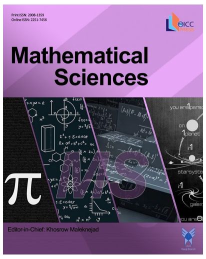

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Research Article

An Improved Extension of Unit Moment Exponential Distribution: Mathematics, Inference, and Applications

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Abstract:

This study contributes to the derivation of a bounded probability model for unit interval data analysis. The proposed model is named the *Sine Unit Moment Exponential (SUME)* distribution. This SUME model has the potential to model both the monotone increase and the bathtub shape for the hazard function. We investigate various statistical properties, including mixture representation, moments, quantile function, mean residual life function, and order statistics. The parameter estimation of the SUME distribution is discussed using six different estimation approaches. A comprehensive simulation study is performed to assess frequentist properties of the considered estimation methodologies. Two different datasets related to failure time and milk production are utilized to evaluate the practicality and flexibility of the proposed distribution over renowned unit interval distributions.

Keywords: Unit ME distribution; Sine-G family; Moments; Inference; Failure time; Milk production data

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1. Introduction

The probability models play critical roles in inference and data analysis in applied areas such as risk management, public health, agricultural sciences, management sciences, radiation, demography, and industrial reliability. In real-world situations, we often come across the uncertainty of phenomena that can be enumerated within specific limits. For instance, proportional qualities are frequently modeled as random variables that follow a certain distribution across the unit interval and range from 0 to 1. To accomplish this goal, a variety of probability models are created and investigated. For example, unit-Birnbaum-Saunders [1], unit-Modified Burr-III [2], unit-XLindley [3], unit-Xgamma [4], generalized exponential uniform [5], unit-Teissier [6], unit-Muth [7], transmuted modified power function [5] unit-Zeghdoudi [8], unit-Haq [9], unit-moment exponential [10], and bounded Lindley exponential distribution [11]

In many applied areas, the lack of appropriate probability models compels researchers to develop or adopt new models to better support their analyses. Accordingly, researchers have continuously concentrated on the derivation of new models by using different generalization techniques. A detailed review on this subject, such as those in [12], makes available comprehensive descriptions of probability models introduced utilizing various techniques. Due to the theoretical constraints of classical probability models and the absence of a "universal" model that can match all types of datasets, researchers usually depend on the selection of the most suitable model from the given collection.

Recently, researchers have introduced transformations or generalized families (G-family) to address these limitations. Among these various trigonometric families for continuous random variables hold a prominent place in distribution theory. Some examples of trigonometric functions-based families are: Sine-G [13], new Sine-G [14], Arctan-X [15], Marshall-Olkin Sine-G [16], Arctan-G [16], Cosine Topp-Leone-G [17], Cot-G [18], Arcsine-G [19], weighted tan-G [20],and hybrid cosine inverse Lomax-G [21]. A comprehensive review of trigonometric families is provided by [22]. The sine-G (S-G) family gets great attention among all these families primarily due to its appealing features.

- No additional parameter(s) to the baseline distribution,
- The use of the sine function imparts distinctive curvature to the resulting distributional shapes,
- The mathematical structure remains relatively simple and tractable, and
- It satisfied the first-order stochastic ordering $F(y) \leq G(y)$ for all $y \in R$, suggesting that, in contrast to the initial baseline distribution, the sin-G family may be used for different modeling applications.

Its cumulative distribution function (CDF) is given by

$$F(\gamma;\Omega) = \sin\left(\frac{\pi}{2} G(y;\Omega)\right), \quad y \in R, \tag{1}$$

where $G(\gamma;\Omega)$ denotes the CDF of an arbitrary continuous baseline distribution indexed by the parameter vector Ω. Differentiating equation (1) with respect to the random variable y yields the associated probability density function (PDF).

$$f(y;\Omega) = \frac{g(y;\Omega)}{2} \cos\left(\frac{\pi}{2} G(y;\Omega)\right), \quad y \in R, \tag{2}$$

where $g(y;\Omega)$ represents the PDF.
Recently, [10] developed the Unit Moment Exponential (UME) distribution. The PDF of the UME distribution can be presented as

$$f(y;\tau) = \frac{1}{\tau^2} \log\left(\frac{1}{y}\right) y^{\frac{1}{\tau}-1}, \quad \tau > 0 \;\&\; 0 < y < 1. \tag{3}$$

The corresponding CDF is

$$F(y;\tau) = \left(1 - \frac{\log(y)}{\tau}\right) y^{\frac{1}{\tau}}. \tag{4}$$

The UME distribution has found extensive application in the modeling of data on the unit interval because of its simplicity and manageable mathematics. The UME distribution, however, can also be characterized by a few shortcomings: it is less flexible in its ability to represent a wide range of shapes, especially skewness and tail behavior, and can fail when a dataset has strong asymmetric characteristics or shape properties. To overcome such shortcomings, we introduced a new trigonometric-based probability distribution called the Sine Unit Moment Exponential (SUME) distribution, a variant of the UME but with a sine transformation added into the model. This change enhances the flexibility of shapes, which means that SUME can be used to better fit skewness and tail behavior, and interpretable parameters are still present. Practically, regarding modeling, the proposed SUME distribution can provide high flexibility in modeling unit-interval data. It can represent positively skewed, negatively skewed, and symmetric distributions and therefore can capture a wide variety of empirical patterns without moving between model families. Further, its hazard rate functionality may take the forms of increasing, bathtub-shaped, and J-shaped, which makes the model applicable in various reliability and lifetime-type phenomena on the unit interval.

The key objectives of our study are.

- The main aim of this study is to extend the UME distribution using the Sine-G family of distributions. The SUME distribution is a flexible model due to the variable shapes of its density function. The SUME distribution can analyze positively and negatively skewed datasets. Notably, it has closed-form expressions of its CDF and supports variable shapes of the failure rate function, including upside-down bathtub and increasing forms.
- A detailed examination of the mathematical characteristics of the SUME distribution, including mixture representation, characterization based on two truncated moments, ordinary moments, quantile function, mean residual life function, entropy, extropy, order statistics, and record values.
- Five distinct parameter estimation approaches are employed for the SUME distribution. Specifically, the estimation approaches considered are maximum likelihood estimation (MLE (τ)), ordinary least squares estimation (OLSE (τ)), Anderson-Darling estimation (ADE (τ)), Cramer-von Mises estimation (CVME (τ)), Maximum Product Spacing (MPSE (τ)), and weighted least squares estimation (WLSE (τ)). In addition, a comprehensive Monte Carlo simulation study is conducted to evaluate the finite sample performance of the derived estimators.
- The SUME distribution demonstrates greater flexibility over considered unit-interval distributions, as illustrated through real-life data applications. We use two unit-interval datasets related to failure time and milk production.

The remainder of the paper is structured as follows. The new bounded distribution is introduced in Section 2. A detailed derivation and examination of the theoretical characteristics is performed in Section 3. In Section 4, the parameter estimation of the proposed distribution is obtained. The SUME distribution and competitive distribution are fitted in Section 5. In the end, the study is concluded in Section 6.

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2. Proposed SUME Distribution

In this section, we introduce the SUME distribution as an improved version of the UME distribution by inserting equations (3) and (4) into (1) and (2). The CDF of the SUME distribution is presented as

$$F(y;\tau) = \sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right), \quad \tau>0,\quad 0<y<1, \tag{5}$$

Remark 2.1 *Let $\tau_1,\tau_2>0$ be two values of parameter and fix $y\in(0,1)$. Put*

$$M(\tau)=\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}},\quad F(y;\tau)=\sin(M(\tau)).$$

First note that $M(\tau)>0$ for all $\tau>0$ because $\left(1-\frac{\log(y)}{\tau}\right)>0$. Differentiating $M(\tau)$ gives

$$\frac{dM}{d\tau}=\frac{\pi}{2}y^{\frac{1}{\tau}}\frac{(\log(y))^2}{\tau^3},$$

So $\frac{dM}{d\tau}>0$ for every $\tau>0$. Hence $M(\tau)$ is strictly increasing in parameter τ; in particular, if $\tau_1>\tau_2$ then $M(\tau_1)>M(\tau_2)$.

Because $F(y;\tau)=\sin(M(\tau))$, the monotonicity of F in is controlled by the monotonicity of *sin* on the interval between $M(\tau_1)$ and $M(\tau_2)$. Concretely,

$$F(y;\tau_1)>F(y;\tau_2)\Longleftrightarrow\sin(M(\tau_1))>\sin(M(\tau_2)).$$

Since sin is increasing exactly where cos > 0, a useful sufficient condition is:

If $M(\tau)$ for τ in the range between τ_2 and τ_1 stays inside an interval on which sinis increasing (for example, if $M(\tau)\in\left[-\frac{\pi}{2}+2k\pi,\frac{\pi}{2}+2k\pi\right]$ for some integer k), then $M(\pi_1)>M(\pi_2)$ implies $F(y;\tau_1)>F(y;\tau_2)$. In particular, if both $M(\tau_1)$ and $M(\tau_2)$ lie in $\left[0,\frac{\pi}{2}\right)$ (the principal interval where sin is increasing), then $\tau_1>_2$ implies $F(y;\tau_1)>F(y;\tau_2)$. For illustration, we plot the CDF curves and present them in Figure 1.

The PDF of the SUME distribution is presented as

$$f(y;\tau)=\frac{\pi}{2\tau^2}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\pi}\right)y^{\frac{1}{\pi}}\right),\quad \pi>0,\ 0<y<1. \tag{6}$$

The survival function and hazard rate function are

$$S(y;\tau)=1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right), \tag{7}$$

and

$$h(y;\tau)=\frac{\frac{\pi}{2\tau^2}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)}{1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)}. \tag{8}$$

The density function and failure rate function provide valuable insights into the analysis capabilities of a bounded distribution. Variable shapes of these measures represent the higher flexibility of the model. Figure 1 shows the PDF, CDF and hazard function for different choices of parameter τ, illustrating how τ affects distribution shape. Figure 1 indicates that parameter τ has an obvious impact on the density and hazard forms of the SUME distribution. Small τ values, when used in the density plot, will cause the curves to be skewed and highly skewed towards the left. As the τ grows, the mode converges to the value of zero, the peak is more defined, and the distribution is more concentrated on the lower end of the support.

The hazard functions indicate that the value of τ that is small results in a bathtub pattern where there is a high risk at the start, there is a decrease in the risk, followed by an increase in the risk towards the end. The earlier dip decreases, and the hazard is largely increasing with increasing τ. This, statistically, designates that the SUME distribution may come up with varying ageing behaviours, such as early failure with stabilization or a progressive increase in risk.

2.1 Alternative Form of PDF

In this section, we present the alternative form of the PDF using series and expansion. Utilizing this alternative form of the density function, various statistical properties can be easily derived. Using the series expansion of Cosine

$$\cos(P)=\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}P^{2i}.$$

So apply equation (6)

$$\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)=\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)^{2i}.$$

This gives

$$f(y;\tau)=\frac{\pi}{2\eta^2}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\times\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}\left[\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right]^{2i},$$

or more compactly

$$f(y;\tau)=\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}\left(\frac{\pi}{2}\right)^{2i+1}\frac{1}{\pi^2}\times\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}+\frac{2i}{\tau}-1}\left(1-\frac{\log(y)}{\tau}\right)^{2i}. \tag{9}$$

Now apply the binomial expansion to $\left(\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)^{2i}$

$$\left(1-\frac{\log(y)}{\tau}\right)^{2i}=\sum_{j=0}^{2i}(-1)^j\binom{2i}{j}\left(\frac{\log(y)}{\tau}\right)^j$$

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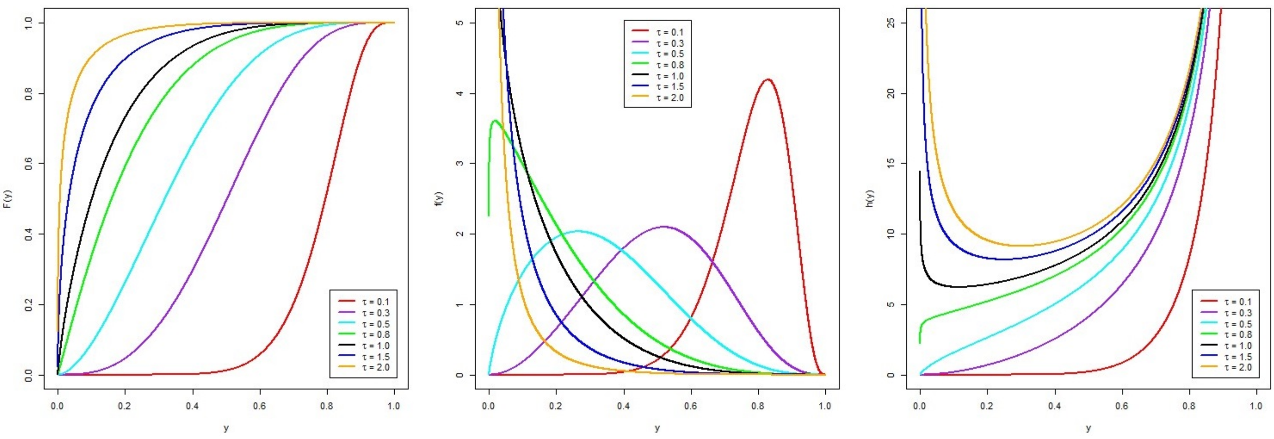

Figure 1. The CDF, PDF and hazard function curves based on different values of τ

Plug into the equation (9)

$$f(y;\tau) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i} \frac{(-1)^{i+j}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1} \times \frac{1}{\tau^{2+j}}\log\left(\frac{1}{y}\right)(\log(y))^{j} y^{\frac{1}{\tau}+\frac{2i}{\tau}-1}. \qquad (10)$$

3. Some Key Mathematical Properties

3.1 Characterization of SUME distribution

The characterization of SUME distribution based on the ratio of two truncated moments is given in Theorem 1, derived utilizing Glänzel's [23]. approach.

Theorem 3.1 *Let random variable X follow $SUME\,(y;\tau)$ supported on $(0,1)$. Define*

$$q_1(y) = 1,$$

$$q_2(y) = 1 - \sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right), \qquad 0<y<1.$$

Let

$$\eta(y) = \frac{\mathbb{E}[\,q_2(Y)\,|Y\ge y]}{\mathbb{E}[\,q_1(Y)\,|Y\ge y]}.$$

Then Y follows the SUME model with parameter if and only of

$$\eta(y) = \frac{1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)}{2}, \qquad 0<y<1.$$

Proof: For $q_1(y)=1$, we have

$$(1-F(y))\,\mathbb{E}\,[q_1(Y)\mid Y\ge y]$$
$$= \int_y^1 \frac{\pi}{2\tau^2}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)dy$$
$$= 1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right), \qquad 0<y<1$$

and similarly, it can be shown that

$$(1-F(y))\,\mathbb{E}\,[\,q_2(Y)\mid Y\ge y]$$
$$= \int_y^1 \left(1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right) \times \frac{\pi}{2\tau^2}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)dy.$$

$$(1-F(y))\,\mathbb{E}\,[\,q_2(Y)\mid Y\ge y]$$
$$= \frac{1}{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right)^2.$$

So that

$$\eta(y) = \frac{\mathbb{E}\,[q_2(Y)\mid Y\ge y]}{\mathbb{E}\,[\,q_1(Y)\mid Y\ge y]} = \frac{1}{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right),$$

As claimed

$$\eta(y)q_1(y) - q_2(y) \qquad (0<y<1),$$
$$= \frac{1}{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right) - \left(1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right)$$
$$= -\frac{1}{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right) \neq 0,$$

So the denominator is Glänzel's recovery formula is nonzero.

Conversely,

$$\acute{s}(y) = \frac{\acute{\eta}(y)q_1(y)}{\eta(y)q_1(y)-q_2(y)}.$$

Here $\eta(y) = \frac{1}{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right)$ so $\eta'(y) = -\frac{\pi}{4}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)$.

Putting in above equation

$$\acute{s}(y) = \frac{\frac{\pi}{4\tau}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\pi}\right)y^{\frac{1}{\pi}}\right)}{\frac{1}{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right)} = \frac{\frac{\tau}{2\tau^2}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)}{1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)},$$

The usual hazard. Integrating and choosing the constant so that $S(0^+)=1$ gives

$$s(y) = -\ln\left(1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right),$$

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Which implies $S(y) = 1 - \sin\left(\frac{\pi}{2}\left(1 - \frac{\log(y)}{\tau}\right) y^{\frac{1}{\tau}}\right)$ and hence identifies $F(y) = \sin\left(\frac{\pi}{2}\left(1 - \frac{\log(y)}{\tau}\right) y^{\frac{1}{\tau}}\right)$ and the density function

$$f(y) = \frac{\pi}{2\tau}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\cos\left(\frac{\pi}{2}\left(1 - \frac{\log(y)}{\tau}\right) y^{\frac{1}{\tau}}\right)$$

uniquely.

This completes the characterization.

3.2 Quantile function

The quantile function of the SUME distribution is given as

$$y_p = \exp\left\{\tau\left[1 - W\left(\frac{2\,archsin\,(p)}{\pi e}\right)\right]\right\}. \tag{11}$$

where $W(.)$ Is the Lambert W function, and this holds for $p \in (0, 1)$.

Proof: The quantile function can be obtained as

$$\sin\left(\frac{\pi}{2}\left(1 - \frac{\log(y)}{\tau}\right) y^{\frac{1}{\tau}}\right) = p,$$

and

$$\left(1 - \frac{\log(y)}{\tau}\right) y^{\frac{1}{\tau}} = \frac{2\,archsin\,(p)}{\pi}.$$

Let's set $y = e^z$, $\log(y) = z$, and $y^{\frac{1}{\tau}} = e^{\frac{z}{\tau}}$

Substitute into the above equation.

$$\left(1 - \frac{z}{\tau}\right) e^{\frac{z}{\tau}} = \frac{2\,archsin\,(p)}{\pi}.$$

Let's simplify further $u = \frac{z}{\tau}$, $z = \tau u$, and $y = e^{\tau u}$, then the equation becomes

$$(1 - u)\, e^u = \frac{2\,archsin\,(p)}{\pi}.$$

We now rearrange this equation to express it in a form suitable for using the Lambert W function.

$$(1 - u)\, e^{u-1} = \frac{2\,archsin\,(p)}{\pi e}$$

$v = 1 - u$, and $u = 1 - v$ and substitute into the above equation

$$ve^{-v} = \frac{2\,archsin\,(p)}{\pi e} \quad \text{and} \quad v = W\left(\frac{2archsin\,(p)}{\pi e}\right)$$

Now recall $v = 1 - u$, $u = 1 - W\left(\frac{2\,archsin(p)}{\pi e}\right)$, but $u = \frac{\log(y)}{\tau}$, so $\frac{\log(y)}{\tau} = 1 - W\left(\frac{2\,archsin(p)}{\pi e}\right)$. Now multiply both sides by

$$\log(y) = \tau\left[-W\left(\frac{2\,archsin\,(p)}{\pi e}\right)\right],$$

and the final closed-form expression using the Lambert W function is

$$y_p = \exp\left\{\tau\left[1 - W\left(\frac{2\,archsin\,(p)}{\pi e}\right)\right]\right\}.$$

3.3 Marginal moments

Using the definition of r-th moments of a random variable Y, we have

$$\mu_r' = \int_0^1 y^r f(y;\tau)\, dy,$$

Now using equation (10)

$$\mu_r' = \int_0^1 y^r \sum_{i=0}^{\infty}\sum_{j=0}^{2i} \frac{(-1)^{i+j}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1} \times \frac{1}{\pi^{2+j}}\log\left(\frac{1}{y}\right)(\log(y))^j y^{\frac{1}{\tau}+\frac{2i}{\tau}-1} dy,$$

Now exchange the sum and the integral (Fubini's theorem)

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{2i} \frac{(-1)^{i+j}}{(2i)!}\binom{2i\left(\frac{\pi}{2}\right)^{2i+1}\frac{1}{\tau^{2+j}}}{j} \times \int_0^1 y^{r+\frac{1}{\tau}+\frac{2i}{\tau}-1}\log\left(\frac{1}{y}\right)(\log(y))^j dy,$$

and the result is

$$\mu_r' = \frac{\pi}{2\tau^2}\sum_{i=0}^{\infty}\sum_{j=0}^{2i} \frac{(-1)^{i+j}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i} \times \frac{\Gamma(j+2)}{\tau^j\left(r + \frac{1}{\tau} + \frac{2i}{\tau}\right)^{j+2}}. \tag{12}$$

The primary four moments of the SUME distribution are derived, from which the standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) are obtained based on selected choices of the parameter τ. The corresponding numerical findings are summarized in Table 1.

3.4 Lower incomplete moments

The lower r-th incomplete moments of the SUME distribution can be computed as

$$\mu_r(t) = \int_0^t y^r f(y;\tau) dy,$$

Substitute the PDF

$$\mu_r(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i} \frac{(-1)^{i+j+1}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}\frac{1}{\pi^{2+j}} \times \int_0^t y^r \log\left(\frac{1}{y}\right)(\log(y))^j y^{\frac{1}{\tau}+\frac{2i}{\tau}-1} dy,$$

$$\mu_r(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i} \frac{(-1)^{i+j}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}\frac{1}{\tau^{2+j}} \times \int_0^t y^{r+\frac{1}{\tau}+\frac{2i}{\tau}-1}(-1)(\log(y))^{j+1} dy. \tag{13}$$

The integral term $\int_0^t y^{r+\frac{1}{\tau}+\frac{2i}{\tau}-1}(\log(y))^{j+1} dy$ is a known incomplete logarithmic gamma function

$$\int_0^t y^{a-1}(\log(y))^b dy = t^a \sum_{m=0}^{b} \frac{(-1)^m b!}{m!\,(b-m)!\,a^{m+1}}(\log(t))^{b-m}$$

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Table 1. Some descriptive statistics of SUME distribution based on some choices of parameter τ

τ	Mean	E (X^2)	E (X^3)	E (X^4)	SD	CV	CS	CK
0.1	0.78050	0.61960	0.49900	0.40700	0.10200	0.13070	-0.77770	3.60350
0.3	0.49900	0.27940	0.16930	0.10870	0.17420	0.34910	-0.08310	2.43720
0.5	0.33560	0.14520	0.07310	0.04070	0.18060	0.53810	0.42020	2.56180
0.8	0.19860	0.06460	0.02700	0.01320	0.15850	0.79830	1.04910	3.75310
1.0	0.14520	0.04070	0.01560	0.00720	0.14020	0.96580	1.43730	5.04950
1.5	0.07310	0.01560	0.00510	0.00210	0.10120	1.38540	2.39940	10.1194
2.0	0.04070	0.00720	0.00210	0.00080	0.07430	1.82430	3.41970	18.4409
2.5	0.02450	0.00380	0.00110	0.00040	0.05620	2.29380	4.53920	31.1285
3.0	0.01560	0.00210	0.00060	0.00020	0.04360	2.79950	5.77740	49.5565
3.5	0.01040	0.00130	0.00030	0.00010	0.03470	3.34470	7.14470	75.3239
4.0	0.00720	0.00080	0.00020	0.00010	0.02830	3.92960	8.64790	110.253
4.5	0.00510	0.00060	0.00010	0.00000	0.02340	4.56000	10.28930	156.342
5.0	0.00380	0.00040	0.00010	0.00000	0.01970	5.23390	12.07220	215.834

The equation (12) becomes

$$\mu_r(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{m=0}^{j+1} \frac{(-1)^{i+j+m}\binom{2i}{j}(j+1)!}{(2i)!m!(j+1-m)!\tau^{2+j}} \times \left(\frac{\pi}{2}\right)^{2i+1} \frac{t^{\left(r+\frac{1}{\tau}+\frac{2i}{\tau}\right)}}{\left(r+\frac{1}{\tau}+\frac{2i}{\tau}\right)^{m+1}} (\log(t))^{j+1-m} \quad (14)$$

The first incomplete moment is obtained by replacing $r = 1$ in equation (14) and follows as

$$\mu_1(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{m=0}^{j+1} \frac{(-1)^{i+j+m}\binom{2i}{j}(j+1)!}{(2i)!m!(j+1-m)!\tau^{2+j}} \times \left(\frac{\pi}{2}\right)^{2i+1} \frac{t^{\left(1+\frac{1}{\tau}+\frac{2i}{\tau}\right)}}{\left(1+\frac{1}{\tau}+\frac{2i}{\tau}\right)^{m+1}} (\log(t))^{j+1-m} \quad (15)$$

Table 1 reports some of our descriptive statistics obtained after making some choices of one parameter τ. It would be seen that the SUME distribution is more focused at zero as the value of τ grows. The mean and the moments reduce slowly and evenly, and the dispersion reduces. The relative dispersion is, however, more because the mean is small. The shape also changes. The skewness shifts to strong right skewness at larger τ rather than mild left skew at smaller τ, and the skewness of the tail is much heavier. Kurtosis increases at an accelerating rate, and the peak is stronger and more likely to have extreme outcomes. In a more concise formulation, the greater the τ, the narrower and taller the right-skewed and heavy-tailed distribution.

3.5 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves are important tools in income inequality analysis, characterizing the cumulative distribution of income or any other non-negative variable modeled by the SUME distribution. Let $\mu = E[X]$ be the mean of the distribution. The Bonferroni curve $B(p)$ is defined as

$$B(p) = \frac{1}{p\mu}\int_0^p xf(x)dx = \frac{1}{p\mu}\mu_1(p)$$

Using the expression for $\mu_1(p)$ given in equation (15)

$$B(p) = \frac{1}{p\mu}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{m=0}^{j+1} \frac{(-1)^{i+j+m}\binom{2i}{j}(j+1)!}{(2i)!m!(j+1-m)!\tau^{2+j}} \times \left(\frac{\pi}{2}\right)^{2i+1} \frac{t^{\left(1+\frac{1}{\tau}+\frac{2i}{\tau}\right)}}{\left(1+\frac{1}{\tau}+\frac{2i}{\tau}\right)^{m+1}} (\log(p))^{j+1-m}\right] \quad (16)$$

Using a similar approach to the one employed in the derivation of the Bonferroni curve, we derive the Lorenz curve $L(p)$, which is defined as:

$$L(p) = \frac{1}{\mu}\int_0^p xf(x)dx = \frac{1}{\mu}\mu_1(p)$$

Substituting the same expression for $\mu_1(p)$, we obtain the Lorenz curve as

$$L(p) = \frac{1}{\mu}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{m=0}^{j+1} \frac{(-1)^{i+j+m}\binom{2i}{j}(j+1)!}{(2i)!m!(j+1-m)!\tau^{2+j}} \times \left(\frac{\pi}{2}\right)^{2i+1} \frac{t^{\left(1+\frac{1}{\tau}+\frac{2i}{\tau}\right)}}{\left(1+\frac{1}{\tau}+\frac{2i}{\tau}\right)^{m+1}} (\log(p))^{j+1-m}\right] \quad (17)$$

3.6 Conditional moments

The conditional moments $E(Y^r|X > t)$ of SUME distribution can be derived as

$$E(Y^r|X > t) = \frac{1}{S(t)}\left[E(Y^r) - \int_0^t y^r f(y;\tau)dy\right].$$

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where $S(t)$ is given in equation (6), $E(Y^r)$ is given in equation (12), and $\int_0^t y^r f(y;\tau)\,dy$ in equation (14).

$$E(Y^r|X>t) = \frac{1}{\left[1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\pi}\right)y^{\frac{1}{\pi}}\right)\right]}$$
$$\times\left[\frac{\pi}{2\pi^2}\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i}\frac{\Gamma(j+2)}{\tau^j\left(r+\frac{1}{\tau}+\frac{2i}{\tau}\right)^{j+2}}\right.$$
$$-\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{m=0}^{j+1}\frac{(-1)^{i+j+m}\binom{2i}{j}(j+1)!}{(2i)!m!(j+1-m)!\tau^{2+j}}\left(\frac{\pi}{2}\right)^{2i+1}$$
$$\left.\times\frac{t^{\left(r+\frac{1}{\tau}+\frac{2i}{\tau}\right)}}{\left(r+\frac{1}{\tau}+\frac{2i}{\tau}\right)^{m+1}}(\log(t))^{j+1-m}\right]. \tag{18}$$

3.7 Generating Functions

Let $Y\sim SUME()$, The moment generating function (MGF) of a random variable y with density $f(y;)$ is defined as:

$$E(e^{ty}) = \int_0^1 e^{ty} f(y;)\,dy,$$

Using Series expansion as given in Equation (10), this leads to

$$E(e^{ty}) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}$$
$$\times\frac{1}{\tau^{2+j}}\int_0^1 e^{ty}\log\left(\frac{1}{y}\right)(\log(y))^j y^{\frac{1}{\tau}+\frac{2i}{\tau}-1}dy,$$

Now expressing e^{tx} using $e^{tx}=\sum_{n=0}^{\infty}\frac{t^n y^n}{n!}$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}\frac{1}{\tau^{2+j}}$$
$$\times\int_0^1\sum_{n=0}^{\infty}\frac{t^n y^n}{n!}\log\left(y^{-1}\right)(\log(y))^j y^{\frac{1}{\tau}+\frac{2i}{\tau}-1}dy,$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{(-1)^{i+j+1}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}\frac{1}{\tau^{2+j}}$$
$$\times\int_0^1 y^n\log(y)(\log(y))^j y^{\frac{1}{\tau}+\frac{2i}{\tau}-1}dy,$$

Using the known integral identity, $\int_0^1(\log(y))^n y^{\alpha-1}dy = (-1)^n\frac{\Gamma(n+1)}{\alpha^{n+1}}$

$$M_y(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{n=0}^{\infty}\frac{(-1)^i}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}$$
$$\times\frac{t^n}{n!}\frac{\Gamma(j+2)}{\tau^{j+2}\left(\frac{1+2i}{\tau}+n\right)^{j+2}}. \tag{19}$$

This is a fully evaluated series expression for the moment generating function of the density function $f(y;)$.

To find the characteristic function $\phi_y(t)$ of the SUME distribution, we follow the same derivation as above, replacing t with it:

$$\phi_y(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{n=0}^{\infty}\frac{(-1)^i}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}$$
$$\times\frac{(it)^n}{n!}\frac{\gamma(j+2)}{\tau^{j+2}\left(\frac{1+2i}{\tau}+n\right)^{j+2}}. \tag{20}$$

3.8 Mean Residual Life and Mean Past Life Function

In survival analysis and reliability theory, the Mean Residual Life (MRL) and Mean Past Life (MPL) functions are important tools for understanding the expected lifetime behavior of a random variable conditional on survival or failure up to a given time point. For a non-negative continuous random variable T with survival function $S(y)$, the MRL is defined as:

$$MRL = E[T-y/T>y] = \frac{\int_y^1(t)f(t)dt}{S(y)} - y$$

where $f(t)$ is the probability density function of the SUME distribution, and the upper limit is 1 due to the bounded support [0,1].

$$MRL = \frac{1}{S(y)}\left[\int_y^1 t\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}\right.$$
$$\left.\times\frac{1}{\tau^{2+j}}\log\left(\frac{1}{t}\right)(\log(t))^j y^{\frac{1}{\tau}+\frac{2i}{\tau}-1}dt\right]-y,$$
$$=\frac{1}{S(y)}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j+1}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}\right.$$
$$\left.\times\frac{1}{\tau^{2+j}}\int_y^1(\log(t))^{j+1}t^{\frac{1}{\tau}+\frac{2i}{\tau}+1-1}dt\right]-y.$$

The integral $\int_y^1(\log(t))^{j+1}t^{\frac{1}{\tau}+\frac{2i}{\tau}+1-1}dt$,is known as the upper incomplete logarithmic gamma integral, which has the analytical expression as

$$\int_y^1 y^{a-1}(\log(y))^b dy = (-1)^b\frac{b!}{\alpha^{b+1}}$$
$$-y^a\sum_{m=0}^{b}\frac{(-1)^m b!}{m!(b-m)!a^{m+1}}(\log(y))^{b-m}$$

So, the final expression for the MRL function becomes,

$$MRL = \frac{1}{S(y)}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j+1}}{(2i)!}\binom{2i}{j}\left(\frac{\pi}{2}\right)^{2i+1}\right.$$
$$\times\frac{1}{\tau^{2+j}}\left[(-1)^{j+1}\frac{(j+1)!}{\alpha^{j+2}}\right. \tag{21}$$
$$\left.\left.-y^{(\alpha)}\sum_{m=0}^{j+1}\frac{(-1)^m(j+1)!}{m!(j+1-m)!(\alpha)^{m+1}}(\log(y))^{j+1-m}\right]\right]-y$$

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Formally, the Mean Past Life at time y is defined as,

$$MPL = E\left[y - T/T \leq y\right] = y - \frac{\int_0^y (t) f(t) dt}{F(y)}$$

where, $\int_0^y (t)\, f(t) dt$ is the first incomplete moment and $F(y)$ is the distribution function of the SUME distribution, using first incomplete moment from equation (15), the MPL of SUME is provided as follows,

$$MPL = y - \frac{1}{F(y)}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{m=0}^{j+1}\frac{(-1)^{i+j+m}\binom{2i}{j}(j+1)!}{(2i)!m!(j+1-m)!\tau^{2+j}}\right.$$
$$\left.\times\left(\frac{\pi}{2}\right)^{2i+1}\frac{t^{\left(1+\frac{1}{\tau}+\frac{2i}{\tau}\right)}}{\left(1+\frac{1}{\tau}+\frac{2i}{\tau}\right)^{m+1}}(\log(y))^{j+1-m}\right] \quad (22)$$

3.9 Stochastic Ordering

Consider the $Y_1 \sim f(y;\tau_1)$ and $Y_2 \sim f(y;\tau_2)$ be the SUME-distributed random variables with density function (6). If $\tau_2 < \tau_1$, then $Y_1 \leq_{lr} Y_2$, i.e., Y_1 is smaller than Y_2 in the likelihood ratio order.

The density ratio is given by

$$\frac{f(y;\tau_1)}{f(y;\tau_2)} = \frac{\tau_2^2}{\tau_1^2}\times y^{\frac{1}{\tau_1}-\frac{1}{\tau_2}}\times\frac{\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau_1}\right)y^{\frac{1}{\tau_1}}\right)}{\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau_2}\right)y^{\frac{1}{\tau_2}}\right)}. \quad (23)$$

Now analyze the monotonicity ratio $\frac{f(y;\tau_1)}{f(y;\tau_2)}$.
Let us for $0 < y < 1$, and suppose $_1 < \tau_2$. Then

$$\frac{1}{\tau_1} > \frac{1}{\tau_2} \Rightarrow y^{\frac{1}{\tau_1}} < y^{\frac{1}{\tau_2}}, \quad \text{since } y \in (0,1).$$

Therefore, the numerator cosine argument grows faster than the denominator as $y \to 1$. Also, the power term $y^{\frac{1}{\tau_1}-\frac{1}{\tau_2}}$ is decreasing in $y \in (0,1)$. The composite effect of both is: A decreasing power function. A cosine ratio that decreases with y. means the ratio is decreasing.

3.10 Order Statistics

Let Y be a random variable following the SUME distribution with CDF $F(y)$ and PDF $f(y)$. When a random sample of size n is drawn from this distribution, the p-th order statistic (i.e., the p-th smallest value in the sample) has its distribution characterized by well-known formulas.

The CDF of the pth order statistic, denoted $F_p(y)$, is given by

$$F_p(y) = \sum_{j=p}^{n}\binom{n}{j}\,[F(y)]^{j-1}[1-F(y)]^{n-j}$$

Where, for the SUME distribution, the cumulative distribution function is expressed in equation (5).

Thus, the CDF of the pth order statistic becomes

$$F_p(y) = \sum_{j=p}^{n}\binom{n}{j}\left[\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right]^{j-1}$$
$$\times\left[1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right]^{n-j} \quad (24)$$

The PDF of the pth order statistic $f_p(y)$is derived from the general result given as

$$f_p(y) = \frac{n!}{(p-1)!(n-p)!}[f(y)][F(y)]^{p-1}[1-F(y)]^{n-p}$$

where $f(y)$ is the density of the original SUME distribution. For the SUME distribution, the PDF is given in equation (6).

Thus, the PDF of the pth order statistic becomes

$$f_p(y) = \frac{n!}{(p-1)!(n-p)!} \quad (25)$$
$$\times\left[\frac{\pi}{2\tau^2}\log\left(\frac{1}{y}\right)y^{\frac{1}{\tau}-1}\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right]$$
$$\times\left[\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right]^{p-1}$$
$$\times\left[1-\sin\left(\frac{\pi}{2}\left(1-\frac{\log(y)}{\tau}\right)y^{\frac{1}{\tau}}\right)\right]^{n-p}$$

This expression represents the PDF of the pth order statistic from a random sample of size n drawn from the SUME distribution. Notably, special cases of interest include:

- The minimum order statistic, obtained by setting $p = 1$, and
- The maximum order statistic, obtained by setting $p = n$.

These cases are particularly useful in reliability analysis and extreme value theory, where the behaviour of the smallest or largest observations in a sample is of primary interest.

4. Parameter Estimation

In this section, we discuss parameter estimation of the SUME distribution. Six competing estimators are examined in this study. The considered estimation approaches: likelihood-based $(\mathrm{MLE}(\tau))$, spacing-based $(\mathrm{MPSE}(\tau))$, goodness-of-fit based $(\mathrm{ADE}(\tau),\ \mathrm{CVME}(\tau))$, and least squares-based $(\mathrm{OLSE}(\tau), \mathrm{WLSE}())$. These optimization tasks were carried out using the quasi-Newton approach developed by (Fletcher, 2000) and implemented in the R program.

4.1 The MLE Method

In this subsection, the unknown parameter τ of the SUME distribution is estimated utilizing the $\mathrm{MLE}(\tau)$ framework. Suppose $y_1, y_2, \ldots, y_n \in (0,1)$ denote an independent random sample generated from the SUME density function given in equation (6). The corresponding log-likelihood function for the parameter τ, constructed from this sample, is expressed as follows:

$$l(\tau)=n\log\left(\frac{\pi}{2}\right)-2n\log(\tau)+\sum_{i=1}^{n}\log\left(\log\left(\frac{1}{y_i}\right)\right)+\left(\frac{1}{\tau}-1\right)\sum_{i=1}^{n}\log(y_i)\sum_{i=1}^{n}\log+\left(\cos\left(\frac{\pi}{2}\left(1-\frac{\log(y_i)}{\tau}\right)y_i^{\frac{1}{\tau}}\right)\right).$$

To find the MLE of $\hat{\tau}$, solves the score equation

$$\frac{\partial l(\tau)}{\partial\tau}=0.$$

Differentiating,

$$\frac{\partial l(\tau)}{\partial\tau}=-\frac{2n}{\tau}-\sum_{i=1}^{n}\frac{\log(y_i)}{\tau^2}-\sum_{i=1}^{n}\tan\left(\frac{\pi}{2}\left(1-\frac{\log(y_i)}{\tau}\right)y_i^{\frac{1}{\tau}}\right)\frac{\pi}{2}y_i^{\frac{1}{\tau}}\times\left(\frac{\log(y_i)}{\tau^2}-\left(1-\frac{\log(y_i)}{\tau}\right)\frac{\log(y_i)}{\tau^2}\right).$$

The parameter estimates based on MLE must be obtained numerically using Newton-Raphson, BFGS, or derivative-free approaches, since the score function is nonlinear and does not admit a closed-form solution.

The existence and uniqueness of the MLE can be obtained through numerical exploration and theoretical arguments under the following sampling schemes:

- Parameter space: $\tau>0$ is an open, convex interval, ensuring no boundary irregularities.
- Based on all simulated samples and various choices of parameter τ, the log-likelihood function is unimodal, displaying a unique maximum. This can be supported by plotting the log-likelihood function over $(0,\infty)$.
- The density function of SUME distribution is smooth in τ; $f(y;\tau)$ and $\partial f/\partial\tau$ are continuous for parameter $\tau>0$.
- The behavior of the log-likelihood function at boundaries: As $\tau\longrightarrow0^+$, $l(\tau)\to-\infty$ because of the term $-2n\log(\tau)$. As $\tau\longrightarrow\infty$, $l(\tau)\to-\infty$ due to the dominance of the cosine factor and the exponent term.

Under the standard regularity conditions, the MLE of parameter satisfies the classical large-sample results:

- Consistency: $\hat{\tau}\xrightarrow{p}\tau$ as $n\to\infty$.
- Asymptotic normality: $\sqrt{n}(\hat{\tau}-\tau)\xrightarrow{d}N\left(0,I^{-1}(\tau)\right)$, where is the Fisher information is $I(\tau)=-\mathbb{E}\left[\frac{\partial^2 l(\tau)}{\partial\tau^2}\right]$. In practice, $I(\tau)$ is approximated by the observed information $I(\hat{\tau})=-\left.\frac{\partial^2 l(\tau)}{\partial\tau^2}\right|_{\tau=\hat{\tau}}$. The asymptotic variance is then $Var(\hat{\tau})=\frac{1}{nI(\tau)}$.

Table 2. Numerical values of $I(\tau)$ and $Var(\hat{\tau})$

τ	n	$I(\tau)$	$Var(\hat{\tau})$
0.5	20	13.384403	0.0037357
0.5	75	13.384403	0.0009962
0.5	150	13.384403	0.0004981
0.5	400	13.384403	0.0001868
1.0	20	3.3461010	0.0149428
1.0	75	3.3461010	0.0039847
1.0	150	3.3461010	0.0019924
1.0	400	3.3461010	0.0007471
1.5	20	1.4871560	0.0336212
1.5	75	1.4871560	0.0089656
1.5	150	1.4871560	0.0044828
1.5	400	1.4871560	0.0016811

Here we compute some numerical values and present them in Table 2.

4.2 Method of MPS Estimation

The Maximum Product of Spacings ($MPSE(\tau)$) method, pioneered by [24], provides an alternative to the traditional maximum likelihood estimation. This estimation approach forms parameter estimates by maximizing the geometric Mean of the differences between consecutive CDF values evaluated at the ordered sample points. Consider $Y_1,Y_2,\ldots,Y_n$ represents an ordered random sample drawn from the SUME distribution with CDF $F_{SUME}(y;\tau)$. The $MPSE(\tau)$ estimates are attained by maximizing the following objective function:

$$Psi(y_i)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(y_i),$$

where $D_i(y_i)=F(y_{i:n})-F(y_{i-1:n})$, $F(y_{0:n})=0$ and $F(y_{n+1:1})=1$.

4.3 Method of AD Estimation

The Anderson-Darling (AD) test provides a useful method for estimating the parameters τ of the SUME distribution. Originally developed by [25], this approach is based on a goodness-of-fit statistic that gives extra weight to discrepancies in the tails of the distribution, making it especially suitable for data where tail behavior is important. The AD estimate, $\hat{\tau}_{AD}$ of the parameters τ is obtained by minimizing

$$AD(y_i;\tau)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(y_{i:n};\tau)+\log S(y_{n+1-i:n};\tau)\right].$$

4.4 Method of CVM Estimation

This method, introduced by [26], is based on minimizing the squared differences between the theoretical CDF $F(x;\tau)$ and the empirical distribution function (EDF) derived from sample data. Given an ordered sample

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$Y_1 \leq Y_2 \leq Y_3 \leq \cdots \leq Y_n$, CvM statistic is defined as

$$CvM\left(y_i;\tau\right) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(y_{i:n}\right) - \frac{2i-1}{2n}\right)^2.$$

The CvM estimates are obtained by maximizing $CvM\left(y_i;\tau\right)$.

4.5 Method of OLS Estimation

OLS Estimation is introduced by [27] as an alternative to the MLE approach; this method focuses on minimizing the discrepancies between the theoretical CDF and EDF. Let $Y_1 \leq Y_2 \leq Y_3 \leq \cdots \leq Y_n$ be the sample data be ordered. The corresponding theoretical CDF values are $F\left(y_{i:n}\right)$, and empirical CDF values are $\frac{i}{n+1}$. The OLS objective function is defined as:

$$OLS\left(y_i;\tau\right) = \sum_{i=1}^{n}\left[F\left(y_{i:n}\right) - \frac{i}{n+1}\right]^2.$$

The OLS estimators of τ are the values that minimize$OLS\left(y_i;\tau\right)$.

4.6 Method of WLS Estimation

The WLS method extends the ordinary least squares approach by assigning weights to the squared differences between the theoretical and empirical cumulative distribution values, giving more influence on observations with smaller variance. The WLS estimates $\hat{\tau}$ are obtained by minimizing the distance given below.

$$WLS\left(y_i;\tau\right) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{(n+1-i)\,i}\left[F\left(y_{i:n}\right) - \frac{i}{n+1}\right]^2.$$

4.7 Simulation Study

In this section, the efficiency of the estimate for all the methods presented above is checked by comparing Absolute Bias, Mean Relative Error and Mean Squared Error for sample sizes 20 (small), 75, 150 (Moderate) and 400 (large). Ten thousand replications are used for checking the efficiency of the estimate.

For simulation, data is generated for the SUME distribution using the random number generator given in equation (11).

The simulation findings show that all the estimation approaches show accuracy with an increase in sample size. For small values of the parameter (0.5-1.5), MLE, ADE, and CVME provide the most accurate estimates, while the MPSE approach performs relatively poorly. For moderate parameter values (2.5-3.0), the MLE and MPSE yield the most efficient approaches, while OLSE, ADE, CVME, and WLSE significantly overestimate the parameter. For a higher parameter value (5.0), MLE and MPSE approaches produce the most accurate estimates. Overall, the MLE estimation method emerges as the most reliable approach across all scenarios.

5. Application

In this section, we evaluate the practical applicability of the proposed SUME distribution by analyzing two real-life datasets: one related to failure time data and the other to milk production data. To ensure comprehensive comparative analysis, we benchmark the performance of the SUME distribution against several well-established probability models, including the unit Bilal (UB) distribution, unit moment exponential (UME) distribution, unit Zighdudi (UZ) distribution, transmuted Lindley distribution, unit Xgamma (UXg) distribution, unit XLindley (UXL) distribution, unit Lindley (UL) distribution, unit Haq (UH) distribution, and log-Xgamma (LXg) distribution. Model parameters are estimated using the method of maximum likelihood estimation. The quality of fit is assessed using various statistical criteria, including the log-likelihood ($\Re_1$), Akaike Information Criterion ($\Re_2$), Bayesian Information Criterion ($\Re_3$), and the Kolmogorov-Smirnov ($\Re_4$) test along with its corresponding p-values ($\Re_5$). Additionally, graphical tools such as probability density plots, cumulative distribution function plots, and Probability-Probability plot (P-P) plots are employed to visually assess the goodness-of-fit of each model.

5.1 Failure Time Data

This dataset consists of the observed lifetimes of 20 items subjected to a controlled life-testing experiment. The items were tested until failure, and the resulting times to failure were recorded and arranged in ascending order [28]. Such ordered lifetime data are frequently used in reliability analysis, survival modeling, and statistical inference to assess the durability and performance characteristics of products or materials. The ordered failure times for the 20 tested items are: 0.0009, 0.0040, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252, 0.1404, 0.1498, 0.1750, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, and 0.6143. We plot some non-parametric plots to explore the shape of the first dataset and present them in Figure 2. Details about the fitted distributions and their corresponding goodness-of-fit metrics are provided in Table 9.

Table 9 summarizes the estimated parameters, standard errors, and goodness-of-fit measures for several probability models fitted to failure time data from 20 tested items. Based on reported fitting statistics, the SUME model demonstrates the best overall fit. It achieves the highest $\Re_1$ value (17.0036) and the highest $\Re_5$value (0.9457), indicating a strong agreement between the model and the observed data. This higher performance is due to the flexibility of the SUME distribution in adapting to the skew and the overall shape of the empirical distribution, which enables it to be very close to the central tendency as well as the tail behavior of the observed failure times. The UME model also performs well, with $\Re_1 = 16.4329$ and $\Re_5 = 0.8120$, but falls slightly short of the SUME model on both measures. Similarly, the UB model attains a high $\Re_5$ value (0.7851).

Models such as UXG, TL, LXL, LXg, UL, UH, and UZ yield lower $\Re_5$values (ranging from 0.0526 to 0.6263) and smaller $\Re_1$ statistics compared to SUME, indicating a comparatively weaker fit. Hence, results indicate that

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Table 3. Parameter estimates via different estimation approaches based on the parameter (τ = 0.50)

	n	MLE (τ)	ADE (τ)	CVME (τ)	OLSE (τ)	WLSE (τ)	MPSE (τ)
$\hat{\tau}$	20	0.50089	0.50355	0.50308	0.50386	0.50378	0.51001
	75	0.49997	0.50038	0.50013	0.50034	0.50033	0.50366
	150	0.50019	0.50053	0.50041	0.50051	0.50047	0.50236
	400	0.49997	0.50001	0.49994	0.49998	0.49998	0.50097
|Bias|	20	0.00089	0.00355	0.00308	0.00386	0.00378	0.01001
	75	0.00003	0.00038	0.00013	0.00034	0.00033	0.00366
	150	0.00019	0.00053	0.00041	0.00051	0.00047	0.00236
	400	0.00003	0.00001	0.00006	0.00002	0.00002	0.00097
MRE	20	0.00178	0.00710	0.00615	0.00773	0.00756	0.02001
	75	0.00007	0.00075	0.00027	0.00069	0.00067	0.00731
	150	0.00038	0.00105	0.00081	0.00102	0.00093	0.00472
	400	0.00006	0.00002	0.00012	0.00004	0.00003	0.00193
MSE	20	0.00379	0.00402	0.00428	0.00433	0.00414	0.00403
	75	0.00096	0.00102	0.00108	0.00108	0.00102	0.00099
	150	0.00051	0.00054	0.00057	0.00057	0.00054	0.00052
	400	0.00018	0.00019	0.00020	0.00020	0.00019	0.00018

Table 4. Parameter estimates via different estimation approaches based on parameter (τ = 1.0)

	n	MLE (τ)	ADE (τ)	CVME (τ)	OLSE (τ)	WLSE (τ)	MPSE (τ)
$\hat{\tau}$	20	1.00151	1.00534	1.00419	1.00576	1.00537	1.01965
	75	1.00073	1.00058	1.00025	1.00066	1.00040	1.00806
	150	1.00119	1.00058	1.00053	1.00074	1.00050	1.00553
	400	1.00151	1.00534	1.00419	1.00576	1.00537	1.01965
|Bias|	20	0.00151	0.00534	0.00419	0.00576	0.00537	0.01965
	75	0.00073	0.00058	0.00025	0.00066	0.00040	0.00806
	150	0.00119	0.00058	0.00053	0.00074	0.00050	0.00553
	400	0.00151	0.00534	0.00419	0.00576	0.00537	0.01965
MRE	20	0.00151	0.00534	0.00419	0.00576	0.00537	0.01965
	75	0.00073	0.00058	0.00025	0.00066	0.00040	0.00806
	150	0.00119	0.00058	0.00053	0.00074	0.00050	0.00553
	400	0.00151	0.00534	0.00419	0.00576	0.00537	0.01965
MSE	20	0.01578	0.01587	0.01672	0.01690	0.01628	0.01681
	75	0.00407	0.00416	0.00443	0.00445	0.00420	0.00420
	150	0.00201	0.00207	0.00220	0.00220	0.00208	0.00207
	400	0.01578	0.01587	0.01672	0.01690	0.01628	0.01681

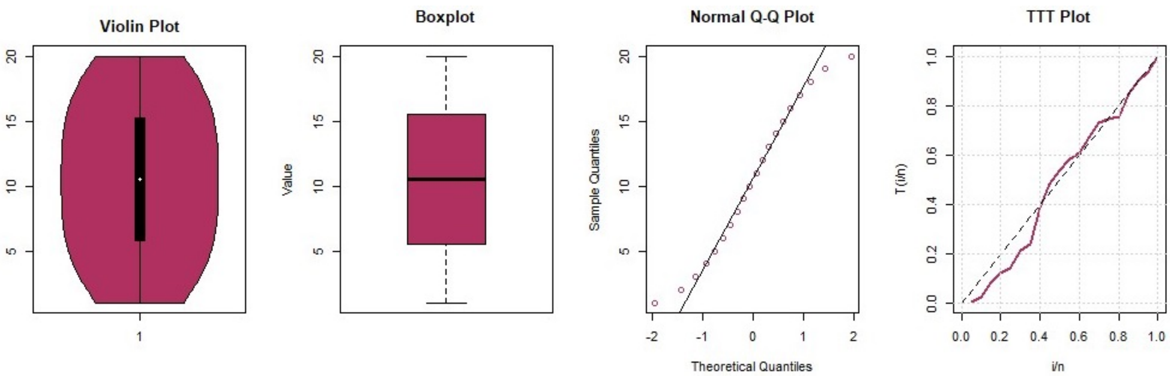

Figure 2. Graphical assessment of failure time data using violin, box, Q-Q, and TTT plots

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Table 5. Parameter estimates via different estimation approaches based on parameter ($\tau = 1.50$)

	n	MLE (τ)	ADE (τ)	CVME (τ)	OLSE (τ)	WLSE (τ)	MPSE (τ)
$\hat{\tau}$	20	1.52441	1.51600	1.51376	1.51617	1.51519	1.55191
	75	1.52129	1.50454	1.50382	1.50446	1.50411	1.53250
	150	1.51871	1.50151	1.50097	1.50128	1.50184	1.52530
	400	1.51891	1.50044	1.50002	1.50014	1.50247	1.52167
$\lvert$Bias$\rvert$	20	0.02441	0.01600	0.01376	0.01617	0.01519	0.05191
	75	0.02129	0.00454	0.00382	0.00446	0.00411	0.03250
	150	0.01871	0.00151	0.00097	0.00128	0.00184	0.02530
	400	0.01891	0.00044	0.00002	0.00014	0.00247	0.02167
MRE	20	0.01627	0.01067	0.00917	0.01078	0.01013	0.03461
	75	0.01419	0.00303	0.00255	0.00297	0.00274	0.02167
	150	0.01248	0.00101	0.00064	0.00085	0.00123	0.01687
	400	0.01261	0.00029	0.00001	0.00009	0.00164	0.01444
MSE	20	0.04254	0.03702	0.03925	0.03974	0.03759	0.04681
	75	0.01211	0.00971	0.01022	0.01025	0.01001	0.01301
	150	0.00585	0.00474	0.00501	0.00502	0.00542	0.00621
	400	0.00233	0.00172	0.00183	0.00183	0.00319	0.00244

Table 6. Parameter estimates via different estimation approaches based on parameter ($\tau = 2.50$)

	n	MLE (τ)	ADE (τ)	CVME (τ)	OLSE (τ)	WLSE (τ)	MPSE (τ)
$\hat{\tau}$	20	2.67236	2.57357	2.50466	2.50520	2.50489	2.71920
	75	2.65854	2.55700	2.50620	2.50648	2.53645	2.67181
	150	2.64367	2.55738	2.50821	2.50835	2.59275	2.64997
	400	2.61698	2.56424	2.51162	2.51167	2.73226	2.61907
$\lvert$Bias$\rvert$	20	0.17236	0.07357	0.00466	0.00520	0.00489	0.21920
	75	0.15854	0.05700	0.00620	0.00648	0.03645	0.17181
	150	0.14367	0.05738	0.00821	0.00835	0.09275	0.14997
	400	0.11698	0.06424	0.01162	0.01167	0.23226	0.11907
MRE	20	0.06894	0.02943	0.00186	0.00208	0.00196	0.08768
	75	0.06341	0.02280	0.00248	0.00259	0.01458	0.06873
	150	0.05747	0.02295	0.00328	0.00334	0.03710	0.05999
	400	0.04679	0.02569	0.00465	0.00467	0.09291	0.04763
MSE	20	0.18936	0.14067	0.10338	0.10272	0.10911	0.21095
	75	0.06683	0.03863	0.02961	0.02956	0.06548	0.07145
	150	0.04186	0.02089	0.01542	0.01541	0.09777	0.04403
	400	0.02500	0.01066	0.00586	0.00586	0.19728	0.02572

Table 7. Parameter estimates via different estimation approaches based on parameter ($\tau = 3.0$)

	n	MLE (τ)	ADE (τ)	CVME (τ)	OLSE (τ)	WLSE (τ)	MPSE (τ)
$\hat{\tau}$	20	3.12720	3.08896	2.99854	2.99785	3.02468	3.17337
	75	3.09390	3.08973	3.01535	3.01523	3.18809	3.10490
	150	3.06628	3.11150	3.03720	3.03716	3.39717	3.07051
	400	3.02563	3.13581	3.05035	3.05034	3.78742	3.02626
|Bias|	20	0.12720	0.08896	0.00146	0.00215	0.02468	0.17337
	75	0.09390	0.08973	0.01535	0.01523	0.18809	0.10490
	150	0.06628	0.11150	0.03720	0.03716	0.39717	0.07051
	400	0.02563	0.13581	0.05035	0.05034	0.78742	0.02626
MRE	20	0.04240	0.02965	0.00049	0.00072	0.00823	0.05779
	75	0.03130	0.02991	0.00512	0.00508	0.06270	0.03497
	150	0.02209	0.03717	0.01240	0.01239	0.13239	0.02350
	400	0.00854	0.04527	0.01678	0.01678	0.26247	0.00875
MSE	20	0.19814	0.19926	0.15243	0.15071	0.20687	0.21470
	75	0.04731	0.05843	0.04779	0.04761	0.25759	0.05029
	150	0.02201	0.03972	0.02832	0.02826	0.43111	0.02311
	400	0.00519	0.02691	0.01269	0.01268	0.81545	0.00536

Table 8. Parameter estimates via different estimation approaches based on parameter ($\tau = 5.0$)

	n	MLE (τ)	ADE (τ)	CVME (τ)	OLSE (τ)	WLSE (τ)	MPSE (τ)
$\hat{\tau}$	20	4.93515	5.53751	5.63527	5.62757	6.04438	4.94801
	75	4.99851	5.68394	5.85342	5.85094	6.67956	4.99858
	150	5.00000	5.68999	5.87072	5.86942	6.79018	5.00000
	400	5.00000	5.69263	5.88000	5.87950	6.88608	5.00000
|Bias|	20	0.06485	0.53751	0.63527	0.62757	1.04438	0.05199
	75	0.00149	0.68394	0.85342	0.85094	1.67956	0.00142
	150	0.00000	0.68999	0.87072	0.86942	1.79018	0.00000
	400	0.00000	0.69263	0.88000	0.87950	1.88608	0.00000
MRE	20	0.01297	0.10750	0.12705	0.12551	0.20888	0.01040
	75	0.00030	0.13679	0.17068	0.17019	0.33591	0.00028
	150	0.00000	0.13800	0.17414	0.17388	0.35804	0.00000
	400	0.00000	0.13853	0.17600	0.17590	0.37722	0.00000
MSE	20	0.10292	0.65314	0.96350	0.95304	1.87420	0.09593
	75	0.00076	0.50388	0.78924	0.78477	2.85602	0.00071
	150	0.00000	0.49259	0.78440	0.78209	3.20870	0.00000
	400	0.00000	0.48554	0.78324	0.78235	3.55811	0.00000

Table 9. Parameter estimates, SEs, and fit criteria for competing models using failure time data

Models	Parameter Estimates		Standard Errors		Fitting measures values				
	$\hat{\tau}$	$\hat{\delta}$	$SE\ (\hat{\tau})$	$SE\ (\hat{\delta})$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$
SUME	0.9818	-	0.1197	-	17.0036	-32.0072	-31.0115	0.1104	0.9457
UME	1.2806	-	0.2025	-	16.4329	-30.8659	-29.8702	0.1351	0.8120
UXG	5.1603		0.9862		15.5559	-29.1118	-28.1161	0.1603	0.6263
UZ	8.6174	-	1.2593	-	2.77730	-3.55470	-2.55890	0.2921	0.0526
TL	0.5112	-	0.1143	-	15.6167	-29.2334	-28.2376	0.1848	0.4481
UXL	0.5494	-	0.0925	-	13.3866	-24.7732	-23.7775	0.2172	0.2615
LXg	0.8459	-	0.1265	-	13.4969	-24.9940	-23.9982	0.2188	0.2540
UL	4.6365	-	0.9012	-	15.0695	-28.1389	-27.1432	0.4002	0.0021
UH	0.6796	-	0.1073		12.1687	-22.3373	-21.3416	0.2560	0.1211
UB	0.1501	2.3020	0.4626	7.0974	16.3691	-28.7382	-26.7468	0.1390	0.7851

the SUME model provides the most accurate and robust representation of the observed survival data, making it the most suitable choice among the models considered.

5.2 Milk Production data

The second dataset reports the total yield from the first lactation of 107 cows belonging to the SINDI breed. These animals are raised on the Carnaúba farm, which is part of Agropecuária Manoel Dantas Ltda (AMDA), located in Taperoá City, Paraíba, Brazil. The data were originally analyzed by [29] and provided as follows:

0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, and 0.2747.

We plot some non-parametric plots to explore the shape of the second dataset and present them in Figure 4. Details about the fitted distributions and their corresponding goodness-of-fit metrics are provided in Table 10.

The Table 10 summarizes the parameter estimates, standard errors, and goodness-of-fit metrics for various probability models applied to the milk production data from the first birth of 107 SINDI cows raised on the Carnaúba farm in Taperoá, Brazil. Among the models evaluated, the SUME model demonstrates the best overall performance, achieving the highest $\Re_1$ value (23.9025) and the highest $\Re_5$ value (0.5021), indicating a superior fit to the data. This higher success can be explained by the fact that the SUME distribution is flexible enough to capture the asymmetry and variability that exist in the production of milk measurements and can therefore closely track both central tendencies and deviations in the data. The UME and UZ models also show reasonable fits, though with lower $\Re_1$ and $\Re_5$ values compared to SUME. On the other hand, the LXL, UL, UH, and LXg models display notably poor performance, especially with $\Re_5$ values of 0.0000, indicating they fail to adequately capture the characteristics of the milk production data. Notably, the LXg model even shows a negative $\Re_1$ value, suggesting a poor fit. Hence, the SUME model provides the most accurate and stable fit for this dataset, making it the most suitable for modeling milk production among the models assessed.

6. Conclusion

In this study, we introduced a new probability distribution to analyze unit interval datasets. The new distribution is named the Sine Unit Moment Exponential distribution. The new model contains increasing and bathtub failure rate shapes. We compute several mathematical characteristics, including characterization, mixture representation, moments, quantile function, mean residual life function, and order statistics. The model is estimated to use five renowned estimation approaches. A detailed simulation study was utilized to illustrate the behavior of these derived estimators. In the end, the performance of the proposed distribution is evaluated using two different datasets, one related to failure time and the second about milk production. The proposed distribution efficiently analyzed these datasets as compared to the considered competitive distributions.

7. Future Work

- Future research can investigate exploring alternative forms of estimation, such as Bayesian estimation, robust estimation, or regularized techniques. These techniques can increase stability, especially when the parameter settings are highly skewed.
- The SUME model can be extended to regression models (e.g., accelerated failure time models, proportional hazards models) so that practitioners can use covariates and apply sit in the framework of survival analysis.
- The SUME distribution could apply to dependent lifetime analysis, or joint risk structure, due to the development of a multivariate or a copula-based version of the SUME distribution being a feasible undertaking.

Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
Data is contained within the article.
Conflict of interests
The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

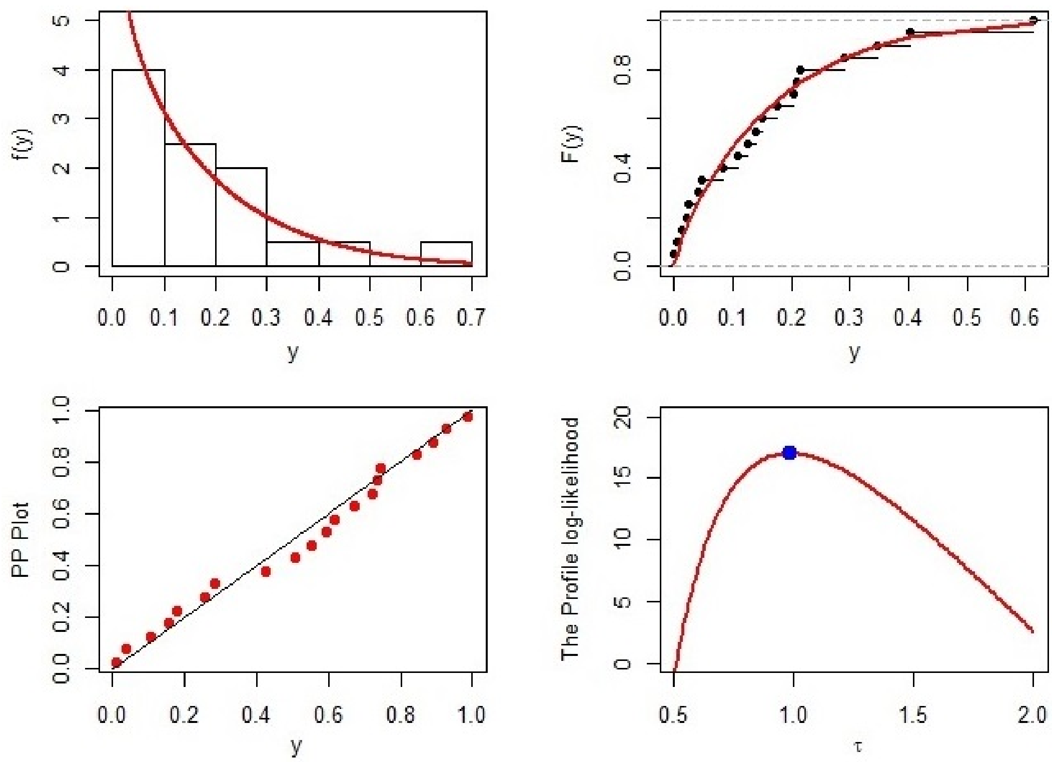

Figure 3. Fit diagnostics for the failure time data

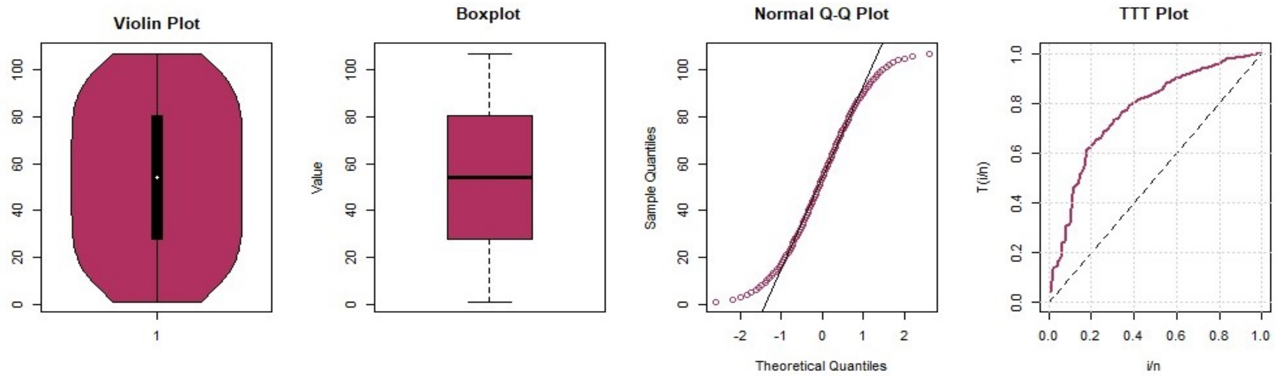

Figure 4. Graphical assessment of milk production data using violin, box, Q-Q, and TTT plots

Table 10. Parameter estimates, SEs, and fit criteria for competing models using milk production data

Models	Parameter Estimates		Standard Errors		Fitting measures values				
	$\hat{\tau}$	$\hat{\delta}$	$SE\,(\hat{\tau})$	$SE\,(\hat{\delta})$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$
SUME	0.3428	-	0.0179	-	23.9025	-45.8049	-43.1322	0.0799	0.5021
UME	0.4495	-	0.0307	-	20.7039	-39.4077	-36.7349	0.1038	0.1991
UXg	2.0568	-	0.1201	-	20.5159	-39.0319	-36.3591	0.0950	0.2882
UZ	2.0802	-	0.2011	-	21.5262	-41.0524	-38.3795	0.0972	0.2638
TL	1.3146	-	0.1052	-	1.51510	-1.0301	-1.64270	0.2338	0.0000
UXL	1.4981	-	0.1053	-	21.7835	-41.5670	-38.8942	0.1417	0.0272
LXg	1.9089	-	0.1391	-	-2.1409	6.2818	8.95470	0.2489	0.0000
UL	1.2001	-	0.0889		25.3803	-48.7610	-46.0881	2.0499	0.0000
UH	1.4701	-	0.1137		-2.6895	7.3791	10.0518	0.2582	0.0000
UB	0.2891	0.3121	2.4962	2.6946	20.6892	-37.3785	-32.0328	0.1080	0.1647

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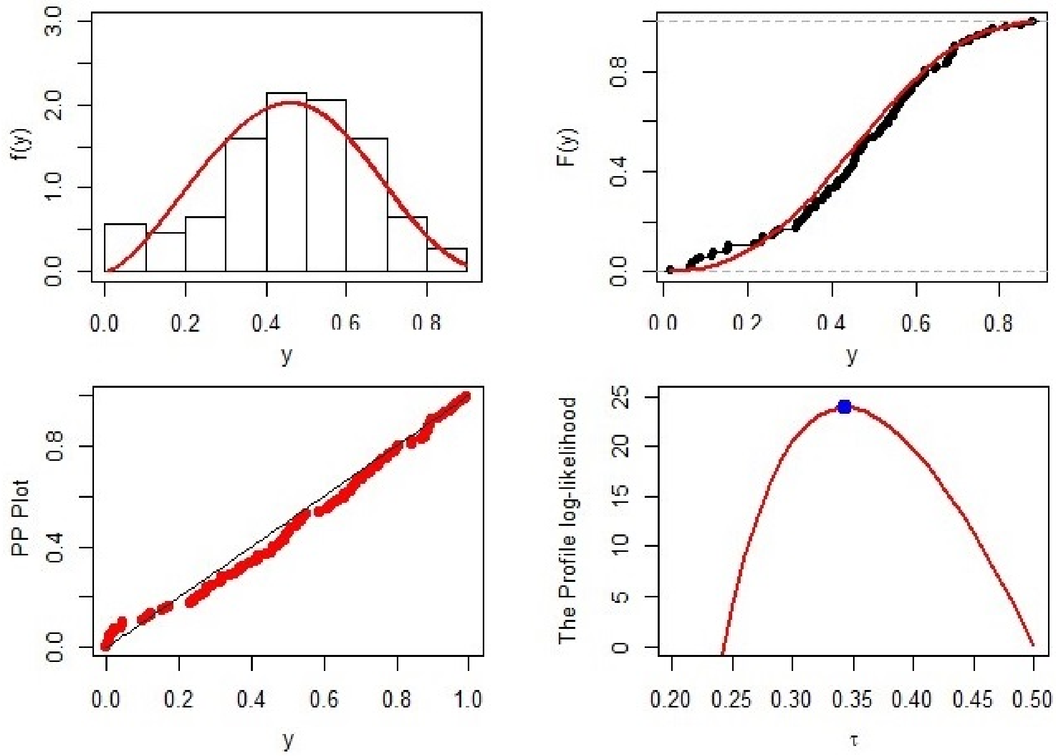

Figure 5. Fit diagnostics for the milk production data

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