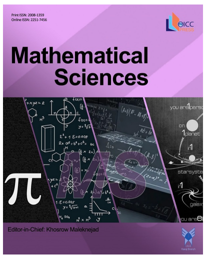

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Research Article

A New Method for Nonlinear Katugampola Fractional Differential Equations

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Abstract:

In this article, we propose a new wavelet method for solving nonlinear Katugampola fractional differential equations on an arbitrary interval. We have introduced a new wavelet, which we named as the Katugampola Gegenbauer wavelet (KGW), and constructed its new operational matrices of Katugampola fractional integrations as well as Katugampola fractional derivatives. The Katugampola Gegenbauer wavelet and its operational matrices are combined with the Adomian polynomials to propose a new method for the solution of nonlinear Katugampola fractional differential equations. The purpose of using the Adomian polynomials is to handle the nonlinearities in the equations. Furthermore, we have provided a detailed methodology for implementing the proposed approach to nonlinear Katugampola fractional differential equations. A detailed error analysis is also performed for the proposed method. The proposed method is implemented on several nonlinear Katugampola fractional differential equations to show the reliability, efficiency, and accuracy of the method.

Keywords: Katugampola fractional differential equations; Katugampola Gegenbauer wavelet; New operational matrices; Adomian polynomials

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1. Introduction

Fractional differential equations involve fractional derivatives and integrals, expanding the concept of integer order derivatives and integrals to non-integer orders, and are used to model more complex models with memory effects. During the last few years, many researchers have investigated fractional differential equations using different definitions of fractional derivatives and integrals. Some of the definitions include: Riemann-Liouville [1], Caputo [1], Hadamard [2], Fabrizio [3], Katugampola [4], reisz [5], and tempered [6] fractional derivatives. Fractional differential equations have applications in various fields of science and engineering. For example, quantum mechanics [4], measles epidemic [7], economic growth model [8], risk theory [9] and dynamical system [10]. For more application of fractional differential equations, we refer the readers to [1, 5].

The Katugampola fractional derivative is the generalization of both the Riemann–Liouville and Caputo derivatives. Katugampola fractional differential equations involve the combination of Katugampola fractional derivatives of different orders. Recently, it has attracted the attention of many researchers because of its vast applications in the field of science and engineering. Some of its application includes: fractional-order dynamical systems, fractional diffusion and wave equations, and mathematical modeling in engineering and physics. Srivastava et al. [11] studied the existence of positive solutions to the Katugampola fractional boundary value problem at resonance. The existence of mild solutions and optimal controls for a class of stochastic Hilfer-Katugampola fractional differential is studied [12]. The existence, uniqueness, equilibrium, and stability of the Hilfer-Katugampola fractional epidemic model for malware propagation are thoroughly examined in [13].

The study of wavelet theory dates back to the mid-20th century and had prominent contributions in mathematical studies [14]. Wavelets are a significant tool for approximating functions and obtaining numerical solutions for fractional differential equations. Wavelets are a family of orthogonal functions; constructed from the translation and dilation of a single function called the mother wavelet, which can be described by [15]:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi(\frac{x-b}{a}), \; a, b, \in \mathbb{R}, \; a \neq 0, \tag{1}$$

where a and b are called scaling and translation parameters, respectively. If $|a| < 1$, the wavelet (1) is the compressed version (smaller support in the time domain) of the mother wavelet and corresponds to mainly higher frequencies. On the contrary, while $|a| > 1$, the wavelet (1) corresponds to lower frequencies and has larger support in the time domain. The parameters are discretized as $a = a_0^{-j}$, $b = nb_0a_0^{-j}$, $a_0 > 1$, $b_0 > 0$ where $j, n \in \mathbb{Z}^+$. Then, we obtained the class of discrete wavelets

$$\psi_{j,n}(x) = |a_0|^{\frac{1}{2}}\psi(a_0^j x - nb_0), \tag{2}$$

where $\psi_{j,n}(x)$ form an orthonormal basis for $L^2(\mathbb{R})$. Several wavelets have been proposed to obtain numerical solutions for fractional differential equations. For example Haar wavelet [16], Generalized Sine-cosine wavelets [17], Legendre wavelets [18], and Krawtchouk wavelets [19].

The purpose of the present method is to introduce a new wavelet which we called "Katugampola Gegenbauer wavelet (KGW)", and propose a method for solving nonlinear Katugampola fractional differential equations. The proposed method is the combination of Katugampola Gegenbauer wavelet along with its new operational matrices and Adomian's polynomials. The purpose of utilizing the Adomian's polynomials is to handle the nonlinearities in the equation. The detailed methodology of the proposed method is provided for the readers' understanding. Error analysis of the proposed method is investigated. The proposed method makes it easy to deal with complex nonlinearities. Several examples have been considered to show the efficiency and accuracy of the KGW method.

The rest of the paper is organized as follows. In Section 2, we have given some basic mathematical preliminaries and properties from fractional calculus. In Section 3, we have introduced Katugampola Gegenbauer wavelets. In Section 4 new operational matrices of Katugampola fractional integration and Caputo-Katugampola fractional derivative are derived and constructed. In Section 5, the detailed procedure of implementation for the proposed method to solve the nonlinear Katugampola fractional differential equations is presented. Error analysis of the KGW method is provided in Section 6. In Section 7, we have presented numerical examples that show the precision of the proposed method. In Section 8, we conclude our work.

2. Preliminaries

We review basic definitions and results of fractional differentiation and integration in the sense of Katugampola[4].

Definition 2.1 *Consider $\alpha \in \mathbb{R}^+$. The Katugampola fractional integral $\left({}^{\kappa}\mathcal{I}_a^{\alpha,\rho} f\right)(x)$ is given by*

$$\left({}^{\kappa}\mathcal{I}_a^{\alpha,\rho} f\right)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \left(x^\rho - \xi^\rho\right)^{\alpha-1}\xi^{\rho-1} f(\xi)d\xi, \tag{3}$$

for $x > a$, if the integral exists.

This integral includes several known fractional integrals as special cases. When $\rho = 1$, it reduces to the Riemann–Liouville fractional integral, and as $\rho \to 0^+$, the Katugampola fractional integral coincides with the Hadamard fractional integral.

Remark 2.2 *For $\rho < 0 < 1$, the kernel becomes less singular, making it suitable for modeling nonlocal processes with weaker influence near the endpoints. When $\rho > 1$, the kernel becomes more singular, which is useful for capturing stronger memory effects. As $\rho \longrightarrow \infty$, the operator tends toward a local integral, thereby reducing the nonlocality of the process.*

Definition 2.3 *If $f(x) \in AC^z[a, b]$ then the Katugampola fractional derivative is defined, for $0 \leq a < x < \infty$, by*

$$\begin{aligned} &\left({}^{\kappa}\mathcal{D}_a^{\alpha,\rho} f\right)(x) \\ &= \frac{\rho^{\alpha-z+1}}{\Gamma(z-\alpha)}\left(x^{1-\rho}\frac{d}{dx}\right)^z \int_a^x \frac{\xi^{\rho-1}}{\left(x^\rho - \xi^\rho\right)^{\alpha-z+1}} f(\xi)d\xi, \end{aligned} \tag{4}$$

where $\rho > 0$ and $z = \lfloor\alpha\rfloor + 1$.

Definition 2.4 *Consider $\alpha > 0$ and $z = \lfloor\alpha\rfloor + 1$. The Caputo-Katugampola fractional derivative is defined by*

$$\begin{aligned} &\left({}^{\kappa}_{c}\mathcal{D}_a^{\alpha,\rho} f\right)(x) \\ &= \frac{\rho^{\alpha-z+1}}{\Gamma(z-\alpha)}\int_a^x \frac{\xi^{\rho-1}}{\left(x^\rho - \xi^\rho\right)^{\alpha-z+1}}\left(\xi^{1-\rho}\frac{d}{d\xi}\right)^z f(\xi)d\xi, \\ &= \left({}^{\kappa}\mathcal{I}_a^{z-\alpha,\rho}\delta_\rho^z f\right)(x), \end{aligned} \tag{5}$$

where $\delta_\rho^z = \left(\xi^{1-\rho}\frac{d}{d\xi}\right)^z$.

Lemma 2.5 *Consider $a \geq 0$. Let $\rho, \; \beta \in \mathbb{R}$, with $\rho > 0$, $\beta > -1$, then we have*

$$^{\kappa}\mathcal{I}_a^{\alpha,\rho}\left(x^\rho - a^\rho\right)^\beta = \frac{\rho^{-\alpha}\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\left(x^\rho - a^\rho\right)^{\beta+\alpha}. \tag{6}$$

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Lemma 2.6 *Consider $a \geq 0$. Let $\rho \in \mathbb{R}$, with $\rho > 0$ and $\beta \in \mathbb{N}$, with $\beta \geq \lceil \alpha \rceil$, then we have*

$$ {}^{\kappa}_{c}\mathcal{D}^{\alpha,\rho}_{a}(x^{\rho}-a^{\rho})^{\beta} = \frac{\rho^{\alpha}\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x^{\rho}-a^{\rho})^{\beta-\alpha}. \quad (7)$$

Lemma 2.7 *Let $\alpha > 0$ and $u \in AC^{z}_{\delta}[a,b]$, then*

$$\left({}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}\, {}^{\kappa}_{c}\mathcal{D}^{\alpha,\rho}_{a}\right)f(x) = f(x) - \sum_{i=0}^{z-1}\frac{\delta^{i} f(a)}{i!}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{i}. \quad (8)$$

Definition 2.8 *For each real number v_1 and v_2, the beta function $\mathcal{B}(v_1, v_2)$ is defined as*

$$\mathcal{B}(v_1,v_2) = \int_{0}^{1} t^{v_1-1}(1-t)^{v_2-1}dt, \quad (9)$$

$$= \frac{\Gamma(v_1)\Gamma(v_2)}{\Gamma(v_1+v_2)}. \quad \mathcal{R}(v_1) > 0,\ \mathcal{R}(v_2) > 0,$$

Definition 2.9 *For each real number v_1 and v_2, the regularized beta function $\mathbb{I}_x(v_1, v_2)$ is defined as*

$$\mathbb{I}_x(v_1,v_2) = \frac{\Gamma(v_1+v_2)}{\Gamma(v_1)\Gamma(v_2)}\int_{0}^{x} t^{v_1-1}(1-t)^{v_2-1}dt. \quad (10)$$

3. Katugampola Gegenbauer Wavelets

The shifted Gegenbauer polynomials, $C^{\sigma}_{m}(x)$ of order m, $m \in \mathbb{Z}^{+}$ are defined for $\sigma > -\frac{1}{2}$ on the interval $[0, 1]$ as

$$C^{\sigma}_{m}(x) = \frac{\Gamma(\sigma+0.5)}{\Gamma(2\sigma)}\sum_{l=0}^{m}\frac{(-1)^{m-l}\Gamma(2\sigma+m+l)}{l!(m-l)!\Gamma(\sigma+l+0.5)}x^{l}. \quad (11)$$

The shifted Gegenbauer polynomials satisfy the orthogonality condition

$$\int_{0}^{1} C^{\sigma}_{m}(x)C^{\sigma}_{q}(x)w^{\sigma}(x)dx = \tau^{\sigma}_{m}\delta_{mq}, \quad (12)$$

where $w^{\sigma}(x) = (1-x)^{\sigma-0.5}x^{\sigma-0.5}$ and $\tau^{\sigma}_{m} = \frac{2^{1-4\sigma}\Gamma(m+2\sigma)\pi}{(m+\sigma)m!(\Gamma(\sigma))^{2}}$.

We define the new Katugampola Gegenbauer wavelet (KGW) on interval $\left[a, (\rho + a^{\rho})^{1/\rho}\right)$, $\rho > 0$, as

$$\Lambda^{\sigma,\rho}_{n,m}(x) = \begin{cases} \frac{2^{\frac{k}{2}}}{\sqrt{\tau^{\sigma}_{m}}}C^{\sigma}_{m}\left(2^{k}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)-n\right), & u_n \leq x < v_n, \\ 0, & \text{elsewhere}, \end{cases} \quad (13)$$

where $u_n = \left(\frac{n\rho}{2^{k}} + a^{\rho}\right)^{\frac{1}{\rho}}$, $v_n = \left(\frac{(n+1)\rho}{2^{k}} + a^{\rho}\right)^{\frac{1}{\rho}}$, $m = 0, 1, \cdots, M-1$, $k = 0, 1, 2, \cdots$, and $n = 0, 1, 2, \cdots, 2^{k}-1$. For each σ a distinct family of wavelets can be obtained. Katugampola Legendre wavelets can be generated for $\sigma = \frac{1}{2}$. For $\sigma = 0$ and $\sigma = 1$, we get the first and second kind Katugampola Chebyshev wavelets, respectively.

Theorem 3.1 *Let $0 \leq a \leq x < (\rho + a^{\rho})^{1/\rho} < \infty$. The KGWs are orthogonal with respect to the weight function $w^{\sigma}_{\rho}(x) = \frac{x^{\rho-1}}{\rho^{2\sigma-1}}\left[(\rho(n+1)-2^{k}(x^{\rho}-a^{\rho}))\cdot(2^{k}(x^{\rho}-a^{\rho})-\rho n)\right]^{\sigma-0.5}$, that is*

$$\int_{u_n}^{v_n}\Lambda^{\sigma,\rho}_{n,m}(x)\Lambda^{\sigma,\rho}_{n,q}(x)w^{\sigma}_{\rho}(x)dx = \delta_{mq}.$$

Proof. Since the shifted Gegenbauer polynomials are orthogonal on the interval $[0, 1]$

$$\int_{0}^{1} C^{\sigma}_{m}(s)C^{\sigma}_{q}(s)w^{\sigma}(s)ds = \tau^{\sigma}_{m}\delta_{mq}, \quad (14)$$

Utilize the following substitution $s = 2^{k}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right) - n$, $ds = 2^{k}x^{\rho-1}dx$ in equation (14) to get equation (15) as

$$\int_{u_n}^{v_n}\frac{2^{\frac{k}{2}}}{\sqrt{\tau^{\sigma}_{m}}}C^{\sigma}_{m}\left(2^{k}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)-n\right)\frac{2^{\frac{k}{2}}}{\sqrt{\tau^{\sigma}_{q}}} \quad (15)$$
$$\times C^{\sigma}_{q}\left(2^{k}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)-n\right)$$
$$\times\frac{\left((\rho+\rho n)-2^{k}(x^{\rho}-a^{\rho})\right)^{\sigma-0.5}}{\left(2^{k}(x^{\rho}-a^{\rho})-\rho n\right)^{-\sigma+0.5}}\frac{x^{\rho-1}}{\rho^{2\sigma-1}}dx = \delta_{mq},$$

or

$$\int_{u_n}^{v_n}\Lambda^{\sigma,\rho}_{n,m}(x)\Lambda^{\sigma,\rho}_{n,q}(x)w^{\sigma}_{\rho}(x)dx = \delta_{mq}. \quad (16)$$

where $w^{\sigma}_{\rho}(x) = \frac{x^{\rho-1}}{\rho^{2\sigma-1}}[(\rho(n+1)-2^{k}(x^{\rho}-a^{\rho}))\cdot(2^{k}(x^{\rho}-a^{\rho})-\rho n)]^{\sigma-0.5}$.

4. Function Approximation and Operational Matrices

The KGW series can be used to approximate any function $y \in L^{2}[a, (\rho + a^{\rho})^{1/\rho})$ as follows:

$$y(x) \approx \sum_{n=0}^{2^{k}-1}\sum_{m=0}^{M-1} h_{n,m}\Lambda^{\sigma,\rho}_{n,m}(x) = \mathbf{H}^{T}\mathbf{\Lambda}^{\sigma,\rho}(x), \quad (17)$$

where $\mathbf{H}$ and $\mathbf{\Lambda}^{\sigma,\rho}(x)$ are of order $\hat{m} = 2^{k}M \times 1$, and are given as

$$\mathbf{H} = [h_{0,0}, h_{0,1}, \cdots, h_{0,M-1},$$
$$h_{1,0}, h_{1,1}, \cdots, h_{1,M-1}, \cdots,$$
$$h_{2^{k}-1,0}, h_{2^{k}-1,1}, \cdots, h_{2^{k}-1,M-1}]$$

$$\mathbf{\Lambda}^{\sigma,\rho}(x) = [\Lambda^{\sigma,\rho}_{0,0}(x), \Lambda^{\sigma,\rho}_{0,1}(x), \cdots, \Lambda^{\sigma,\rho}_{0,M-1}(x),$$
$$\Lambda^{\sigma,\rho}_{1,0}(x), \Lambda^{\sigma,\rho}_{1,1}(x), \cdots, \Lambda^{\sigma,\rho}_{1,M-1}(x), \cdots,$$
$$\Lambda^{\sigma,\rho}_{2^{k}-1,0}(x), \Lambda^{\sigma,\rho}_{2^{k}-1,1}(x), \cdots, \Lambda^{\sigma,\rho}_{2^{k}-1,M-1}(x)]$$

Expand KGW wavelets at the collocation points, $x_j = \left(a^{\rho} + \rho\left(\frac{2j-1}{2\hat{m}}\right)\right)^{\frac{1}{\rho}}$, $j = 1, 2, \cdots, \hat{m}$, to get the KGW matrix as

$$\boldsymbol{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}} = \left[\boldsymbol{\Lambda}^{\sigma,\rho}\left(a^{\rho}+\frac{\rho}{2\hat{m}}\right), \boldsymbol{\Lambda}^{\sigma,\rho}\left(a^{\rho}+\frac{3\rho}{2\hat{m}}\right), \dots, \boldsymbol{\Lambda}^{\sigma,\rho}\left(a^{\rho}+\frac{\rho(2\hat{m}-1)}{2\hat{m}}\right)\right]$$

or

$$\boldsymbol{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}} = \begin{bmatrix} \Lambda^{\sigma,\rho}_{0,0}\left(a^{\rho}+\frac{\rho}{2\hat{m}}\right) & \Lambda^{\sigma,\rho}_{0,0}\left(a^{\rho}+\frac{3\rho}{2\hat{m}}\right) & \cdots & \Lambda^{\sigma,\rho}_{0,0}\left(a^{\rho}+\frac{\rho(2\hat{m}-1)}{2\hat{m}}\right) \\ \Lambda^{\sigma,\rho}_{0,1}\left(a^{\rho}+\frac{\rho}{2\hat{m}}\right) & \Lambda^{\sigma,\rho}_{0,1}\left(a^{\rho}+\frac{3\rho}{2\hat{m}}\right) & \cdots & \Lambda^{\sigma,\rho}_{0,1}\left(a^{\rho}+\frac{\rho(2\hat{m}-1)}{2\hat{m}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda^{\sigma,\rho}_{0,M-1}\left(a^{\rho}+\frac{\rho}{2\hat{m}}\right) & \Lambda^{\sigma,\rho}_{0,M-1}\left(a^{\rho}+\frac{3\rho}{2\hat{m}}\right) & \cdots & \Lambda^{\sigma,\rho}_{0,M-1}\left(a^{\rho}+\frac{\rho(2\hat{m}-1)}{2\hat{m}}\right) \\ \Lambda^{\sigma,\rho}_{1,0}\left(a^{\rho}+\frac{\rho}{2\hat{m}}\right) & \Lambda^{\sigma,\rho}_{1,0}\left(a^{\rho}+\frac{3\rho}{2\hat{m}}\right) & \cdots & \Lambda^{\sigma,\rho}_{1,0}\left(a^{\rho}+\frac{\rho(2\hat{m}-1)}{2\hat{m}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda^{\sigma,\rho}_{2^k-1,M-1}\left(a^{\rho}+\frac{\rho}{2\hat{m}}\right) & \Lambda^{\sigma,\rho}_{2^k-1,M-1}\left(a^{\rho}+\frac{3\rho}{2\hat{m}}\right) & \cdots & \Lambda^{\sigma,\rho}_{2^k-1,M-1}\left(a^{\rho}+\frac{\rho(2\hat{m}-1)}{2\hat{m}}\right) \end{bmatrix}. \tag{18}$$

In particular, for $a = 1$, $\sigma = 0.3$, $\rho = 2.3$, $k = 2$, and $M = 3$, the KGW matrix is given as

$$\boldsymbol{\Lambda}^{0.7,1.5}_{12\times12} = \begin{bmatrix} 1.6238 & 1.6238 & 1.6238 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.6238 & 1.6238 & 1.6238 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.6238 & 1.6238 & 1.6238 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6238 & 1.6238 & 1.6238 \\ -1.7456 & 0 & 1.7456 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.7456 & 0 & 1.7456 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.7456 & 0 & 1.7456 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.7456 & 0 & 1.7456 \\ 0.3029 & -1.9469 & 0.3029 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3029 & -1.9469 & 0.3029 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3029 & -1.9469 & 0.3029 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3029 & -1.9469 & 0.3029 \end{bmatrix} \tag{19}$$

4.1 Operational matrices of Katugampola fractional integration

To derive the operational matrix of Katugampola fractional integration, we apply the Katugampola fractional integral operator of order α, on (13), to get

$$^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}\Lambda^{\sigma,\rho}_{n,m}(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{a}^{x}\left(x^{\rho}-\xi^{\rho}\right)^{\alpha-1}\xi^{\rho-1}\Lambda^{\sigma,\rho}_{n,m}(\xi)d\xi, \qquad u^{\rho}_{n} \le x < v^{\rho}_{n}.$$

For $x \in [u^{\rho}_{n}, v^{\rho}_{n})$, we have

$$^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}\Lambda^{\sigma,\rho}_{n,m}(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{2^{\frac{k}{2}}}{\sqrt{\tau^{\sigma}_{m}}} \times \int_{u^{\rho}_{n}}^{x}\left(x^{\rho}-\xi^{\rho}\right)^{\alpha-1}\xi^{\rho-1}C^{\sigma}_{m}\left(2^{k}\left(\frac{\xi^{\rho}-a^{\rho}}{\rho}\right)-n\right)d\xi.$$

By using (11), we get

$$^{\kappa}P^{\alpha,\sigma,\rho}_{n,m}(x) = \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^{\sigma}_{m}}\Gamma(2\sigma)} \times \sum_{l=0}^{m}\frac{(-1)^{m-l}2^{kl}\rho^{-l}\Gamma(2\sigma+m+l)}{l!(m-l)!\Gamma(\sigma+l+0.5)} \times \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{u^{\rho}_{n}}^{x}\left(x^{\rho}-\xi^{\rho}\right)^{\alpha-1}\xi^{\rho-1}\left(\xi^{\rho}-u^{\rho}_{n}\right)^{l}d\xi. \tag{20}$$

Substituting $t = \frac{\xi^{\rho}-u^{\rho}_{n}}{x^{\rho}-u^{\rho}_{n}}$ in (20) yields

$$^{\kappa}P^{\alpha,\sigma,\rho}_{n,m}(x) = \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^{\sigma}_{m}}\Gamma(2\sigma)} \times \sum_{l=0}^{m}\frac{(-1)^{m-l}2^{kl}\Gamma(2\sigma+m+l)}{\rho^{l+\alpha}(m-l)!\Gamma(\sigma+l+0.5)\Gamma(l+\alpha+1)}\left(x^{\rho}-u^{\rho}_{n}\right)^{l+\alpha}. \tag{21}$$

For $x \ge v^{\rho}_{n}$, we have

$$^{\kappa}P^{\alpha,\sigma,\rho}_{n,m}(x) = \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^{\sigma}_{m}}\Gamma(2\sigma)} \times \sum_{l=0}^{m}\frac{(-1)^{m-l}2^{kl}\rho^{-l}\Gamma(2\sigma+m+l)}{l!(m-l)!\Gamma(\sigma+l+0.5)}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \times \int_{u^{\rho}_{n}}^{v^{\rho}_{n}}\left(x^{\rho}-\xi^{\rho}\right)^{\alpha-1}\xi^{\rho-1}\left(\xi^{\rho}-u^{\rho}_{n}\right)^{l}d\xi. \tag{22}$$

Substituting $t = \frac{\xi^{\rho}-u^{\rho}_{n}}{x^{\rho}-u^{\rho}_{n}}$ in (22) yields

$$^{\kappa}P^{\alpha,\sigma,\rho}_{n,m}(x) = \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^{\sigma}_{m}}\Gamma(2\sigma)} \times \sum_{l=0}^{m}\frac{(-1)^{m-l}2^{kl}\rho^{-l}\Gamma(2\sigma+m+l)}{l!(m-l)!\Gamma(\sigma+l+0.5)} \times \frac{\rho^{-\alpha}}{\Gamma(\alpha)}\left(x^{\rho}-u^{\rho}_{n}\right)^{l+\alpha}\int_{0}^{w^{\rho}_{n}}(1-t)^{\alpha-1}t^{l}dt, \tag{23}$$

$$= \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^{\sigma}_{m}}\Gamma(2\sigma)} \times \sum_{l=0}^{m}\frac{(-1)^{m-l}2^{kl}\Gamma(2\sigma+m+l)}{\rho^{l+\alpha}(m-l)!\Gamma(\sigma+l+0.5)\Gamma(l+\alpha+1)} \times \left(x^{\rho}-u^{\rho}_{n}\right)^{l+\alpha}\mathbb{I}_{w^{\rho}_{n}}(l+1,\alpha), \tag{24}$$

where $^{\kappa}P^{\alpha,\sigma,\rho}_{n,m}(x) = {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{u^{\rho}_{n}}\Lambda^{\sigma,\rho}_{n,m}(x)$, $w^{\rho}_{n} = \frac{v^{\rho}_{n}-u^{\rho}_{n}}{x^{\rho}-u^{\rho}_{n}}$ and $\mathbb{I}_{w^{\rho}_{n}}$ is the regularized beta function.

In vector form,

$$
\begin{aligned}
&{}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{m,n}(x) = \\
&[\,{}^{\kappa}P^{\alpha,\sigma,\rho}_{0,0}(x), {}^{\kappa}P^{\alpha,\sigma,\rho}_{0,1}(x),\cdots, {}^{\kappa}P^{\alpha,\sigma,\rho}_{0,M-1}(x),\\
&{}^{\kappa}P^{\alpha,\sigma,\rho}_{1,0}(x), {}^{\kappa}P^{\alpha,\sigma,\rho}_{1,1}(x),\cdots, {}^{\kappa}P^{\alpha,\sigma,\rho}_{1,M-1}(x),\cdots,\\
&{}^{\kappa}P^{\alpha,\sigma,\rho}_{2^k-1,0}(x), {}^{\kappa}P^{\alpha,\sigma,\rho}_{2^k-1,1}(x),\cdots, {}^{\kappa}P^{\alpha,\sigma,\rho}_{2^k-1,M-1}(x)]^T.
\end{aligned}
$$

Expand ${}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{m,n}(x)$ at the collocation points x_j, $j = 1, 2, \cdots, \hat{m}$, to get the KGW operational matrix of fractional integration of order $\alpha > 0$, as

$$
{}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{\hat{m}\times\hat{m}} = \left[\,{}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}(x_1),\ {}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}(x_2),\cdots,\ {}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}(x_{\hat{m}})\right].
$$

For fix values of $\alpha = 1.51$, $\rho = 3.6$, $\sigma = 0.1$, $a = 2$, $k = 1$ and $M = 4$, we obtain

$$
{}^{\kappa}\mathbf{P}^{1.51,0.1,3.6}_{8\times8} = \begin{bmatrix}
0.0103 & 0.0543 & 0.1174 & 0.1951 & 0.2748 & 0.3317 & 0.3794 & 0.4215\\
0 & 0 & 0 & 0 & 0.0103 & 0.0543 & 0.1174 & 0.1951\\
-0.0138 & -0.0564 & -0.0874 & -0.0876 & -0.0607 & -0.0498 & -0.0435 & -0.0393\\
0 & 0 & 0 & 0 & -0.0138 & -0.0564 & -0.0874 & -0.0876\\
0.0108 & 0.0119 & -0.0327 & -0.0873 & -0.1042 & -0.1209 & -0.1364 & -0.1506\\
0 & 0 & 0 & 0 & 0.0108 & 0.0119 & -0.0327 & -0.0873\\
-0.0059 & 0.0244 & 0.0566 & 0.0356 & 0.0252 & 0.0216 & 0.0191 & 0.0173\\
0 & 0 & 0 & 0 & -0.0059 & 0.0244 & 0.0566 & 0.0356
\end{bmatrix}.
$$

4.2 Operational matrices of Caputo-Katugampola fractional derivative of order β

We need another operational matrix of Caputo-Katugampola fractional derivative while solving nonlinear Caputo-Katugampola fractional differential equations. For operational matrix of Caputo-Katugampola fractional derivative, we apply the Caputo-Katugampola fractional derivative of order β on (13), to get

$$
{}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{a}\Lambda^{\sigma,\rho}_{n,m}(x) = \frac{\rho^{\beta-z+1}}{\Gamma(z-\beta)}\int_a^x \frac{\xi^{\rho-1}}{(x^\rho-\xi^\rho)^{\beta-z+1}}\left(\xi^{1-\rho}\frac{d}{d\xi}\right)^z\Lambda^{\sigma,\rho}_{n,m}(\xi)d\xi,\quad u^\rho_n \le x < v^\rho_n.
$$

For $x \in [u^\rho_n, v^\rho_n)$, we have

$$
{}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{u^\rho_n}\Lambda^{\sigma,\rho}_{n,m}(x) = \frac{\rho^{\beta-z+1}}{\Gamma(z-\beta)}\frac{2^{\frac{k}{2}}}{\sqrt{\tau^\sigma_m}}\times \int_{u^\rho_n}^x \frac{\xi^{\rho-1}}{(x^\rho-\xi^\rho)^{\beta-z+1}}\left(\xi^{1-\rho}\frac{d}{d\xi}\right)^z C^\sigma_m\left(2^k\left(\frac{\xi^\rho-a^\rho}{\rho}\right)-n\right)d\xi.
$$

By using (11), we get

$$
{}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{u^\rho_n}\Lambda^{\sigma,\rho}_{n,m}(x) = \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^\sigma_m}\Gamma(2\sigma)}\times \sum_{l=0}^m \frac{(-1)^{m-l}2^{kl}\Gamma(2\sigma+m+l)}{\rho^l l!(m-l)!\Gamma(\sigma+l+0.5)}\times \frac{\rho^{\beta-z+1}}{\Gamma(z-\beta)}\int_{u^\rho_n}^x \frac{\xi^{\rho-1}}{(x^\rho-\xi^\rho)^{\beta-z+1}}\left(\xi^{1-\rho}\frac{d}{d\xi}\right)^z(\xi^\rho-a^\rho)^l d\xi. \tag{25}
$$

Since

$$
(\xi^{1-\rho}\frac{d}{d\xi})^z(\xi^\rho-u^\rho_n)^l = \frac{\rho^z\Gamma(l+1)(x^\rho-u^\rho_n)^{l-z}}{\Gamma(l-z+1)}. \tag{26}
$$

Using eq. (26) and substituting $t = \frac{\xi^\rho-u^\rho_n}{x^\rho-u^\rho_n}$ in (25) yields

$$
{}^{\kappa}R^{\beta,\sigma,\rho}_{n,m}(x) = \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^\sigma_m}\Gamma(2\sigma)}\times \sum_{l=0}^m \frac{(-1)^{m-l}2^{kl}\Gamma(2\sigma+m+l)}{\rho^{l-\beta}(m-l)!\Gamma(\sigma+l+0.5)\Gamma(l-\beta+1)}(x^\rho-u^\rho_n)^{l-\beta},\quad l \ge \lceil\beta\rceil. \tag{27}
$$

For $x \ge v^\rho_n$, we have

$$
{}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{u^\rho_n}\Lambda^{\sigma,\rho}_{n,m}(x) = \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^\sigma_m}\Gamma(2\sigma)}\times \sum_{l=0}^m \frac{(-1)^{m-l}2^{kl}\Gamma(2\sigma+m+l)}{\rho^l l!(m-l)!\Gamma(\sigma+l+0.5)}\times \frac{\rho^{\beta-z+1}}{\Gamma(z-\beta)}\int_{u^\rho_n}^{v^\rho_n} \frac{\xi^{\rho-1}}{(x^\rho-\xi^\rho)^{\beta-z+1}}\left(\xi^{1-\rho}\frac{d}{d\xi}\right)^z(\xi^\rho-a^\rho)^l d\xi. \tag{28}
$$

Using eq. (26) and substituting $t = \frac{\xi^\rho-u^\rho_n}{x^\rho-u^\rho_n}$ in (28) yields

$$
{}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{u^\rho_n}\Lambda^{\sigma,\rho}_{n,m}(x) = \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^\sigma_m}\Gamma(2\sigma)}\times \sum_{l=0}^m \frac{(-1)^{m-l}2^{kl}\rho^{-l}\Gamma(2\sigma+m+l)}{l!(m-l)!\Gamma(\sigma+l+0.5)}\times \frac{\rho^\beta\Gamma(l+1)}{\Gamma(z-\beta)\Gamma(l-z+1)}(x^\rho-u^\rho_n)^{l-\beta}\times \int_0^{w^\rho_n}(1-t)^{z-\beta-1}t^{l-z}dt, \tag{29}
$$

$$
{}^{\kappa}R^{\beta,\sigma,\rho}_{n,m}(x) = \frac{2^{\frac{k}{2}}\Gamma(\sigma+0.5)}{\sqrt{\tau^\sigma_m}\Gamma(2\sigma)}\times \sum_{l=0}^m \frac{(-1)^{m-l}2^{kl}\Gamma(2\sigma+m+l)}{\rho^{l-\beta}(m-l)!\Gamma(\sigma+l+0.5)\Gamma(l-\beta+1)}\times (x^\rho-u^\rho_n)^{l-\beta}\,\mathbb{I}_{w^\rho_n}(l-z+1,z-\beta),\quad l \ge \lceil\beta\rceil, \tag{30}
$$

where ${}^{\kappa}R^{\beta,\sigma,\rho}_{n,m}(x) = {}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{u_n}\Lambda^{\sigma,\rho}_{n,m}(x)$, $w^{\rho}_{n} = \frac{v^{\rho}_{n}-u^{\rho}_{n}}{x^{\rho}_{n}-u^{\rho}_{n}}$ and $\mathbb{I}_{w^{\rho}_{n}}$ is the regularized beta function.
In vector form,

$$\begin{aligned}
{}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}_{m,n}(x) =\\
&[{}^{\kappa}R^{\beta,\sigma,\rho}_{0,0}(x),\ {}^{\kappa}R^{\beta,\sigma,\rho}_{0,1}(x),\cdots,\ {}^{\kappa}R^{\beta,\sigma,\rho}_{0,M-1}(x),\\
&{}^{\kappa}R^{\beta,\sigma,\rho}_{1,0}(x),\ {}^{\kappa}R^{\beta,\sigma,\rho}_{1,1}(x),\cdots,{}^{\kappa}R^{\beta,\sigma,\rho}_{1,M-1}(x),\cdots,\\
&{}^{\kappa}R^{\beta,\sigma,\rho}_{2^k-1,0}(x),\ {}^{\kappa}R^{\beta,\sigma,\rho}_{2^k-1,1}(x),\cdots,\ {}^{\kappa}R^{\beta,\sigma,\rho}_{2^k-1,M-1}(x)]^{T}.
\end{aligned}$$

Expand ${}^{\kappa}R^{\beta,\sigma,\rho}_{n,m}(x)$ at the collocation points $x_j,\ j = 1,2,\cdots,\hat{m}$, to get the operational matrix of Caputo-Katugmapola fractional derivative of order $\beta > 0$, as

$${}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}_{\hat{m}\times\hat{m}} = \left[\ {}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}(x_1),\ {}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}(x_2),\cdots,\ {}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}(x_{\hat{m}})\right].$$

For fixed values of $\beta = 1.3,\ \rho = 1.9,\ \sigma = 0.7,\ k = 1$ and $M = 4$, we have

$${}^{\kappa}\mathbf{R}^{1.3,0.7,1.9}_{8\times 8} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
31.3049 & 67.5457 & 96.5810 & 122.2310 & 114.4363 & 100.1754 & 91.9458 & 86.1589\\
0 & 0 & 0 & 0 & 31.3049 & 67.5457 & 96.5810 & 122.2310\\
-158.5498 & -224.1333 & -151.8061 & 21.3470 & 66.7599 & 37.0952 & 25.5070 & 19.2189\\
0 & 0 & 0 & 0 & -158.5498 & -224.1333 & -151.8061 & 21.3470
\end{bmatrix}.$$

5. Procedure of implementation

In this section, we have discussed the proposed procedure of implementation of the KGW method for the solution of nonlinear Katugampola fractional differential equation with initial conditions. Consider the following nonlinear Katugampola fractional differential equation with variable coefficients

$$\begin{aligned}
&{}^{\kappa}_{c}\mathcal{D}^{\alpha,\rho}_{a}y(x) + q(x)\,{}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{a}y(x) \\
&\quad + b(x)y(x) - f(x,y(x)) = g(x),\\
&\quad a \le x \le (\rho + a^{\rho})^{1/\rho}, 1 < \alpha \le 2, 0 < \beta \le 1,
\end{aligned} \tag{31}$$

$$y(a) = h_1,\ {}^{\kappa}_{c}\mathcal{D}^{1,\rho}_{a}y(a) = h_2, \tag{32}$$

where the functions $q(x), b(x), g(x)$ are continuous on $[a,(\rho + a^{\rho})^{1/\rho}]$. The nonlinear mapping $y \mapsto f(x,y(x))$ is continuous in y for each $x \in [a,(\rho + a^{\rho})^{1/\rho}]$ and satisfies a Lipschitz condition in y: there exists $L > 0$ such that for all $x \in [a,(\rho + a^{\rho})^{1/\rho}]$ and all y_1, y_2,

$$\left|f(x,y_1) - f(x,y_2)\right| \le L\left|y_1 - y_2\right|.$$

We apply the Katugampola integral operator, ${}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}$of order α on (31) and use the initial conditions to get

$$\begin{aligned}
&y(x) + {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}q(x)\,{}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{a}y(x)\\
&\quad + {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}b(x)y(x) = {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}f(x,y(x))\\
&\quad + {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}g(x) + h_1 + h_2\Big(\frac{x^{\rho}-a^{\rho}}{\rho}\Big).
\end{aligned} \tag{33}$$

Approximate the unknown function $y(x)$ by the truncated Adomian series, and approximate the nonlinear functional $f(x,y(x))$ by using an Adomian polynomials as

$$y(x) = \sum_{l=0}^{i} y_l(x), \tag{34}$$

$$f(x,y(x)) = \sum_{l=0}^{i-1} A_l(y_0,y_1,\cdots,y_i),\ i\in\mathbb{N},$$

where $A_0, A_1,\cdots,A_{i-1}$ can be calculated by

$$A_l = \frac{1}{l!}\frac{d^l}{d\lambda^l}\Big[f\Big(\sum_{l'=0}^{l}\lambda^{l'}y_{l'}(x)\Big)\Big]\Big|_{\lambda=0}. \tag{35}$$

Substituting $y(x)$ and $f(t,y(x))$ from (34) into (33), we obtain

$$\begin{aligned}
&\sum_{l=0}^{i} y_l(x) + \sum_{l=0}^{i}\ {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}q(x)\,{}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{a}y_l(x)\\
&+\sum_{l=0}^{i}\ {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}b(x)y_l(x) = \sum_{l=0}^{i-1}\ {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}A_l(y_0,y_1,\cdots,y_i)\\
&\qquad + {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}g(x) + h_1 + h_2\Big(\frac{x^{\rho}-a^{\rho}}{\rho}\Big).
\end{aligned} \tag{36}$$

Since the zeroth approximation of $y(x)$ is

$$y_0(x) = {}^{\kappa}\mathcal{I}^{\alpha,\rho}_{a}g(x) + h_1 + h_2\Big(\frac{x^{\rho}-a^{\rho}}{\rho}\Big). \tag{37}$$

Approximate $g(x)$ and A_l by the truncated KGW series as

$$g(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} e_{n,m}\Lambda^{\sigma,\rho}_{n,m}(x), \tag{38}$$

$$A_l \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c^{l}_{n,m}\Lambda^{\sigma,\rho}_{n,m}(x).$$

Apply the Katugampola fractional integral operator of order α on equation (38), to get

$$^{\kappa}\mathcal{I}_a^{\alpha,\rho} g(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} e_{n,m}\, {}^{\kappa}\mathcal{I}_a^{\alpha,\rho}\Lambda_{n,m}^{\sigma,\rho}(x), \tag{39}$$
$$^{\kappa}\mathcal{I}_a^{\alpha,\rho} A_l \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c^l_{n,m}\, {}^{\kappa}\mathcal{I}_a^{\alpha,\rho}\Lambda_{n,m}^{\sigma,\rho}(x).$$

Expand (39) at the collocation points $x_j = \Big(a^\rho + \rho\big(\frac{2j-1}{2\hat{m}}\big)\Big)^{\frac{1}{\rho}}$, $j = 1, 2, \cdots, \hat{m}$, to get

$$\mathbf{G}_{1\times\hat{m}} = \mathbf{E}_{1\times\hat{m}}\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}}, \tag{40}$$
$$\mathbf{A}^l_{1\times\hat{m}} = \mathbf{C}_l\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}},$$

or

$$\mathbf{E}_{1\times\hat{m}} = \mathbf{G}_{1\times\hat{m}} \times (\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}})^{-1}, \tag{41}$$
$$\mathbf{C}_l = \mathbf{A}^l_{1\times\hat{m}} \times (\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}})^{-1},$$

where

$$\mathbf{E}_{1\times\hat{m}} = [e_{0,0}, e_{0,1}, \cdots, e_{0,M-1}, e_{1,0}, e_{1,1}, \cdots, e_{1,M-1}, \cdots, e_{2^k-1,0}, e_{2^k-1,1}, \cdots, e_{2^k-1,M-1}]^T,$$

$$\mathbf{C}_i = [c_{0,0}, c_{0,1}, \cdots, c_{0,M-1}, c_{1,0}, c_{1,1}, \cdots, c_{1,M-1}, \cdots, c_{2^k-1,0}, c_{2^k-1,1}, \cdots, c_{2^k-1,M-1}]^T,$$

$$\mathbf{G}_{1\times\hat{m}} = [g(x_1), g(x_2), \cdots, g(x_j)]^T,$$

and

$$\mathbf{A}^l_{1\times\hat{m}} = [A_l|_{(x_1)}, A_l|_{(x_2)}, \cdots, A_l|_{(x_j)}]^T.$$

Substitute the values of $\mathbf{E}_{1\times\hat{m}}$ and $\mathbf{C}_i$ in (39) to get $^{\kappa}\mathcal{I}_a^{\alpha,\rho} g(x)$ and $^{\kappa}\mathcal{I}_a^{\alpha,\rho} A_l(x)$. Put the values of $^{\kappa}\mathcal{I}_a^{\alpha,\rho} g$ in equation (37) to get the zeroth approximation $y_0(x)$ at the collocation points. We can write equation (36) as

$$y_{l+1}(x) + {}^{\kappa}\mathcal{I}_a^{\alpha,\rho} q(x)\, {}^{\kappa}_{c}\mathcal{D}_a^{\beta,\rho} y_l(x) + {}^{\kappa}\mathcal{I}_a^{\alpha,\rho} b(x) y_l(x) = {}^{\kappa}I_a^{\alpha} A_l(y_0, y_1, \cdots, y_i), \quad l = 0, 1, 2, \cdots, i-1. \tag{42}$$

Approximate $y_l(x)$ and $q(x)\,{}^{\kappa}D_a^{\beta}$ by the truncated KGW series as

$$y_l(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} h^l_{n,m}\Lambda^{\sigma,\rho}_{n,m}(x) \approx \mathbf{H}_l\, \mathbf{\Lambda}^{\sigma,\rho}(x)^T, \tag{43}$$

$$q(x)\, {}^{\kappa}_{c}\mathcal{D}_a^{\beta,\rho} y_l(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} Q^l_{n,m}\Lambda^{\sigma,\rho}_{n,m}(x) \approx \mathbf{Q}_l\, \mathbf{\Lambda}^{\sigma,\rho}(x)^T. \tag{44}$$

where $\mathbf{Q}_l$=$[Q^l_{0,0}, Q^l_{0,1}, \cdots, Q^l_{0,M-1}, Q^l_{1,0}, Q^l_{1,1}, \cdots, Q^l_{1,M-1}, \cdots, Q^l_{2^k-1,0}, Q^l_{2^k-1,1}, \cdots, Q^l_{2^k-1,M-1}]$.

Apply Katugampola differential operator of order β and Katugampola integral operator of order α on (43) and (44) respectively, we get

$$^{\kappa}_{c}\mathcal{D}_a^{\beta,\rho} y_l(x) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} h^l_{n,m}\, {}^{\kappa}_{c}\mathcal{D}_a^{\beta,\rho}\Lambda^{\sigma,\rho}_{n,m}(x) = \mathbf{H}_l\, {}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}_{n,m}(x)^T, \tag{45}$$

$$^{\kappa}\mathcal{I}_a^{\alpha,\rho} q(x)\, {}^{\kappa}_{c}\mathcal{D}_a^{\beta,\rho} y_l(x) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} Q^l_{n,m}\, {}^{\kappa}\mathcal{I}_a^{\alpha,\rho}\Lambda^{\sigma,\rho}_{n,m}(x) = \mathbf{Q}_l\, {}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{n,m}(x)^T. \tag{46}$$

Substitute (45) in (44) we get

$$q(x)\mathbf{H}_l\, {}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}_{n,m}(x)^T = \mathbf{Q}_l\, \mathbf{\Lambda}^{\sigma,\rho}(x)^T. \tag{47}$$

To get the value of $\boldsymbol{Q}_l$, we evaluate (47) at the collocation points to obtain

$$\boldsymbol{Q}_l = \mathbf{H}_l\, \tilde{\boldsymbol{q}}_{\hat{m}\times\hat{m}}\, {}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}_{\hat{m}\times\hat{m}}(\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}})^{-1}. \tag{48}$$

Equation (46) implies that

$$^{\kappa}\mathcal{I}_a^{\alpha,\rho} q(\bar{x})\, {}^{\kappa}_{c}\mathcal{D}_a^{\beta,\rho} y_l(\bar{x}) = \mathbf{H}_l\Big(\tilde{\boldsymbol{q}}_{\hat{m}\times\hat{m}}\, {}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}_{\hat{m}\times\hat{m}}\Big)(\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}})^{-1}\, {}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{\hat{m}\times\hat{m}}. \tag{49}$$

By following a similar procedure, we get

$$^{\kappa}\mathcal{I}_a^{\alpha,\rho} b(\bar{x}) y_l(\bar{x}) = \mathbf{H}_l\Big(\tilde{\boldsymbol{B}}_{\hat{m}\times\hat{m}}\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}}\Big)(\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}})^{-1}\, {}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{\hat{m}\times\hat{m}}, \tag{50}$$

where $\bar{x} = [x_1, x_2, \cdots, x_{\hat{m}}]$, $x_j = \Big(a^\rho + \rho\big(\frac{2j-1}{2\hat{m}}\big)\Big)^{\frac{1}{\rho}}$, $j = 1, 2, \cdots, \hat{m}$,

$$\tilde{\mathbf{q}}_{\hat{\mathbf{m}}\times\hat{\mathbf{m}}} = \begin{bmatrix} \tilde{q}(x_1) & 0 & \cdots & 0 \\ 0 & \tilde{q}(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{q}(x_{\hat{m}}) \end{bmatrix},$$
$$\tilde{\mathbf{B}}_{\hat{\mathbf{m}}\times\hat{\mathbf{m}}} = \begin{bmatrix} \tilde{B}(x_1) & 0 & \cdots & 0 \\ 0 & \tilde{B}(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{B}(x_{\hat{m}}) \end{bmatrix},$$

and the matrices $\Lambda^{\sigma,\rho}_{\hat{m}\times\hat{m}}$, ${}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}_{\hat{m}\times\hat{m}}$, ${}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{\hat{m}\times\hat{m}}$, are operational matrices derived in section 4.

Expand (42) at the collocation points and substitute (39), (49) and (50) to get

$$y_{l+1}(\bar{x}) + \mathbf{H}_l\Big(\tilde{\boldsymbol{q}}_{\hat{m}\times\hat{m}}\, {}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}_{\hat{m}\times\hat{m}} + \tilde{\boldsymbol{B}}_{\hat{m}\times\hat{m}}\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}}\Big)(\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}})^{-1}\, {}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{\hat{m}\times\hat{m}} = \mathbf{C}_l\, {}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{\hat{m}\times\hat{m}}, \tag{51}$$

or

$$y_{l+1}(\bar{x}) = \mathbf{C}_l\, {}^{\kappa}\mathbf{P}^{\alpha,\sigma,rho}_{\hat{m}\times\hat{m}} - \mathbf{H}_l\Big(\tilde{\boldsymbol{q}}_{\hat{m}\times\hat{m}}\, {}^{\kappa}\mathbf{R}^{\beta,\sigma,\rho}_{\hat{m}\times\hat{m}} + \tilde{\boldsymbol{B}}_{\hat{m}\times\hat{m}}\mathbf{\Lambda}^{\sigma,rho}_{\hat{m}\times\hat{m}}\Big)(\mathbf{\Lambda}^{\sigma,\rho}_{\hat{m}\times\hat{m}})^{-1}\, {}^{\kappa}\mathbf{P}^{\alpha,\sigma,\rho}_{\hat{m}\times\hat{m}}, \tag{52}$$

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where $l = 0, 1, \cdots, i-1$, $\mathbf{C}_l$ is given in (41), and the coefficients $\mathbf{H}_l$ can be obtained by solving (43) at each iteration. From (52), we get $y_1(x), y_2(x), \cdots, y_i(x)$ and then substitute in (34) to get the solution $y(x)$ at the collocation points.

6. Error Analysis

In this section, we performed the error analysis for the proposed KGW method.

Theorem 6.1 Let $0 \le a < b < \infty$. Suppose ${}^{\kappa}_{c}\mathcal{D}^{z\alpha,\rho}_{a} f \in C[a,b]$ for $z = 0,1,2,\cdots,M$. Then for $\alpha = (0,1]$, the Katugampola Taylor-Lagrange formula involving the Katugampola Caputo-type fractional derivatives writes

$$f(x) = \sum_{z=0}^{M-1} \frac{{}^{\kappa}_{c}\mathcal{D}^{z\alpha,\rho}_{a} f(a)}{\rho^{z\alpha}\Gamma(z\alpha+1)} \left(x^\rho - a^\rho\right)^{z\alpha} + \frac{{}^{\kappa}_{c}\mathcal{D}^{M\alpha,\rho}_{a} f(\xi)}{\rho^{-M\alpha}\Gamma(M\alpha+1)} \left(x^\rho - a^\rho\right)^{M\alpha}, \tag{53}$$

or

$$\left| f(x) - \sum_{z=0}^{M-1} \frac{{}^{\kappa}_{c}\mathcal{D}^{z\alpha,\rho}_{a} f(a)}{\rho^{z\alpha}\Gamma(z\alpha+1)} \left(x^\rho - a^\rho\right)^{z\alpha} \right| \le Z \frac{\left(x^\rho - a^\rho\right)^{M\alpha}}{\rho^{-M\alpha}\Gamma(M\alpha+1)}, \tag{54}$$

where $Z \ge \left| {}^{\kappa}_{c}\mathcal{D}^{M\alpha,\rho}_{a} f(\xi) \right|$.

Proof. For proof, we refer the readers to Ref. [20].

Theorem 6.2 Let $\bigvee^{\sigma,\rho}(x)$ be the subspace of $L^2[a, (\rho+a^\rho)^{1/\rho}]$ generated by $\left[\Lambda^{\sigma,\rho}_{0,0}(x), \Lambda^{\sigma,\rho}_{0,1}(x), \cdots, \Lambda^{\sigma,\rho}_{0,M-1}(x), \Lambda^{\sigma,\rho}_{1,0}(x), \Lambda^{\sigma,\rho}_{1,1}(x), \cdots, \Lambda^{\sigma,\rho}_{1,M-1}(x), \cdots, \Lambda^{\sigma,\rho}_{2^k-1,0}(x), \Lambda^{\sigma,\rho}_{2^k-1,1}(x), \cdots, \Lambda^{\sigma,\rho}_{2^k-1,M-1}(x)\right]$. Suppose that ${}^{\kappa}_{c}\mathcal{D}^{z\alpha,\rho}_{a} y(x) \in C[a, (\rho+a^\rho)^{1/\rho}]$ for $z = 0,1,2,\cdots,M$. If $y_M(x)$, given in eq. (17), is the best approximation of $y(x)$ from $\bigvee^{\sigma,\rho}(x)$, then we have

$$\|y(x) - y_M(x)\| \le \frac{Z((n+1)\rho + a^\rho)^{\frac{1-\rho}{2\rho}}}{2^{k(M\alpha+\sigma+\frac{1}{2\rho}-\frac{1}{2})}\rho^{\frac{1}{2}-\sigma}\Gamma(M\alpha+1)} \times \sqrt{\frac{\Gamma(\sigma+0.5)\Gamma(2M\alpha+\sigma+0.5)}{\Gamma(2M\alpha+2\sigma+1)}} \times \left[{}_2F_1\left(\gamma, \sigma+0.5; 2M\alpha+2\sigma+1; \varpi^\rho_n\right) \right]^{\frac{1}{2}}, \tag{55}$$

where $\varpi^\rho_n = \frac{v^\rho_n - u^\rho_n}{v^\rho_n}$, $\gamma = \frac{\rho-1}{\rho}$.

Let $g_M(x) = \sum\limits_{z=0}^{M-1} \frac{(x^\rho - a^\rho)^{z\alpha}}{\rho^{z\alpha}\Gamma(z\alpha+1)} \, {}^{\kappa}_{c}\mathcal{D}^{z\alpha,\rho}_{a} f(a)$, then from Theorem 6.1, we have

$$\left| y(x) - g_M(x) \right| \le Z \frac{\left(x^\rho - a^\rho\right)^{M\alpha}}{\rho^{M\alpha}\Gamma(M\alpha+1)}. \tag{56}$$

Since $y_M(x) \cong \sum\limits_{n=0}^{2^k-1}\sum\limits_{m=0}^{M-1} h_{n,m}\Lambda^{\sigma,\rho}_{n,m}(x) = \mathbf{H}^T\mathbf{\Lambda}^{\sigma,\rho}(x)$, is the best approximation of $y(x)$ from $\bigvee^{\sigma,\rho}(x)$, given in (17), and $g_M(x) \in \bigvee^{\sigma,\rho}(x)$, then we get

$$\begin{aligned} \|y(x) - y_M(x)\|^2 &\le \|y(x) - g_M(x)\|^2 \\ &\le \frac{Z^2}{\rho^{2M\alpha}(\Gamma(M\alpha+1))^2} \times \int_{u^\rho_n}^{v^\rho_n} \left(x^\rho - u^\rho_n\right)^{2M\alpha+\sigma-0.5} \times \left[(v^\rho_n - u^\rho_n) - (x^\rho - u^\rho_n)\right]^{\sigma-0.5} dx, \end{aligned} \tag{57}$$

or

$$\begin{aligned} \|y(x) - y_M(x)\|^2 &\le \|y(x) - g_M(x)\|^2 \\ &\le \frac{Z^2}{\rho^{2M\alpha}(\Gamma(M\alpha+1))^2} \times \int_{u^\rho_n}^{v^\rho_n} \left(v^\rho_n - x^\rho\right)^{\sigma-0.5} \times \left[(v^\rho_n - u^\rho_n) - (v^\rho_n - x^\rho)\right]^{2M\alpha+\sigma-0.5} dx, \end{aligned} \tag{58}$$

Substituting $\frac{v^\rho_n - x^\rho}{v^\rho_n - u^\rho_n} = t$, $dx = -\frac{\left(v^\rho_n - u^\rho_n\right)\left(v^\rho_n - (v^\rho_n - u^\rho_n)t\right)^{\frac{1-\rho}{\rho}}}{\rho} dt$ in (58), yields

$$\|y(x) - y_M(x)\|^2 \le \frac{Z^2 (v^\rho_n)^{-\gamma}(v^\rho_n - u^\rho_n)^{2M\alpha+2\sigma}}{\rho^{2M\alpha+1}(\Gamma(M\alpha+1))^2} \times \int_0^1 t^{\sigma+0.5-1}(1-t)^{(2M\alpha+2\sigma+1)-(\sigma+0.5)-1}(1-\varpi^\rho_n t)^{-\gamma} dt, \tag{59}$$

where $\varpi^\rho_n = \frac{v^\rho_n - u^\rho_n}{v^\rho_n}$, $|\varpi^\rho_n| < 1$ and $\gamma = \frac{\rho-1}{\rho}$.
Since

$$\mathcal{B}(b, c-b)\, {}_2F_1(a,b;c;z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx, \quad \mathcal{R}(c) > \mathcal{R}(b) > 0,\ |z| < 1, \tag{60}$$

where $\mathcal{B}$ is the beta function and ${}_2F_1$ is the hypergeometric function. If $a = \frac{\rho-1}{\rho}$, $b = \sigma+0.5$, $c = 2M\alpha+2\sigma+1$ and $z = \frac{v^\rho_n - u^\rho_n}{v^\rho_n}$, then using (60) in (59) gives

$$\|y(x) - y_M(x)\|^2 \le \frac{Z^2 (v^\rho_n)^{-\gamma}(v^\rho_n - u^\rho_n)^{2M\alpha+2\sigma}}{\rho^{2M\alpha+1}(\Gamma(M\alpha+1))^2} \times \frac{\Gamma(\sigma+0.5)\Gamma(2M\alpha+\sigma+0.5)}{\Gamma(2M\alpha+2\sigma+1)} \times {}_2F_1\left(\gamma, \sigma+0.5; 2M\alpha+2\sigma+1; \varpi^\rho_n\right), \tag{61}$$

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or

$$\|y(x)-y_M(x)\| \le \frac{Z((n+1)\rho+a^\rho)^{\frac{1-\rho}{2\rho}}}{2^{k(M\alpha+\sigma+\frac{1}{2\rho}-\frac{1}{2})}\rho^{\frac{1}{2}-\sigma}\Gamma(M\alpha+1)} \times \sqrt{\frac{\Gamma(\sigma+0.5)\Gamma(2M\alpha+\sigma+0.5)}{\Gamma(2M\alpha+2\sigma+1)}} \times \left[{}_2F_1(\gamma,\sigma+0.5;2M\alpha+2\sigma+1;\varpi_n^\rho)\right]^{\frac{1}{2}}. \tag{62}$$

Lemma 6.3 [21] If the derivatives of the nonlinear function $f(x, y)$ are uniformly bounded on a given interval $J \subset \mathbb{R}$, that is,

$$\sup_{x\in J} |f^{(k)}(x, y_0)| \le M,$$

the corresponding Adomian polynomials satisfy the inequality

$$A_n \le \frac{(n+1)^n M^{n+1}}{(n+1)!}.$$

This bound provides a quantitative estimate of the growth of the Adomian polynomials and serves as a useful analytical tool in establishing the convergence properties of the Adomian Decomposition Method.

Remark 6.4 Following Cherruault [22], the convergence of the Adomian Decomposition Method is ensured if the nonlinear operator $N(y)$ is Lipschitz continuous or satisfies a contraction condition, i.e., there exists a constant $L < 1$ such that

$$\|N(y_1)-N(y_2)\| \le L\|y_1-y_2\|.$$

Under this condition, the ADM series converges absolutely to the unique solution of the problem. In the present work, the nonlinear term $f(x, y(x))$ satisfies this condition, which guarantees the convergence of the decomposition series.

According to the analysis above, we conclude that the solution by the present method converges to the exact solution of (31), when k, M and i approach ∞.

7. Illustrative examples

In this section, we apply the proposed method to nonlinear Katugampola fractional differential equations.

Example 7.1 Consider the following nonlinear Katugampola fractional initial value problem

$$\begin{gathered} {}^{\kappa}_{c}\mathcal{D}^{\alpha,\rho}_{a} y(x) + q(x)\, {}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{a} y(x) - \log(y) = f(x), \\ x\in[a,b],\ 1<\alpha\le 2,\ 0<\beta\le 1, \\ y(a) = 2\Gamma(\alpha),\quad {}^{\kappa}_{c}\mathcal{D}^{1,\rho}_{a} y(a) = 0, \end{gathered} \tag{63}$$

where $b = (\rho + a^\rho)^{1/\rho}$, $\rho > 0$, $q(x) = \cos(x)$, $f(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}(\frac{x^\rho}{\rho})^{\nu-\alpha} + `q(x)\frac{\Gamma(\nu+1)}{\Gamma(\nu-\beta+1)}(\frac{x^\rho}{\rho})^{\nu-\beta} - \log\left((\frac{x^\rho-a^\rho}{\rho})^\nu + 2\Gamma(\alpha)\right)$, $\nu = \alpha + 4$.

The exact solution of (63) is $y(x) = (\frac{x^\rho-a^\rho}{\rho})^\nu + 2\Gamma(\alpha)$.

Exact and approximate solutions are shown in Figure 1, with $\rho = 2.4$, $\sigma = 1.9$, $k = 3$, $M = 3$, $i = 7$, and

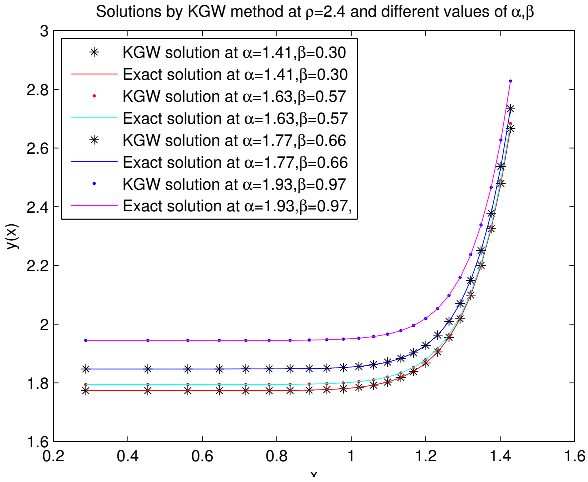

Figure 1. Numerical solutions of (63) when $k = 3$, $M = 3$, $i = 7$, $\rho = 2.4$, $\sigma = 1.9$ and at different values of α, β

different values of α, β . It shows that solution obtained by the proposed method are in full agreement with exact solution. Figure 2 (a) is used to plot the approximate solution by the present method along the exact solution and their corresponding absolute error. It shows that absolute error decreases with the increase of Adomian's iteration i and wavelet parameters k, M. For Figure 2 (a), we utilize the following values of the parameter $\rho = 1.4$, $\sigma = 0.5$, $\alpha = 1.33$, and $\beta = 0.61$. Table 1 and Table 2 shows that the maximum absolute and absolute error decreases by increasing wavelet parameter $\hat{m}$ and number of Adomian's iterations i for $\sigma = 1.2$, $\alpha = 1.93$, and $\beta = 0.97$. We have considered the different values of ρ for the construction of Table 1 and fixed value of $\rho = 1.4$ for Table 2.

Example 7.2 Consider the following nonlinear Katugampola fractional initial value problem

$$\begin{gathered} {}^{\kappa}_{c}\mathcal{D}^{\alpha,\rho}_{a} y(x) + q(x)\, {}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{a} y(x) - (\sin(y))^2 = f(x), \\ x\in[a,b],\ 1<\alpha\le 2,\ 0<\beta\le 1, \\ y(a) = \tfrac{\rho}{2},\quad {}^{\kappa}_{c}\mathcal{D}^{1,\rho}_{a} y(a) = 0, \end{gathered} \tag{64}$$

where $b = (\rho + a^\rho)^{1/\rho}$, $\rho > 0$, $q(x) = 2x$, $f(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}(\frac{x^\rho}{\rho})^{\nu-\alpha} + q(x)\frac{\Gamma(\nu+1)}{\Gamma(\nu-\beta+1)}(\frac{x^\rho}{\rho})^{\nu-\beta} - \sin\left((\frac{x^\rho-a^\rho}{\rho})^\nu + \frac{\rho}{2}\right)^2$, $\nu = \alpha + 1$. The exact solution of (64) is $y(x) = (\frac{x^\rho-a^\rho}{\rho})^\nu + \frac{\rho}{2}$.

Exact and numerical solutions by the KGW method are presented in graphical form in Figure 3 by using $\rho = 1.5$, $\sigma = 0.6$,, $k = 3$, $M = 3$, $i = 7$, and different values of α, β.. It shows that exact solution and the solution obtained by the present method are in good agreement. Figure 4 shows the graph of numerical solution by present method for various values of ρ. Figure 5 (a) shows that absolute error decreases by increasing iteration of the Adomian polynomials i and the wavelet parameters k, M. For Table 3, we utilize the foloowing value of the parameters $\rho = 1.6$, $\sigma = 1.4$, $\alpha = 1.44$, and

Table 1. Maximum absolute error when $\sigma = 2.7$, $\alpha = 1.63$, $\beta = 0.30$, and different values of ρ

ρ	$\hat{m} = 12$	$\hat{m} = 24$	$\hat{m} = 32$	$\hat{m} = 64$
	$i = 3$	$i = 4$	$i = 5$	$i = 6$
0.5	4.624977e-04	8.39334e-04	1.04222e-04	2.86145e-05
1.0	1.9696212e-03	4.66616e-04	8.17115e-05	1.86368e-05
1.5	3.0938136e-03	2.13355e-04	4.58914e-05	8.67903e-06
2.0	3.3539119e-03	1.40647e-04	2.45588e-05	4.32693e-06
2.5	3.3612523e-03	1.25825e-04	1.37870e-05	2.51601e-06

Table 2. Absolute error by proposed method when $\rho = 1.4$, $\sigma = 1.2$, $\alpha = 1.93$, and $\beta = 0.97$

	$\hat{m} = 12$	$\hat{m} = 16$	$\hat{m} = 24$	$\hat{m} = 32$	$\hat{m} = 64$
	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$
x	y_{abs}	y_{abs}	y_{abs}	y_{abs}	y_{abs}
0.1	2.50965e-05	2.47260e-06	4.51490e-07	1.63585e-08	1.29082e-09
0.2	4.17773e-05	1.65366e-06	2.86409e-07	9.12140e-08	3.96064e-09
0.3	2.33156e-05	6.12523e-06	9.06771e-07	1.82156e-07	1.11360e-08
0.4	6.85994e-06	5.65979e-06	2.77055e-06	5.17402e-07	2.97663e-08
0.5	8.80959e-05	2.27651e-06	6.64945e-06	9.32042e-07	6.38240e-08
0.6	2.50258e-04	2.13170e-06	1.52751e-06	1.79886e-06	1.19512e-07
0.7	6.76480e-04	3.34506e-05	3.86576e-05	2.83119e-06	2.17162e-07
0.8	1.31970e-03	1.17372e-04	7.39533e-05	3.58728e-06	3.50193e-07
0.9	2.03562e-03	2.28700e-04	1.41260e-04	4.80483e-06	5.38901e-07
1.0	3.28273e-03	4.27891e-04	2.72916e-04	6.51829e-06	6.80120e-07

Table 3. Absolute error by proposed method when $\rho = 1.6$, $\sigma = 1.4$, $\alpha = 1.44$, and $\beta = 0.22$

	$k = 2, M = 3$	$k = 3, M = 3$
	$i = 4$	$i = 5$
x	y_{abs}	y_{abs}
0.1	1.12575e-03	4.36295e-05
0.2	8.51581e-05	1.73707e-05
0.3	1.24799e-04	6.45032e-07
0.4	1.42275e-05	8.73419e-06
0.5	7.72692e-05	2.18924e-05
0.6	1.50927e-04	4.40773e-05
0.7	2.91744e-04	7.94300e-05
0.8	5.00552e-04	1.28296e-04
0.9	8.07404e-04	1.83740e-04
1.0	1.27650e-03	2.28602e-04

$\beta = 0.22$. It shows that absolute error decreases while increasing values of k, M, and i.

Example 7.3 Consider the nonlinear Katugampola fractional differential equation

$$
\begin{aligned}
&{}^{\kappa}_{c}\mathcal{D}^{2,\rho}_{0} y(x) + {}^{\kappa}_{c}\mathcal{D}^{\alpha,\rho}_{0} y(x) + {}^{\kappa}_{c}\mathcal{D}^{\beta,\rho}_{0} y(x) + y^3(x) = f(x),\\
&1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \quad x \in [0,1],\\
&y(0) = 0, \quad {}^{\kappa}\mathcal{D}^{1,\rho}_{0} y(0) = 0,
\end{aligned} \tag{65}
$$

where $f(x) = \frac{2\rho^2}{\Gamma(2)} x^\rho + \frac{2\rho^\alpha}{\Gamma(4-\alpha)} (x^\rho)^{3-\alpha} + \frac{2\rho^\beta}{\Gamma(4-\beta)} (x^\rho)^{3-\beta} + \left(\frac{1}{3}(x^\rho)^3\right)^3$. Exact solution of problem (65) is $y(x) = \frac{1}{3}(x^\rho)^3$.

Table 4. Comparison of absolute error by the proposed method with Haar wavelet method [23] when $\rho = 1.45$, $\sigma = 0.6$, $i = 11$, $\eta = 2$, $\alpha = 1.55$, $\beta = 0.75$.

x	Haar [23] $\hat{m} = 32$	y_{abs} $\hat{m} = 24$
0.1	1.25591e-07	1.08369e-05
0.3	1.55214e-06	2.92444e-07
0.5	1.55037e-05	3.66629e-07
0.7	5.49232e-05	1.25874e-07
0.9	1.11750e-04	5.09783e-06

Table 4 is used to compare the results obtained by the present method with the Haar wavelet method [23]. It shows that the absolute error by the KGWs method y_{abs} and Haar wavelets method for fixed values of $\rho = 1.45$, $\sigma = 0.6$, $\eta = 2$, $\alpha = 1.55$, and $\beta = 0.75$. According to the Table 3, our results are better than the results produced by the Haar wavelet method.

Remark 7.4 *The domain of variable x in the figures changes for various choices of ρ.*

8. Conclusion

We have introduced the Katugampola Gegenbauer wavelet specifically designed to solve nonlinear Katugampola fractional differential equations. We have constructed new operational matrices for Katugampola fractional integrals and derivatives. We have combined Katugampola Gegenbauer wavelets with the Adomian

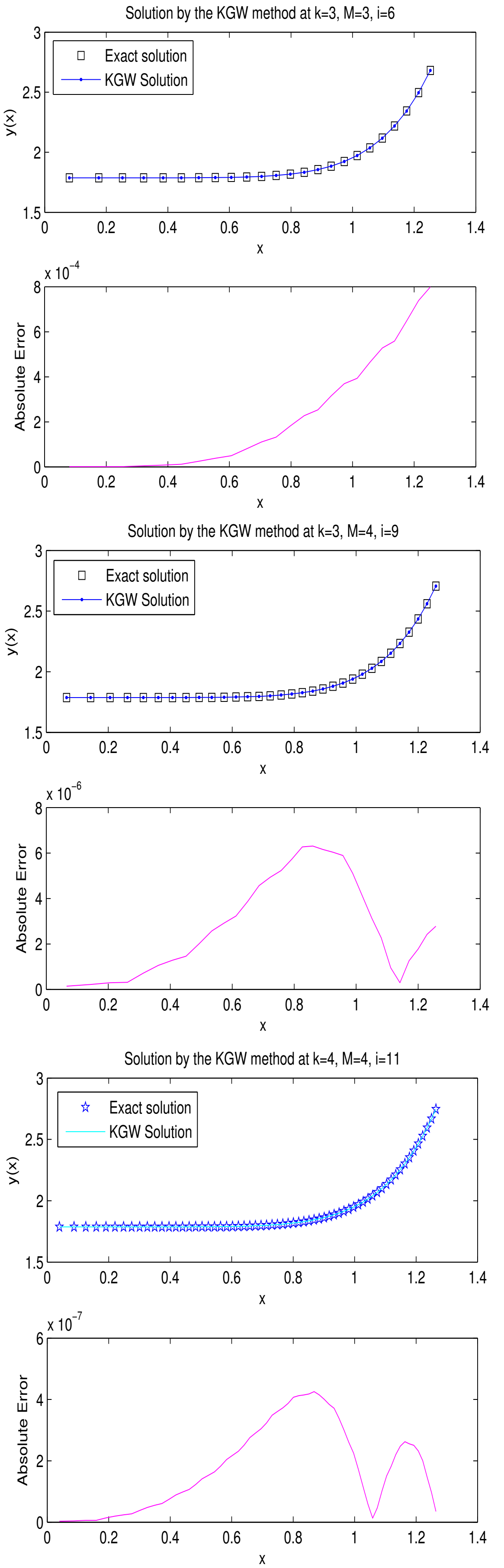

Figure 2. KGW solution, exact solution and absolute error when $\rho = 1.4$, $\sigma = 0.5$, $\alpha = 1.33$, $\beta = 0.61$ and at different values of k, M, and i

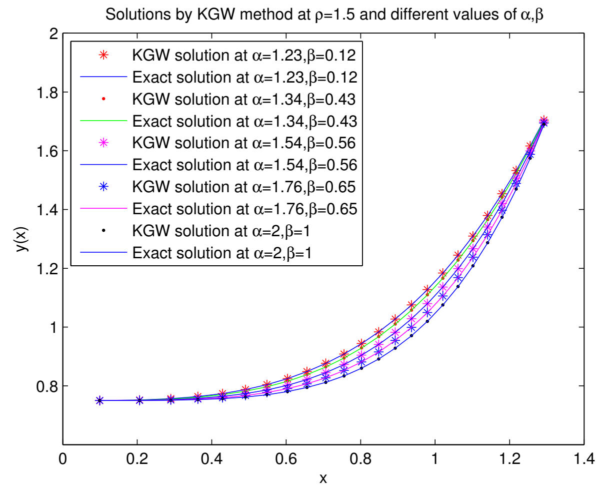

Figure 3. Numerical solutions of (64) when $k = 3$, $M = 3$, $i = 7$, $\rho = 1.5$, $\sigma = 0.6$, and at different values of α, β

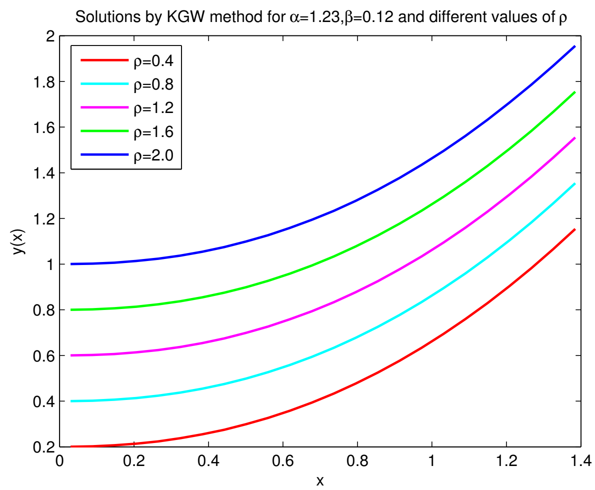

Figure 4. Numerical solutions for $\sigma = 3.1$, $\alpha = 1.23$, $\beta = 0.12$ and different values of ρ

polynomials to propose the KGW method for solving the non-linear Katugampola fractional differential equations. We have performed a detailed error analysis for the present method, which showed that the absolute error reduces while increasing the values of the parameters i, k and M. The detailed procedure for the implementation of the method is also provided.

Figure 1 and Figure 3 showed that the results produced by the present method are in good agreement with the exact solutions. According to Table 1, Table 2, Table 3, and figures Figure 2 (a), Figure 5 (a), the results produced by the KGW method getting more accurate when we increase the values of the parameters, i, k and M, as derived in the error analysis of the method. Furthermore, our method provides better results compared to the Haar wavelet method [23], as given in Table 4.

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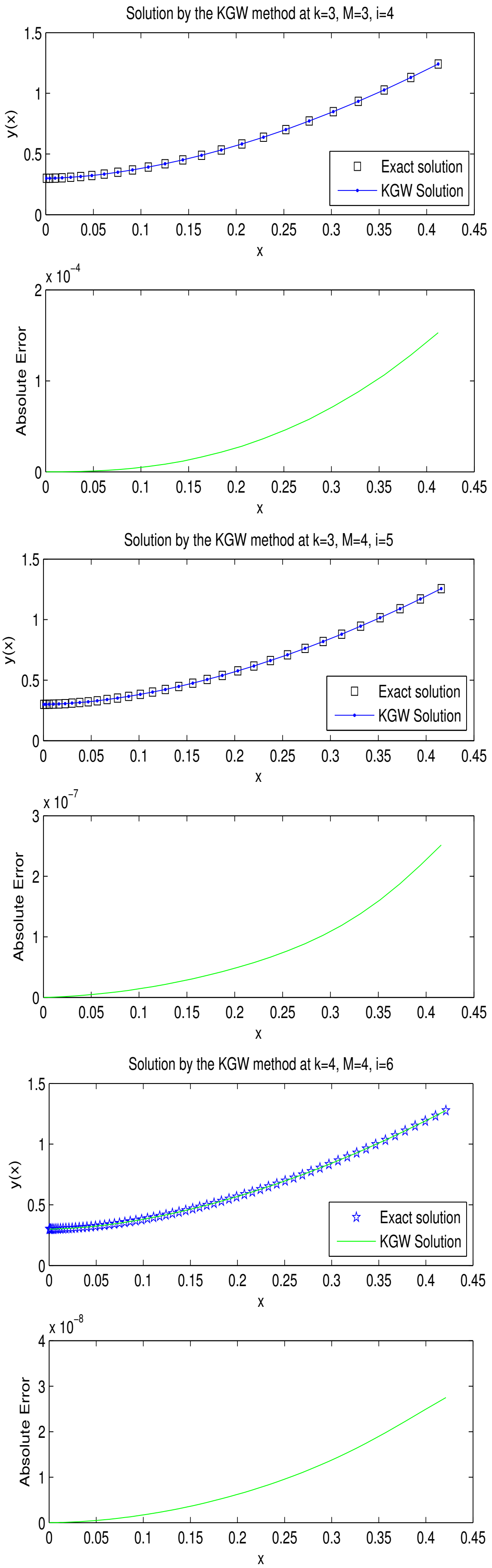

Figure 5. KGW solution, exact solution and absolute error when $\rho = 0.6$, $\sigma = 2.2$, $\alpha = 1.89$, $\beta = 0.41$ and at different values of k, M, and i

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Authors contributions

All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests

The authors declare that there are no conflicts of interest in this work.

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