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Research Article

A Novel Extension of the Marshall-Olkin Distribution: Reliability, Mathematical Properties, and Applications in Different Sciences

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Abstract:

This paper introduces a new and flexible family of continuous probability distributions, referred to as the Exponentiated Chen Marshall–Olkin family. The linear representation of the proposed model is derived, and several of its statistical properties, including moments, quantile function, Rényi entropy, and reliability measures, are investigated. Parameter estimation for this family is discussed using the maximum likelihood method under both complete and right-censored samples, while three distance-based estimation approaches are also considered.

A particular sub-model of this family, called the Exponentiated Chen Marshall-Olkin Weibull distribution, is also proposed and studied in detail. Its mathematical characteristics and related sub-models are explored, and four different estimation techniques-maximum likelihood, least squares, weighted least squares, and Anderson-Darling-are employed to estimate the unknown parameters.

Furthermore, a comprehensive simulation study is conducted to assess the bias and mean square error of the estimators, followed by applications to real health and engineering datasets. The empirical results demonstrate that the Exponentiated Chen Marshall–Olkin family provides excellent flexibility for modeling data exhibiting skewness, heavy tails, reliability characteristics, and non-monotonic hazard rates, confirming its potential as a powerful tool in reliability and lifetime data analysis.

Keywords: Marshall-Olkin family of distributions; Exponentiated-G family; Chen-G family; Generated Family; Maximum Likelihood Estimation

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1. Introduction

In recent years, several extended and compound probability distributions have been developed to provide greater flexibility for modeling skewed, heavy-tailed, and non-monotonic hazard rate data observed in engineering, environmental, and biomedical studies. For instance, the Gamma-generated family [1] enhances tail behavior control and modeling flexibility in lifetime data, while other constructions, such as the transformed MG-extended exponential distribution [2], provide additional flexibility. Similarly, a generalized inverse Gaussian distribution [3] demonstrates further advances in flexible lifetime modeling. These models have been widely applied in reliability and survival analysis, where incomplete and censored observations frequently arise in practice (see also [4]; however, many existing distributional generators enhance flexibility only in a single aspect, such as tail behavior or hazard rate shape, and often lack the ability to simultaneously control both the left and right tails of the distribution. In addition, several well-known families face limitations in adequately representing complex hazard rate structures, including bathtub-shaped and multi-phase patterns, which are frequently encountered in practical lifetime data.

To address these limitations, we propose a novel four-parameter family named the Exponentiated Chen Marshall-Olkin-G (ECMO-G) family of distributions. This model combines the flexibility of the Chen-G and

Marshall-Olkin-G generators within the Exponentiated-G framework [5]. Unlike many existing constructions that extend a baseline distribution in a single direction, the proposed ECMO-G family introduces additional shape parameters that provide enhanced control over tail behavior while allowing a wide range of hazard rate shapes within a unified modeling framework. As a result, the proposed ECMO-G family generalizes several well-known models as special cases, including the Weibull, Lomax, Chen, and Marshall-Olkin distributions.

From a mathematical perspective, this construction offers extended tail flexibility through the inclusion of shape parameters that jointly regulate the left and right tails of the distribution, which is particularly important for modeling asymmetric lifetime data. At the same time, the proposed formulation remains analytically tractable, allowing closed-form derivations for key statistical measures such as moments, entropy, quantile functions, and order statistics.

The ECMO-G family is introduced not merely as an additional distribution but as a flexible and practically motivated modeling framework. Its usefulness is supported by both theoretical development and empirical evidence. Specifically, key statistical properties are derived in explicit form, a particular sub-model is investigated in detail, and parameter estimation is developed using maximum likelihood methods under both complete and censored observations. The performance of the estimators is further examined through Monte Carlo simulations. Moreover, applications to two real datasets and one simulated dataset demonstrate that the proposed family provides improved goodness-of-fit compared to commonly used competing models, according to standard information criteria such as AIC and related measures. These results highlight the practical relevance and added value of the proposed family for modeling complex lifetime data.

To motivate the proposed construction and clarify its connection with existing generators, we briefly review the Exponentiated-G, Chen-G, and Marshall-Olkin-G families.

1.1 Exponentiated-G Family of Distributions

Starting with a baseline distribution with a probability distribution function (PDF) $j(x;\beta)$, cumulative distribution function (CDF) $J(x;\beta)$, and parameter β. [6] proposed a new flexible family of distributions by adding an extra shape parameter γ. For $x > 0$ the Exponentiated-G (Exp-G) family of distributions has a PDF

$$f(x;\beta) = \gamma j(x;\beta)\, J(x;\beta)^{\gamma-1}, \tag{1}$$

and CDF

$$F(x;\beta) = J(x;\beta)^{\gamma}. \tag{2}$$

where $\gamma > 0$ is the shape parameter.

The Exp-G has been applied in extending baseline distributions, including the exponentiated exponential distribution [7], the exponentiated odd Lindley-X power series distribution [8], the exponentiated Weibull-logarithmic transformation distribution [9], the exponentiated generalized Weibull exponential distribution [10], the exponentiated gamma Burr-type X family [11], the discrete exponentiated Chen distribution [12], and the exponentiated half Logistic-G family [13].

1.2 Marshall-Olkin-G Family of Distributions

[14] defined a new semi-parametric model by adding a parameter to a family of distributions. Let $\overline{G}(x;\beta)$ be the survival function of the existing distribution with parameter β. Then, the CDF and PDF of the Marshall-Olkin-G (MO-G) family of distributions can be obtained by

$$H(x;\beta) = \frac{G(x;\beta)}{1-\overline{\alpha}\,\overline{G}(x;\beta)}, \tag{3}$$

and

$$h(x;\beta) = \frac{\alpha g(x;\beta)}{\left(1-\overline{\alpha}\,\overline{G}(x;\beta)\right)^2}. \tag{4}$$

where $-\infty \le x \le \infty$, $\alpha > 0$, $\overline{\alpha} = 1-\alpha$, and if $\alpha = 1$, then $\overline{H}(x;\beta) = \overline{G}(x;\beta)$.

[15] also proposed two-parameter exponential and three-parameter Weibull distributions in their pioneering work. Additionally, [16] demonstrated the application of the Marshall-Olkin technique in various areas, including reliability theory, time series modeling, and stress-strength analysis.

Furthermore, several studies have been conducted to develop the MO-G family of distributions. Examples include the Marshall-Olkin unit-exponentiated-half-logistic distribution [17], the Marshall-Olkin exponentiated inverse Rayleigh distribution [18], the Marshall-Olkin power Rayleigh distribution [19], the generalized Marshall-Olkin exponentiated exponential distribution [20], the Marshall-Olkin Zubair-G family [21] and the Marshall-Olkin alpha power inverse exponential distribution [22].

1.3 Chen-G Family of Distributions

Several classes of distributions are extended using one or more shape parameters in addition to those in the baseline. [23] proposed a new wider family called the Chen-G family of distributions. For a baseline CDF $H(x;\beta)$ with PDF $h(x;\beta)$ and parameter vector β, they defined the Chen-G family of distributions, which has a CDF

$$J(x;\beta) = A\left[1-e^{\tau\left(1-e^{(H(x;\beta))^{\theta}}\right)}\right], \tag{5}$$

and the PDF

$$\begin{aligned} j(x;\beta) &= \tau\theta A h(x;\beta)\,(H(x;\beta))^{\theta-1} e^{(H(x;\beta))^{\beta}} \\ &\quad \times e^{\tau\left(1-e^{(H(x;\beta))^{\theta}}\right)}. \end{aligned} \tag{6}$$

where $A = \frac{1}{1-e^{\tau(1-e)}}$ is a normalizing constant, $x > 0$, $\tau > 0$ is the scale parameter and $\theta > 0$ is the shape parameter.

10.57647/mathsci.2026.66781

Further developments include generalized Chen variants, such as the entropy-transformed Chen distribution [24], the modified generalized Chen distribution [25], and extended Chen-Poisson lifetime distribution [26].

Although numerous distributions have been proposed for the analysis of lifetime and reliability data, there remains a practical need for a unified and flexible modeling framework that can accommodate skewness, heavy tails, and complex hazard rate behavior without introducing unnecessary model complexity. In this study, we propose a new family of continuous distributions by combining the Exp-G, MO-G, and Chen-G families, called the ECMO-G family of distributions. The proposed family is not intended as merely another alternative distribution, but rather as a general framework that unifies several classical models while offering enhanced flexibility through a parsimonious parameterization.

The proposed model, owing to its closed-form distribution functions and analytical tractability, allows the investigation of key statistical properties and reliability measures. Additionally, the hazard rate function of the proposed distribution can take various shapes, such as increasing, decreasing, bathtub-shaped, inverted bathtub-shaped, and increasing-decreasing-increasing, making it suitable for modeling complex lifetime data.

Multiple estimation methods are employed for parameter estimation, including likelihood-based approaches that account for censoring, as well as distance-based methods, and the performance of the estimators is assessed through Monte Carlo simulations. The results demonstrate satisfactory finite-sample behavior and consistency across the considered evaluation criteria. Furthermore, two real-life datasets, along with one dataset generated through a simulation algorithm, are analyzed to illustrate the applicability of the proposed model in comparison with several competing distributions.

Unlike existing Chen-G or Marshall-Olkin-G extensions, which primarily enhance flexibility through a single generating mechanism, the proposed ECMO framework enables the simultaneous regulation of tail behavior and hazard rate shape within a unified construction. This joint flexibility allows the model to accommodate complex lifetime data exhibiting both heavy-tailed characteristics and non-monotonic hazard structures, which are not adequately captured by single-generator extensions. As a result, the ECMO family provides a more comprehensive yet tractable modeling framework for reliability and survival data.

The remainder of the paper is organized as follows. In Section 2, the new family and its special models are defined, and a useful mixture representation for the PDF and CDF is provided. The statistical properties and maximum likelihood estimation (MLE) of the model parameters under both complete and censored data are also discussed. In Section 3, the Exponentiated Chen Marshall-Olkin Weibull distribution is formally introduced, and its mathematical properties are presented. Section 4 contains a Monte Carlo simulation study to examine the performance of the proposed estimators. In Section 5, two real-life datasets and one simulated dataset are analyzed and compared with several baseline models. Finally, concluding remarks are given in Section 6.

2. The Exponentiated Chen Marshall-Olkin G Family of Distributions

We now propose a new extended family of distributions. First, the CDF and PDF of the MO-G family in Equations (3) and (4) are inserted into the Chen formulation in Equation (6), resulting in the Chen MO-G (CMO-G) distribution. The PDF of the CMO-G is given by

$$
\begin{aligned}
j(x;\beta) &= \tau\theta A \frac{\alpha g(x;\beta)}{(1-\overline{\alpha}\overline{G}(x;\beta))^2}\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta-1} \\
&\quad \times e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}} e^{\tau\left(1-e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}}\right)} \qquad (7)
\end{aligned}
$$

Similarly, by using Equation (3) in Equation (5), we obtain the CDF of the CMO-G distribution.

$$
J(x;\beta) = A\left[1 - e^{\tau\left(1-e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}}\right)}\right] \qquad (8)
$$

Now, by inserting Equations (7) and (8) into Equation (1), the PDF of the Exponentiated Chen Marshall-Olkin-G family of distributions is obtained as

$$
\begin{aligned}
f(x;\beta) &= \gamma\tau\theta\alpha A^{\gamma} g(x;\beta)\left(G(x;\beta)\right)^{\theta-1} \qquad (9)\\
&\quad \times\left[1-\overline{\alpha}\overline{G}(x;\beta)\right]^{-(\theta+1)} e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}} \\
&\quad \times e^{\tau\left(1-e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}}\right)}\left[1-e^{\tau\left(1-e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}}\right)}\right]^{\gamma-1}
\end{aligned}
$$

Similarly, by inserting Equation (8) into Equation (2), the CDF of the ECMO-G family is obtained as

$$
F(x;\beta) = \left\{A\left[1 - e^{\tau\left(1-e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}}\right)}\right]\right\}^{\gamma}. \qquad (10)
$$

where $x > 0$, $\gamma, \theta, \alpha > 0$, and $|\tau| > 0$.

For specific parameter values, the PDF in Equation (9) reduces to

1. The CMO-G family of distributions, for $\gamma = 1$.
2. The EC-G family of distributions [27], for $\alpha = 1$.
3. The EMO-G family of distributions [28], for $\tau = 1$ and $\theta = 1$.
4. The Chen-G family of distribution [23], for $\alpha = 1$ and $\gamma = 1$.
5. The Exp-G family of distributions [6], for $\alpha = 1$, $\tau = 1$, and $\theta = 1$.
6. The MO-G family of distributions [14], for $\gamma = 1$, $\tau = 1$, and $\theta = 1$.

10.57647/mathsci.2026.66781

2.1 Series Expansion

Obtaining explicit analytical expressions for certain mathematical quantities directly from the PDF (Equation 9) and the CDF (Equation 10) is often difficult or even impossible. Therefore, series expansions are commonly employed to derive simpler forms of these expressions. In this subsection, we present such simplified forms, which facilitate the derivation of various structural properties of the ECMO-G family of distributions. By repeated application of the binomial, Taylor, and generalized binomial series, the following form of the density function for the proposed family of distributions is obtained.

$$f\left(x;\beta\right)=\sum_{j,k,m,s=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s}t_h\varphi_{\theta(m+1)+p}\ , \qquad (11)$$

where

$$t_h=(-1)^{j+l+p}\binom{\gamma-1}{j}\binom{k}{l}\binom{\theta\left(m+1\right)+s}{s}\binom{s}{p} \times\frac{\tau^k(j+1)^k(l+1)^m}{k!m!\left[\theta\left(m+1\right)+p\right]}\gamma\tau\theta\alpha\overline{\alpha}^s A^\gamma, \qquad (12)$$

and

$$\varphi_{\theta(m+1)+p}=\left[\theta(m+1)+p\right]g(x;\beta)G(x;\beta)^{\theta(m+1)+p-1}.$$

The expansion of the PDF of the ECMO-G distribution is a linear combination of the Exp-G distributions. Similarly, the CDF of the ECMO-G distribution in Equation (10) can be expressed in a comparable simplified form as follows

$$F\left(x;\beta\right)=\sum_{j,k,m,s=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s}t_h\phi_{\theta(m+1)+p}\ , \qquad (13)$$

where

$$\phi_{\theta(m+1)+p}=G\left(x;\beta\right)^{\theta(m+1)+p}\ .$$

is the CDF of the Exp-G family of distributions with the power parameter $\theta\left(m+1\right)+p$. This means that the ECMO-G density can be expressed as a mixture of Exp-G densities. Therefore, several of its properties can be derived directly from those of the Exp-G model.

2.2 Distributional Properties

In this subsection, we discuss the mathematical properties of the ECMO-G family of distributions. Established algebraic expansions to determine some structural properties of the ECMO-G distribution can be more efficient than computing them directly by numerical integration of its density function.

2.2.1 Moment

Some of the most important features of a distribution can be studied through moments. Descriptive statistics of a distribution, such as mean, variance, skewness, kurtosis, and other parametric measures that describe the shape and behavior of the distribution, can be derived from it. In this subsection, analytical expressions for the moments of the proposed family of distributions are obtained and summarized in the following theorem.

Theorem 2.1 *Assume that X is a random variable that adheres to the ECMO-G family of distributions, and let r denote a positive integer. The r^{th} moment of the ECMO-G distribution can be determined as*

$$E\left[X^r\right]=\sum_{j,k,m,s=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s}t_{h\,\theta(m+1)+p}\ ,$$

where t_h is introduced in Equation (12) and

$$_{\theta(m+1)+p}=\int_0^\infty x^r\ \varphi_{\theta(m+1)+p}dx\ .$$

is the moment of the Exp-G distribution with the power parameter $\theta\left(m{+}1\right)+p$.

Proof of Theorem 2.1 Using the definition of the r^{th} moment, we have

$$E\left[X^r\right]=\delta'_r=\int_0^\infty x^r\ f\left(x;\beta\right)dx,$$

Substituting Equation (11) into the above integral yields

$$E\left[X^r\right]=\sum_{j,k,m,s=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s}t_h\int_0^\infty x^r\varphi_{\theta(m+1)+p}dx,$$

Therefore,

$$E[X^r]=\sum_{j,k,m,s=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s}t_hI_{\theta(m+1)+p}.$$

Remark 2.2 *The n^{th} central moment of X, given as F_n, is*

$$F_n=\sum_{j,k,m,s=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s}\sum_{r=0}^{n}r_nI_{\theta(m+1)+p}\ .$$

where, $r_n=(-1)^{q-r}\left(\delta'_1\right)^{n-r}\times t_h$.

Proof of Remark 2.2 Based on the definition of the central moment as

$$F_n=E\left[\left(X-\delta'_1\right)^n\right]=\sum_{r=0}^{n}\binom{n}{r}\left(-\delta'_1\right)^{n-r}E\left[X^r\right].$$

and by substituting the expression given in Theorem 1, the proof is completed.
The variance (V), skewness (S), and kurtosis (K) can be obtained from the ordinary moments using well-known relationships $V{=}\delta'_2{-}\left(\delta'_1\right)^2$, $S{=}\frac{\delta'_3}{(\delta'_2)^{\frac{3}{2}}}$, and $K{=}\frac{\delta'_4}{(\delta'_2)^2}$.

2.2.2 Moment Generating Function

The moment generating function (MGF) is a useful tool for studying the distributional properties of a random variable. It can be employed to derive moments, analyze sums of independent random variables, and facilitate approximations in statistical inference. In this subsection, we obtain the MGF of the new family of distributions.

10.57647/mathsci.2026.66781

Theorem 2.3 *The MGF of the ECMO-G family of distributions is given by*

$$M_x(t) = \sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} t_h \Gamma_{\theta(m+1)+p}.$$

where $\Gamma_{\theta(m+1)+p}$ *is the MGF of the Exp-G distribution with the power parameter* $\theta(m+1)+p$.

Proof of Theorem 2.3 Starting from the definition of the MGF, we have

$$M_x(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x;\beta)\,dx,$$

By successive applications of the series expansion and substitution into the above integral, we obtain

$$M_x(t) = \sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} t_h \int_0^{\infty} e^{tx} [\theta(m+1) +p]\, g(x;\beta)\, G(x;\beta)^{\theta(m+1)+p-1} dx.$$

Then, we obtain the desired result. The above integral is convergent if $G(x)$ has finite moments.

2.2.3 Quantile Function

In this subsection, we derive the quantile function of the proposed family of distributions, which is useful for random variate generation, simulation studies, and descriptive analysis. The explicit analytical expression is given in the following theorem.

Theorem 2.4 *The quantile function for the ECMO-G family of distributions is defined by* $F(Q(u)) = u$ *and can be expressed as*

$$Q(u) = G_{\beta}^{-1}\left(\frac{\alpha(\ln[1-\frac{1}{\tau}\ln(1-\frac{u^{\frac{1}{\gamma}}}{A})])^{\frac{1}{\theta}}}{1-\bar{\alpha}(\ln[1-\frac{1}{\tau}\ln(1-\frac{u^{\frac{1}{\gamma}}}{A})])^{\frac{1}{\theta}}}\right)$$

Proof of Theorem 2.4 The CDF given in Equation (10) can be written as

$$F(Q(u)) = \left\{A[1-e^{\tau(1-e^{\left(\frac{G(x;\beta)}{1-\bar{\alpha}\bar{G}(x;\beta)}\right)^{\theta}})}]\right\}^{\gamma} = u,$$

then,

$$e^{\tau(1-e^{\left(\frac{G(x;\beta)}{1-\bar{\alpha}\bar{G}(x;\beta)}\right)^{\theta}})} = 1 - \frac{u^{\frac{1}{\gamma}}}{A},$$

and

$$\frac{G(x;\beta)}{1-\bar{\alpha}\bar{G}(x;\beta)} = (\ln[1-\frac{U^{\frac{1}{\gamma}}}{A})])^{\frac{1}{\theta}} = R(u),$$

so, we have

$$G(x;\beta) = \frac{\alpha R(u)}{1-\overline{\alpha} R(u)}.$$

and hence the desired result is obtained.

The above equation has no closed-form solution, so a numerical technique such as the Newton-Raphson method must be used to obtain the quantile. If we put $q = 0.5$, one obtains the median. A random variate X from the ECMO-G family can be generated as x_q according to the last equation, where $U \sim U(0,1)$, $0 \le u \le 1$.

2.2.4 Rényi Entropy

Entropy is a fundamental concept in probability and information theory and is widely used to quantify uncertainty associated with a random variable. Among the various entropy measures, Rényi entropy is particularly useful due to its flexibility and generality. In this subsection, the Rényi entropy of the proposed family is derived.

Theorem 2.5 *The Rényi entropy of order* π $(\pi > 0,\ \pi \neq 1)$ *for the ECMO-G family of distributions is given by*

$$I_R(\pi) = \frac{1}{1-\pi} \log \int_0^{\infty} \sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} u_h \Gamma^{\pi} dx,$$

where

$$u_h = (-1)^{j+l+p} \binom{\pi(\gamma-1)}{j} \binom{k}{l} \binom{[\theta(m+\pi)+\pi]+n-1}{s} \times \binom{n}{p} \frac{\tau^k (j+\pi)^k (l+\pi)^m}{k!m!} \gamma^{\pi} \tau^{\pi} \theta^{\pi} \alpha^{\pi} \overline{\alpha}^s A^{\pi\gamma}.$$

and $\Gamma = g(x;\beta)\, G(x;\beta)^{\frac{\beta(m+\pi)+p}{\pi}-1}$ *is the PDF of the Exp-G distribution with parameter* $\frac{\theta(m+\pi)+p}{\pi}$.

Proof of Theorem 2.5 By definition, the Rényi entropy of order π is defined as

$$I_R(\pi) = \frac{1}{1-\pi} \log\left[\int_0^{\infty} f(x;\beta)\,dx\right],$$

substituting the expression of the PDF given in Equation (9) into the above integral and applying repeated Taylor and generalized binomial expansions, we obtain the following relation.

$$I_R(\pi) = \frac{1}{1-\pi} \log \sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} (-1)^{j+l+p} \times \binom{\pi(\gamma-1)}{j} \binom{k}{l} \binom{[\theta(m+\pi)+\pi]+n-1}{s} \binom{n}{p} \times \frac{\tau^k (j+\pi)^k (1+\pi)^m}{k!m!} \gamma^{\pi} \tau^{\pi} \theta^{\pi} \alpha^{\pi} \overline{\alpha}^s A^{\pi\gamma} \times \left[\int_0^{\infty} g(x;\beta) G(x;\beta)^{\frac{\theta(m+\pi)+p}{\pi}-1} dx\right]^{\pi}.$$

This means that the Rényi entropy of the ECMO-G family of distributions can be expressed as a mixture of Exp-G family of distributions.

2.2.5 Reliability Measures

In this subsection, we introduce several key reliability measures for the proposed distribution. Each of these measures provides a slightly different perspective on the system's failure behavior and overall risk, helping to build a more comprehensive picture of how the system performs over time.

The survival function describes the probability that a system will operate beyond a certain time. The hazard rate function represents the instantaneous risk of failure at a given time, conditional on survival up to that time. The reversed hazard rate characterizes the rate of failure before a given time. The cumulative hazard function summarizes the accumulated risk of failure over time. The odds function compares the likelihood of failure to survival, and the Mills ratio provides information on tail behavior and rare-event characteristics.

The explicit expressions of these measures for the ECMO-G family are presented in the following theorem.

Theorem 2.6 *Let X be a random variable following the ECMO-G family of distributions. The corresponding reliability measures are given by*

- *Survival Function*

$$S(x;\beta) = 1 - F(x) = 1 - [A(1-W)]^{\gamma},$$

- *Hazard Rate Function*

$$h(x;\beta) = \frac{f(x;\beta)}{S(x;\beta)}$$
$$= \frac{\gamma\tau\theta\alpha A^{\gamma} g(x;\beta)(G(x;\beta))^{\theta-1}\left[1-\overline{\alpha}\overline{G}(x;\beta)\right]^{-(\beta+1)}}{1-[A(1-W)]^{\gamma}} \times VW[1-W]^{\gamma-1},$$

- *Reversed Hazard Rate Function*

$$Rh(x;\beta) = \frac{f(x;\beta)}{F(x;\beta)}$$
$$= \frac{\gamma\tau\theta\alpha g(x;\beta)(G(x;\beta))^{\theta-1}\left[1-\overline{\alpha}\overline{G}(x;\beta)\right]^{-(\theta+1)} VW}{1-W},$$

- *Cumulative Hazard Rate Function*

$$\begin{aligned} H(x;\beta) &= -\ln(S(x;\beta)) \\ &= -\ln\left[1-(A(1-W))^{\gamma}\right], \end{aligned}$$

- *Odds Function*

$$O_f(x;\beta) = \frac{F(x;\beta)}{S(x;\beta)} = \frac{[A(1-W)]^{\gamma}}{1-[A(1-W)]^{\gamma}},$$

- *Mills Ratio*

$$M_r(x;\beta) = \frac{S(x;\beta)}{f(x;\beta)}$$
$$= \frac{1-[A(1-W)]^{\gamma}}{\gamma\tau\theta\alpha A^{\gamma} g(x;\beta)(G(x;\beta))^{\theta-1}\left[1-\overline{\alpha}\overline{G}(x;\beta)\right]^{-(\theta+1)}} \times \frac{1}{VW[1-W]^{\gamma-1}}.$$

where $V = e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}}$ *and* $W = e^{\tau(1-V)}$.

2.2.6 Order Statistics

Order statistics are widely used in statistical theory and practice, particularly in reliability analysis and life-testing studies. In this subsection, we derive the distributional form of the order statistics associated with the proposed model.

Theorem 2.7 *Let* $X_1, \ldots, X_n$ *be a random sample of size n from the ECMO-G distribution, and let* $X_{(1)} \leq \cdots \leq X_{(n)}$ *denote the corresponding order statistics. Then, the PDF of the* i^{th} *order statistic* $X_{(i)}, 1 \leq i \leq n$*, is given by*

$$f_{X_{(i)}}(x) = \frac{1}{B(i, n-i+1)} \sum_{v=0}^{n-i} (-1)^{v} \binom{n-i}{v} \times \sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} C_{h,v+i-1} Z^{\theta(m+1)+p}.$$

Proof of Theorem 2.7 By the definition of the order statistics, we have

$$f_{X_{(i)}}(x;\beta) = \frac{1}{B(i,n-i+1)} f(x;\beta) \times \sum_{v=0}^{n-i} (-1)^{v} \binom{n-i}{v} F(x;\beta)^{v+i-1},$$

Therefore, using Equation (13), we can write

$$F(x;\beta)^{v+i-1} = \left(\sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} t_h \left[R^{m+1+\frac{p}{\theta}} \right]^{\theta} \right)^{v+i-1},$$

hence, from Gradshteyn and Ryzhik (2007), we have

$$\left(\sum_{k=0}^{\infty} f_k u^k \right)^n = \sum_{k=0}^{\infty} C_{n,k} u^k,$$

where coefficient $C_{n,k}$ is

$$C_{n,k} = \left(k(C_{n,0})^{\frac{1}{n}}\right)^{-1} \sum_{l=1}^{k} (l(n+1) - (C_{n,l})^{\frac{1}{n}} (C_{n,k-m}),$$

Then, by series expansion, the probability density function of $X_{(i)}$ is represented as

$$f_{X_{(i)}}(x) = \frac{1}{B(i, n-i+1)} \sum_{v=0}^{n-i} (-1)^{v} \binom{n-i}{v} \times \sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} C_{h,v+i-1} Z^{\theta(m+1)+p}.$$

where $Z^{\theta(m+1)+p}$ is the PDF of the Exp-G distribution with power $\theta(m+1)+p$.

Remark 2.8 *By substituting* $i = 1$ *and* $i = n$ *into the expression given in Theorem 6, the probability density functions of the first* $X_{(1)}$ *and last* $X_{(n)}$ *order statistics* *of the ECMO-G are obtained as*

$$f_{X_{(1)}}(x) = n\sum_{v=0}^{n-1}(-1)^v\binom{n-1}{v} \times \sum_{j,k,m,s=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s} C_{h,v} Z^{\theta(m+1)+p},$$

and

$$f_{X_{(n)}}(x) = n\sum_{v=0}^{n-i}(-1)^v\binom{n-i}{v} \times \sum_{j,k,m,s=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s} C_{h,v+n-1} Z^{\theta(m+1)+p}.$$

2.2.7 Estimation Method

In this section, several parameter estimation procedures are developed for the ECMO-G family of distributions. The methods include maximum likelihood estimation (MLE) under both complete and right-censored observation, as well as least squares estimation (LSE), weighted least squares estimation (WLSE), and Anderson-Darling estimation (ADE). For each method, the corresponding objective functions and estimating equations are derived.

2.2.7.1 Maximum Likelihood Estimation

This method is employed due to its desirable theoretical properties and widespread use in statistical inference. It estimates the model parameters by maximizing the likelihood function constructed from the observed data.

Let $X_1, \ldots, X_n$ be a random sample of size n from the ECMO-G family with parameter vector $\Omega = (\gamma,\ \tau,\ \theta,\ \alpha,\ \beta)$. Under complete observations, the log-likelihood function is given by

$$l(x_i;\gamma,\tau,\theta,\alpha,\beta) = l = n\ln\gamma + n\gamma\ln A + n\ln\tau + n\ln\theta + n\ln\alpha + \sum_{i=1}^{n}\ln g(x_i;\beta) + (\theta-1)\sum_{i=1}^{n}\ln G(x_i;\beta) - (\theta+1)\sum_{i=1}^{n}\ln T_i + \sum_{i=1}^{n}U_i^{\theta} + \sum_{i=1}^{n}\tau(1-V_i) + (\gamma-1)\sum_{i=1}^{n}\ln(1-W_i).$$

By taking the partial derivatives of the log-likelihood function with respect to Ω, the corresponding score equations are obtained as follows.

with respect to γ

$$\frac{\partial l}{\partial\gamma} = \frac{n}{\gamma} + n\ln A + \sum_{i=1}^{n}\ln(1-W_i),$$

with respect to τ

$$\frac{\partial l}{\partial\tau} = n\gamma\frac{A'}{A} + \frac{n}{\tau} + \sum_{i=1}^{n}(1-V_i) - (\gamma-1)\sum_{i=1}^{n}\frac{(1-V_i)W_i}{1-W_i},$$

with respect to θ

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln G(x_i;\beta) - \sum_{i=1}^{n}\ln T_i + \sum_{i=1}^{n}[\ln U_i]U_i^{\theta} - \sum_{i=1}^{n}\tau[\ln U_i]U_i^{\theta}V_i + (\gamma-1)\sum_{i=1}^{n}\frac{\tau[\ln U_i]U_i^{\theta}V_iW_i}{1-W_i},$$

with respect to α

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} - (\theta+1)\sum_{i=1}^{n}\frac{\overline{G}(x_i;\beta)}{T_i} - \sum_{i=1}^{n}\theta\frac{\overline{G}(x_i;\beta)}{T_i}U_i^{\theta} - \sum_{i=1}^{n}\tau\theta\frac{\overline{G}(x_i;\beta)}{T_i}U_i^{\theta}V_i + (\gamma-1)\sum_{i=1}^{n}\frac{\tau\theta\overline{G}(x_i;\beta)U_i^{\theta}V_iW_i}{T_i(1-W_i)},$$

with respect to β

$$\frac{\partial l}{\partial\beta} = \sum_{i=1}^{n}\frac{g'(x_i;\beta)}{g(x_i;\beta)} + (\theta-1)\sum_{i=1}^{n}\frac{G'(x_i;\beta)}{G(x_i;\beta)} - (\theta+1)\sum_{i=1}^{n}\frac{\overline{\alpha}G'(x_i;\beta)}{T_i} + \sum_{i=1}^{n}\theta\frac{\alpha G'(x_i;\beta)}{T_i^2}U_i^{\theta-1} - \sum_{i=1}^{n}\tau\theta\frac{\alpha G'(x_i;\beta)}{T_i^2}U_i^{\theta-1}V_i + (\gamma-1) \sum_{i=1}^{n}\frac{\tau\theta\alpha G'(x_i;\beta)U_i^{\theta-1}V_iW_i}{T_i^2(1-W_i)}.$$

In the presence of censoring, the likelihood function is constructed by combining the PDF for uncensored observations with the survival function for censored observations. For right-censored data, the likelihood function is given by

$$L = \prod_{i=1}^{r} f(x_i;\beta)\prod_{i=r+1}^{n} S(x_i^{+};\beta).$$

where r denotes the number of uncensored observations. For i= 1, . . . , r, $f(x_i;\beta)$ represents the contribution of uncensored observations, while for $i = r+1, \ldots, n$, $S(x_i^{+};\beta)$ denotes the contribution of censored observations.

The corresponding censored log-likelihood function is

$$l(x_i, x_i^{+};\Omega^*) = l^* = r\ln\gamma + r\gamma\ln A + r\ln\tau + r\ln\theta + r\ln\alpha + \sum_{i=1}^{r}\ln g(x_i;\beta) + (\theta-1)\sum_{i=1}^{r}\ln G(x_i;\beta) - (\theta+1)\sum_{i=1}^{r}\ln T_i + \sum_{i=1}^{r}U_i^{\theta} + \sum_{i=1}^{r}\tau(1-V_i) + (\gamma-1)\sum_{i=1}^{r}\ln(1-W_i) + \sum_{i=r+1}^{n}\ln[1-(A[1-W_i^{+}])^{\gamma}].$$

The score equations are obtained by differentiating the censored log-likelihood with respect to unknown parameters.

with respect to γ

$$\frac{\partial l^*}{\partial\gamma}=\frac{r}{\gamma}+r\ln A+\sum_{i=1}^{r}\ln(1-W_i)-\sum_{i=r+1}^{n}\frac{\ln[A[1-W_i^+]](A[1-W_i^+])^{\gamma}}{1-(A[1-W_i^+])^{\gamma}},$$

with respect to τ

$$\frac{\partial l^*}{\partial\tau}=r\gamma\frac{A'}{A}+\frac{r}{\tau}+\sum_{i=1}^{r}(1-V_i)-(\gamma-1)\sum_{i=1}^{r}\frac{(1-V_i)W_i}{1-W_i}$$

$$-\sum_{i=r+1}^{n}\frac{\gamma A' A^{\gamma-1}\left[1-W_i^+\right]^{\gamma}-\gamma A^{\gamma}\left(1-V_i^+\right)W_i^+\left[1-W_i^+\right]^{\gamma-1}}{1-\left(A\left[1-W_i^+\right]\right)^{\gamma}},$$

with respect to θ

$$\frac{\partial l^*}{\partial\theta}=\frac{r}{\theta}+\sum_{i=1}^{r}\ln G(x_i;)-\sum_{i=1}^{r}\ln T_i+\sum_{i=1}^{r}[\ln U_i]\,{U_i}^{\theta}$$

$$-\sum_{i=1}^{r}\tau\,[\ln U_i]\,U_i^{\theta}V_i+(\gamma-1)\sum_{i=1}^{r}\frac{\tau\,[\ln U_i]\,{U_i}^{\theta}V_iW_i}{1-W_i}$$

$$-\sum_{i=r+1}^{n}\frac{\gamma A^{\gamma}\tau\left[\ln U_i^+\right]\left(U_i^+\right)^{\theta}V_i^+W_i^+\left[1-W_i^+\right]^{\gamma-1}}{1-\left(A\left[1-W_i^+\right]\right)^{\gamma}},$$

with respect to α

$$\frac{\partial l^*}{\partial\alpha}=\frac{r}{\alpha}-(\theta+1)\sum_{i=1}^{r}\frac{\overline{G}(x_i;\beta)}{T_i}-\sum_{i=1}^{r}\theta\frac{\overline{G}(x_i;\beta)}{T_i}{U_i}^{\theta}$$

$$-\sum_{i=1}^{r}\tau\theta\frac{\overline{G}(x_i;\beta)}{T_i}{U_i}^{\theta}V_i+(\gamma-1)\sum_{i=1}^{r}\frac{\tau\theta\overline{G}(x_i;\beta)\,{U_i}^{\theta}V_iW_i}{T_i(1-W_i)}$$

$$+\sum_{i=r+1}^{n}\frac{\gamma A^{\gamma}\tau\theta\overline{G}({x_i}^+;\beta)\left(U_i^+\right)^{\theta}V_i^+W_i^+\left[1-W_i^+\right]^{\gamma-1}}{T_i^+\left[1-\left(A\left[1-W_i^+\right]\right)^{\gamma}\right]},$$

with respect to β

$$\frac{\partial l^*}{\partial\beta}=\sum_{i=1}^{r}\frac{g'(x_i;\beta)}{g(x_i;\beta)}+(\theta-1)\sum_{i=1}^{r}\frac{G'(x_i;\beta)}{G(x_i;\beta)}$$

$$-(\theta+1)\sum_{i=1}^{r}\frac{\overline{\alpha}G'(x_i;\beta)}{T_i}+\sum_{i=1}^{r}\theta\frac{\alpha G'(x_i;\beta)}{T_i^2}{U_i}^{\theta-1}$$

$$-\sum_{i=1}^{r}\tau\theta\frac{\alpha G'(x_i;\beta)}{T_i^2}{U_i}^{\theta-1}V_i$$

$$+(\gamma-1)\sum_{i=1}^{r}\frac{\tau\theta\alpha G'(x_i;\beta)\,{U_i}^{\theta-1}V_iW_i}{T_i^2(1-W_i)}$$

$$-\sum_{i=r+1}^{n}\frac{\gamma A^{\gamma}\tau\theta\alpha G'({x_i}^+;\beta)\left(U_i^+\right)^{\theta-1}V_i^+W_i^+\left[1-W_i^+\right]^{\gamma-1}}{\left(T_i^+\right)^2\left[1-\left(A\left[1-W_i^+\right]\right)^{\gamma}\right]}.$$

where $A=\frac{1}{1-e^{\tau(1-e)}}$, $A'=\frac{(1-e)e^{\tau(1-e)}}{\left(1-e^{\tau(1-e)}\right)^2}$, $T_i=1-\overline{\alpha}\overline{G}(x_i;\beta)$, $U_i=\left(\frac{G(x_i;\beta)}{T_i}\right)$, $V_i=e^{(U_i)^{\theta}}$, and $W_i=e^{\tau(1-V_i)}$, and T_i^+, U_i^+, V_i^+, and W_i^+ are obtained by replacing x_i with x_i^+.

In both complete and censored cases, the resulting likelihood equations are nonlinear and do not admit closed-form solutions. Consequently, the MLEs are obtained using numerical optimization techniques. Although the EM algorithm can be considered for censored data, its implementation for the proposed multi-parameter model leads to substantial computational complexity. Therefore, likelihood-based estimation via direct numerical optimization was adopted.

2.2.7.2 Least Squares Estimation

The LSE method is based on minimizing the discrepancy between the empirical distribution function and the theoretical CDF of the model. This approach provides a simple alternative to likelihood-based estimation. Assume that $X_1,...,X_n$ is a random sample from the ECMO-G family of distributions, and let $X_{(1)},...,X_{(n)}$ denote the corresponding order statistics. The LSEs are obtained by minimizing the objective function

$$LSE\,(x_i;\ \gamma,\ \tau,\ \theta,\ \alpha,\ \beta)=\sum_{i=1}^{n}\left[F\,(x_{i:n};\ \gamma,\ \tau,\ \theta,\ \alpha,\ \beta)-\frac{i}{n+1}\right]^2=\sum_{i=1}^{n}\left[\left\{A[1-e^{\tau(1-e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}})}]\right\}^{\gamma}-\frac{i}{n+1}\right]^2.$$

2.2.7. Weighted Least Squares Estimation

To improve the efficiency of the LSE, a weighted version is also considered. In this approach, different weights are assigned to each squared difference in order to account for the variability of the empirical distribution function. The WLSEs are obtained by minimizing

$$WLSE\,(x_i;\ \gamma,\tau,\theta,\alpha,\beta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i\,(n-i+1)}\left[F\,(x_{i:n};\gamma,\tau,\theta,\alpha,\beta)-\frac{i}{n+1}\right]^2=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i\,(n-i+1)}\times\left[\left\{A[1-e^{\tau(1-e^{\left(\frac{G(x;\beta)}{1-\overline{\alpha}\overline{G}(x;\beta)}\right)^{\theta}})}]\right\}^{\gamma}-\frac{i}{n+1}\right]^2.$$

2.2.7.4 Anderson-Darling estimation

The ADE method is a minimum distance approach that assigns greater weight to the tails of the distribution. This property makes it particularly suitable for lifetime and reliability data. The ADEs are obtained by minimizing the Anderson–Darling objective function, given by

$$ADE\,(x_i;\gamma,\tau,\theta,\alpha,\beta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\times\left[\ln F\,(x_{i:n};\gamma,\tau,\theta,\alpha,\beta)+\ln\overline{F}\,(x_{i:n};\gamma,\tau,\theta,\alpha,\beta)\right].$$

For the LSE, WLSE, and ADE methods introduced above, the corresponding parameter estimates are obtained by differentiating the associated objective functions with respect to the parameter vector and setting the resulting equations equal to zero. This process leads to systems of nonlinear estimating equations, which do not admit closed-form solutions. therefore, the parameter estimates are computed using numerical optimization techniques.

3. Exponentiated Chen Marshall-Olkin Weibull Distribution

The proposed family can be applied to all baseline distributions that satisfy the properties of a CDF. In this section, we present a special case of the ECMO-G by considering the Weibull distribution as the baseline model.

Let X be a random variable that denotes the lifetime of a system. Assume that X follows a Weibull distribution with parameters μ and λ. The corresponding PDF and CDF are given, respectively, by

$$f(x;\mu,\lambda) = \mu^{\lambda}\lambda x^{\lambda-1}e^{-(\mu x)^{\lambda}}, \tag{14}$$

and

$$F(x;\mu,\lambda) = 1 - e^{-(\mu x)^{\lambda}}. \tag{15}$$

where $\mu > 0$ and $\lambda > 0$.

By replacing Equation (14) and Equation (15) as the baseline PDF and CDF in Equations (9) and (10), the PDF and CDF of the Exponentiated Chen Marshall-Olkin Weibull (ECMOW) distribution are obtained as

$$\begin{aligned} f(x;\beta) &= \gamma^{\lambda}A^{\gamma}x^{\lambda-1}e^{-(\mu x)^{\lambda}}\left(1-e^{-(\mu x)^{\lambda}}\right)^{\theta-1} \\ &\times\left[1-\bar{\alpha}e^{-(\mu x)^{\lambda}}\right]^{-(\theta+1)} e^{\left(\frac{1-e^{-(\mu x)^{\lambda}}}{1-\bar{\alpha}e^{-(\mu x)^{\lambda}}}\right)^{\theta}} \\ &\times e^{\tau\left(1-e^{\left(\frac{1-e^{-(\mu x)^{\lambda}}}{1-\bar{\alpha}e^{-(\mu x)^{\lambda}}}\right)^{\theta}}\right)}\left[1-e^{\tau\left(1-e^{\left(\frac{1-e^{-(\mu x)^{\lambda}}}{1-\bar{\alpha}e^{-(\mu x)^{\lambda}}}\right)^{\theta}}\right)}\right]^{\gamma-1}, \end{aligned}$$

and

$$F(x;\beta) = \left\{A\left[1-e^{\tau\left(1-e^{\left(\frac{1-e^{-(\mu x)^{\lambda}}}{1-\bar{\alpha}e^{-(\mu x)^{\lambda}}}\right)^{\theta}}\right)}\right]\right\}^{\gamma}.$$

where $\gamma, \theta, \alpha, \mu, \lambda > 0$ and $|\tau| > 0$.

The PDF plots of the ECMOW distribution for various parameter values are shown in Figure 1 It is evident from Figure 1 that the ECMOW distribution exhibits a wide range of shapes, including left-skewed, right-skewed, unimodal, and reversed J-shaped forms. Moreover, the PDF and CDF expressions of several submodels derived from the ECMOW distribution are summarized in Table 1.

3.1 Mixture Representation of the ECMOW Distribution

The series expansion of the ECMOW distribution is presented here. Substituting the Weibull PDF and CDF into the equation from subsection 2.1 and applying a power series, we obtain

$$f(x;\beta) = \sum_{j,k,m,s=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s}\sum_{q=0}^{\theta(m+1)+p-1}\sum_{r=0}^{\theta(m+1)+p} e_h x^{(r+1)\lambda-1},$$

where

$$\begin{aligned} e_h &= (-1)^{j+l+p+q+r}\binom{\gamma-1}{j}\binom{k}{l}\binom{\theta(m+1)+s}{s} \\ &\times\binom{s}{p}\binom{\theta(m+1)+p-1}{q} \\ &\times\frac{\tau^k(j+1)^k(l+1)^m}{k!m!r!}(\theta(m+1)+p)^r\,\gamma\tau\theta\alpha\bar{\alpha}^s\lambda\mu^{\lambda(r+1)}A^{\gamma}. \end{aligned}$$

3.2 Moments of the ECMOW Distribution

The r^{th} moment of the ECMOW distribution can be obtained using Theorem 1, power series expansions, and

$$\int_0^{\infty} x^{r+\lambda-1}\,e^{-(t+1)(\mu x)^{\lambda}}dx = \frac{\Gamma(\frac{r}{\lambda}+1)}{((t+1)\mu^{\lambda})^{\frac{r}{\lambda}+1}},$$

as

$$\begin{aligned} E[X^r] = &\sum_{j,k,m,s,t=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s} t_h(-1)^t\binom{\theta(m+1)+p-1}{t} \\ &\times[\theta(m+1)+p]\,\mu^{\lambda}\frac{\Gamma(\frac{r}{\lambda}+1)}{((t+1)\mu^{\lambda})^{\frac{r}{\lambda}+1}}. \end{aligned}$$

Table 2 presents the numerical results for the first four moments, variance, skewness, kurtosis, and coefficient of variation for the ECMOW distribution, evaluated under different parameter combinations. As the values of parameters α and λ increase, the first four moments and variance also increase, while skewness and kurtosis decrease. This indicates that the distribution becomes more symmetric and less heavy-tailed. The overall shape of the distribution is strongly influenced by the model parameters, particularly in terms of skewness and kurtosis. Moreover, the table shows that skewness remains positive across all parameter settings, with higher skewness observed when parameters γ, α, and λ take lower values. This explains the right-skewed nature of the distribution's density plots. In contrast, parameter τ appears to have no significant effect on skewness, kurtosis, or the coefficient of variation.

3.3 Moment Generating Function of the ECMOW Distribution

From the expression in Theorem 2 and Equations (14) and (15), and using power series expansions along with the following identity

$$\int_0^{\infty} e^{tx}x^{\lambda-1}e^{-(q+1)(\mu x)^{\lambda}}dx = \sum_{z=0}^{\infty}\frac{t^z\Gamma(\frac{z}{\lambda}+1)}{z!\lambda(q+1)^{\frac{z}{\lambda}+1}\mu^{z+\lambda}},$$

the MGF of the ECMOW distribution is given by

$$\begin{aligned} M_X(t) = &\sum_{j,k,m,s,q,z=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s} t_h(-1)^q \\ &\times\binom{\theta(m+1)+p-1}{q}[\theta(m+1)+p]\,\frac{t^z\Gamma(\frac{z}{\lambda}+1)}{z!(q+1)^{\frac{z}{\lambda}+1}\mu^z}. \end{aligned}$$

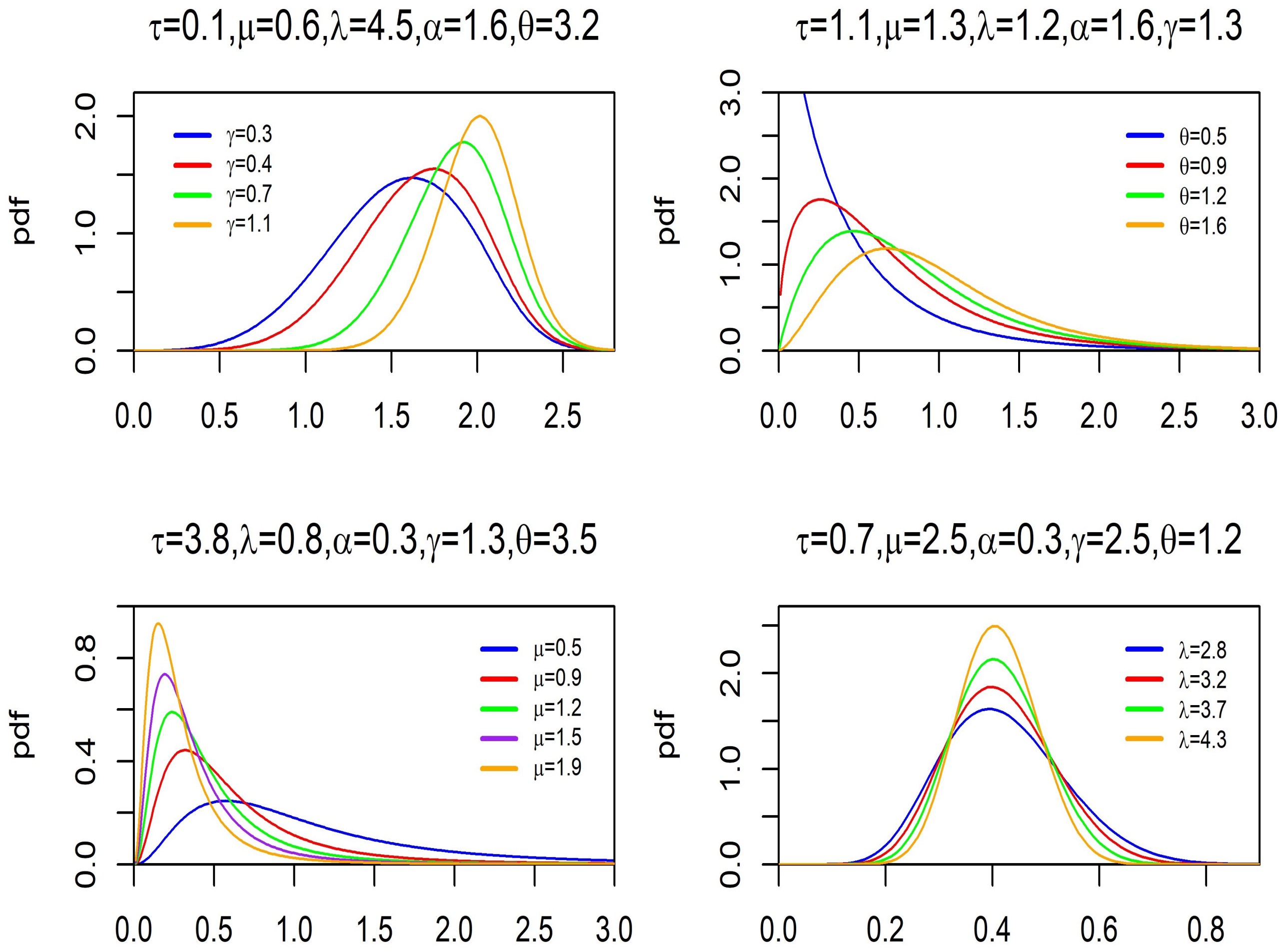

Figure 1. Plots of the PDF of the ECMOW distribution for different parameter values

Table 1. PDF and CDF of the sub-models of the ECMOW distribution

Distribution	PDF	CDF
Weibull	$\mu^{\lambda}\lambda x^{\lambda-1}e^{-(\mu x)^{\lambda}}$	$1-e^{-(\mu x)^{\lambda}}$
Exponentiated Weibull	$\gamma\mu^{\lambda}\lambda x^{\lambda-1}e^{-(\mu x)^{\lambda}}[1-e^{-(\mu x)^{\lambda}}]^{\gamma-1}$	$[1-e^{-(\mu x)^{\lambda}}]^{\gamma}$
Chen Weibull	$A\tau\theta\mu^{\lambda}\lambda x^{\lambda-1}e^{-(\mu x)^{\lambda}}(P^*)^{\theta-1}e^{(P^*)^{\theta}}e^{\tau(1-e^{(P^*)^{\theta}})}$	$A[1-e^{\tau(1-e^{(P^*)^{\theta}})}]$
Marshall-Olkin Weibull	$\frac{\alpha\mu^{\lambda}\lambda x^{\lambda-1}e^{-(\mu x)^{\lambda}}}{\left[1-\overline{\alpha}e^{-(\mu x)^{\lambda}}\right]^2}$	$\frac{1-e^{-(\mu x)^{\lambda}}}{1-\overline{\alpha}e^{-(\mu x)^{\lambda}}}$
Chen Marshal-Olkin Weibull	$A\tau\theta\frac{\alpha\mu^{\lambda}\lambda x^{\lambda-1}e^{-(\mu x)^{\lambda}}}{\left(1-\overline{\alpha}e^{-(\mu x)^{\lambda}}\right)^2}(Q^*)^{\theta-1}e^{(Q^*)^{\theta}}e^{\tau(1-e^{(Q^*)^{\theta}})}$	$A[1-e^{\tau(1-e^{(Q^*)^{\theta}})}]$
Exponentiated Chen Weibull	$\gamma A^{\gamma}\tau\theta\mu^{\lambda}\lambda x^{\lambda-1}e^{-(\mu x)^{\lambda}}(P^*)^{\theta-1}e^{(P^*)^{\theta}}e^{\tau(1-e^{(P^*)^{\theta}})}[1-e^{\tau(1-e^{(P^*)^{\theta}})}]^{\gamma-1}$	$(A[1-e^{\tau(1-e^{(P^*)^{\theta}})}])^{\gamma}$
Exponentiated Marshal-Olkin Weibull	$\gamma\frac{\alpha\mu^{\lambda}\lambda x^{\lambda-1}e^{-(\mu x)^{\lambda}}}{\left(1-\overline{\alpha}e^{-(\mu x)^{\lambda}}\right)^2}\left(\frac{1-e^{-(\mu x)^{\lambda}}}{1-\overline{\alpha}e^{-(\mu x)^{\alpha}}}\right)^{\gamma-1}$	$\left(\frac{1-e^{-(\mu x)^{\lambda}}}{1-\overline{\alpha}e^{-(\mu x)^{\lambda}}}\right)^{\gamma}$

Where $P^* = 1-e^{-(\mu x)^{\lambda}}$ and $Q^* = \left(\frac{1-e^{-(\mu x)^{\lambda}}}{1-\overline{\alpha}e^{-(\mu x)^{\lambda}}}\right)$.

10.57647/mathsci.2026.66781

Table 2. Numerical values of some descriptive measures of the ECMOW with $\mu = 1.5$ and $\theta = 0.5$

α	τ	λ	γ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	SK	KU	CV
0.3	0.7	1.7	1.3	0.3454	0.3443	0.6414	1.8006	0.2250	3.4388	22.0851	1.3730
			1.8	0.4456	0.3824	0.4957	0.8691	0.1838	2.0492	9.5514	0.9621
		4.2	1.3	0.7075	0.8990	1.7670	4.8754	0.3983	2.2564	11.4871	0.8919
			1.8	0.8036	0.9024	1.3178	2.3921	0.2566	1.3844	6.0972	0.6304
	1.6	1.7	1.3	0.1511	0.0659	0.0537	0.0659	0.0430	3.4388	22.0851	1.3730
			1.8	0.1949	0.0731	0.0415	0.0318	0.0351	2.0492	9.5516	0.9621
		4.2	1.3	0.3095	0.1720	0.1479	0.1786	0.0762	2.2564	11.4871	0.8919
			1.8	0.3515	0.1727	0.1103	0.0876	0.0491	1.3844	6.0972	0.6304
1.1	0.7	1.7	1.3	0.7842	1.0519	2.1973	6.4314	0.4369	2.3784	11.9719	0.8429
			1.8	0.8745	1.0248	1.5568	2.9538	0.2601	1.5500	6.7393	0.5832
		4.2	1.3	1.3316	2.3661	5.5379	16.6387	0.5928	1.7703	8.1993	0.5781
			1.8	1.3187	2.0238	3.6024	7.3929	0.2848	1.2000	5.3613	0.4046
	1.6	1.7	1.3	0.3430	0.2013	0.1840	0.2356	0.0836	2.3784	11.9719	0.8429
			1.8	0.3825	0.1961	0.1303	0.1082	0.0497	1.5500	6.7393	0.5832
		4.2	1.3	0.5826	0.4529	0.4637	0.6095	0.1134	1.7703	8.1993	0.5781
			1.8	0.5769	0.3873	0.3016	0.2708	0.0545	1.2000	5.3613	0.4046
2	0.7	1.7	1.3	1.0574	1.6673	3.7656	11.4010	0.5492	2.0662	9.6436	0.7008
			1.8	1.1034	1.5048	2.5115	5.0325	0.2873	1.4091	5.9778	0.4857
		4.2	1.3	1.6711	3.4687	8.9587	28.4887	0.6760	1.6229	7.2853	0.4920
			1.8	1.5659	2.7432	5.3876	11.8570	0.2910	1.1500	5.0934	0.3444
	1.6	1.7	1.3	0.4626	0.3191	0.3153	0.4176	0.1051	2.0662	9.6436	0.7008
			1.8	0.4827	0.2880	0.2103	0.1843	0.0549	1.4091	5.9778	0.4857
		4.2	1.3	0.7311	0.6639	0.7502	1.0437	0.1294	1.6229	7.2853	0.4920
			1.8	0.6851	0.5250	0.4511	0.4344	0.0556	1.1500	5.0934	0.3444

3.4 Quantile Function of the ECMOW Distribution

The quantile x_q of the ECMOW distribution is given by

$$Q(u) = \frac{1}{\mu}\left[-\ln(1 - \frac{\alpha(\ln[1 - \frac{1}{\tau}\ln(1 - \frac{u^{\frac{1}{\gamma}}}{A})\,]\,)^{\frac{1}{\theta}}}{1 - \overline{\alpha}(\ln[1 - \frac{1}{\tau}\ln(1 - \frac{u^{\frac{1}{\gamma}}}{A})\,]\,)^{\frac{1}{\theta}}})\right]^{\frac{1}{\gamma}}.$$

This expression can be used to generate random variates from the ECMOW distribution.

3.5 Rényi Entropy of the ECMOW Distribution

The Rényi entropy of the ECMOW distribution, using the expression from Theorem 4 and Equations (14) and (15) along with the power series, is obtained as

$$I_R(\pi) = \frac{1}{1-\pi}\log\int_0^\infty \sum_{j,k,m,s,q=0}^{\infty}\sum_{l=0}^{k}\sum_{p=0}^{s} u_h(-1)^q \times \binom{\theta(m+\pi)+p-\pi}{q}\mu^{\pi-1}\lambda^{\pi-1} \times \frac{\Gamma\left(\left(1-\frac{1}{\lambda}\right)(\pi-1)+1\right)}{(q+\pi)^{\pi\left(1-\frac{1}{\lambda}\right)+\frac{1}{\lambda}}}$$

3.6 Reliability Measures of the ECMOW Distribution

In this subsection, we present the reliability characteristics of the ECMOW distribution. Analytical expressions for the relevant functions are provided below.

- Survival function

$$S_{ECMOW}(x;\beta) = 1 - [A(1-W)]^\gamma,$$

- Hazard rate function

$$h_{ECMOW}(x;\beta) = \frac{\gamma\tau\theta\alpha\lambda\mu^\lambda A^\gamma x^{\lambda-1} S(1-S)^{\theta-1} T^{-(\theta+1)} \times V \times W \times [1-W]^{\gamma-1}}{1-[A(1-W)]^\gamma},$$

- Reversed hazard rate function

$$Rh_{ECMOW}(x;\beta) = \frac{\gamma\tau\theta\alpha\lambda\mu^\lambda x^{\lambda-1} S(1-S)^{\theta-1} T^{-(\theta+1)} \times V \times W}{1-W},$$

- Cumulative hazard rate function

$$H_{ECMOW}(x;\beta) = -\ln\left[1-(A(1-W))^\gamma\right],$$

- Odds function

$$O_{f_{ECMOW}}(x;\beta) = \frac{[A(1-W)]^\gamma}{1-[A(1-W)]^\gamma},$$

- Mills ratio

$$Mr_{ECMOW}(x;\beta) = \frac{1-[A(1-W)]^\gamma}{\gamma\tau\theta\alpha\lambda\mu^\lambda A^\gamma x^{\lambda-1} S(1-S)^{\theta-1} T^{-(\theta+1)} \times V \times W \times [1-W]^{\gamma-1}}.$$

10.57647/mathsci.2026.66781

where $S = e^{-(\mu x)^{\lambda}}$, $T = 1 - \overline{\alpha}S$, $V = e^{\left(\frac{1-S}{T}\right)^{\theta}}$, and $W = e^{\tau(1-V)}$.

Figure 2 shows the hazard rate function plots for different parameter values. As these figures demonstrate, the ECMOW distribution can accommodate decreasing, decreasing-constant, increasing, increasing-constant, inverted bathtub-shaped, and increasing-decreasing-increasing hazard rate functions.

3.7 Order Statistics of the ECMOW Distribution

The PDF and CDF of the i^{th} order statistics of the ECMOW distribution are obtained from the expression given in Theorem 6 as follows.

$$f_{X_{(i)}}(x) = \frac{1}{B(i, n-i+1)} \sum_{v=0}^{n-i} (-1)^v \binom{n-i}{v} \times \sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} C_{h,v+i-1} \left[\theta(m+1)+p\right] \mu^{\lambda} \lambda x^{\lambda-1} \times e^{-(\mu x)^{\lambda}} \left(1 - e^{-(\mu x)^{\lambda}}\right)^{\theta(m+1)+p-1}.$$

The PDF of the first $X_{(1)}$ and last $X_{(n)}$ order statistics of the ECMOW distribution are given by

$$f_{X_{(1)}}(x) = n \sum_{v=0}^{n-1} (-1)^v \binom{n-1}{v} \times \sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} C_{h,v} \left[\theta(m+1)+p\right] \mu^{\lambda} \lambda x^{\lambda-1} \times e^{-(\mu x)^{\lambda}} \left(1 - e^{-(\mu x)^{\lambda}}\right)^{\theta(m+1)+p-1},$$

and

$$f_{X_{(n)}}(x) = n \sum_{v=0}^{n-i} (-1)^v \binom{n-i}{v} \times \sum_{j,k,m,s=0}^{\infty} \sum_{l=0}^{k} \sum_{p=0}^{s} C_{h,v+n} \left[\theta(m+1)+p\right] \mu^{\lambda} \lambda x^{\lambda-1} \times e^{-(\mu x)^{\lambda}} \left(1 - e^{-(\mu x)^{\lambda}}\right)^{\theta(m+1)+p-1}.$$

3.8 Estimation Methods of the ECMOW Distribution

In this subsection, we address the estimation of the unknown parameters of the ECMOW distribution using several approaches, including MLE under complete and censored observations, as well as three minimum distance-based methods: LSE, WLSE, and ADE estimators.

3.8.1 Maximum Likelihood Estimation of the ECMOW Distribution

Let $X_1, \ldots, X_n$ be a random sample of size n from the ECMOW distribution with parameter vector$\Omega = (\gamma, \tau, \theta, \alpha, \lambda, \mu)$. Under complete observations, the log-likelihood function is given by

$$\begin{aligned} l(x_i; \gamma, \tau, \theta, \alpha, \lambda, \mu) = l &= n\ln\gamma + n\gamma\ln A + n\ln\tau \\ &+ n\ln\theta + n\ln\alpha + n\lambda\ln\mu + n\ln\lambda + (\lambda-1)\sum_{i=1}^{n}\ln x_i \\ &- \sum_{i=1}^{n}\mu^{\lambda}{x_i}^{\lambda} + (\theta-1)\sum_{i=1}^{n}\ln(1-S_i) \\ &- (\theta+1)\sum_{i=1}^{n}\ln T_i + \sum_{i=1}^{n}{U_i}^{\theta} + \sum_{i=1}^{n}\tau(1-V_i) \\ &+ (\gamma-1)\sum_{i=1}^{n}\ln(1-W_i). \end{aligned}$$

For censored data, the log-likelihood function combines the ECMOW density for uncensored observations with the survival function for censored observations.

$$\begin{aligned} l(x_i, {x_i}^{+}; \Omega^*) = l^* &= r\ln\gamma + r\gamma\ln A + r\ln\tau + r\ln\theta \\ &+ r\ln\alpha + r\lambda\ln\mu + r\ln\lambda + (\lambda-1)\sum_{i=1}^{r}\ln x_i \\ &- \sum_{i=1}^{r}(\mu x_i)^{\lambda} + (\theta-1)\sum_{i=1}^{r}\ln(1-S_i) \\ &- (\theta+1)\sum_{i=1}^{r}\ln T_i + \sum_{i=1}^{r}{U_i}^{\theta} + \sum_{i=1}^{r}\tau(1-V_i) \\ &+ (\gamma-1)\sum_{i=1}^{r}\ln(1-W_i) \\ &+ \sum_{i=r+1}^{n}\ln[1-(A[1-W_i^{+}])^{\gamma}]. \end{aligned}$$

where $S_i = e^{-(\mu x_i)^{\lambda}}$, $T_i = 1 - \overline{\alpha}S_i$, $U_i = \left(\frac{1-S_i}{T_i}\right)$, $V_i = e^{{U_i}^{\theta}}$, $W_i = e^{\tau(1-V_i)}$, and S_i^{+}, T_i^{+}, U_i^{+}, V_i^{+}, W_i^{+} are obtained by replacing x_i with x_i^{+}.

Differentiating the log-likelihood with respect to the parameters and setting the score equations to zero yields the likelihood equations. As these equations are nonlinear, the MLEs are obtained via numerical optimization.

3.8.2 Least Squares Estimation of the ECMOW Distribution

Let $X_1, \ldots, X_n$ be a random sample from the ECMOW distribution, with order statistics $X_{(1)}, \ldots, X_{(n)}$. The LSEs of the unknown parameters $\Omega = (\gamma,\ \tau,\ \theta,\ \alpha,\ \lambda,\ \mu)$ are obtained by numerically minimizing the objective function.

$$LSE\,(x_i;\ \gamma,\ \tau,\ \theta,\ \alpha,\ \lambda,\ \mu) = \sum_{i=1}^{n}\left[\left\{A\left[1 - e^{\tau\left(1 - e^{\left(\frac{1-e^{-(\mu x)^{\lambda}}}{1-\overline{\alpha}e^{-(\mu x)^{\lambda}}}\right)^{\theta}}\right)}\right]\right\}^{\gamma} - \frac{i}{n+1}\right]^2,$$

By solving the following equations simultaneously

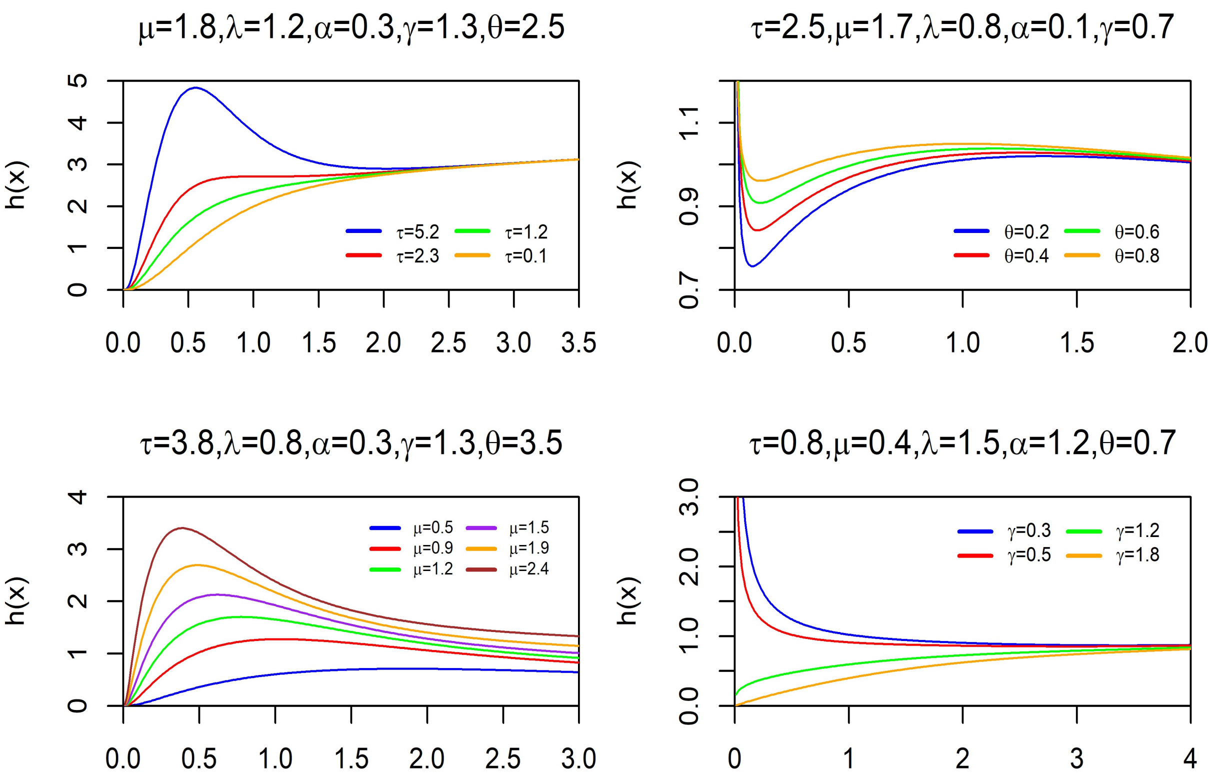

Figure 2. Plots of the hazard rate function for the ECMOW distribution with various parameter values

for the different parameters, the LSEs can be obtained.

$$\frac{\partial LSE\left(x_i;\ \gamma,\ \tau,\ \theta,\ \alpha,\ \lambda,\ \mu\right)}{\partial\Omega_i} = \sum_{i=1}^{n} F'_{\Omega_i}\left(x_{i:n};\Omega\right)\left(F\left(x_{i:n};\Omega\right)-\frac{i}{n+1}\right)=0.$$

where $\Omega_1=\gamma$, $\Omega_2=\tau$, $\Omega_3=\theta$, $\Omega_4=\alpha$, $\Omega_5=\lambda$, and $\Omega_6=\mu$.

3.8.3 Weighted Least Squares Estimation of the ECMOW Distribution

Similarly, the WLSEs for the parameters $\Omega = (\gamma,\ \tau,\ \theta,\ \alpha,\ \lambda,\ \mu)$ can be determined by minimizing the following objective function.

$$WLSE\left(x_i;\ \gamma,\ \tau,\ \theta,\ \alpha,\ \lambda,\ \mu\right)=\sum_{i=1}^{n}\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)} \times\left[\left\{A\left[1-e^{\tau\left(1-e^{\left(\frac{1-e^{-(\mu x)^{\lambda}}}{1-\overline{\alpha}e^{-(\mu x)^{\lambda}}}\right)^{\theta}}\right)}\right]\right\}^{\gamma}-\frac{i}{n+1}\right]^2,$$

By solving the following nonlinear equations, the WLSs of the parameters are calculated.

$$\frac{\partial WLSE\left(x_i;\ \gamma,\ \tau,\ \theta,\ \alpha,\ \lambda,\ \mu\right)}{\partial\Omega_i} = \sum_{i=1}^{n}\Omega\left(i,n\right)F'_{\Omega_i}\left(x_{i:n};\Omega\right)\left(F\left(x_{i:n};\Omega\right)-\frac{i}{n+1}\right)=0.$$

where $\Omega\left(i,n\right)=\frac{(n+1)^2(n+2)}{i(n-i+1)}$, $\Omega_1=\gamma$, $\Omega_2=\tau$, $\Omega_3=\theta$, $\Omega_4=\alpha$, $\Omega_5=\lambda$, and $\Omega_6=\mu$.

3.8.4 Anderson–Darling Estimation of the ECMOW Distribution

The ADEs of $\Omega=(\gamma,\ \tau,\ \theta,\ \alpha,\ \lambda,\ \mu)$ from the ECMOW distribution are obtained by minimizing the following objective function.

$$ADE\left(x_i;\gamma,\tau,\theta,\alpha,\lambda,\mu\right)=-n -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F\left(x_{i:n};\Omega\right)+\ln\overline{F}\left(x_{i:n};\Omega\right)\right],$$

Estimates are then determined by solving the following equations simultaneously.

$$\frac{\partial ADE\left(x_i;\gamma,\tau,\theta,\alpha,\lambda,\mu\right)}{\partial\Omega_i} = \sum_{i=1}^{n}(2i-1)\left[\frac{F'_{\Omega_i}\left(x_{i:n};\Omega\right)}{F\left(x_{i:n};\Omega\right)}+\frac{\overline{F}'_{\Omega_i}\left(x_{i:n};\Omega\right)}{\overline{F}\left(x_{i:n};\Omega\right)}\right]=0.$$

where $\Omega_1=\gamma$, $\Omega_2=\tau$, $\Omega_3=\theta$, $\Omega_4=\alpha$, $\Omega_5=\lambda$, and $\Omega_6=\mu$.

4. Simulation Study

In this section, a Monte Carlo simulation study is carried out to assess the finite-sample performance of the MLE, LSE, WLSE, and ADE estimators for the parameters of

10.57647/mathsci.2026.66781

the ECMOW distribution. For each scenario, 2000 replications are performed, and in each replication, random samples of sizes 50, 200, and 800 are generated from the ECMOW distribution under three different sets of true parameter values. All simulations and computations are implemented using the R statistical software. The estimators are evaluated in terms of their mean estimates, bias, and root mean square error (RMSE). For a generic parameter ω, the bias and RMSE are computed as

$$Bias = \frac{1}{2000}\sum_{i=1}^{2000}\left(\hat{\Omega}_i - \Omega\right)$$

and

$$RMSE = \sqrt{\frac{1}{2000}\sum_{i=1}^{2000}\left(\hat{\Omega}_i - \Omega\right)^2}.$$

Table 3, Table 4, Table 5 present the mean estimates of the parameters along with their corresponding bias and RMSE values. As the sample size increases, the mean estimates generally approach the true parameter values, while the bias values move closer to zero and the RMSEs tend to decrease, confirming consistency. For smaller sample sizes, minor fluctuations in bias and RMSE are observed, which diminish as the sample size grows.

Overall, the distance-based estimators (LSE, WLSE, and ADE) exhibit competitive finite-sample performance in terms of bias and RMSE under the considered parameter settings. This behavior may be influenced by the highly nonlinear likelihood surface and the multi-parameter structure of the ECMOW model in small and moderate samples. In particular, ADE shows competitive performance in tail-sensitive scenarios due to its emphasis on distributional tails.

Nevertheless, the MLE demonstrates stable convergence, and its bias and RMSE decrease as the sample size increases, which is consistent with its well-known asymptotic efficiency under standard regularity conditions. These results suggest that all considered estimators exhibit consistent behavior, and the observed differences are mainly a finite-sample phenomenon.

5. Application

In this section, we compare the goodness-of-fit of the ECMOW distribution with its sub-models and other competing distributions from the literature. Applications are provided to two real-life datasets and one simulated dataset from different domains, demonstrating the flexibility of the ECMOW distribution relative to alternative models. The competitive distributions considered include the Exponentiated Chen Exponential Distribution (ECE) and Exponentiated Chen Weibull Distribution (ECW) [27], the Marshall-Olkin Marshall-Olkin Weibull Distribution (MOMOW) [29], the Exponentiated Exponential Weibull Distribution (EEW) [30], and the Exponentiated Kumaraswamy Weibull Distribution (EKW) [31]. The CDFs of the competing distributions considered in this study are presented in Table 6.

5.1 Treatment Data

The first dataset, reported by [32], represents the survival times of patients who received chemotherapy treatment alone. The data consist of 46 survival times for 46 patients, as follows: 1.485, 1.553, 1.099, 1.219, 1.271, 0.047, 0.115, 0.197, 0.203, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 3.658, 3.743, 3.978, 0.260, 0.282, 0.534, 0.540, 0.570, 0.641, 0.644, 1.326, 1.447, 1.581, 1.589, 2.178, 0.121, 0.132, 0.164, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 0.696, 0.841, 0.863, 4.003, 4.033.

5.2 P3 Benchmark Data (Algorithmic Output)

The second dataset consists of 22 values generated by the P3 algorithm for the EPRI synthetic system, as originally reported by [33]. These values are algorithmic outputs (unit capacity factors) and do not constitute an observational random sample. This dataset has been widely used in the literature as a benchmark for distributional comparisons. The dataset is listed as follows: 0.010, 0.014, 0.019, 0.026, 0.036, 0.044, 0.056, 0.067, 0.078, 0.097, 0.118, 0.118, 0.207, 0.334, 0.399, 0.503, 0.557, 0.716, 0.759, 0.800, 0.853, 0.874.

Although the second dataset has a relatively small sample size, the observations correspond to independent experimental units collected under homogeneous conditions, justifying the i.i.d. assumption commonly adopted in reliability studies.

5.3 Fiber Data

The third dataset, sourced from [34], relates to the strengths of 1.5 cm glass fibers. There are 63 observations, which are given as: 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.024.

Three datasets are analyzed in this study. The treatment dataset consists of observed survival times from patients and represents a real-world i.i.d. sample, allowing standard inferential goodness-of-fit procedures, the fiber dataset contains laboratory measurements of fiber strength and also represents a real observational sample suitable for inferential analysis, and the P3 dataset comprises algorithmically generated values representing system performance and is used for illustrative and comparative distributional analysis rather than standard inferential procedures.

To better understand the characteristics of the three datasets, a variety of graphical tools are presented in Figure 3, Figure 7, and Figure 11, including histograms, violin plots, box plots with strip charts, quantile-quantile (QQ) plots, and total time in test (TTT) plots. The histograms and violin plots reveal clear departures from symmetry, indicating skewness and heavy-tailed behavior. Box plots further highlight variability and potential outliers, while the QQ plots demon-

10.57647/mathsci.2026.66781

Table 3. Monte Carlo simulation results for the ECMOW distribution with true parameter values ($\gamma = 1.8$, $\theta = 1.3$, $\alpha = 0.2$, $\mu = 0.8$, $\lambda = 2$, $\tau = 1.5$)

Sample Size	Parameter	MLE			LSE			WLSE			ADE		
		Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE
50	γ	2.420	0.620	1.462	1.892	0.092	0.606	1.988	0.188	0.791	1.958	0.158	0.610
200		2.159	0.359	1.060	1.889	0.089	0.445	1.937	0.137	0.561	1.892	0.092	0.450
800		1.956	0.156	0.641	1.861	0.061	0.299	1.871	0.071	0.278	1.850	0.050	0.257
50	θ	2.264	0.964	2.157	1.548	0.248	0.498	1.606	0.306	0.750	1.592	0.292	0.654
200		1.570	0.270	1.495	1.465	0.165	0.365	1.499	0.199	0.543	1.463	0.163	0.542
800		1.330	0.030	0.959	1.395	0.095	0.265	1.373	0.073	0.295	1.361	0.061	0.260
50	α	0.074	-0.126	0.174	0.176	-0.024	0.258	0.113	-0.087	0.180	0.112	-0.088	0.182
200		0.092	-0.108	0.155	0.168	-0.022	0.153	0.146	-0.054	0.130	0.149	-0.051	0.132
800		0.145	-0.055	0.140	0.189	-0.011	0.097	0.187	-0.013	0.096	0.186	-0.014	0.096
50	μ	0.857	0.057	0.559	0.728	-0.072	0.339	0.700	-0.100	0.401	0.694	-0.106	0.351
200		0.846	0.046	0.406	0.781	-0.019	0.286	0.774	-0.026	0.306	0.758	-0.042	0.279
800		0.847	0.047	0.329	0.802	0.002	0.208	0.806	0.006	0.198	0.794	-0.006	0.182
50	λ	2.313	0.313	0.704	2.044	0.044	0.444	2.061	0.061	0.409	2.096	0.096	0.394
200		2.315	0.310	0.642	1.991	-0.039	0.265	2.016	0.016	0.313	2.038	0.038	0.310
800		2.214	0.214	0.525	1.971	-0.029	0.199	1.993	-0.007	0.237	2.002	0.002	0.217
50	τ	0.634	-0.866	1.216	1.466	-0.034	0.785	1.459	-0.041	0.880	1.480	-0.020	1.013
200		0.831	-0.669	1.107	1.504	0.024	0.696	1.492	-0.018	0.759	1.521	0.011	0.831
800		1.058	-0.442	0.792	1.533	0.013	0.484	1.498	-0.002	0.463	1.514	0.009	0.442

Table 4. Monte Carlo simulation results for the ECMOW distribution with true parameter values ($\gamma = 3$, $\theta = 2$, $\alpha = 0.2$, $\mu = 0.8$, $\lambda = 1.4$, $\tau = 0.7$)

Sample Size	Parameter	MLE			LSE			WLSE			ADE		
		Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE
50	γ	3.061	0.231	1.693	2.833	-0.167	0.504	2.956	-0.044	0.616	2.920	-0.080	0.521
200		3.197	0.197	1.370	2.909	-0.091	0.291	2.985	-0.035	0.418	2.959	-0.041	0.372
800		3.178	0.178	0.895	2.967	-0.033	0.148	3.019	0.019	0.280	2.995	-0.005	0.272
50	θ	3.095	1.095	2.830	1.930	-0.070	0.538	2.076	0.076	0.774	2.041	0.041	0.558
200		2.551	0.551	2.185	1.942	-0.058	0.311	2.050	0.050	0.560	2.011	0.021	0.442
800		2.232	0.232	1.225	1.978	-0.022	0.148	2.034	0.034	0.342	2.012	0.012	0.328
50	α	0.146	-0.054	0.268	0.308	0.108	0.367	0.246	0.046	0.310	0.243	0.043	0.311
200		0.122	-0.048	0.199	0.300	0.100	0.308	0.248	0.038	0.266	0.257	0.037	0.273
800		0.135	-0.035	0.137	0.279	0.079	0.233	0.246	0.026	0.204	0.254	0.024	0.0217
50	μ	0.681	-0.119	0.520	0.765	-0.045	0.483	0.702	-0.098	0.442	0.704	-0.096	0.397
200		0.752	-0.048	0.487	0.844	0.034	0.386	0.811	0.071	0.372	0.791	-0.059	0.282
800		0.791	-0.009	0.411	0.847	0.027	0.233	0.830	0.050	0.208	0.829	0.029	0.213
50	λ	1.849	0.449	0.901	1.470	0.070	0.356	1.530	0.130	0.452	1.522	0.122	0.331
200		1.707	0.307	0.529	1.386	-0.054	0.231	1.436	0.036	0.244	1.440	0.040	0.239
800		1.583	0.183	0.397	1.363	-0.037	0.165	1.392	-0.008	0.169	1.396	-0.004	0.179
50	τ	0.411	-0.289	0.830	0.836	0.136	0.851	0.850	0.150	0.882	0.851	0.151	0.901
200		0.344	-0.156	0.689	0.776	0.076	0.703	0.699	-0.071	0.660	0.729	0.029	0.656
800		0.365	-0.115	0.532	0.778	0.058	0.430	0.701	0.001	0.394	0.708	0.008	0.410

Table 5. Monte Carlo simulation results for the ECMOW distribution with true parameter values ($\gamma = 4$, $\theta = 2.5$, $\alpha = 0.1$, $\mu = 0.4$, $\lambda = 1.5$, $\tau = 0.9$)

Sample Size	Parameter	MLE			LSE			WLSE			ADE		
		Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE
50	γ	3.988	-0.072	0.876	3.883	-0.117	0.373	3.880	-0.120	0.473	3.906	-0.094	0.386
200		4.055	0.055	0.627	3.957	-0.043	0.176	3.972	-0.028	0.312	3.961	-0.039	0.225
800		4.099	0.039	0.431	3.978	-0.022	0.074	3.998	-0.002	0.230	3.985	-0.015	0.133
50	θ	2.580	0.080	1.174	2.391	-0.109	0.392	2.360	-0.140	0.639	2.432	-0.068	0.434
200		2.591	0.071	1.048	2.456	-0.044	0.192	2.468	-0.032	0.429	2.472	-0.028	0.222
800		2.609	0.059	0.966	2.468	-0.032	0.113	2.481	-0.019	0.270	2.480	-0.010	0.166
50	α	0.082	-0.018	0.180	0.196	0.096	0.303	0.160	0.070	0.246	0.160	0.060	0.251
200		0.092	-0.008	0.159	0.182	0.082	0.247	0.161	0.051	0.202	0.168	0.058	0.225
800		0.090	-0.003	0.076	0.150	0.050	0.156	0.139	0.039	0.124	0.144	0.044	0.135
50	μ	0.369	-0.031	0.314	0.455	0.055	0.356	0.401	0.051	0.290	0.408	0.028	0.270
200		0.407	0.007	0.301	0.484	0.044	0.306	0.443	0.043	0.229	0.448	0.018	0.223
800		0.402	0.002	0.218	0.446	0.036	0.154	0.429	0.029	0.110	0.432	0.009	0.117
50	λ	1.824	0.324	0.492	1.466	-0.034	0.341	1.547	0.047	0.316	1.543	0.043	0.313
200		1.705	0.205	0.393	1.429	-0.021	0.267	1.467	-0.033	0.235	1.466	-0.034	0.243
800		1.612	0.112	0.291	1.440	-0.010	0.179	1.456	-0.024	0.159	1.454	-0.026	0.170
50	τ	0.558	-0.342	0.681	0.989	0.089	0.753	0.947	0.047	0.737	0.955	0.055	0.771
200		0.599	-0.201	0.654	0.917	0.057	0.519	0.945	0.035	0.499	0.932	0.032	0.517
800		0.719	-0.181	0.421	0.936	0.036	0.236	0.956	0.026	0.230	0.949	0.029	0.253

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Table 6. The CDFs of the competitive models

Model	CDF
Exponentiated Chen Exponential	$\left(A\left[1-e^{\tau\left(1-e^{\left(1-e^{-\mu x}\right)^{\theta}}\right)}\right]\right)^{\gamma}$
Exponentiated Chen Weibull	$\left[A\left(1-e^{\tau\left(1-e^{\left(1-e^{-(\mu x)^{\lambda}}\right)^{\theta}}\right)}\right)\right]^{\gamma}$
Marshall-Olkin Marshall-Olkin Weibull	$\frac{1-e^{-\lambda x^{\beta}}}{\alpha\theta+\left(\alpha\overline{\theta}+\overline{\alpha}\right)\left(1-e^{-\lambda x^{\beta}}\right)}$
Exponentiated Exponential Weibull	$\left[1-e^{-b(\mu x)^{\lambda}}\right]^{\gamma}$
Exponentiated Kumaraswamy Weibull	$\left[1-\left[1-\left(1-e^{-(\mu x)^{\lambda}}\right)^{a}\right]^{b}\right]^{\gamma}$

Table 7. MLEs and goodness-of-fits measures for the first dataset

Model	Par.	Est	S.E	-L	AIC	CAIC	BIC	HQIC	A^*	W^*	KS	P-Value
ECMOW	γ	20.240	27.779	53.483	118.967	121.121	129.939	123.077	0.329	0.051	0.090	0.812
	μ	0.290	0.018									
	θ	0.022	0.012									
	λ	7.417	0.032									
	τ	2.907	0.742									
	α	23.661	29.823									
ECW	γ	15.455	16.411	56.027	122.055	123.555	131.198	125.48	0.414	0.062	0.093	0.785
	μ	0.237	0.025									
	θ	0.026	0.012									
	λ	6.821	0.047									
	τ	2.512	0.657									
ECE	γ	2.276	3.107	58.67	125.340	126.316	132.655	128.080	0.459	0.068	0.093	0.784
	μ	0.528	0.047									
	θ	1.414	1.436									
	τ	2.482	1.460									
MOMOW	μ	0.387	0.289	58.416	124.833	125.809	132.148	127.574	0.469	0.070	0.092	0.795
	λ	1.261	0.245									
	α	0.629	22.969									
	β	0.629	22.969									
EEW	γ	1.924	2.332	58.665	125.33	126.306	132.645	128.07	0.467	0.070	0.0952	0.763
	μ	0.960	63.848									
	b	1.335	65.837									
	λ	0.741	0.455									
EKW	γ	12.276	17.831	57.357	124.715	126.215	133.858	128.14	0.513	0.078	0.108	0.609
	μ	0.176	0.063									
	b	2.013	0.771									
	λ	6.056	0.122									
	a	0.025	0.025									

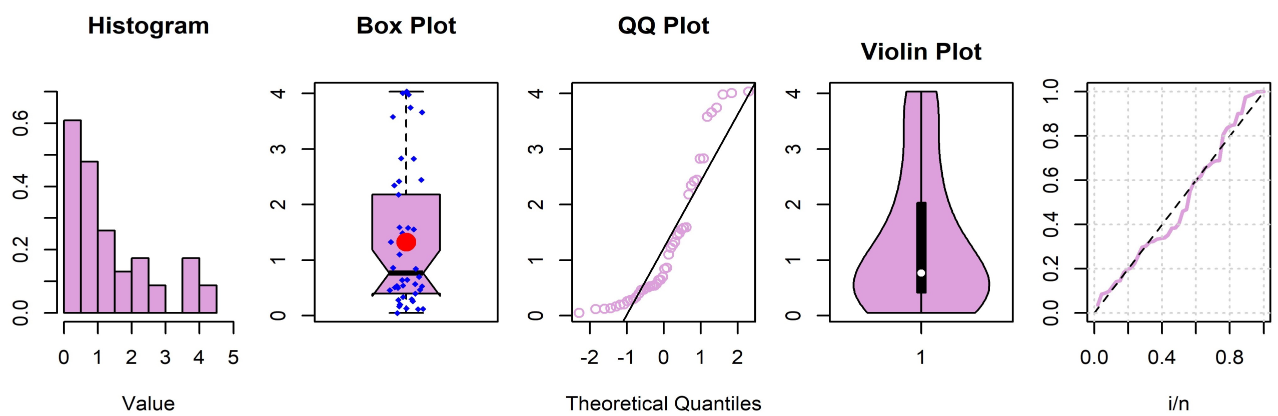

Figure 3. Basic non-parametric plots for the first dataset

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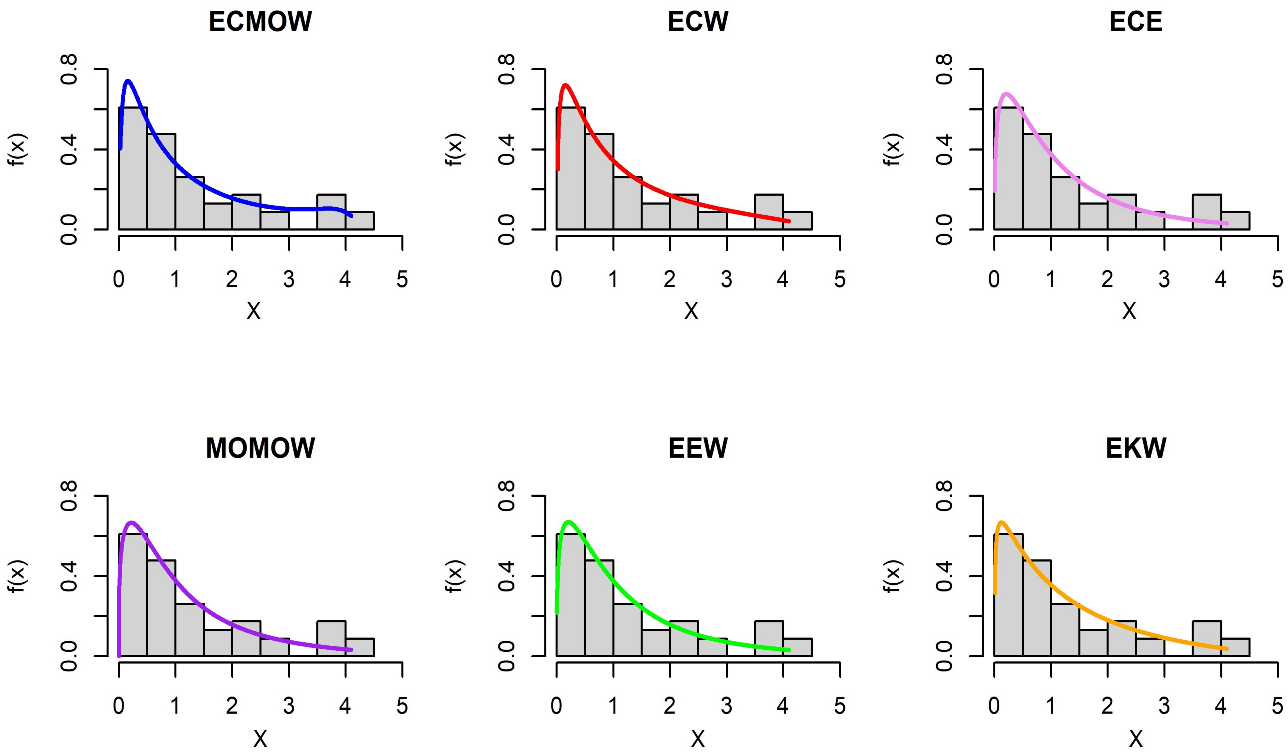

Figure 4. Estimated PDFs of the competing models for the first dataset

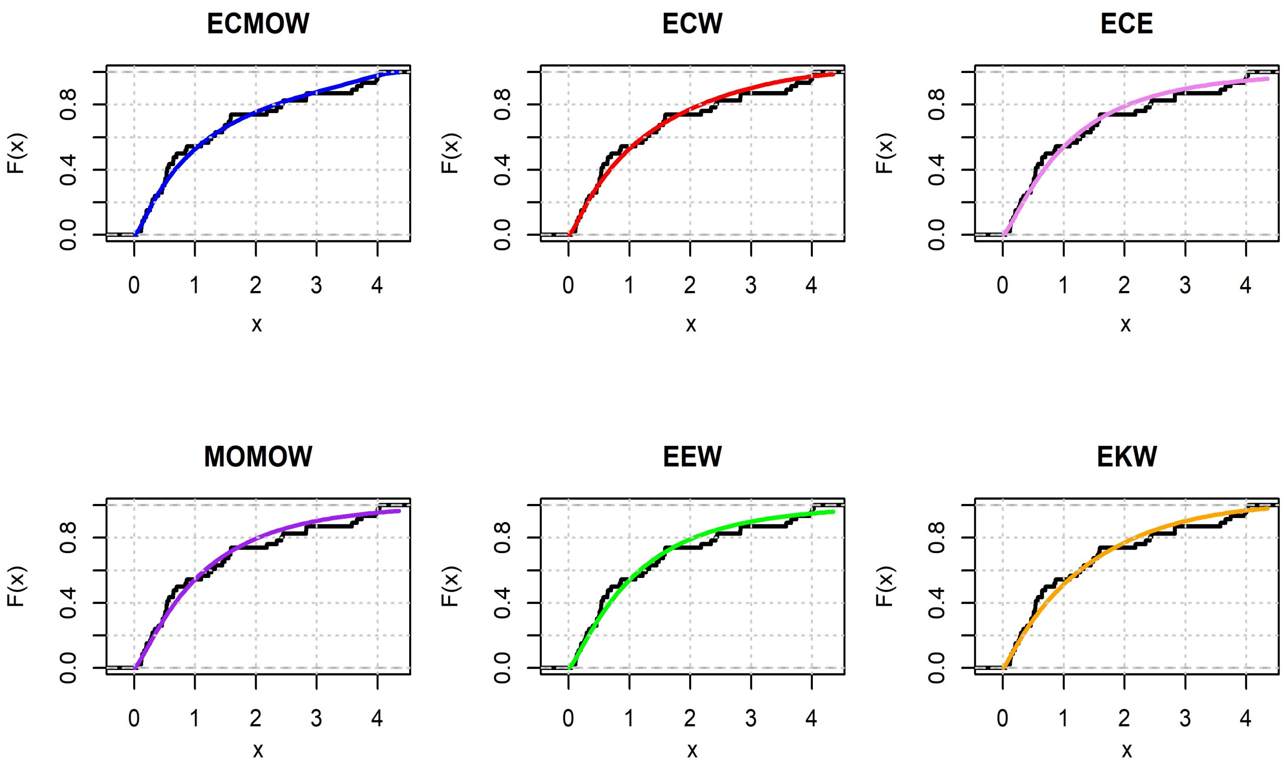

Figure 5. Estimated CDFs of the competing models for the first dataset

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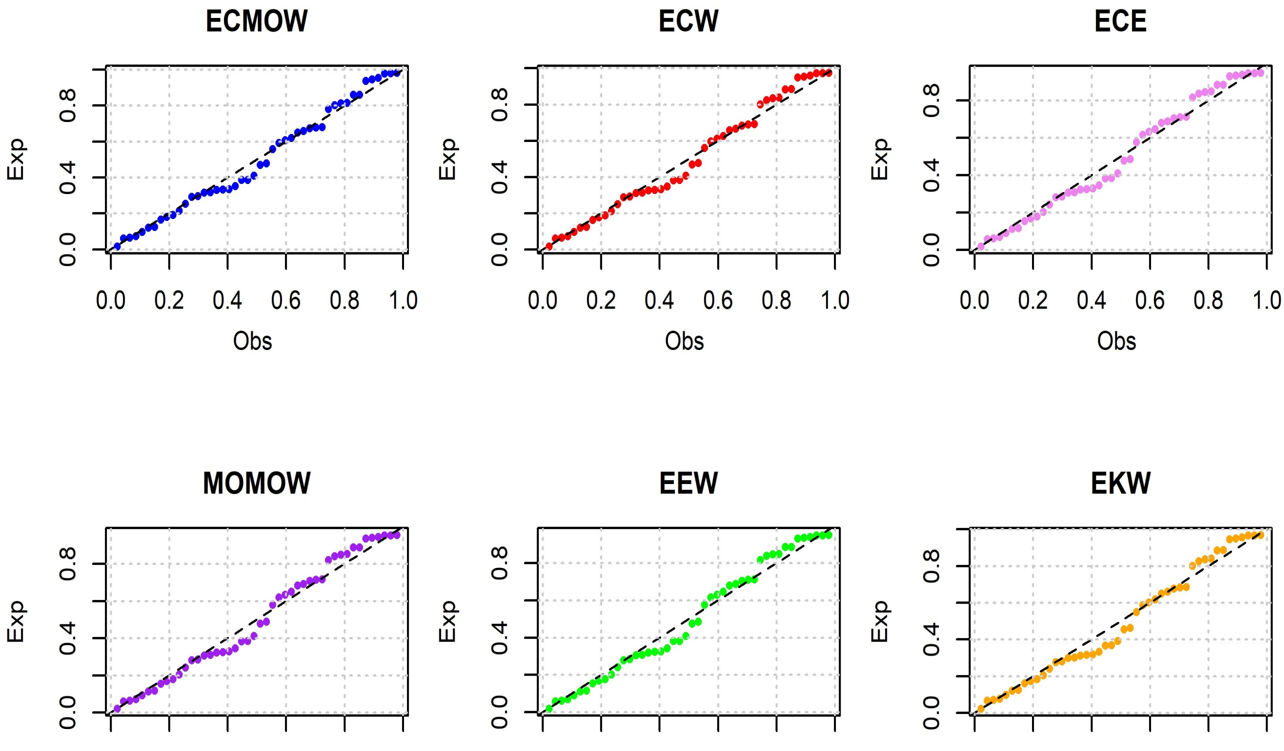

Figure 6. PP plots of the competing models for the first dataset

Table 8. MLEs and goodness-of-fits measures for the second dataset

Model	Par.	Est	S.E	-L	AIC	CAIC	BIC	HQIC	A^*	W^*	KS	P-Value
ECMOW	γ	40.667	81.877	-10.177	-8.355	-2.755	-1.809	-6.813	0.350	0.058	0.143	0.755
	μ	1.403	0.034									
	θ	0.012	0.008									
	λ	7.225	0.038									
	τ	2.821	1.232									
	α	66.869	74.837									
ECW	γ	10.471	14.491	-7.274	-4.548	-0.798	0.906	-3.263	0.564	0.090	0.156	0.651
	μ	1.097	0.033									
	θ	0.017	0.012									
	λ	7.306	0.033									
	τ	1.776	0.953									
ECE	γ	1.155	4.010	-4.855	-1.711	0.641	2.652	-0.683	0.642	0.099	0.150	0.703
	μ	0.114	0.539									
	θ	0.742	1.634									
	τ	14.318	47.633									
MOMOW	μ	2.350	1.415	-4.951	-1.903	0.449	2.461	-0.874	0.615	0.094	0.145	0.712
	λ	0.928	0.256									
	α	2.031	99.742									
	β	0.270	13.287									
EEW	γ	0.075	0.016	-6.409	-4.818	-2.465	-0.454	-3.790	0.755	0.124	0.204	0.313
	μ	0.928	0.0002									
	b	3.938	0.008									
	λ	7.071	0.026									
EKW	γ	0.156	0.128	-6.864	-3.729	0.020	1.725	-2.444	0.755	0.124	0.209	0.287
	μ	1.434	0.021									
	b	0.239	0.160									
	λ	7.551	0.021									
	a	0.412	0.368									

10.57647/mathsci.2026.66781

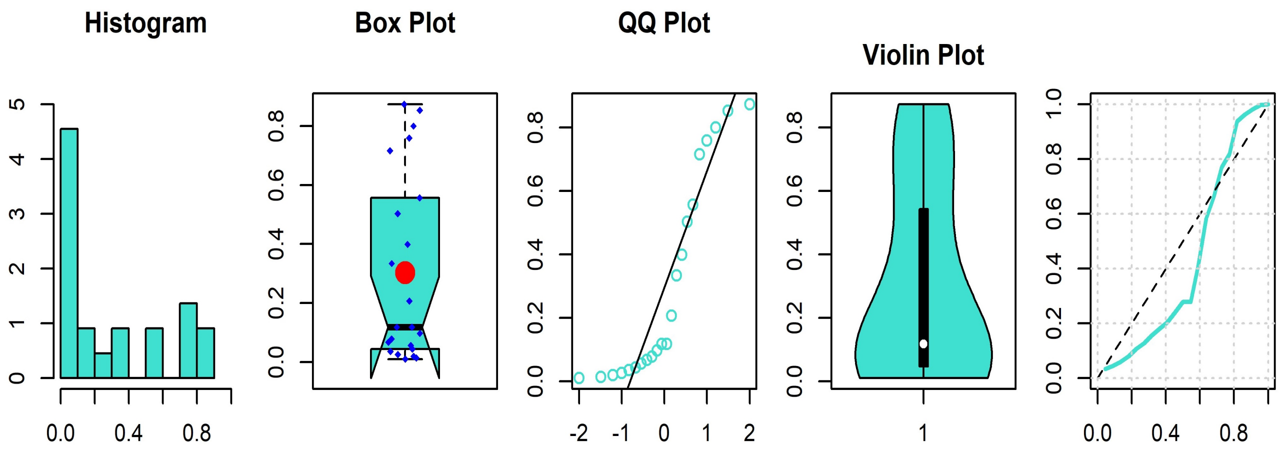

Figure 7. Basic non-parametric plots for the second dataset

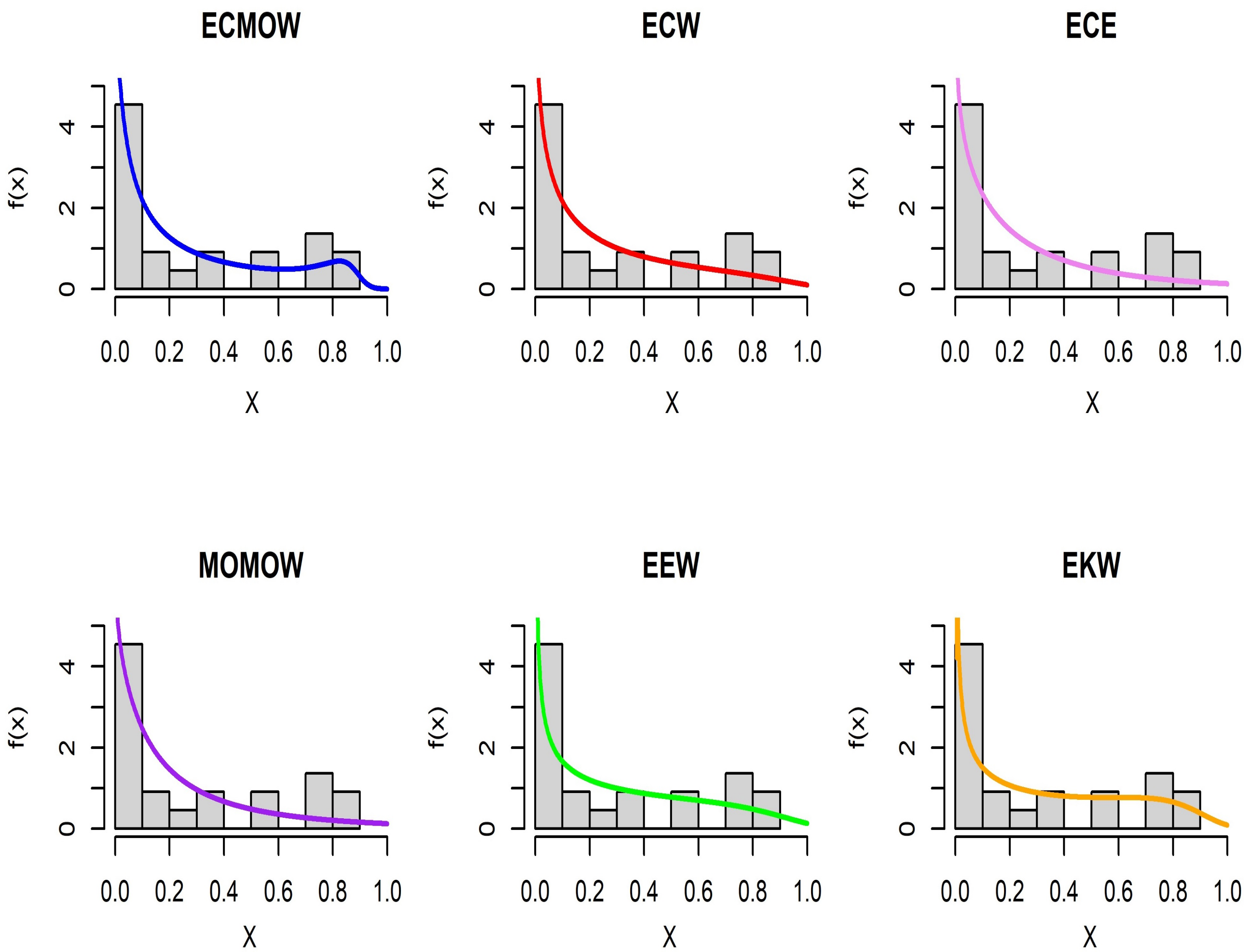

Figure 8. Estimated PDFs of the competing models for the second dataset

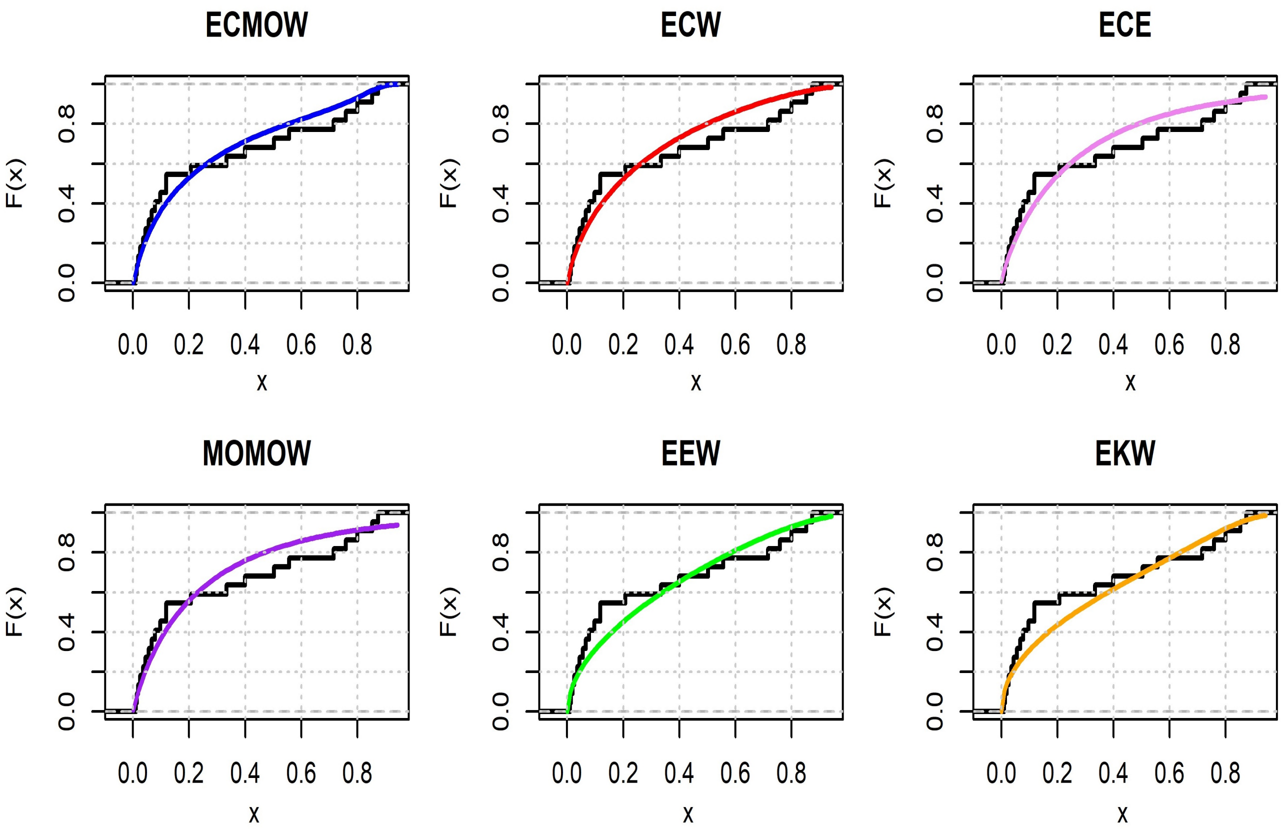

Figure 9. Estimated CDFs of the competing models for the second dataset

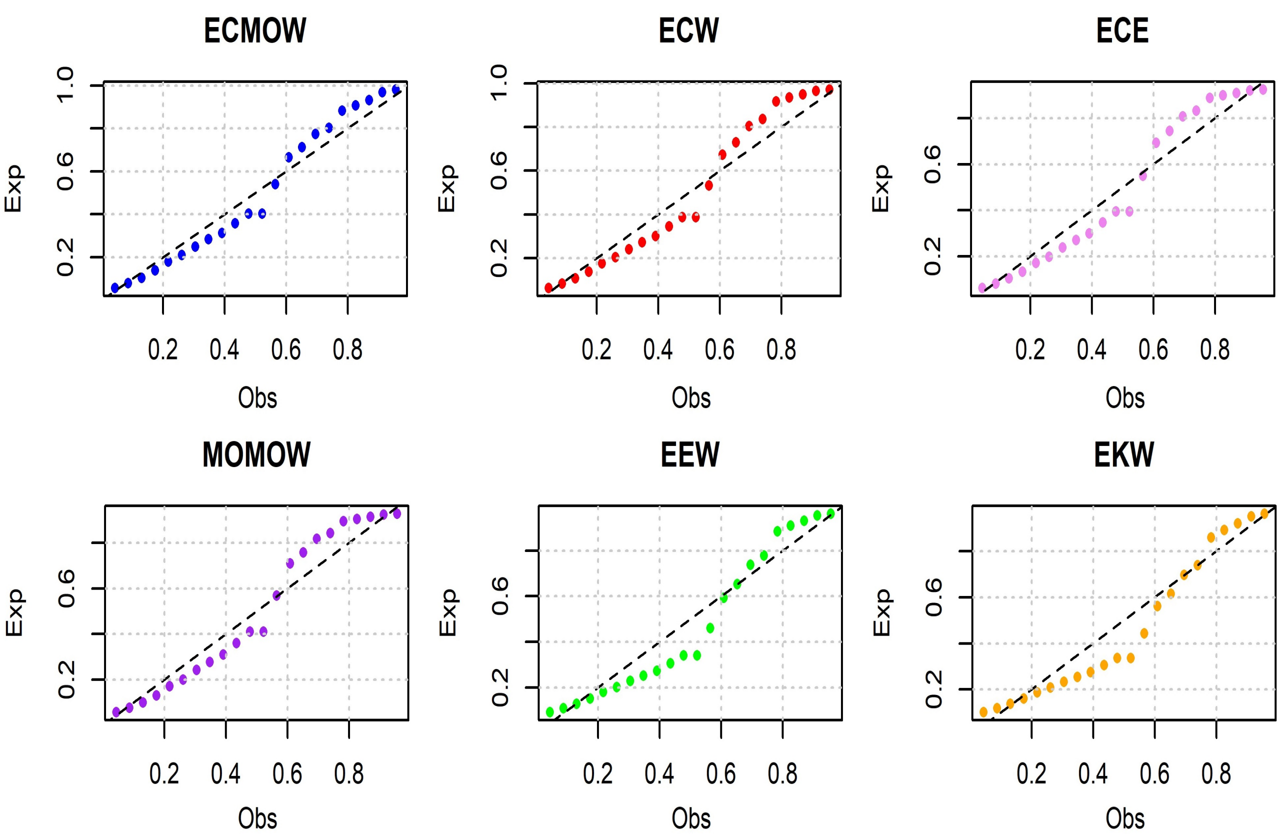

Figure 10. PP plots of the competing models for the second dataset

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Table 9. MLEs and goodness-of-fits measures for the third dataset

Model	Par.	Est	S.E	-L	AIC	CAIC	BIC	HQIC	A*	W*	KS	P-Value
ECMOW	γ	0.551	0.639	11.172	34.344	35.844	47.203	39.402	0.419	0.072	0.101	0.539
	μ	0.815	0.167									
	θ	1.555	1.842									
	λ	3.495	1.123									
	τ	1.681	1.431									
	α	19.286	24.915									
ECW	γ	0.404	0.305	13.360	36.721	37.774	47.437	40.936	0.884	0.159	0.140	0.164
	μ	0.528	0.047									
	θ	1.415	1.435									
	λ	8.364	3.274									
	τ	2.482	1.460									
ECE	γ	0.377	0.082	15.784	39.568	40.258	48.063	42.94	1.438	0.261	0.171	0.049
	μ	0.999	0.102									
	θ	25.118	4.571									
	τ	88.595	39.769									
MOMOW	μ	0.003	0.0008	20.796	49.593	50.282	58.165	52.964	2.434	0.443	0.175	0.041
	λ	7.867	0.701									
	α	0.261	0.125									
	β	0.304	0.146									
EEW	γ	1.279	0.671	16.085	40.170	40.860	48.743	43.542	1.546	0.282	0.171	0.048
	μ	1.811	0.774									
	b	0.007	0.003									
	λ	4.776	1.525									
EKW	γ	8.498	4.855	13.385	36.770	37.823	47.486	40.985	0.791	0.142	0.128	0.247
	μ	0.979	0.032									
	b	0.248	0.047									
	λ	4.238	0.034									
	a	0.010	0.016									

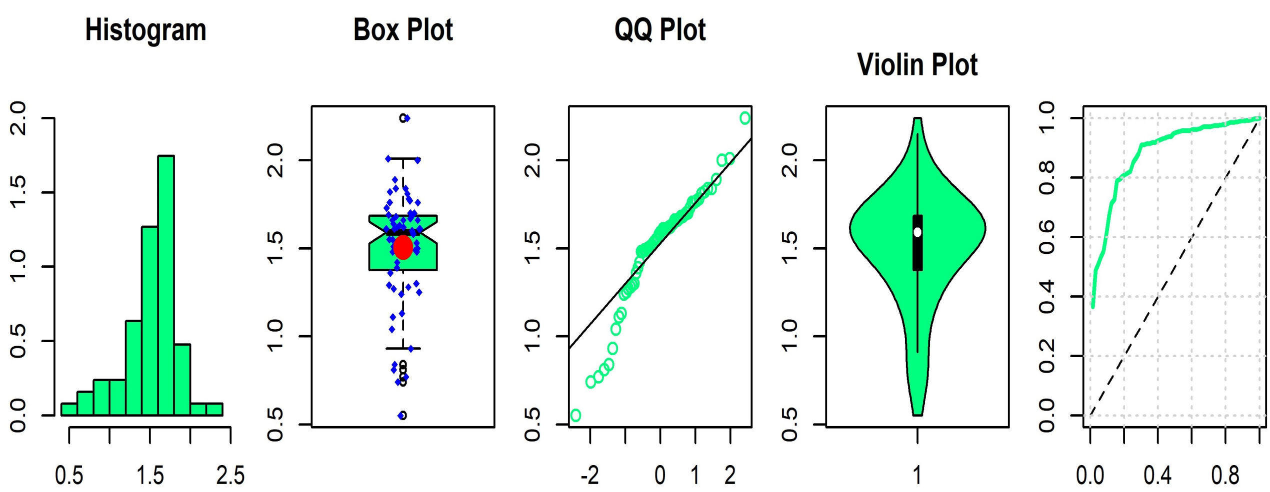

Figure 11. Basic non-parametric plots for the third dataset

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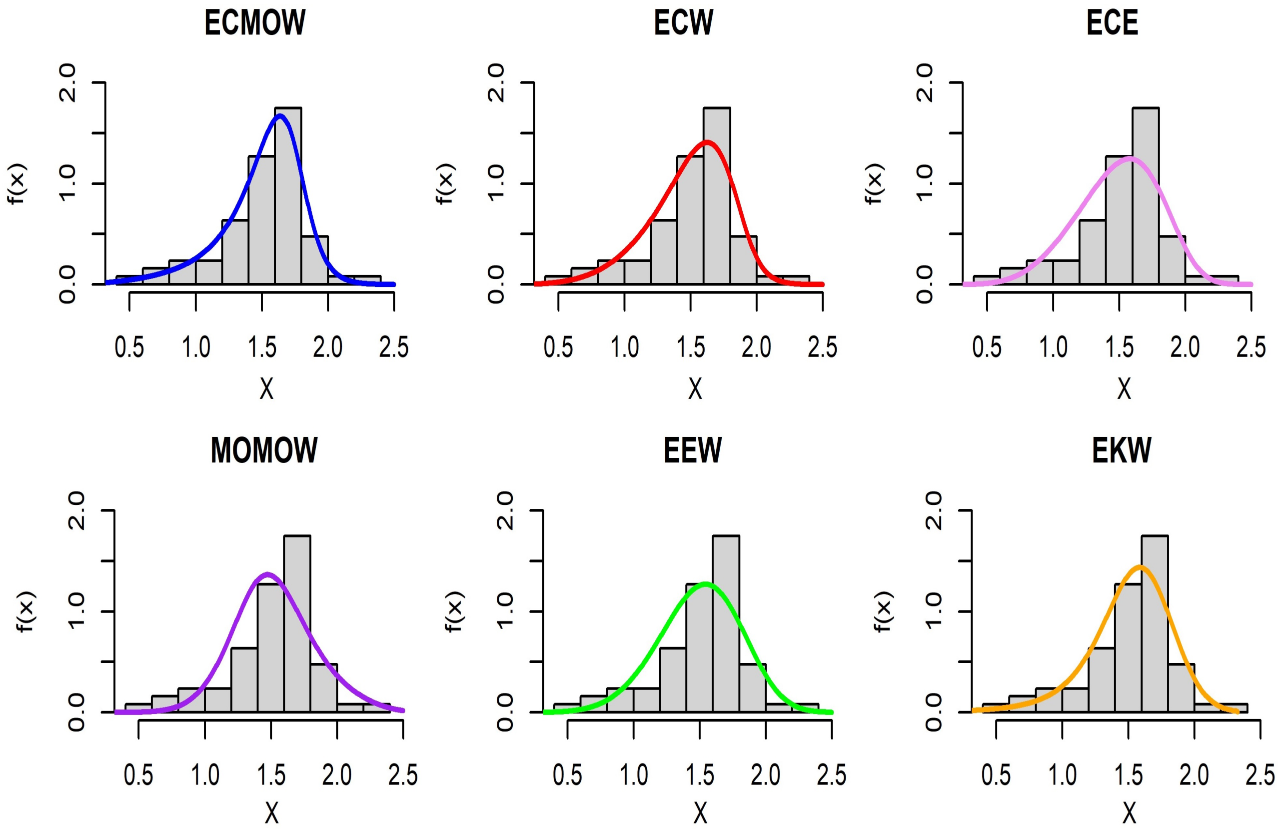

Figure 12. Estimated PDFs of the competing models for the third dataset

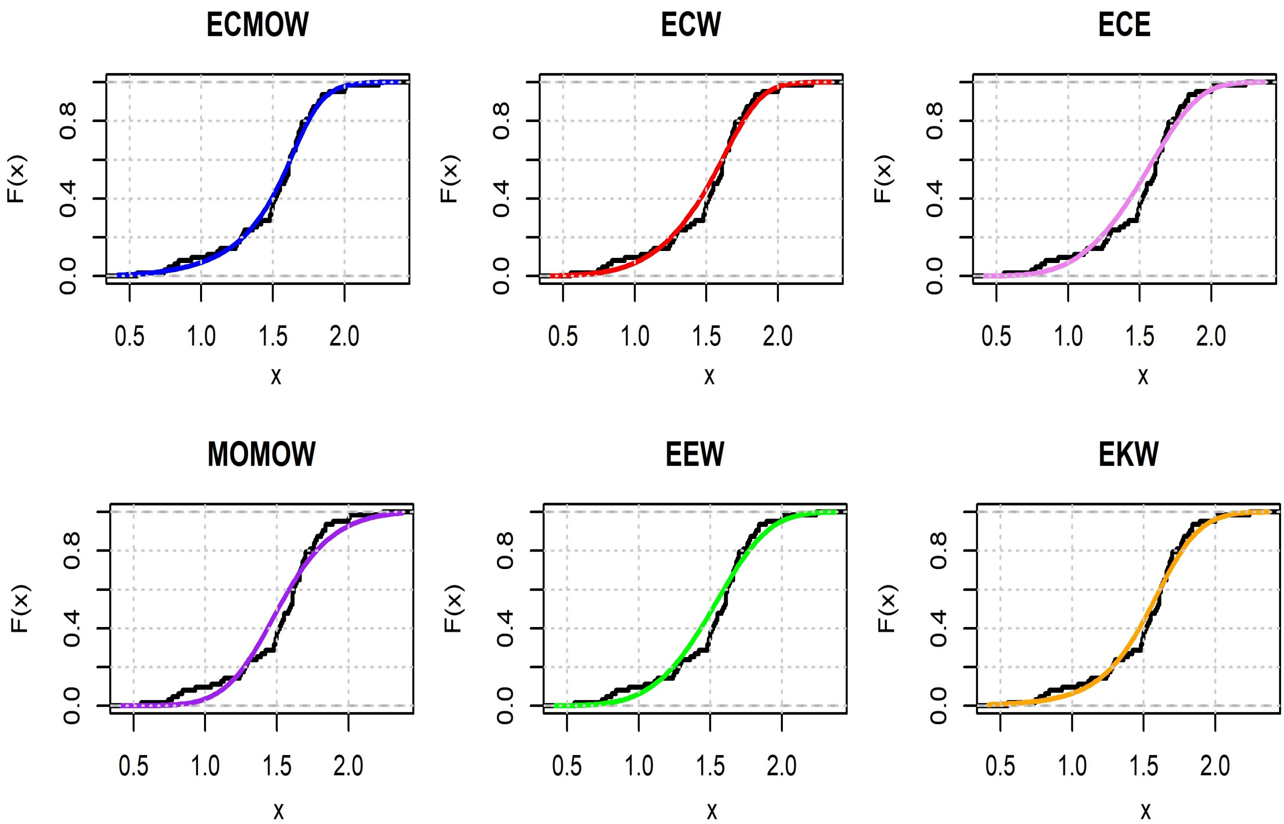

Figure 13. Estimated CDFs of the competing models for the third dataset

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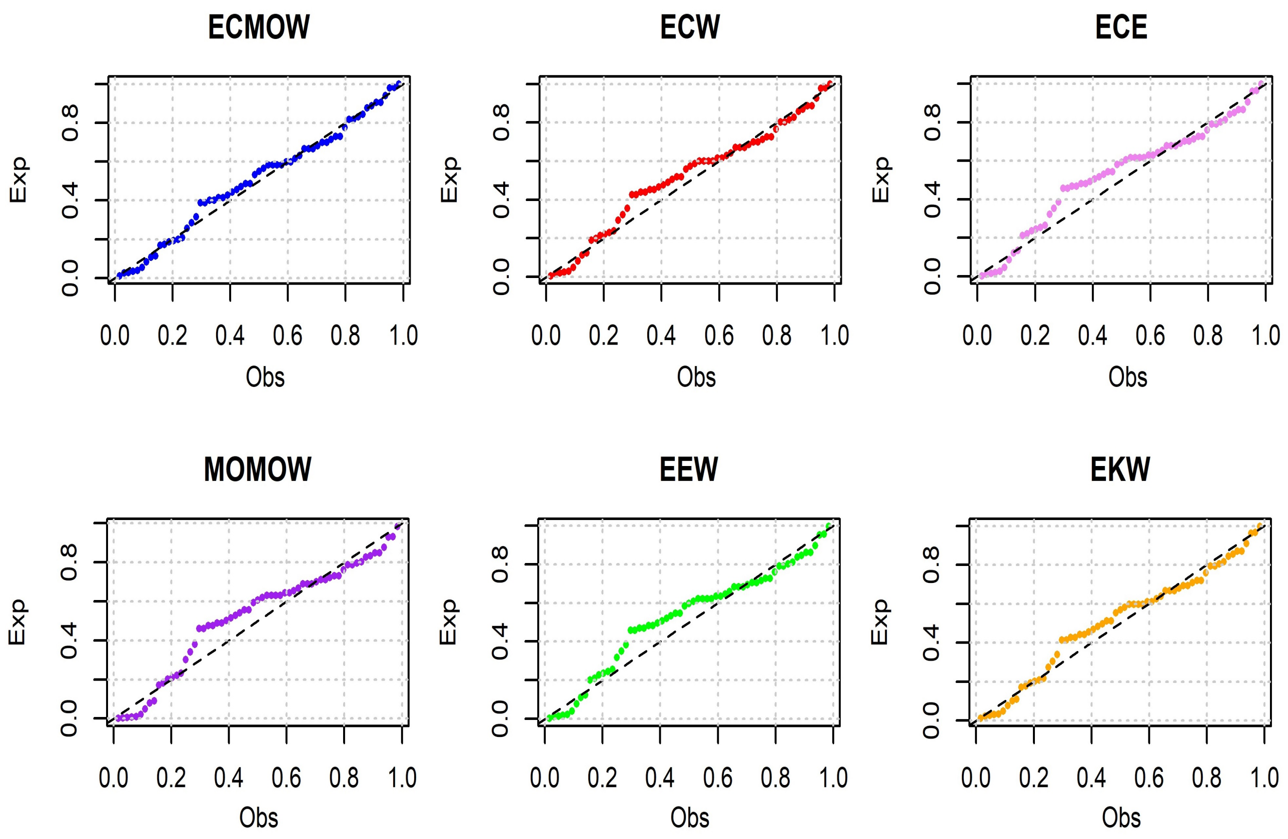

Figure 14. PP plots of the competing models for the third dataset

strate deviations from normality, supporting the need for flexible lifetime distributions. Additionally, the shapes of the TTT plots suggest non-monotonic hazard rate patterns, motivating the use of models capable of capturing complex hazard structures.

Parameter estimation for all competing models was performed using the MLE method. The resulting estimates and standard errors (SEs) are reported in Table 7, Table 8, Table 9, alongside several goodness-of-fit criteria, including Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*), Cramer-von Mises (W^*), Kolmogorov–Smirnov (KS) statistics, and associated p-values. Across all datasets, the proposed ECMOW model consistently yields the lowest values of AIC, BIC, CAIC, HQIC, A^*, W^*, and KS, accompanied by comparatively larger p-values, indicating a superior overall fit relative to the competing distributions.

Figure 4, Figure 8, and Figure 12 display the estimated PDFs for all fitted models. The PDF of the ECMOW distribution closely follows the empirical frequency patterns, particularly in the tails, where other models exhibit noticeable discrepancies. Similarly, the estimated CDFs shown in Figure 5, Figure 9, and Figure 13 demonstrate that the ECMOW model provides an excellent approximation to the empirical CDF across the entire data range, accurately capturing both central tendencies and tail behavior.

Further visual confirmation is provided by the PP plots in Figure 6, Figure 10, and Figure 14. For the ECMOW model, the points lie very close to the reference line, whereas competing models show systematic deviations. These visual assessments corroborate the numerical goodness-of-fit results and confirms that the ECMOW distribution offers a more accurate and reliable representation of the datasets.

Overall, the graphical analyses presented in Figure 3, Figure 4, Figure 5, Figure 6, Figure 7, Figure 8, Figure 9, Figure 10, Figure 11, Figure 12, Figure 13, Figure 14 consistently indicate that the ECMOW distribution provides the best fit across all datasets. The histograms, violin plots, and box plots highlight skewness, tail behavior, and variability, while the QQ and TTT plots confirm deviations from normality and non-monotonic hazard patterns. Across the PDFs, CDFs, and PP plots, the ECMOW model closely tracks the empirical data with minimal deviations compared to competing models. Combined with the numerical goodness-of-fit statistics, these results demonstrate that the ECMOW distribution effectively captures the key features of the datasets and outperforms alternative distributions.

6. Conclusions

In this paper, we proposed the Exponentiated Chen Marshall-Olkin (ECMO) family as a unified and flexible framework for modeling lifetime and reliability data.

10.57647/mathsci.2026.66781

The main contribution of this work lies in integrating multiple generating mechanisms within a single construction, which enables the simultaneous regulation of skewness, tail behavior, and diverse hazard rate shapes, including non-monotonic and multi-phase patterns frequently observed in real-world applications. This joint flexibility distinguishes the ECMO family from existing single-generator extensions and allows it to address important limitations of classical lifetime models within a parsimonious parameterization.

From a theoretical perspective, several fundamental statistical properties of the proposed family were derived in closed or semi-closed form, highlighting its analytical tractability and facilitating inference. A particular sub-model, namely the Exponentiated Chen Marshall-Olkin Weibull (ECMOW) distribution, was examined in detail to demonstrate the applicability of the general framework and to provide further insight into its structural characteristics.

Parameter estimation was investigated using both likelihood-based and distance-based approaches under complete and censored observations. The Monte Carlo simulation results indicated satisfactory finite-sample performance of all considered estimators, with decreasing bias and RMSE as the sample size increased, confirming their consistency. The empirical applications to real datasets further illustrated that the proposed models provide competitive goodness-of-fit compared with commonly used alternatives, particularly in settings involving complex hazard rate behavior and tail characteristics relevant to reliability analysis.

Overall, the ECMO family offers a flexible yet tractable modeling framework that effectively bridges theoretical development and practical applicability in lifetime and reliability studies.

Future Research

Several directions for future research naturally arise from this work. First, extending the ECMO framework to regression settings would allow the incorporation of covariate information, enabling a more comprehensive analysis of lifetime data in applied reliability and survival studies. Similarly, the development of time-series or dynamic versions of the proposed models could broaden their applicability to dependent lifetime data.

Second, while likelihood-based estimation under censored observations was considered in this study, further investigation of alternative computational strategies, such as EM-based algorithms, may improve numerical efficiency for more complex censoring schemes or higher-dimensional parameterizations.

Finally, multivariate extensions of the ECMO family and the incorporation of dependence structures through copula-based or hierarchical formulations represent promising avenues for future research. Such developments could substantially expand the scope of the proposed framework and enhance its usefulness in modern applied statistics.

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Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.
Conflict of interests
The authors declare that they have no conflicts of interest.

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