin(0))^{\beta-1}}{\Gamma(\beta)}x_0, \frac{(1 + sech^2(0))y_0}{1 + sech^2(t)}\right) \in C((0, \frac{\pi}{2}], \mathbb{R}) \times C((0, \frac{\pi}{2}], \mathbb{R}),$$

such that for any $t \in (0, \frac{\pi}{2}]$, *we have*

$$\begin{aligned}
&\left|\frac{(\sin(t) - \sin(0))^{\beta-1}}{\Gamma(\beta)}x_0 + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t \cos(s) \times \frac{tanh(s + x(s) + y(s))(1 + sech^2(s))}{20\sqrt{\sin(t) - \sin(s)}}ds - x(t)\right| \\
&= \left|\frac{1}{\Gamma(\frac{1}{2})}\int_0^t \cos(s) \times \frac{tanh(s + x(s) + y(s))(1 + sech^2(s))}{20\sqrt{\sin(t) - \sin(s)}}ds\right| \\
&\le \frac{1}{10\sqrt{\pi}}\left|\int_0^t \frac{\cos(s)ds}{\sqrt{\sin(t) - \sin(s)}}\right|,
\end{aligned}$$

let $\alpha = \sin(s)$ *and* $d\alpha = \cos(s)ds$, *we have*

$$\begin{aligned}
&\le \frac{1}{10\sqrt{\pi}}\left|\int_0^{\sin(t)} \frac{d\alpha}{\sqrt{\sin(t) - \alpha}}\right| \\
&= \frac{1}{10\sqrt{\pi}}\left|2(-\sqrt{\sin(t)})\right| \\
&\le \frac{1}{5\sqrt{\pi}} \le \sum_{s=0}^{\infty}\frac{|\sin(t)|^{\frac{s}{2}}(-1)^s}{\Gamma(\frac{s}{2}+1)},
\end{aligned}$$

that is, we get

$$\begin{aligned}
&\left|\frac{(\sin(t) - \sin(0))^{\beta-1}}{\Gamma(\beta)}x_0 + \frac{1}{\Gamma(0.5)}\int_0^t \cos(s) \times \frac{tanh(s + x(s) + y(s))(1 + sech^2(s))}{20\sqrt{\sin(t) - \sin(s)}}ds - x(t)\right| \\
&\le E_{\frac{1}{2}}(-|\sin^{\frac{1}{2}}(t)|). \qquad (30)
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\left|\frac{(1+sech^2(0))y_0}{1 + sech^2(t)} + \frac{1}{(1+sech^2(t))\Gamma(\frac{1}{2})}\int_0^t(1+sech^2(s)) \times\cos(s)\frac{\tan^{-1}(s + x(s) + y(s))}{20(1 + sech^2(s))\sqrt{\sin(t) - \sin(s)}}ds - y(t)\right| \\
&= \frac{1}{(1+sech^2(t))\Gamma(\frac{1}{2})}\left|\int_0^t\cos(s)\frac{\tan^{-1}(s+x(s)+y(s))}{20\sqrt{\sin(t)-\sin(s)}}ds\right| \\
&\le \frac{1}{20(1 + sech^2(t))\sqrt{\pi}}\left|\int_0^t \frac{\cos(s)}{\sqrt{\sin(t) - \sin(s)}}ds\right|,
\end{aligned}$$

let $\alpha = \sin(s)$ *and* $d\alpha = \cos(s)ds$, *we can write*

$$\begin{aligned}
&\le \frac{1}{20(1 + sech^2(t))\sqrt{\pi}}\left|\int_0^{\sin(t)} \frac{d\alpha}{\sqrt{\sin(t) - \alpha}}\right| \\
&\le \frac{1}{20(1 + sech^2(t))\sqrt{\pi}}\left|2(-\sqrt{\sin(t)})\right| \\
&\le \frac{1}{10\sqrt{\pi}(1 + sech^2(t))} \\
&\le \frac{1}{(1 + sech^2(t))}\sum_{s=0}^{\infty}\frac{|\sin(t)|^{\frac{s}{2}}(-1)^s}{\Gamma(\frac{s}{2}+1)},
\end{aligned}$$

that is, we get

$$\begin{aligned}
&\left|\frac{(1+sech^2(0))y_0}{1+sech^2(t)} + \frac{1}{(1+sech^2(t))\Gamma(\frac{1}{2})}\int_0^t(1+sech^2(s)) \times\cos(s)\frac{\tan^{-1}(t+x(t)+y(t))}{20(1+sech^2(s))\sqrt{\sin(t)-\sin(s)}}ds-y(t)\right| \\
&\le \frac{E_{\frac{1}{2}}(-|(\sin(t))^{\frac{1}{2}}|)}{1 + sech^2(t)}. \qquad (31)
\end{aligned}$$

From relations (30) *and* (31), *we conclude that* (x, y) *satisfies in the inequity* (5).

Now, we check condition (M_1') *for the system* (29). *For any arbitrary* $(h(t), z(t)), (x(t), y(t)) \in C((0, \frac{\pi}{2}], \mathbb{R}) \times C((0, \frac{\pi}{2}], \mathbb{R})$ *and every* $t \in (0, \frac{\pi}{2}]$, *we have*

$$\begin{aligned}
&(1 + sech^2(t))\left|\frac{tanh(t + x(t) + y(t))}{20} - \frac{tanh(t + h(t) + z(t))}{20}\right| \\
&\le \frac{(1 + sech^2(t))}{20}|t + x(t) + y(t) - (t + h(t) + z(t))| \\
&\le \frac{1}{10}\left(|x(t) - h(t)| + (1 + sech^2(t))|y(t) - z(t)|\right). \qquad (32)
\end{aligned}$$

And

$$\begin{aligned}
&(1 + sech^2(t))^{-1}\left|\frac{\tan^{-1}(t + x(t) + y(t))}{20} - \frac{\tan^{-1}(t + h(t) + z(t))}{20}\right| \qquad (33) \\
&\le \frac{(1 + sech^2(t))^{-1}}{20}|x(t) + y(t) - h(t) - z(t)| \\
&\le \frac{1}{20}\left((1 + sech^2(t))^{-1}|x(t) - h(t)| + |y(t) - z(t)|\right).
\end{aligned}$$

According to Lemma (4.1), *for* $0 < w < 1$, *the following relations are obtained:*

$$\mathbb{I}_{0^+}^{\frac{1}{2};\sin(t)}E_{\frac{1}{2}}(-|(\sin(t) - \sin(0))^{\frac{1}{2}}|) \le wE_{\frac{1}{2}}(-|(\sin(t) - \sin(0))^{\frac{1}{2}}|), \qquad (34)$$

$${}_{1+sech^2(t)}\mathbb{I}_{0^+}^{\frac{1}{2};\sin(t)}\frac{E_{\frac{1}{2}}(-|(\sin(t) - \sin(0))^{\frac{1}{2}}|)}{1 + sech^2(t)} \le w\frac{E_{\frac{1}{2}}(-|(\sin(t) - \sin(0))^{\frac{1}{2}}|)}{1 + sech^2(t)}. \qquad (35)$$

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Due to $\lambda = \dfrac{3w}{20} < 1$ *and Corollary 4.3, we derive that there exists* $(p, q) \in C((0, \frac{\pi}{2}], \mathbb{R}) \times C((0, \frac{\pi}{2}], \mathbb{R})$ *such that it satisfies in the system (29), which is unique and we also have*

$$\begin{cases} |x(t) - p(t)| \le \left(\dfrac{c_1}{1-\lambda}\right) E_{\frac{1}{2}}\left(-|(\sin(t))^{\frac{1}{2}}|\right), \\ |y(t) - q(t)| \le \left(\dfrac{c_2}{1-\lambda}\right) \dfrac{E_{\frac{1}{2}}\left(-|(\sin(t))^{\frac{1}{2}}|\right)}{1 + sech^2(t)}, \end{cases} \tag{36}$$

for any $t \in (0, \dfrac{\pi}{2}]$.

Example 5.2 *Assume the following fractional differential system.*

$$\begin{cases} {}^{\mathscr{H}}\mathbb{D}_{0^+}^{\frac{1}{2},\frac{1}{3};\sin(t)} x(t) \\ \qquad = \dfrac{(x(t)+y(t))(2+\cos(t))}{60}, \qquad t \in \left(0, \frac{\pi}{2}\right], \\ {}^{c}_{2+\cos(t)}\mathbb{D}_{0^+}^{\frac{1}{2},\sin(t)} y(t) \\ \qquad = \dfrac{x(t)+y(t)}{60(2+\cos(t))}, \qquad t \in \left(0, \frac{\pi}{2}\right], \\ \mathbb{I}_{0^+}^{\frac{1}{3};\sin(t)} x(0^+) = 0, \\ y(0) = 0, \end{cases} \tag{37}$$

where $\mathbb{I}_{0^+}^{\frac{1}{3};\sin(t)}(.)$ *is the Riemann-Liouville fractional integral of order* $\dfrac{1}{3}$ *according to the function* $\sin(t)$. *Furthermore,* ${}^{\mathscr{H}}\mathbb{D}_{0^+}^{\frac{1}{2},\frac{1}{3};\sin(t)}(.)$ *is the* $\sin(t)$*-Hilfer fractional derivative of type* $\dfrac{1}{3}$ *with order* $\dfrac{1}{2}$ *and* ${}^{c}_{2+\cos(t)}\mathbb{D}_{0^+}^{\frac{1}{2},\sin(t)}(.)$ *is the weighted generalized Caputo fractional derivative (WCFD) of order* $\dfrac{1}{2}$.

First, we choose $(x(t), y(t)) = (0.25, 0)$, *for any* $t \in (0, \frac{\pi}{2}]$, *we have*

$$\begin{aligned} &\left|\frac{1}{\Gamma(\frac{1}{2})}\int_0^t \cos(s)\frac{0.25(2+\cos(s))}{60\sqrt{\sin(t)-\sin(s)}}ds - 0.25\right| \\ &\le \left|\frac{3(0.25)}{60\sqrt{\pi}}\int_0^t \frac{\cos(s)ds}{\sqrt{\sin(t)-\sin(s)}} - 0.25\right|, \end{aligned}$$

let $\alpha = \sin(s)$ *and* $d\alpha = \cos(s)ds$, *we have*

$$\begin{aligned} &\le \left|\frac{0.25}{20\sqrt{\pi}}\int_0^{\sin(t)} \frac{d\alpha}{\sqrt{\sin(t)-\alpha}} - 0.25\right| \\ &= \left|\frac{2(0.25)}{20\sqrt{\pi}}(-\sqrt{\sin(t)}) - 0.25\right| \\ &\le \frac{0.25}{10\sqrt{\pi}} + 0.25 = 0.264 \le \sum_{s=0}^{\infty} \frac{|\sin(t)|^{\frac{s}{2}}(-1)^s}{\Gamma(\frac{s}{2}+1)}, \end{aligned}$$

that is, we get

$$\begin{aligned} &\left|\frac{1}{\Gamma(\frac{1}{2})}\int_0^t \cos(s)\frac{0.25(2+\cos(s))}{60\sqrt{\sin(t)-\sin(s)}}ds - 0.25\right| \\ &\qquad \le E_{\frac{1}{2}}(-|(\sin(t))^{\frac{1}{2}}|). \end{aligned} \tag{38}$$

Also, we have

$$\begin{aligned} &\left|\frac{1}{(2+\cos(t))\Gamma(\frac{1}{2})}\int_0^t (2+\cos(s))\cos(s) \right. \\ &\qquad \left. \times \frac{0.25}{60\sqrt{\sin(t)-\sin(s)}(2+\cos(t))}ds\right| \\ &= \frac{0.25}{60(2+\cos(t))\Gamma(\frac{1}{2})}\left|\int_0^t \frac{\cos(s)}{\sqrt{\sin(t)-\sin(s)}}ds\right|, \end{aligned}$$

let $\alpha = \sin(s)$ *and* $d\alpha = \cos(s)ds$, *we can write*

$$\begin{aligned} &\le \frac{0.25}{60(2+\cos(s))\sqrt{\pi}}\left|\int_0^{\sin(t)} \frac{d\alpha}{\sqrt{\sin(t)-\alpha}}\right| \\ &\le \frac{0.25}{60(2+\cos(s))\sqrt{\pi}}\left|2(-\sqrt{\sin(t)})\right| \\ &\le \frac{0.25}{30\sqrt{\pi}(2+\cos(s))} \\ &\le \frac{1}{(2+\cos(s))}\sum_{s=0}^{\infty}\frac{|\sin(t)|^{\frac{s}{2}}(-1)^s}{\Gamma(\frac{s}{2}+1)}, \end{aligned}$$

that is, we get

$$\begin{aligned} &\left|\frac{1}{(2+\cos(t))\Gamma(\frac{1}{2})}\int_0^t (2+\cos(s))\cos(s) \right. \\ &\qquad \left. \times \frac{0.25}{60\sqrt{\sin(t)-\sin(s)}(2+\cos(t))}ds\right| \\ &\le \frac{E_{\frac{1}{2}}(-|(\sin(t))^{\frac{1}{2}}|)}{2+\cos(t)}. \end{aligned} \tag{39}$$

Now, we check condition (M_1') *for the system (37). For any arbitrary* $(h, z), (x, y) \in C((0, \frac{\pi}{2}], \mathbb{R}) \times C((0, \frac{\pi}{2}], \mathbb{R})$ *and every* $t \in (0, \frac{\pi}{2}]$, *we can write*

$$\begin{aligned} &(2+\cos(t))\left|\frac{x(t)+y(t)}{60} - \frac{h(t)+z(t))}{60}\right| \\ &\le \frac{(2+\cos(t))}{60}\,|x(t)+y(t)-(h(t)+z(t))| \\ &\le \frac{1}{20}\left(|x(t)-h(t)| + (2+\cos(t))|y(t)-z(t)|\right), \end{aligned} \tag{40}$$

and

$$\begin{aligned} &(2+\cos(t))^{-1}\left|\frac{x(t)+y(t)}{60} - \frac{h(t)+z(t)}{60}\right| \\ &\le \frac{(2+\cos(t))^{-1}}{60}\,|x(t)+y(t)-h(t)-z(t)| \\ &\le \frac{1}{60}\left((2+\cos(t))^{-1}|x(t)-h(t)| + |y(t)-z(t)|\right). \end{aligned} \tag{41}$$

According to Lemma (4.1), for $0 < w < 1$, *the following* *relations are obtained:*

$$\mathbb{I}_{0^+}^{\frac{1}{2};\sin(t)} E_{\frac{1}{2}}(-|(\sin(t)-\sin(0))^{\frac{1}{2}}|) \le w E_{\frac{1}{2}}(-|(\sin(t)-\sin(0))^{\frac{1}{2}}|), \tag{42}$$

$$_{2+\cos(t)}\mathbb{I}_{0^+}^{\frac{1}{2};\sin(t)} \frac{E_{\frac{1}{2}}(-|(\sin(t)-\sin(0))^{\frac{1}{2}}|)}{2+\cos(t)} \le w \frac{E_{\frac{1}{2}}(-|(\sin(t)-\sin(0))^{\frac{1}{2}}|)}{2+\cos(t)}. \tag{43}$$

Due to $\lambda = \frac{2w}{30} < 1$ *and Corollary 4.3, we derive that for the system (37) there exists the unique solution* $(p, q) \in C((0, \frac{\pi}{2}], \mathbb{R}) \times C((0, \frac{\pi}{2}], \mathbb{R})$, *and we also have*

$$\begin{cases} |x(t) - p(t)| \le \left(\frac{c_1}{1-\lambda}\right) E_{\frac{1}{2}}\left(-|(\sin(t))^{\frac{1}{2}}|\right), \\ |y(t) - q(t)| \le \left(\frac{c_2}{1-\lambda}\right) \frac{E_{\frac{1}{2}}\left(-|(\sin(t))^{\frac{1}{2}}|\right)}{2+\cos(t)}, \end{cases} \tag{44}$$

for any $t \in \left(0, \frac{\pi}{2}\right]$.

6. Numerical results

The numerical method, based on complete metric spaces and the Banach fixed-point theorem, enables the analysis of the existence, uniqueness, and stability of solutions, as discussed in references such as [21, 22, 23]. This section describes a numerical method for solving system (37) by applying the Banach fixed-point method (Fixed-Point Iteration). This method is an iterative procedure based on the Banach fixed-point Theorem. The algorithm of Fixed-Point Iteration method is as follows:

1. Find the fixed point of the system under consideration ($f(x) = x$).
2. Choose an initial point.
3. Use the recurrence relation $x_{n+1} = f(x_n)$ and Iterate repeatedly.
4. Continue the iterations until the desired accuracy (tolerance) is achieved.

Now we apply this algorithm to system (37).

1. In Example 5.2, we discussed the existence of the unique solution for system (37) such that

$$(p(t), q(t)) = \left(\frac{1}{\Gamma(\frac{1}{2})}\int_0^t \cos(s)\frac{(p(s)+q(s))(2+\cos(s))}{60\sqrt{\sin(t)-\sin(s)}}ds, \right.$$
$$\frac{1}{(2+\cos(t))\Gamma(\frac{1}{2})}\int_0^t (2+\cos(s))\cos(s)$$
$$\left. \times \frac{p(s)+q(s)}{60\sqrt{\sin(t)-\sin(s)}(2+\cos(t))}ds\right).$$

2. We choose initial points $p_0(t) = 0.25$, $q_0(t) = 0$ for $t \in (0, \frac{\pi}{2}]$.

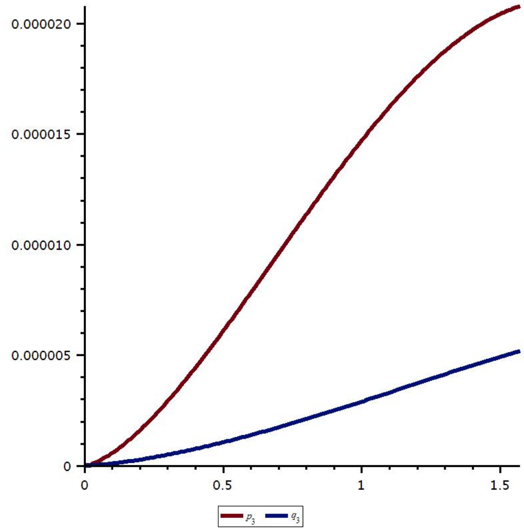

Figure 1. Approximation values for Example 5.2

3. We consider the recurrence relation $(p_{n+1}(t), q_{n+1}(t)) = H(p_n(t), q_n(t))$.
4. We compute the following points:

$$(p_1(t), q_1(t)) = H(p_0(t), q_0(t)),$$
$$(p_2(t), q_2(t)) = H(p_1(t), q_1(t)),$$
$$(p_3(t), q_3(t)) = H(p_2(t), q_2(t)).$$

The functions $p_i(t)$ and $q_i(t)$ for $i = 1, 2, 3$ are computed iteratively using the recurrence relation described above, starting from the initial points $(p_0(t), q_0(t))$. All computations have been performed with an upper bounded error of 10^{-4}.

Approximate values for any point $t \in [0, 1]$ are shown in Figure 1, and the upper bounded error for some points are given in Table 1.

7. Conclusion

In this study, we established the necessary conditions for the existence, uniqueness, and stability of solutions to the fractional differential system (1) using the alternative fixed point theorem. Furthermore, by presenting concrete examples and employing the iterative fixed point method of Banach, we investigated the practical applicability and numerical feasibility of the theoretical results. In particular, the numerical solution of one of the examples confirmed the convergence behavior of the iterative algorithm, as well as the accuracy and efficiency of the proposed method. Numerical results not only verified the correctness of the theoretical findings but also highlighted the potential of this approach in approximating solutions of complex fractional systems with high accuracy. Overall, the results of this study confirm the efficiency and favorable performance of the proposed method, establishing it as a powerful tool for analyzing and solving challenging problems in fractional calculus and their applications. In future research,

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Table 1. Upper bounded error of Example 5.2

t	$p_3(t)$	$q_3(t)$	$\lvert p_3(t)-p_2(t)\rvert$	$\lvert q_3(t)-q_2(t)\rvert$
0	0	0	0	0
0.1	1.74×10^{-6}	1.94×10^{-7}	8.6×10^{-5}	9.6×10^{-6}
0.2	4.86×10^{-6}	5.46×10^{-7}	1.6×10^{-4}	1.9×10^{-5}
0.3	8.73×10^{-6}	9.93×10^{-7}	2.4×10^{-4}	2.8×10^{-5}
0.4	1.30×10^{-5}	1.50×10^{-6}	3.2×10^{-4}	3.7×10^{-5}
0.5	1.74×10^{-5}	2.06×10^{-6}	3.9×10^{-4}	4.6×10^{-5}
0.6	2.18×10^{-5}	2.65×10^{-6}	4.5×10^{-4}	5.4×10^{-5}
0.7	2.59×10^{-5}	3.26×10^{-6}	5.0×10^{-4}	6.3×10^{-5}
0.8	2.97×10^{-5}	3.87×10^{-6}	5.5×10^{-4}	7.1×10^{-5}
0.9	3.29×10^{-5}	4.47×10^{-6}	5.8×10^{-4}	7.9×10^{-5}
1	3.56×10^{-5}	5.06×10^{-6}	6.1×10^{-4}	8.7×10^{-5}

this framework can be extended to systems with time delays, variable-order fractional derivatives, or stochastic perturbations.

Authors contributions
ST, methodology, writing-original draft preparation. MBG, supervision and project administration. RS, methodology, project administration, supervision and editing-original draft preparation. HT, project administration and editing-original draft preparation. All authors have read and approved the final manuscript.

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Conflict of interests
The authors declare that they have no competing interests.

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