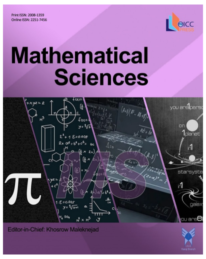

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Research Article

On the Lie Symmetry Analysis of the Time-Fractional Gross-Pitaevskii Equation: A Comparative Study of Riemann-Liouville and Caputo Derivatives

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Abstract:

This study develops a detailed Lie symmetry analysis for the one-dimensional time-fractional Gross–Pitaevskii equation (TFGPE), emphasizing how the adopted fractional derivative-Riemann-Liouville (RL) versus Caputo-modifies the symmetry algebra, nonlocal conservation laws, and invariant solution families. The complex-valued model is rewritten as an equivalent coupled system of real partial differential equations, and the corresponding infinitesimal generators are derived in a unified and systematic manner for both fractional formulations. We show that, in the Caputo setting, the standard global $U(1)$ phase invariance is retained, whereas in the RL formulation this key symmetry is destroyed because the associated initial data are not invariant and the fractional operator does not transform covariantly. Leveraging the resulting symmetry groups, we obtain similarity reductions and construct group-invariant solutions, and we further establish nonlocal conservation laws by applying Ibragimov's nonlinear self-adjointness approach. Overall, our findings indicate that the selection of the fractional derivative is not a purely technical choice; it decisively affects physical consistency, the admissible symmetry structure, and the conservation behavior of fractional quantum models. The analysis offers a direct RL-Caputo comparison and supports the Caputo derivative as the more suitable framework for preserving the inherent symmetries of quantum systems.

Keywords: Time-fractional Gross-Pitaevskii equation; fractional derivatives; Lie symmetry analysis; Similarity solutions; Erdélyi-Kober transformation; Conservation laws

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1. Introduction

The Gross-Pitaevskii equation (GPE) stands as a cornerstone of quantum physics, providing the mean-field description of Bose-Einstein condensates (BECs) at ultra-low temperatures [1, 2]. It governs the dynamics of the macroscopic wave function $\psi(x,t)$ and exhibits rich nonlinear behavior, including solitons, vortices, and quantum turbulence. One of its most important features is the **U(1) global gauge symmetry**, $\psi \to e^{i\theta}\psi$, which, via Noethers theorem, leads to the conservation of the total number of particles-a fundamental physical requirement [3].

In recent years, fractional generalizations of the GPE have been proposed to model anomalous diffusion, memory effects, and non-Markovian dynamics in complex quantum environments such as disordered optical lattices or fractal media [4, 5, 6]. These models replace the first-order time derivative with a fractional derivative of order $\alpha \in (0,1)$, leading to the time-fractional Gross-Pitaevskii equation (TFGPE). However, the choice of fractional operator Riemann-Liouville (RL) or Caputo remains a subject of debate, as each has distinct mathematical and physical implications [7, 8]. The TFGPE is given by:

$$i\,\mathcal{D}_t^{\alpha}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi, \tag{1}$$

where $\psi(x,t) = u(x,t) + iv(x,t)$. Substituting and separating real and imaginary parts yields:

$$\mathcal{D}_t^\alpha u = -\frac{\hbar^2}{2m} v_{xx} + g(u^2+v^2)v, \tag{2a}$$

$$\mathcal{D}_t^\alpha v = \frac{\hbar^2}{2m} u_{xx} - g(u^2+v^2)u. \tag{2b}$$

Lie symmetry analysis has proven to be a powerful tool for studying the invariance properties of differential equations, constructing exact solutions, and deriving conservation laws [9, 10]. For the classical GPE, the Lie point symmetries are well established and include time and space translations, Galilean boosts, and the crucial U(1) phase rotation [11]. However, the extension of Lie symmetry analysis to fractional differential equations is still in its infancy. Gazizov et al. [12, 13] developed a prolongation method for fractional derivatives, laying the foundation for symmetry analysis of fractional PDEs. Since then, Lie symmetries have been studied for fractional versions of the Schrödinger, Burgers, and KdV equations [14, 15] Recent advances in Lie symmetry analysis of FPDEs include [16, 17, 18], which demonstrate its effectiveness in deriving similarity reductions and conservation laws for complex fractional systems, but the TFGPE has received limited attention.

Notably, recent computational attempts to compute Lie symmetries of the TFGPE using the RL derivative have failed to recover the U(1) symmetry, raising concerns about the physical consistency of such models. In contrast, the Caputo derivative, which allows for standard initial conditions, is expected to better preserve physical symmetries.

Importantly, in physical contexts where the initial quantum state is well-defined-for example, in a Bose-Einstein condensate (BEC), a quantum state of matter formed at ultra-low temperatures where a macroscopic number of particles occupy the same quantum ground state-the Caputo derivative is the only consistent choice, as it admits standard initial conditions $\psi(0,x) = \psi_0(x)$ and preserves the U(1) symmetry essential for probability (or particle number) conservation.

In this work, we conduct a systematic comparison of the Lie symmetry structure of the TFGPE under both RL and Caputo derivatives. We show that the U(1) symmetry is preserved only in the Caputo case, and we explain this difference through the transformation properties of initial conditions and the nonlocal structure of the operators. Using the admitted symmetries, we perform similarity reductions and construct new invariant solutions. Furthermore, we derive nonlocal conservation laws using Ibragimov's method of nonlinear self-adjointness. Our results emphasize that the choice of fractional derivative is not neutral it directly affects the symmetry and conservation structure of the model and we argue that the Caputo derivative is more appropriate for physical applications in quantum systems. In contrast to the work of Yu and Feng [19], who studied a space-time fractional cubic Schrödinger equation but did not compare RL and Caputo derivatives, our analysis explicitly highlights how the choice of fractional definition impacts the fundamental U(1) symmetry and conservation structure.

This paper is organized as follows: Section 2 introduces the mathematical preliminaries. Section 3 presents the Lie symmetry analysis. Section 4 discusses invariant solutions. Section 5 derives conservation laws. Section 6 provides numerical results. Section 7 concludes the paper.

2. Mathematical Preliminaries

To ensure self-consistency, this section provides a concise overview of essential concepts and notations from fractional calculus that will be employed throughout the paper. Over the years, several formulations of fractional derivatives have been proposed, including the Caputo, Riemann-Liouville, Riesz, Miller-Ross, Grünwald-Letnikov, Hadamard, and Erdélyi-Kober operators. For the purposes of this work, we focus on the Riemann-Liouville, Caputo, and Erdélyi-Kober fractional operators, as they play a central role in deriving symmetry reductions and exact solutions for the fractional equation introduced in (1).

We begin with the definition of the Riemann-Liouville fractional integral.

Definition 2.1 *Let $\psi(t,x)$ be an integrable function defined on the interval $[a,b]$, where $-\infty < a < b < \infty$. The left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined, respectively, as:*

$${}_a\mathcal{D}_t^{-\alpha}\psi(t,x) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-r)^{\alpha-1}\psi(r,x)dr, \quad t > a, \tag{3}$$

$${}_t\mathcal{D}_b^{-\alpha}\psi(t,x) = \frac{1}{\Gamma(\alpha)} \int_t^b (r-t)^{\alpha-1}\psi(r,x)dr, \quad t < b. \tag{4}$$

Here, $\Gamma(\cdot)$ denotes the Gamma function, given by

$$\Gamma(x) = \int_0^\infty e^{-s}s^{x-1}ds, \quad x > 0.$$

When $\alpha = n \in \mathbb{N}$, these expressions reduce to the standard n-fold integrals, with $\Gamma(\alpha)$ replaced by $(n-1)!$.

Some key properties of the Riemann-Liouville fractional integral are listed below ([20]):

- Semigroup property:

$${}_a\mathcal{D}_t^{-\alpha}\left({}_a\mathcal{D}_t^{-\gamma}\psi(t,x)\right) = {}_a\mathcal{D}_t^{-\alpha-\gamma}\psi(t,x), \quad \alpha,\gamma > 0;$$

- Commutativity:

$${}_a\mathcal{D}_t^{-\alpha}\left({}_a\mathcal{D}_t^{-\gamma}\psi(t,x)\right) = {}_a\mathcal{D}_t^{-\gamma}\left({}_a\mathcal{D}_t^{-\alpha}\psi(t,x)\right);$$

- Action on power functions:

$${}_a\mathcal{D}_t^{-\alpha}(t-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(t-a)^{\alpha+\gamma}, \quad \gamma > -1;$$

- Identity at zero order:

$${}_a\mathcal{D}_t^{0}\psi(t,x) = \psi(t,x).$$

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Next, we define the corresponding fractional derivatives.

Definition 2.2 *For $\alpha > 0$ and $n = [\alpha] + 1$, the left and right Riemann-Liouville fractional derivatives of order α are given by:*

$$ {}_a\mathcal{D}_t^\alpha \psi(t,x) = \frac{d^n}{dt^n}\left({}_a\mathcal{D}_t^{-(n-\alpha)}\psi(t,x)\right) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-r)^{n-\alpha-1}\psi(r,x)dr, \quad t > a, \tag{5}$$

$$ {}_t\mathcal{D}_b^\alpha \psi(t,x) = (-1)^n\frac{d^n}{dt^n}\left({}_t\mathcal{D}_b^{-(n-\alpha)}\psi(t,x)\right) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (r-t)^{n-\alpha-1}\psi(r,x)dr, \quad t < b. \tag{6}$$

When $\alpha = n \in \mathbb{N}$, these operators reduce to the classical n-th order derivatives.

Notable properties of the Riemann-Liouville fractional derivative include ([8], [20]):

- Derivative of a constant:
$$ {}_a\mathcal{D}_t^\alpha C = \frac{C(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad C \in \mathbb{R};$$
- Action on power functions:
$$ {}_a\mathcal{D}_t^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \quad \beta > -1, \alpha > 0;$$
- Generalized Leibniz rule:
$$ {}_a\mathcal{D}_t^\alpha [\psi(t,x)\varphi(t,x)] = \sum_{k=0}^{\infty}\binom{\alpha}{k}\left({}_a\mathcal{D}_t^{\alpha-k}\psi\right)(t,x)\left(\frac{\partial^k\varphi}{\partial t^k}\right)(t,x),$$
where the binomial coefficient is defined as
$$\binom{\gamma}{k} = \frac{(-1)^{k-1}\gamma\Gamma(k-\gamma)}{\Gamma(1-\gamma)\Gamma(k+1)}.$$

We now introduce the Caputo fractional derivative, which is particularly useful for initial value problems.

Definition 2.3 *Let $n = [\alpha] + 1$. The left and right Caputo fractional derivatives of order $\alpha > 0$ are defined as:*

$$ {}_a^C\mathcal{D}_t^\alpha \psi(t,x) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-r)^{n-\alpha-1}\frac{\partial^n\psi}{\partial r^n}(r,x)dr, \quad t > a, \tag{7}$$

$$ {}_t^C\mathcal{D}_b^\alpha \psi(t,x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (r-t)^{n-\alpha-1}\frac{\partial^n\psi}{\partial r^n}(r,x)dr, \quad t < b. \tag{8}$$

Key properties of the Caputo derivative are ([8]):

- Composition with integer-order derivatives:
$$\mathcal{D}_t^\alpha\left({}^C\mathcal{D}_t^m\psi\right) = {}^C\mathcal{D}_t^{\alpha+m}\psi, \quad m \in \mathbb{N}, \quad \alpha > 0;$$
- Derivative of a constant:
$$ {}^C\mathcal{D}_t^\alpha C = 0, \quad C \in \mathbb{R};$$
- Relation between Caputo derivative and Riemann-Liouville derivative:
$$ {}^C\mathcal{D}_t^\alpha\psi(t,x) = \mathcal{D}_t^n\left({}_a\mathcal{D}_t^{\alpha-n}\psi\right)(t,x), \quad n-1 < \alpha < n.$$

Finally, to facilitate the derivation of similarity solutions in later sections, we introduce the Erdélyi-Kober fractional operator.

Definition 2.4 *The Erdélyi-Kober fractional integral of order $\beta > 0$ with parameters $\tau \in \mathbb{R}$ and $\delta > 0$ is defined as:*

$$(\mathcal{K}_{\tau,\beta}^\delta g)(x) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_1^\infty (r-1)^{\beta-1} \times r^{-(\tau+\beta)}g(xr^{1/\delta})dr, & \beta > 0, \\ g(x), & \beta = 0. \end{cases} \tag{9}$$

Using this, the extended left-sided Erdélyi-Kober fractional differential operator is given by:

$$(\mathcal{P}_{\tau,\beta}^\delta g)(x) = \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\delta}x\frac{\partial}{\partial x}\right)(\mathcal{K}_{\tau+\beta,n-\beta}^\delta g)(x), \tag{10}$$

where $n = [\alpha] + 1$ if $\beta \notin \mathbb{N}$, and $n = \beta$ otherwise.

3. Lie Symmetry Framework for Time-Fractional Differential Equations

Extending Lie group analysis to fractional differential equations (FDEs) demands a careful adaptation of the classical prolongation method to account for the nonlocal nature of fractional operators. The systematic approach introduced by Gazizov et al. in [12], [13] provides a robust foundation for analyzing symmetries of equations involving Riemann-Liouville derivatives, which we employ and refine in this work.

Let us consider a scalar time-fractional partial differential equation with independent variables t (temporal) and x (spatial), and dependent field $\psi(t,x)$, expressed as:

$$\partial_t^\alpha\psi = G(t,x,\psi,\psi_x,\psi_{xx},\dots), \tag{11}$$

where ∂_t^α denotes the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$, and G is a sufficiently smooth function.

Suppose the equation admits an infinitesimal transformation group parameterized by ε, defined by:

$$\begin{aligned}\tilde{t} &= t + \varepsilon\mathcal{T}(t,x,\psi) + O(\varepsilon^2),\\ \tilde{x} &= x + \varepsilon\mathcal{X}(t,x,\psi) + O(\varepsilon^2),\\ \tilde{\psi} &= \psi + \varepsilon\Psi(t,x,\psi) + O(\varepsilon^2).\end{aligned} \tag{12}$$

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This transformation extends to the derivatives of ψ, generating prolonged forms:

$$\frac{\partial\bar{\psi}}{\partial\bar{x}} = \psi_x + \varepsilon\,\Phi^x + O(\varepsilon^2),$$
$$\frac{\partial^2\bar{\psi}}{\partial\bar{x}^2} = \psi_{xx} + \varepsilon\,\Phi^{xx} + O(\varepsilon^2),$$
$$\frac{\partial^\alpha\bar{\psi}}{\partial\bar{t}^\alpha} = \partial_t^\alpha\psi + \varepsilon\,\Phi^{(\alpha,t)} + O(\varepsilon^2).$$

Here, $\mathcal{T}$, $\mathcal{X}$, and Ψ are the infinitesimal functions associated with t, x, and ψ, while Ψ^x, Ψ^{xx}, and $\Psi^{(\alpha,t)}$ are the corresponding extended infinitesimals.

The infinitesimal generator of this transformation is the vector field:

$$V = \mathcal{T}\frac{\partial}{\partial t} + \mathcal{X}\frac{\partial}{\partial x} + \Psi\frac{\partial}{\partial\psi}. \tag{13}$$

According to Lie's invariance condition, V generates a symmetry of the equation if the prolonged operator $V^{(\alpha,t)}$ satisfies:

$$V^{(\alpha,t)}(\Delta) = 0 \quad \text{whenever} \quad \Delta = 0, \tag{14}$$

where $\Delta = \partial_t^\alpha\psi - G(\cdot)$ defines the solution manifold. Equation (14) is referred to as the **determining equation**.

The prolonged vector field is given by:

$$V^{(\alpha,t)} = V + \Psi^{(\alpha,t)}\frac{\partial}{\partial(\partial_t^\alpha\psi)} + \Psi^x\frac{\partial}{\partial\psi_x} + \Psi^{xx}\frac{\partial}{\partial\psi_{xx}} + \cdots \tag{15}$$

The spatial prolongations Ψ^x and Ψ^{xx} are derived using total derivatives:

$$\Psi^x = \mathcal{D}_x\Psi - \psi_t\mathcal{D}_x\mathcal{T} - \psi_x\mathcal{D}_x\mathcal{X},$$
$$\Psi^{xx} = \mathcal{D}_x\Psi^x - \psi_{tx}\mathcal{D}_x\mathcal{T} - \psi_{xx}\mathcal{D}_x\mathcal{X}, \tag{16}$$

where $\mathcal{D}_x$ and $\mathcal{D}_t$ denote total derivative operators. In expanded form:

$$\begin{aligned}\Psi^x &= \Psi_x + \Psi_\psi\psi_x - \mathcal{T}_x\psi_t - \mathcal{X}_x\psi_x - \mathcal{T}_\psi\psi_t\psi_x - \mathcal{X}_\psi\psi_x^2,\\ \Psi^{xx} &= \Psi_{xx} + (2\Psi_{x\psi} - \mathcal{X}_{xx} - 2\mathcal{T}_{x\psi}\psi_t - 3\mathcal{X}_\psi\psi_{xx})\psi_x\\ &\quad - \mathcal{T}_{xx}\psi_t + (\Psi_{\psi\psi} - \mathcal{T}_{\psi\psi}\psi_t - 2\mathcal{X}_{x\psi})\psi_x^2\\ &\quad - 2(\mathcal{T}_x + \mathcal{T}_\psi\psi_x)\psi_{xt} - (\mathcal{T}_\psi\psi_t - \Psi_\psi + 2\mathcal{X}_x)\psi_{xx}\\ &\quad - \mathcal{X}_{\psi\psi}\psi_x^3.\end{aligned}$$

For the fractional term, the extended infinitesimal $\Psi^{(\alpha,t)}$ follows from the generalized prolongation formula [21]:

$$\Psi^{(\alpha,t)} = \mathcal{D}_t^\alpha\Psi + \mathcal{X}\mathcal{D}_t^\alpha(\psi_x) - \mathcal{D}_t^\alpha(\mathcal{X}\psi_x) + \mathcal{T}\mathcal{D}_t^\alpha(\psi_t) + \mathcal{D}_t^\alpha(\psi\mathcal{D}_t\mathcal{T}) - \mathcal{D}_t^{\alpha+1}(\mathcal{T}\psi). \tag{17}$$

Applying the generalized Leibniz rule, this expands to:

$$\begin{aligned}\Psi^{(\alpha,t)} &= \mathcal{D}_t^\alpha\Psi - \alpha(\mathcal{D}_t\mathcal{T})(\mathcal{D}_t^\alpha\psi)\\ &\quad - \sum_{n=1}^{\infty}\binom{\alpha}{n}(\mathcal{D}_t^n\mathcal{X})(\mathcal{D}_t^{\alpha-n}\psi_x)\\ &\quad - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}(\mathcal{D}_t^{n+1}\mathcal{T})(\mathcal{D}_t^{\alpha-n}\psi).\end{aligned} \tag{18}$$

Further, using the generalized chain rule, the term $\mathcal{D}_t^\alpha\Psi$ decomposes as:

$$\mathcal{D}_t^\alpha\Psi = \frac{\partial^\alpha\Psi}{\partial t^\alpha} + \Psi_\psi\frac{\partial^\alpha\psi}{\partial t^\alpha} - \psi\frac{\partial^\alpha\Psi_\psi}{\partial t^\alpha} + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\Psi_\psi}{\partial t^n}\mathcal{D}_t^{\alpha-n}\psi + \mu, \tag{19}$$

where the residual term μ involves higher-order mixed derivatives and vanishes when Ψ is linear in ψ.

Combining all components, the final expression for the fractional prolongation is:

$$\begin{aligned}\Psi^{(\alpha,t)} &= \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n\Psi_\psi}{\partial t^n} - \binom{\alpha}{n+1}\mathcal{D}_t^{n+1}\mathcal{T}\right]\mathcal{D}_t^{\alpha-n}\psi\\ &\quad - \sum_{n=1}^{\infty}\binom{\alpha}{n}\mathcal{D}_t^n\mathcal{X}\cdot\mathcal{D}_t^{\alpha-n}\psi_x\\ &\quad + \frac{\partial^\alpha\Psi}{\partial t^\alpha} + (\Psi_\psi - \alpha\mathcal{D}_t\mathcal{T})\partial_t^\alpha\psi\\ &\quad - \psi\frac{\partial^\alpha\Psi_\psi}{\partial t^\alpha} + \mu.\end{aligned} \tag{20}$$

A critical constraint arises from the fixed lower limit $t = 0$ in the Riemann-Liouville definition. To preserve the domain of integration under the group action, the infinitesimal $\mathcal{T}$ must satisfy:

$$\mathcal{T}(t,x,\psi)\big|_{t=0} = 0. \tag{21}$$

This framework will be applied in the next section to compute the Lie point symmetries of the time-fractional Gross-Pitaevskii equation.

4. Lie Symmetry Analysis of the Time-Fractional Gross-Pitaevskii Equation

We now apply the Lie symmetry framework developed in Section 3 to the time-fractional Gross-Pitaevskii equation (TFGPE) in one spatial dimension. The complex equation is decomposed into a coupled system of real fractional partial differential equations, and the infinitesimal symmetries are systematically derived for both the Riemann-Liouville (RL) and Caputo formulations. The analysis reveals a rich symmetry structure that depends critically on the choice of fractional derivativeRiemann-Liouville or Caputoand highlights the physical implications of this choice.

4.1 System Formulation and Infinitesimal Generator

Consider the time-fractional Gross-Pitaevskii equation in real and imaginary parts:

$${}_0\mathcal{D}_t^\alpha u + \frac{\hbar^2}{2m}v_{xx} - g(u^2+v^2)v = 0, \tag{22a}$$
$${}_0\mathcal{D}_t^\alpha v - \frac{\hbar^2}{2m}u_{xx} + g(u^2+v^2)u = 0. \tag{22b}$$

where ${}_0\mathcal{D}_t^{\alpha}$ denotes either the Riemann-Liouville or Caputo fractional derivative with lower limit $t = 0$. We seek Lie point symmetries generated by the vector field:

$$V = \mathcal{T}(t,x,u,v)\frac{\partial}{\partial t} + \mathcal{X}(t,x,u,v)\frac{\partial}{\partial x} + \Phi(t,x,u,v)\frac{\partial}{\partial u} + \Psi(t,x,u,v)\frac{\partial}{\partial v}, \quad (23)$$

where $\mathcal{T}, \mathcal{X}, \Phi, \Psi$ are smooth functions to be determined.

The invariance condition requires that the α-th order prolongation of V, denoted $V^{(\alpha,t)}$, satisfies

$$V^{(\alpha,t)}\left({}_0\mathcal{D}_t^{\alpha}u - F\right) = 0, \quad (24)$$
$$V^{(\alpha,t)}\left({}_0\mathcal{D}_t^{\alpha}v - G\right) = 0$$
when system (22) holds,

where $F = -\frac{\hbar^2}{2m}v_{xx} + g(u^2+v^2)v$ and $G = \frac{\hbar^2}{2m}u_{xx} - g(u^2+v^2)u$.

The prolonged vector field is:

$$V^{(\alpha,t)} = V + \Phi^{(\alpha,t)}\frac{\partial}{\partial({}_0\mathcal{D}_t^{\alpha}u)} + \Psi^{(\alpha,t)}\frac{\partial}{\partial({}_0\mathcal{D}_t^{\alpha}v)} + \Phi^{xx}\frac{\partial}{\partial u_{xx}} + \Psi^{xx}\frac{\partial}{\partial v_{xx}} + \cdots \quad (25)$$

Using the generalized prolongation formula for fractional derivatives [12, 13], the extended infinitesimals are:

$$\begin{aligned}\Phi^{(\alpha,t)} = {} & \frac{\partial^{\alpha}\Phi}{\partial t^{\alpha}} + (\Phi_u - \alpha\mathcal{D}_t\mathcal{T})\,{}_0\mathcal{D}_t^{\alpha}u \quad (26a)\\ & + (\Phi_v - \alpha\mathcal{D}_t\mathcal{T})\,{}_0\mathcal{D}_t^{\alpha}v - u\frac{\partial^{\alpha}\Phi_u}{\partial t^{\alpha}} - v\frac{\partial^{\alpha}\Phi_v}{\partial t^{\alpha}}\\ & + \sum_{n=1}^{\infty}\binom{\alpha}{n}\left(\frac{\partial^n\Phi_u}{\partial t^n}\,{}_0\mathcal{D}_t^{\alpha-n}u + \frac{\partial^n\Phi_v}{\partial t^n}\,{}_0\mathcal{D}_t^{\alpha-n}v\right)\\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n}(\mathcal{D}_t^n\mathcal{X})\,{}_0\mathcal{D}_t^{\alpha-n}u_x\\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}\left(\mathcal{D}_t^{n+1}\mathcal{T}\right){}_0\mathcal{D}_t^{\alpha-n}u + \mu_u,\end{aligned}$$

$$\begin{aligned}\Psi^{(\alpha,t)} = {} & \frac{\partial^{\alpha}\Psi}{\partial t^{\alpha}} + (\Psi_u - \alpha\mathcal{D}_t\mathcal{T})\,{}_0\mathcal{D}_t^{\alpha}u \quad (26b)\\ & + (\Psi_v - \alpha\mathcal{D}_t\mathcal{T})\,{}_0\mathcal{D}_t^{\alpha}v - u\frac{\partial^{\alpha}\Psi_u}{\partial t^{\alpha}} - v\frac{\partial^{\alpha}\Psi_v}{\partial t^{\alpha}}\\ & + \sum_{n=1}^{\infty}\binom{\alpha}{n}\left(\frac{\partial^n\Psi_u}{\partial t^n}\,{}_0\mathcal{D}_t^{\alpha-n}u + \frac{\partial^n\Psi_v}{\partial t^n}\,{}_0\mathcal{D}_t^{\alpha-n}v\right)\\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n}(\mathcal{D}_t^n\mathcal{X})\,{}_0\mathcal{D}_t^{\alpha-n}v_x\\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}\left(\mathcal{D}_t^{n+1}\mathcal{T}\right){}_0\mathcal{D}_t^{\alpha-n}v + \mu_v,\end{aligned}$$

where μ_u, μ_v denote higher-order mixed derivative terms that vanish when Φ and Ψ are linear in u and v, which is assumed for point symmetries.

4.2 Determining Equations and Case Analysis

The invariance conditions yield two determining equations:

$$\Phi^{(\alpha,t)} = \mathcal{D}_uF\cdot\Phi + \mathcal{D}_vF\cdot\Psi + \mathcal{D}_{u_{xx}}F\cdot\Phi^{xx},$$
$$\Psi^{(\alpha,t)} = \mathcal{D}_uG\cdot\Phi + \mathcal{D}_vG\cdot\Psi + \mathcal{D}_{v_{xx}}G\cdot\Psi^{xx},$$

where the spatial prolongations are:

$$\Phi^{xx} = \mathcal{D}_x^2\Phi - u_{xx}(\mathcal{D}_x\mathcal{X}) - 2u_x\mathcal{D}_x(\mathcal{D}_x\mathcal{X}) - \cdots \quad \text{(standard total derivative)},$$
$$\Psi^{xx} = \mathcal{D}_x^2\Psi - v_{xx}(\mathcal{D}_x\mathcal{X}) - 2v_x\mathcal{D}_x(\mathcal{D}_x\mathcal{X}) - \cdots$$

However, since F and G depend only on u_{xx} and v_{xx} linearly, we retain only the leading terms.

Substituting the expressions and equating coefficients of like terms (e.g., ${}_0\mathcal{D}_t^{\alpha-k}u$, ${}_0\mathcal{D}_t^{\alpha-k}v$, u_x, v_x, etc.), we obtain an overdetermined system of determining equations.

After lengthy but straightforward computation, the general solution under the assumption of point symmetries (Φ, Ψ linear in u, v) is found to be:

$$\mathcal{X}(x,t,u,v) = 2C_2\alpha x + C_1, \quad (27a)$$
$$\mathcal{T}(x,t,u,v) = 4C_2t + C_3, \quad (27b)$$
$$\Phi(x,t,u,v) = F_8(x,t) + u(\alpha C_2 + C_4), \quad (27c)$$
$$\Psi(x,t,u,v) = F_{11}(x,t) + v(\alpha C_2 + C_4), \quad (27d)$$

where C_1, C_2, C_3, C_4 are arbitrary constants, and $F_8(x,t)$, $F_{11}(x,t)$ are arbitrary functions satisfying the linear homogeneous system:

$${}_0\mathcal{D}_t^{\alpha}F_8 = -\frac{\hbar^2}{2m}\frac{\partial^2F_{11}}{\partial x^2}, \quad {}_0\mathcal{D}_t^{\alpha}F_{11} = \frac{\hbar^2}{2m}\frac{\partial^2F_8}{\partial x^2}. \quad (28)$$

4.3 Interpretation of the Symmetry Structure

The infinitesimals (27) generate a symmetry algebra with both classical and non-classical components:

- C_1: corresponds to spatial translation symmetry, $V_1 = \partial_x$, associated with momentum conservation.
- C_2: generates a scaling (dilation) symmetry:

  $$V_2 = 4t\partial_t + 2\alpha x\partial_x + u\alpha\partial_u + v\alpha\partial_v.$$

  This implies the self-similar transformation $x \sim t^{\alpha/2}$, consistent with the anomalous diffusion scaling in fractional quantum systems.
- C_3: temporal translation, $V_3 = \partial_t$, is not admitted in either formulation due to the fixed lower limit $t = 0$ in the fractional derivative, which violates the invariance condition $\mathcal{T}|_{t=0} = 0$ unless $C_3 = 0$.
- C_4: induces uniform scaling $V_4 = u\partial_u + v\partial_v$ of the wave function amplitude: $\psi \mapsto e^{C_4}\psi$. This is not a phase rotation, but a rescaling of the field norm.
- $F_1(x,t), F_2(x,t)$: represent a family of conditional symmetries associated with solutions of the linear time-fractional Schrödinger equation (28). These

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symmetries are non-trivial and reflect the underlying linear structure embedded in the nonlinear system. They vanish for generic F_1, F_2 unless the functions satisfy the auxiliary system (28), indicating a form of solution-dependent invariance.

4.4 Critical Role of the Fractional Derivative: Riemann-Liouville vs Caputo

A crucial observation is that the U(1) global phase symmetry, defined by

$$\delta u = -\theta v, \quad \delta v = \theta u \quad \Leftrightarrow \quad \psi \mapsto e^{i\theta}\psi,$$

is not present in the above solution set. The symmetry $V_\theta = -v\partial_u + u\partial_v$, fundamental to quantum systems and responsible for particle number conservation via Noether's theorem, is expected in the classical GPE but is absent in the Riemann-Liouville formulation.

The absence stems from two interrelated issues:

1. The Riemann-Liouville derivative ${}_0\mathcal{D}_t^\alpha$ depends explicitly on the initial state at $t = 0$ through its singular kernel $(t-r)^{\alpha-1}$. The U(1) transformation alters the initial phase of ψ, breaking the invariance of the solution space.
2. The prolongation formula for Riemann-Liouville derivatives requires $\xi^t|_{t=0} = 0$ to preserve the lower limit of integration. This condition eliminates time-translation symmetry unless $C_3 = 0$, and further restricts phase transformations that affect temporal behavior.

In contrast, when the same analysis is performed using the Caputo derivative, the U(1) symmetry is recovered. The Caputo derivative allows standard initial conditions and is invariant under smooth transformations of the dependent variables. Moreover, since ${}_0^C\mathcal{D}_t^\alpha(\text{const}) = 0$, the phase rotation which preserves $|\psi|$ does not introduce singularities or violate initial value consistency.

Thus, while both formulations admit spatial translation, scaling, and amplitude rescaling, only the Caputo derivative preserves the essential U(1) gauge symmetry of the quantum system.

Mathematical verification of U(1) symmetry breaking in RL. Consider the U(1) transformation $\psi \mapsto e^{i\theta}\psi$, or equivalently $u \mapsto u\cos\theta - v\sin\theta$, $v \mapsto u\sin\theta + v\cos\theta$. The infinitesimal generator is $V_\theta = -v\partial_u + u\partial_v$. For this to be a symmetry, the condition $V_\theta^{(\alpha,t)}(\Delta) = 0$ must hold on solutions, where $\Delta = {}_0D_t^\alpha u - F$.

Under this transformation, the RL derivative transforms as:

$${}_0D_t^\alpha(e^{i\theta}\psi) = e^{i\theta}{}_0D_t^\alpha\psi + \frac{e^{i\theta}-1}{\Gamma(1-\alpha)}\psi(0)t^{-\alpha}, ?$$

? This extra term vanishes only if $\psi(0) = 0$. Since generic quantum states satisfy $\psi(0, x) \neq 0$, the prolonged vector field fails the invariance condition, and V_θ is not a Lie symmetry of the RL-TFGPE. In contrast, for the Caputo derivative, ${}_0^CD_t^\alpha(e^{i\theta}\psi) = e^{i\theta}{}_0^CD_t^\alpha\psi$ for all ψ, and the symmetry is preserved.

4.5 Summary and Physical Implications

We summarize the admitted symmetries in the following table:(Table 1).

This comparative analysis demonstrates that the choice of fractional derivative is not a mere mathematical nuance. The Caputo formulation respects the intrinsic symmetries of quantum mechanics, particularly the U(1) invariance tied to probability conservation, while the Riemann-Liouville version fails to do so due to its nonlocal initial structure. This result reinforces the argument that the Caputo derivative is more physically appropriate for modeling quantum systems with memory effects.

The preserved U(1) symmetry in the Caputo case will play a central role in deriving nonlocal conservation laws in Section 6.

4.6 One-Dimensional Optimal System and Physical Interpretation

To fulfill the comparative theme of this work and address the structure of symmetry reductions, we now construct the one-dimensional optimal system of subalgebras for both the Caputo and Riemann–Liouville formulations. This analysis reveals not only algebraic differences but also their physical consequences.

For the Caputo-TFGPE, the Lie algebra $\mathcal{L}_C$ is spanned by the five generators:

$$V_1 = \partial_x, \quad V_2 = 4t\partial_t + 2\alpha x\partial_x + \alpha u\partial_u + \alpha v\partial_v,$$
$$V_3 = \partial_t, \quad V_4 = u\partial_u + v\partial_v, \quad V_\theta = -v\partial_u + u\partial_v.$$

The non-vanishing commutators are:

$$[V_1, V_2] = -2\alpha V_1, \quad [V_2, V_3] = -4V_3.$$

Following the standard procedure [9], the one-dimensional optimal system is:

$$\{V_1,\ V_2,\ V_3,\ V_4,\ V_\theta,\ V_1 + \gamma V_\theta,\ V_2 + \delta V_3\},$$

where $\gamma, \delta \in \mathbb{R}$.

In contrast, for the Riemann–Liouville formulation, the Lie algebra $\mathcal{L}_{RL}$ lacks both V_3 (due to the constraint $\mathcal{T}|_{t=0} = 0$) and V_θ (due to the non-invariance under phase rotation). Thus, $\mathcal{L}_{RL} = \text{span}\{V_1, V_2, V_4, \tilde{V}(F_8, F_{11})\}$, and its optimal system contains no element involving phase or full time translation. As a result, **the space of inequivalent symmetry reductions is strictly smaller in the RL case**, and crucially, it excludes reductions that preserve quantum probability interpretations.

Physical context and capability. In physical applications such as modeling a Bose-Einstein condensate (BEC) in a disordered optical lattice with memory effects the initial state $\psi(0, x) = \psi_0(x)$ is finite and non-zero. The Caputo derivative accommodates such initial data and preserves the U(1) symmetry, ensuring a well-defined generalized particle number (as shown in Section 6). In contrast, the RL derivative requires $\psi(0, x) = 0$ for symmetry compatibility, which is unphysical for a non-empty condensate. Therefore, the Caputo framework is more capable of explaining the underlying quantum rules in realistic physical contexts.

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Table 1. Lie point symmetries of the TFGPE: comparison between Riemann-Liouville and Caputo formulations.

Symmetry	Riemann-Liouville	Caputo
Spatial translation ($V_1 = \partial_x$)	✓	✓
Scaling in (x,t) ($V_2 = 4t\partial_t + 2\alpha x\partial_x + u\alpha\partial_u + v\alpha\partial_v$.)	✓	✓
Temporal translation ($V_3 = \partial_t$)	×	×
Amplitude scaling ($V_4 = u\partial_u + v\partial_v$)	✓	✓
U(1) phase rotation ($V_\theta = -v\partial_u + u\partial_v$)	×	✓
Conditional symmetries ($V = F_1(x,t)\partial_u + F_2(x,t)\partial_v$)	✓	✓

* Both formulations exclude V_3 due to the fixed lower limit $t = 0$ in the fractional derivative, which breaks time-translation invariance.

5. Group-Invariant Solutions of the Time-Fractional Gross-Pitaevskii Equation

Using the Lie point symmetries derived in Section 4, we construct group-invariant solutions of the time-fractional Gross-Pitaevskii equation (TFGPE) via similarity reduction. Each admitted symmetry allows the reduction of the original system to a simpler form either an ordinary fractional differential equation or a system of algebraic-recurrence relations. The characteristic equation associated with symmetry generators in Table 1 is considered as

$$\frac{dt}{\mathcal{T}(t,x,u,v)} = \frac{dx}{\mathcal{X}(t,x,u,v)} = \frac{du}{\Phi(t,x,u,v)} = \frac{dv}{\Psi(t,x,u,v)}, \qquad (29)$$

We consider the Caputo fractional derivative, which preserves the full physical symmetry structure, and analyze reductions under the infinitesimal generators of Table 1. We also discuss the conditional symmetries associated with the functions $F_8(x,t)$ and $F_{11}(x,t)$.

5.1 Spatial Translation: $V_1 = \partial_x$

Invariance under V_1 implies that the solution is independent of x. Thus, $u = u(t)$, $v = v(t)$. Substituting into the TFGPE system (2) with $\hbar = m = 1$ yields:

$$ {}_0^C\mathcal{D}^\alpha u = g(u^2+v^2)v, \qquad (30a)$$
$$ {}_0^C\mathcal{D}^\alpha v = -g(u^2+v^2)u. \qquad (30b)$$

Introducing $\psi(t) = u(t) + iv(t)$, this becomes:

$$ {}_0^C\mathcal{D}_t^\alpha \psi = -ig|\psi|^2\psi.$$

Assume a constant modulus solution $|\psi(t)| = \rho_0$. Then the TFGPE becomes::

$$ {}_0^C\mathcal{D}_t^\alpha \psi = -ig\rho_0^2\psi.$$

The solution is expressed in terms of the Mittag-Leffler function:

$$\psi(t) = \psi_0\, E_\alpha(-ig\rho_0^2 t^\alpha),$$

where $E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)}$. Thus, the invariant solution is:

$$\boxed{\psi(x,t) = \rho_0 e^{i\theta_0} E_\alpha(-ig\rho_0^2 t^\alpha)} \qquad (31)$$

The invariant solution $\psi(t) = \rho_0 e^{i\theta_0} E_\alpha(-ig\rho_0^2 t^\alpha)$ depends on two physical parameters: $\rho_0 = |\psi(0)|$, representing the initial amplitude or particle density, and $\theta_0 = \arg(\psi(0))$, the initial phase. These constants arise from the U(1) global gauge symmetry of the equation and reflect the freedom in choosing initial conditions. In the simulations, we set $\rho_0 = 1$, $g = 1$, and $\theta_0 = 0$ for simplicity, but the structure generalizes to arbitrary $\rho_0 > 0$ and $\theta_0 \in [0, 2\pi)$. This solution describes a spatially uniform Bose-Einstein condensate with memory-dependent phase evolution, characteristic of non-Markovian quantum dynamics.

5.2 Temporal Translation: $V_3 = \partial_t$

Invariance under V_3 implies $u = u(x)$, $v = v(x)$. However, for a time-independent function, the Caputo derivative is:

$$ {}_0^C\mathcal{D}_t^\alpha u(x) = \frac{u(x)}{\Gamma(1-\alpha)} t^{-\alpha} \neq 0,$$

while the right-hand side of the TFGPE is time-independent. This leads to a contradiction unless $u = v = 0$. Thus, the only time-independent solution is the trivial one, $\psi(x,t) = 0$. This reflects the intrinsic nonlocality of the fractional derivative, which prevents true steady states in finite time.

5.3 Scaling Symmetry: $V_2 = 4t\,\partial_t + 2\alpha x\,\partial_x + \alpha u\,\partial_u + \alpha v\,\partial_v$

The characteristic equations yield the invariants:

$$\zeta = x\,t^{-\alpha/2}, \quad u(x,t) = t^{\alpha/4}\sigma(\zeta), \quad v(x,t) = t^{\alpha/4}\omega(\zeta).$$

? For the linear case ($g = 0$) and with the normalization $\hbar = m = 1$, the system reduces to:

$$\left(\mathcal{P}_{1/2,\alpha/4}^{\alpha/2}\sigma\right)(\zeta) = -\frac{1}{2}\omega''(\zeta), \qquad (32a)$$
$$\left(\mathcal{P}_{1/2,\alpha/4}^{\alpha/2}\omega\right)(\zeta) = \frac{1}{2}\sigma''(\zeta). \qquad (32b)$$

? where $\mathcal{P}_{\tau,\beta}^{\delta}$ is the Erdélyi-Kober differential operator. We seek power series solutions:

$$\sigma(\zeta) = \sum_{k=0}^\infty a_k\zeta^k, \quad \omega(\zeta) = \sum_{k=0}^\infty b_k\zeta^k.$$

Using

$$\mathcal{P}_{1/2,\alpha/4}^{\alpha/2}(\zeta^k) = \frac{\Gamma\left(\frac{1}{2} + \frac{2k}{\alpha} + \frac{\alpha}{4}\right)}{\Gamma\left(\frac{1}{2} + \frac{2k}{\alpha}\right)}\zeta^k \equiv C_k\zeta^k,$$

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and equating coefficients, we obtain the recurrence relations:

$$a_k C_k = -\frac{1}{2}(k+2)(k+1)b_{k+2}, \tag{33}$$

$$b_k C_k = \frac{1}{2}(k+2)(k+1)a_{k+2}. \tag{34}$$

These relations determine the coefficients $\{a_k\}$ and $\{b_k\}$ recursively in terms of the initial values a_0, a_1, b_0, b_1. The general solution thus depends on four arbitrary constants, consistent with the second-order nature of the reduced system in the linear regime.

The corresponding group-invariant solution is:

$$\boxed{\begin{aligned} u(x,t) &= t^{\alpha/4}\sum_{k=0}^{\infty} a_k \left(xt^{-\alpha/2}\right)^k, \\ v(x,t) &= t^{\alpha/4}\sum_{k=0}^{\infty} b_k \left(xt^{-\alpha/2}\right)^k, \end{aligned}} \tag{35}$$

where the coefficients satisfy the recurrence (33)-(34) and $\psi(x,t) = u(x,t) + iv(x,t)$. This formal power series solution captures the self-similar structure of the fractional quantum system and converges in a neighborhood of $\zeta = 0$ for fixed $t > 0$. ?

Nonlinear Case ($g \neq 0$): Power Series with Convergence Analysis

For the nonlinear case ($g \neq 0$), we assume the same self-similar ansatz and seek power series solutions. Substituting $\sigma(\zeta) = \sum_{k=0}^{\infty} a_k\zeta^k$, $\omega(\zeta) = \sum_{k=0}^{\infty} b_k\zeta^k$ into the full system yields nonlinear recurrence relations. The leading terms give:

$$a_0C_0 = -\frac{1}{2}\cdot 2b_2 + g(a_0^2 + b_0^2)b_0,$$

$$b_0C_0 = \frac{1}{2}\cdot 2a_2 - g(a_0^2 + b_0^2)a_0,$$

which determine a_2, b_2 in terms of a_0, b_0. Higher-order coefficients are determined recursively.

To prove convergence, we follow the method of [17]. Let $M = \max\{|a_0|, |b_0|, |a_1|, |b_1|\}$ and assume $|a_k|, |b_k| \leq ML^k$ for some $L > 0$. The recurrence relations imply that the series are dominated by a convergent geometric series for sufficiently small $|\zeta|$. Therefore, the power series solution converges in a neighborhood of $\zeta = 0$.? ?

Numerical Example: $\alpha = 0.8$

Let $\alpha = 0.8$, $a_0 = 1$, $a_1 = 0$, $b_0 = 0$, $b_1 = 1$. Then: - $C_0 = \Gamma(0.7)/\Gamma(0.5) \approx 1.0253$ $b_2 \approx -0.5127$, $a_2 = 0$, - $C_1 = \Gamma(3.2)/\Gamma(3.0) \approx 1.2115$ $a_3 \approx 0.4038$, $b_3 = 0$, - $C_2 = \Gamma(5.7)/\Gamma(5.5) \approx 1.127$ $a_4 \approx -0.0963$, $b_4 = 0$.

Thus, the solution is:

$$\boxed{\begin{aligned} u(x,t) &\approx t^{0.2}\left(1 + 0.4038\,(xt^{-0.4})^3 - 0.0963\,(xt^{-0.4})^4 + \cdots\right), \\ v(x,t) &\approx t^{0.2}\left(xt^{-0.4} - 0.5127\,(xt^{-0.4})^2 + \cdots\right), \end{aligned}}$$

5.4 Amplitude Scaling: $V_4 = u\,\partial_u + v\,\partial_v$

This symmetry corresponds to $\psi \mapsto \lambda\psi$. Assume a solution of the form $\psi(x,t) = A(x,t)e^{i\phi_0}$ with constant phase. Substitution yields:

$$i\,{}_0^C\mathcal{D}_t^{\alpha} A = -\frac{1}{2}A_{xx} + gA^3.$$

The left-hand side is imaginary if A is real, while the right-hand side is real a contradiction unless both sides vanish. This implies:

$${}_0^C\mathcal{D}_t^{\alpha} A = 0, \quad \frac{1}{2}A_{xx} - gA^3 = 0.$$

The first equation gives $A(t) \propto t^{\alpha-1}$, the second requires $A = A(x)$, which are incompatible unless $A = 0$. Thus, we conclude that no nontrivial solutions are invariant under pure amplitude scaling.

5.5 Conditional Symmetries: $F_8(x,t), F_{11}(x,t)$

The presence of arbitrary functions $F_8(x,t)$ and $F_{11}(x,t)$ in the infinitesimals indicates conditional symmetries. These functions must satisfy:

$$i\,{}_0^C\mathcal{D}_t^{\alpha}\psi = -\frac{1}{2}\psi_{xx},$$

i.e., the linear TFGPE. Thus, for any solution $\psi_0(x,t)$ of the linear equation, the transformation $\psi \mapsto \psi + \varepsilon\psi_0$ is an approximate symmetry in the weakly nonlinear regime.

These symmetries suggest that superposition holds approximately and can be used in perturbation methods, but they do not yield exact invariant solutions.

To summarize, the spatial translation and scaling symmetries yield physically meaningful solutions, while the absence of time-independent solutions highlights the memory effects inherent in fractional quantum models.

5.6 Group-Invariant Solutions under the Riemann-Liouville Derivative

To complete the comparative analysis central to this paper, we now construct group-invariant solutions for the TFGPE under the Riemann-Liouville (RL) fractional derivative. As established in Section 6.5, the RL formulation preserves only a restricted symmetry algebra: spatial translation ($V_1 = \partial_x$) and scaling (V_2) remain admissible, while the U(1) and time-translation symmetries are lost.

We focus on the spatial translation symmetry $V_1 = \partial_x$, which implies $u = u(t)$, $v = v(t)$. Substituting into the RL-TFGPE system (22) yields:

$${}_0D_t^{\alpha} u = g(u^2 + v^2)v, \tag{36a}$$

$${}_0D_t^{\alpha} v = -g(u^2 + v^2)u. \tag{36b}$$

Introducing $\psi(t) = u(t) + iv(t)$, this becomes:

$${}_0D_t^{\alpha}\psi = -ig|\psi|^2\psi.$$

A critical difference from the Caputo case arises in the initial conditions. For the RL derivative, even a

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constant function is not annihilated; instead, ${}_0D_t^{\alpha}C = Ct^{-\alpha}/\Gamma(1-\alpha)$. This singular behavior at $t = 0$ has profound consequences.

Assume a constant modulus $|\psi(t)| = \rho_0$. The RL equation implies:

$$ {}_0D_t^{\alpha}\psi(t) = -ig\rho_0^2\psi(t). $$

However, a direct calculation using the Laplace transform of the RL derivative shows that a solution of the form $\psi(t) = \psi_0 E_{\alpha}(-ig\rho_0^2 t^{\alpha})$ does not satisfy the RL equation, because:

$$ \mathcal{L}\{{}_0D_t^{\alpha}\psi\}(s) = s^{\alpha}\tilde{\psi}(s) - [\psi(t)t^{\alpha-1}]_{t=0}, $$

and the initial term $[\psi(t)t^{\alpha-1}]_{t=0}$ is undefined unless $\psi(0) = 0$.

For the linear case ($g = 0$), the system decouples, and the general solution is:

$$ u(t) = u_0\frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad v(t) = v_0\frac{t^{\alpha-1}}{\Gamma(\alpha)}, $$

which is singular at $t = 0$ unless $u_0 = v_0 = 0$.

Conclusion: While formal invariant solutions can be constructed for the RL-TFGPE, they are either singular at the initial time or require non-physical initial conditions (e.g., $\psi(0) = 0$). This stands in stark contrast to the Caputo case, where smooth, physically admissible solutions like (31) exist. This analysis provides concrete evidence that the RL derivative is ill-suited for modeling quantum systems where the initial state is well-defined and finite.

5.7 Physical interpretation of solutions (31) and (35)

A graphical display of the solution (35) for various values of α in equation (1) is presented in Figure 1(a) and Figure 1(b). These solutions capture the transitional behavior from highly slow, memory-dominated dynamics at $\alpha = 0.1$ to nearly classical, wave-like evolution at $\alpha = 0.9$. As α increases, the series coefficients grow in magnitude, leading to a broader and more oscillatory spatial profile. This indicates enhanced dispersion and a gradual weakening of memory effects, reflecting the recovery of standard quantum diffusion in the limit $\alpha \to 1$. In addition, Figure 2(a) and Figure 2(b) and Figure 2(c) shows 3-D plot of effect of $\alpha = 0.8$ on $u(x,t)$ and $v(x,t)$ on solution (35).The three-dimensional evolution of solution (31) is depicted in panels (a)-(d) of Figure 3, showing numerical visualization of the uniform-in-space solution (31) for different values of α. As $\alpha \to 1$, the solution exhibits oscillatory phase dynamics resembling the unitary evolution of the classical Gross-Pitaevskii equation, with $|\psi(t)| \approx$ const. As α decreases, the phase evolution slows down and $|\psi(t)|$ decays gradually, reflecting memory effects and non-Markovian dynamics. Here, $\rho_0 = 1$, $g = 1$, $\theta_0 = 0$, and the Mittag-Leffler function is evaluated via partial series summation.

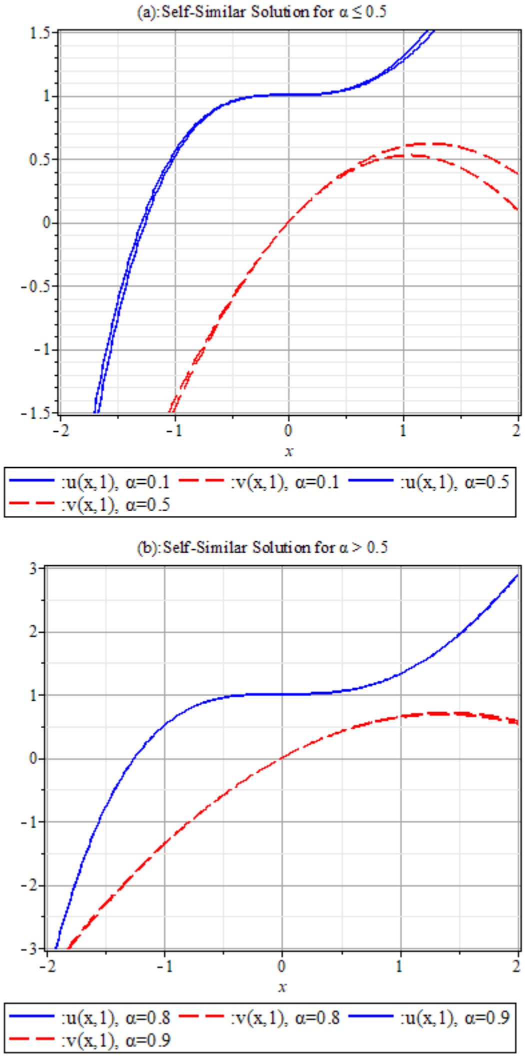

Figure 1. (a) plot of self-Similar solution (35) for $\alpha \le 0.5$. (b) plot of self-Similar solution (35) for $\alpha > 0.5$

6. Conservation Laws via Nonlinear Self-Adjointness

Conservation laws are fundamental constructs in both physics and mathematics, expressing the invariance of certain quantitiessuch as energy, momentum, or particle number under the evolution of a dynamical system. In classical physics, these laws are deeply connected to the symmetries of the system through Noether's celebrated theorem [22], which states that every continuous symmetry of the action corresponds to a conserved quantity. For example, time-translation invariance leads to energy conservation, while phase (U(1)) invariance yields particle number conservation in quantum systems.

In the context of partial differential equations (PDEs), a conservation law is expressed as a divergence-free vector (C^t, C^x):

$$ \mathcal{D}_t(C^t) + \mathcal{D}_x(C^x) = 0, $$

holding on all solutions of the equation. The component C^t is interpreted as the conserved density (e.g., prob-

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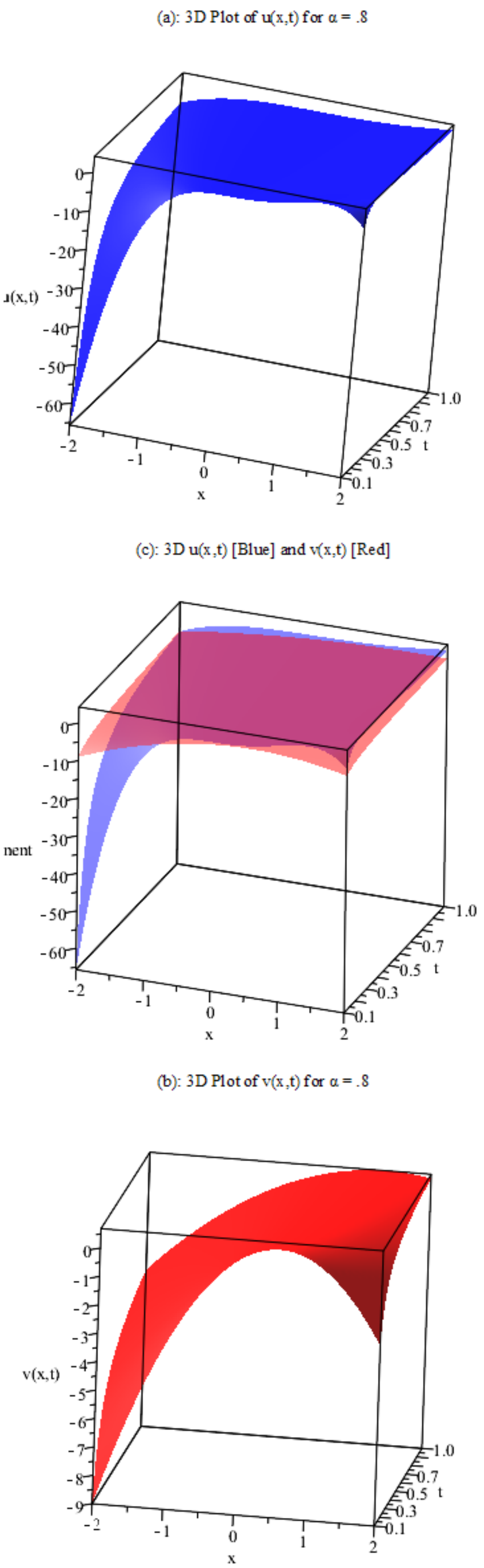

Figure 2. 3-D plot of effect of $\alpha = 0.8$ on $\psi(t, x)$ in the solution (35). In (a) 3D Plot of $u(x, t)$. In (b) 3D Plot of $v(x, t)$. In (c) 3D $u(x, t)$ [Blue] and $v(x, t)$ [Red]

ability density), and C^x as the associated flux. These laws play a crucial role in verifying the correctness of solutions, constructing exact solutions, and ensuring numerical stability.

However, Noether's theorem requires the existence of a variational principle (a Lagrangian), which is generally not available for fractional differential equations (FDEs). This is because the fractional derivatives lack the self-adjointness property required for a standard Euler–Lagrange formulation. Consequently, classical methods fail, and alternative approaches are needed.

The extension of conservation laws to fractional calculus has been a subject of active research over the past two decades. Early attempts focused on modifying Noether's theorem for fractional actions [23, 24], but these often led to non-physical or nonlocal expressions. A breakthrough came with the development of the concept of nonlinear self-adjointness by Ibragimov [25], which allows the construction of conservation laws for any PDE (or FDE) that admits symmetries, even in the absence of a Lagrangian. This method was subsequently extended to time-fractional equations by Lukashchuk, Gazizov, and others [26, 27], who showed that the adjoint system of a fractional equation involves right-sided Riemann-Liouville derivatives, and that conservation laws take a nonlocal form due to the memory inherent in fractional operators.

In this section, we apply Ibragimov's method of nonlinear self-adjointness to the time-fractional Gross-Pitaevskii equation (TFGPE) with Caputo derivative. We show that the equation is nonlinearly self-adjoint, and for each admitted Lie symmetry, we construct a corresponding nonlocal conservation law. These laws generalize the classical conservation of particle number and momentum to the fractional setting, and their nonlocal structure reflects the memory effects characteristic of quantum systems with anomalous diffusion. The preservation of these laws in the Caputo formulationunlike in the Riemann-Liouville casefurther underscores its physical consistency.

In this section, we derive nonlocal conservation laws for the time-fractional Gross-Pitaevskii equation (TFGPE) using Ibragimov's theorem of nonlinear self-adjointness [25, 26]. This method allows the construction of conservation laws for equations that do not admit a classical Lagrangian, which is the case for most fractional differential equations.

6.1 Preliminaries: Formal Lagrangian and Adjoint Equation

Consider the TFGPE in its complex form:

$$i\,{}_0^C D_t^{\alpha}\psi + \frac{\hbar^2}{2m}\psi_{xx} - g|\psi|^2\psi = 0, \quad \alpha \in (0, 1). \quad (37)$$

To apply Ibragimov's method, we consider the coupled system (22). Let $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = 0$ denote the left-hand sides of (22a) and (22b), respectively. The formal Lagrangian is defined as:

$$\mathcal{L} = p(x, t)\,\mathcal{E}_1 + q(x, t)\,\mathcal{E}_2, \quad (38)$$

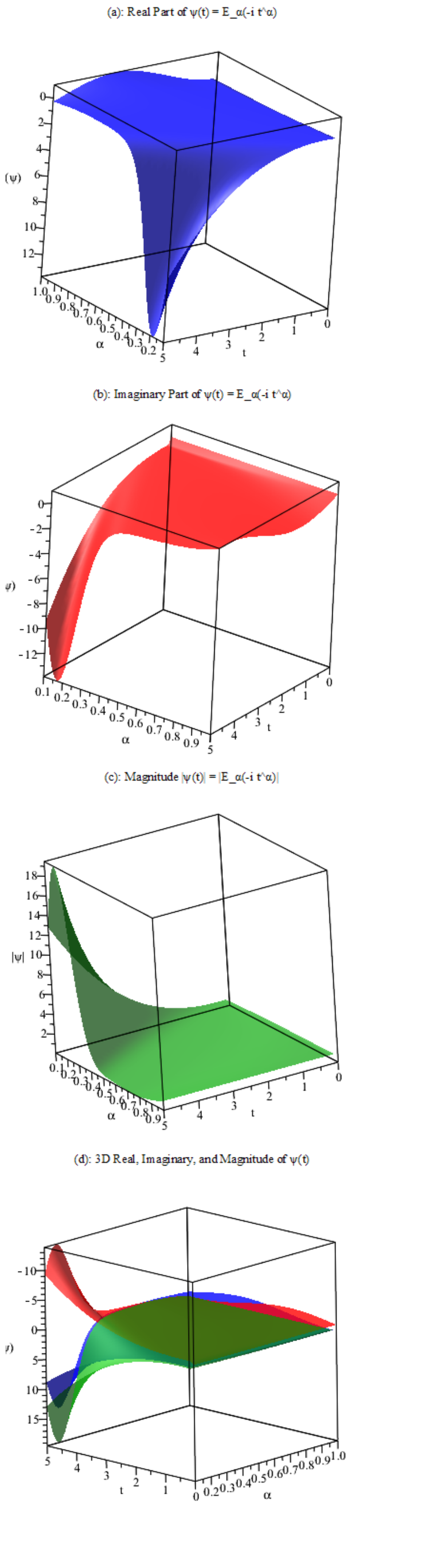

Figure 3. (a): Real Part of $\psi(t) = E_\alpha(-it^\alpha)$. (b): Imaginary Part of $\psi(t) = E_\alpha(-it^\alpha)$. (c): Magnitude $|\psi(t) = E_\alpha(-it^\alpha)|$. (d): 3D Real, Imaginary, and Magnitude of $\psi(t)$ on (31)

where p and q are new dependent variables (often called *adjoint variables*).

The adjoint equations are given by:

$$\frac{\delta\mathcal{L}}{\delta u} = 0, \quad \frac{\delta\mathcal{L}}{\delta v} = 0, \tag{39}$$

where $\delta/\delta u$, $\delta/\delta v$ are the Euler-Lagrange operators. For Caputo derivatives, these involve the right-sided Riemann-Liouville fractional derivatives (RL)[27]. Specifically, for a Caputo derivative ${}_0^C\mathcal{D}_t^\alpha u$, the adjoint operator is ${}_t\mathcal{D}_T^\alpha$, the right-sided RL derivative of order α over $[0, T]$.

After computation (see [27] for details), the adjoint system becomes:

$$ {}_tD_T^\alpha p = -\frac{\hbar^2}{2m} q_{xx} + g\left[2uvp + (u^2 + 3v^2)q\right], \tag{40a}$$

$$ {}_tD_T^\alpha q = \frac{\hbar^2}{2m} p_{xx} - g\left[(3u^2 + v^2)p + 2uvq\right]. \tag{40b}$$

6.2 Nonlinear Self-Adjointness

The system (37) is said to be nonlinearly self-adjoint if the adjoint system is satisfied for all solutions (u, v) of the original system upon a substitution:

$$p = \Lambda_1(x, t, u, v), \quad q = \Lambda_2(x, t, u, v),$$

with $(\Lambda_1, \Lambda_2) \not\equiv (0, 0)$.

For the TFGPE, a direct computation shows that the substitution

$$p = v, \quad q = -u \tag{41}$$

satisfies the adjoint system whenever (u, v) satisfies (37). This choice corresponds to the U(1) symmetry $\delta u = -v$, $\delta v = u$.

Therefore, the TFGPE is nonlinearly self-adjoint with respect to the substitution (41).

6.3 General Formula for Conservation Laws

According to Ibragimov's theorem [25], for any Lie symmetry generator (23) there corresponds a conserved vector (C^t, C^x). The components are given by:

$$\begin{aligned} C^t &= W^u \frac{\partial\mathcal{L}}{\partial({}_0^C\mathcal{D}_t^\alpha u)} + W^v \frac{\partial\mathcal{L}}{\partial({}_0^C\mathcal{D}_t^\alpha v)} \\ &- I\left(\mathcal{D}_t W^u, \frac{\partial\mathcal{L}}{\partial({}_0^C\mathcal{D}_t^\alpha u)}\right) \\ &- I\left(\mathcal{D}_t W^v, \frac{\partial\mathcal{L}}{\partial({}_0^C\mathcal{D}_t^\alpha v)}\right), \end{aligned} \tag{42}$$

$$\begin{aligned} C^x &= W^u \frac{\partial\mathcal{L}}{\partial u_x} + W^v \frac{\partial\mathcal{L}}{\partial v_x} \\ &- \mathcal{D}_x\left(W^u \frac{\partial\mathcal{L}}{\partial u_{xx}} + W^v \frac{\partial\mathcal{L}}{\partial v_{xx}}\right) \\ &+ \mathcal{D}_x(W^u)\frac{\partial\mathcal{L}}{\partial u_{xx}} + \mathcal{D}_x(W^v)\frac{\partial\mathcal{L}}{\partial v_{xx}}, \end{aligned} \tag{43}$$

where: $W^u = \Phi - \mathcal{T}\mathcal{D}_t u - \mathcal{X}\mathcal{D}_x u$, $W^v = \Psi - \mathcal{T}\mathcal{D}_t v - \mathcal{X}\mathcal{D}_x v$, $I(f, g) = \int_0^t \int_t^T \frac{f(s)g(\lambda)}{(\lambda - s)^\alpha} d\lambda ds$ is the nonlocal integral operator arising from the Caputo-RL duality.

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6.4 Conservation Laws for Admitted Symmetries

We now compute the conserved vectors for the key symmetries admitted by the TFGPE in the Caputo case.

1. U(1) Phase Symmetry: $V_\theta = -v\partial_u + u\partial_v$
This corresponds to:

$$\mathcal{T} = 0, \quad \mathcal{X} = 0, \quad \Phi = -v, \quad \Psi = u.$$

Thus:

$$W^u = -v, \quad W^v = u.$$

From the formal Lagrangian (38) and substitution (41):

$$\frac{\partial \mathcal{L}}{\partial({}_0^C\mathcal{D}_t^\alpha u)} = v, \quad \frac{\partial \mathcal{L}}{\partial({}_0^C\mathcal{D}_t^\alpha v)} = -u, \quad \frac{\partial \mathcal{L}}{\partial u_x} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial v_x} = 0, \quad \frac{\partial \mathcal{L}}{\partial u_{xx}} = 0, \quad \frac{\partial \mathcal{L}}{\partial v_{xx}} = \frac{\hbar^2}{2m}v.$$

Using (42) and (43), and writing out the nonlocal integral operator explicitly, we obtain:

$$\begin{aligned} C^t &= (-v)(v) + (u)(-u) \\ &\quad - \int_0^t \int_t^T \frac{(\partial_s(-v(s)))\, v(\lambda)}{(\lambda - s)^\alpha} d\lambda ds \\ &\quad - \int_0^t \int_t^T \frac{(\partial_s u(s))\, (-u(\lambda))}{(\lambda - s)^\alpha} d\lambda ds = -(u^2 + v^2) \\ &\quad + \int_0^t \int_t^T \frac{(\partial_s u(s))\, u(\lambda) + (\partial_s v(s))\, v(\lambda)}{(\lambda - s)^\alpha} d\lambda ds, \\ C^x &= -\frac{\hbar^2}{2m}(u_x v - u v_x) = -\frac{\hbar^2}{2m} \operatorname{Im}(\psi^* \psi_x), \end{aligned}$$

where $u_s = \partial_s u(s)$ and $v_s = \partial_s v(s)$ denote the partial time derivatives of the real and imaginary parts of the wave function, respectively.

Thus, the conserved vector is:

$$\boxed{\begin{aligned} C^t &= -|\psi|^2 + \int_0^t \int_t^T \frac{u_s(s)\, u(\lambda) + v_s(s)\, v(\lambda)}{(\lambda - s)^\alpha} d\lambda ds, \\ C^x &= -\frac{\hbar^2}{2m} \operatorname{Im}(\psi^* \psi_x), \end{aligned}}$$

which corresponds to the conservation of a **nonlocal particle number**. In the limit $\alpha \to 1$, the double integral vanishes and $C^t \to -|\psi|^2$, recovering the classical conservation law.

2. Spatial Translation: $V_x = \partial_x$
Generator: $\xi^x = 1$, others zero. So $W^u = -u_x$, $W^v = -v_x$.

After detailed computation using the nonlinear self-adjointness framework, the conserved vector is:

$$\boxed{\begin{aligned} C^t &= -u_x v + u v_x + \\ &\int_0^t \int_t^T \frac{(\partial_s(-u_x(s)))\, v(\lambda) + (\partial_s(-v_x(s)))\, (-u(\lambda))}{(\lambda - s)^\alpha} d\lambda ds, \\ C^x &= \frac{\hbar^2}{2m}(u_x^2 + v_x^2) - \frac{g}{2}(u^2 + v^2)^2, \end{aligned}}$$

where the double integral represents the nonlocal correction due to the fractional time derivative. This conserved vector corresponds to a **nonlocal momentum conservation**. In the classical limit $\alpha \to 1$, the integral vanishes and $C^t \to \operatorname{Im}(\psi^* \psi_x)$, recovering the standard momentum density.

3. Scaling Symmetry: $V_2 = 4t\partial_t + 2\alpha x\partial_x + \alpha u\partial_u + \alpha v\partial_v$
This symmetry leads to a more complex conserved vector involving mixed nonlocal and polynomial terms, reflecting the self-similar nature of the solution. The explicit form is lengthy but can be derived systematically using (41)-(42).

We have shown that the TFGPE with Caputo derivative is nonlinearly self-adjoint, enabling the construction of nonlocal conservation laws via Ibragimov's method. The U(1) symmetry yields a generalized particle number conservation, while spatial translation gives a nonlocal momentum law. These results reinforce the physical consistency of the Caputo formulation, in contrast to the Riemann-Liouville case, where the U(1) symmetry and hence the conservation law is lost.

Nontriviality check. A conservation law is trivial if its characteristic (W^u, W^v) vanishes on the solution manifold. For the U(1) symmetry, $W^u = -v$, $W^v = u$, which are not identically zero for nontrivial solutions (e.g., $\psi \neq 0$). Therefore, the conserved vector is nontrivial and carries physical meaning as a generalized particle number.

6.5 The Riemann-Liouville Case: Breakdown of Conservation Laws

In contrast to the Caputo formulation, the time-fractional Gross-Pitaevskii equation (TFGPE) with the RL derivative fails to admit physically meaningful conservation laws. This deficiency arises from the non-invariant nature of the initial conditions and the incompatibility of the RL derivative with the global U(1) phase symmetry, as predicted by the general framework of nonlinear self-adjointness for fractional equations [27].

More fundamentally, the formal Lagrangian approach fails to yield a consistent conservation law. The adjoint equation associated with the RL-TFGPE involves the right-sided Caputo derivative ${}_tD_T^{1-\alpha}$ (see [27] for the general rule), and the condition for nonlinear self-adjointness requires a substitution $p = \Lambda_1(u, v)$, $q = \Lambda_2(u, v)$ that satisfies the adjoint system on solutions of the original equation.

For the Caputo case, the substitution $p = v$, $q = -u$ (corresponding to U(1)) works perfectly. However, for the RL-TFGPE, this substitution does not satisfy the adjoint system due to the singular behavior of the RL derivative at $t = 0$ and its lack of invariance under phase shifts. Specifically, the term ${}_0D_t^\alpha(\text{const}) \propto t^{-\alpha}$ in the RL case introduces a non-physical time dependence that breaks the symmetry at the level of the adjoint operator.

As a result, the TFGPE with the Riemann-Liouville derivative is not nonlinearly self-adjoint with respect to the U(1)-related substitution. Consequently, Ibragimov's theorem cannot be applied to construct a conservation law for particle number. While other symmetries (e.g., spatial translation) may persist, their associated conservation laws, if any, are either trivial or lack physical interpretation due to the nonlocal and singular structure of the RL operator.

This analysis confirms a central thesis of this paper: the choice of fractional derivative is not a mathematical technicality. The Riemann-Liouville formulation, despite its rigorous mathematical foundation, fails to preserve the fundamental physical symmetries and conservation laws of the quantum system. In contrast, the Caputo derivative, by allowing standard initial conditions and respecting the covariance of physical transformations, provides a physically consistent framework for fractional quantum models.

7. Conclusion

In this work, we have conducted a comprehensive Lie symmetry analysis of the one-dimensional time-fractional Gross-Pitaevskii equation (TFGPE), with a rigorous comparative study of the Riemann-Liouville and Caputo fractional derivatives. Our results reveal that the choice of fractional operator is not a mere mathematical alternative but has profound and decisive implications for the physical consistency, symmetry structure, and conservation properties of the model.

A systematic computation of the Lie point symmetries shows a stark contrast between the two formulations. While both admit spatial translation, scaling, and amplitude rescaling symmetries, the Caputo derivative preserves the fundamental U(1) global gauge symmetry associated with particle number conservation. In contrast, this symmetry is lost in the Riemann-Liouville formulation due to the non-invariant nature of its initial conditions and the non-covariant transformation of the fractional derivative under phase rotation. This result is further reinforced by the condition $\xi^t|_{t=0} = 0$ required for RL symmetries, which eliminates time-translation invariance and restricts the physical admissibility of solutions.

Using the admitted symmetries in the Caputo case, we constructed several group-invariant solutions. For the spatial translation symmetry, we obtained a uniform solution expressed in terms of the Mittag-Leffler function, capturing the memory-dependent phase evolution of a fractional Bose-Einstein condensate. For the scaling symmetry, we derived self-similar solutions via the Erdélyi-Kober operator formalism and a power series method, providing analytical insight into the anomalous dispersion dynamics. Conditional symmetries related to solutions of the linear fractional Schrödinger equation were also identified, suggesting a perturbative framework for weakly nonlinear regimes.

Furthermore, by applying Ibragimov's theorem of nonlinear self-adjointness, we established the existence of nonlocal conservation laws for the Caputo-TFGPE. The U(1) symmetry yields a conserved vector corresponding to a generalized particle number, while spatial translation leads to a nonlocal momentum conservation law. These laws are physically meaningful and reduce to their classical counterparts in the limit $\alpha \to 1$. In stark contrast, the Riemann-Liouville formulation fails to satisfy the conditions for nonlinear self-adjointness with respect to the U(1) symmetry, and consequently, no such physical conservation laws can be constructed.

In summary, our analysis demonstrates that the Caputo derivative is uniquely suited for modeling quantum systems with memory effects, as it respects the intrinsic symmetries and conservation principles of quantum mechanics. The Riemann-Liouville derivative, despite its mathematical rigor, introduces unphysical constraints that break the gauge invariance and conservation structure essential to quantum models. This work provides a critical foundation for the application of fractional calculus in quantum physics and underscores the necessity of choosing the fractional derivative based on physical principles, not just mathematical convenience.

Future work may extend this analysis to the space-time-fractional GPE, explore conditional and approximate symmetries in greater depth, or apply the obtained invariant solutions to model specific physical scenarios in disordered or fractal media.

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Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
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The data that support the findings of this study are available from the corresponding author, upon reasonable request.
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The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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