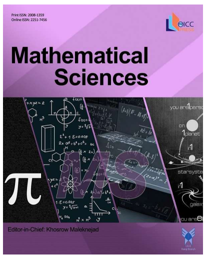

Mathsci. 2025; 19(3): Article 16

Mathematical Science

https://doi.org/10.57647/mathsci.2025.1903.16

Discrete Poisson Ramos–Louzada Exponential Distribution with Mathematical Properties, and Applications

Abdullah M. Alomair[1] , Muhammad Ahsan-ul-Haq[2,3*]

[1]*Department of Quantitative Methods, School of Business, King Faisal University, Al-Ahsa, Saudi Arabia*
[2]*Quality Enhancement Cell, National College of Arts, Lahore, Pakistan*
[3]*College of Statistical Sciences, University of the Punjab, Lahore, Pakistan*

Corresponding authors: ahsanshani36@gmail.com

Research Article

Received:
19 August 2025

Revised:
3 September 2025

Accepted:
18 September 2025

Published in Issue:
30 September 2025

Abstract

A new two-parameter mixed Poisson distribution is introduced and explored in this study. The new model is named the Poisson Ramos Louzada Exponential distribution. Various statistical characteristics of the new count distribution are derived and studied, including moments, generating function, overdispersion, failure rate, reversed hazard function, cumulative hazard function, Mills ratio, odd function, and order statistics. A novel regression model is also introduced based on this distribution. The parameters of the new probability model are estimated using the maximum likelihood estimation approach. The estimation behaviour of these derived estimators is studied using a Monte Carlo simulation study. The flexibility and adaptability of the new distribution have been confirmed using two datasets related to radiation and corn borer. The results reveal that the proposed distribution is efficient and competitive with existing count models.

Keywords: Ecology data; Mixed Poisson distribution; Moments; Maximum likelihood; Overdispersion; Radiation

Cite this article: Alomair, A. M., & Ahsan-ul-Haq, M. (2025). Discrete Poisson Ramos–Louzada Exponential Distribution with Mathematical Properties, and Applications. *MathSci 19* (03), 1-20. https://doi.org/10.57647/mathsci.2025.1903.16

1. Introduction

In recent years, the development of flexible probability models has gained increasing attention across various scientific and engineering disciplines, including radiation, insurance, public health, ecology, biology, engineering, economics, and environmental sciences (Ahsan-ul-Haq, Memon, et al., 2022). Such data often represents the number of occurrences of an event within a given time, space, or experimental unit. Examples include the number of insurance claims, equipment failures, disease cases, the number of deaths, phone calls, doctor visits in a hospital ward, or customer arrivals (Afify et al., 2022; A. M. Alomair, Nasir, et al., 2025). Poisson (Poi) distribution serves as the standard model for analyzing count data due to its simplicity, interpretability, and theoretical foundation within the framework of stochastic processes. Despite its wide applicability, the Poisson distribution rests on a restrictive assumption: the equality of its mean and variance, known as equi-dispersion. However, empirical datasets rarely satisfy this property. In many practical situations, data exhibit over-dispersion (variance exceeds the mean) or, less commonly, under-dispersion (variance is less than the mean). Over-dispersion can arise from unobserved heterogeneity, contagion effects, or clustering, while under-dispersion may occur in processes with inhibitory or limiting mechanisms. When the Poisson model is applied to such data, it typically leads to underestimated standard errors, biased parameter estimates, and unreliable inference. To overcome these shortcomings, numerous generalizations and extensions of the Poisson distribution have been proposed. Notable examples include the negative binomial distribution, which models over-dispersion through a gamma mixing distribution, and the generalized Poisson and Conway–Maxwell Poisson distributions, which introduce additional parameters to capture dispersion effects. Another important approach involves compound Poisson models, where the Poisson parameter itself follows continuous distribution. This strategy allows greater flexibility in modeling complex data structures by introducing randomness into the rate parameter. The choice of mixing distribution plays a crucial role in determining the flexibility and interpretability of the resulting model. Various discrete count models have been developed based on the Poisson mixture process. Some of the contributions in this area are the Poisson XLindley distribution by Ahsan-ul-Haq, Al-Bossly, et al. (2022), Poisson transmuted Ailamujia by Adetunji and Sabri (2023), and Poisson Xrani by Borbye et al. (2024). Other previous developments are the Poisson Ishita by Hassan et al. (2019) and, Poisson moment exponential distribution by Ahsan-ul-Haq (2022).

Other developments are the Poisson transmuted moment exponential by Alrumayh and Khogeer (2023), Poisson length-biased Chris-Jerry by M. A. Alomair et al. (2025), Poisson Komal by A. M. Alomair and Ahsan-ul-Haq (2025a), and Poisson quasi-XLindley by F. M. Alghamdi et al. (2024). Other more recent developments are the Poisson Haq by (A. M. Alomair, Ayyaz, et al., 2025), Poisson-Juchez by (Naz et al., 2025), Poisson Chris-Jerry by (Sindhu et al., 2025), Poisson Xrama by A. M. Alomair and Ahsan-ul-Haq (2025b), Poisson Copoun by (Mahnashi & Zaagan, 2025), Poisson Xrani by (Borbye et al., 2024), and Poisson Entropy-based weighted exponential by A. Alomair and Ahsan-ul-Haq (2023).In this context, the Ramos Louzada–Exponential (RLE) distribution, recently introduced by Aljohani et al. (2025) offers an attractive mixing component due to its analytical tractability and flexibility in modeling various lifetime behaviors. The RLE distribution extends the exponential model by incorporating two parameters, enabling it to capture both unimodal and decreasing hazard rate patterns. Its probability density function (PDF) is defined as

$$g(y;\eta,\vartheta)=\left(\frac{\eta}{\vartheta^{2}(\vartheta-1)}\right)(\vartheta^{2}-2\vartheta+\eta y)e^{-\left(\frac{\eta}{\vartheta}\right)y},\quad y>0,\quad \eta>0,\vartheta\geq 2. \tag{1}$$

The corresponding cumulative distribution function (CDF) is given by

$$G(y;\eta,\vartheta)=1-\left(1+\frac{\eta y}{\vartheta(\vartheta-1)}\right)e^{-\left(\frac{\eta}{\vartheta}\right)y} \tag{2}$$

The primary motivation for this work is to incorporate extra flexibility at the mixing component level, enhancing the family of Poisson mixture models for analysing count variables with excess zeros and overdispersion issues. To achieve that, an additional structural complexity is incorporated through the mixing mechanism.

For that reason, a two-parameter count model named the PRLE distribution is derived through the composition of the Poisson distribution and the Ramos-Louzada Exponential distribution. The proposed PRLE distribution differs from the classical Poisson distribution, since it includes an additional complexity in the form of the Ramos-Louzada mechanism. The PRLE distribution not only captures a variety of dispersion patterns but also retains analytical tractability, making it more suitable for practical data analysis and inferential aspects. Its mathematical properties are systematically derived and studied, and a corresponding count regression model is developed to analyse over-dispersed count data. In addition to the distributional development, this work also proposes a new count regression model based on the PRLE distribution, in which covariate information is incorporated into the mean structure through an appropriate link function.

The model parameters are estimated using the renowned maximum likelihood estimation approach. A Monte Carlo simulation study is utilized to illustrate the behaviour of estimators. The PRLE distribution allows more flexibility in terms of dispersion compared to Poisson and other competitive discrete distributions. The remaining study is organized as follows: Section 2 is based on the derivation and shape analysis of the new model, while Section 3 details the mathematical characteristics of the proposed model.

The parameter estimation, along with an extensive simulation study, is detailed in Section 4. A new count regression model based on this model is introduced in Section 5. Section 6 outlines the real data application of the proposed distribution. The concluding remarks and suggestions based on future research are presented in Section 7.

2. Derivation of poisson RLE distribution

A random variable (RV) Y follows to PRLE distribution if it obeys the following stochastic representations.

$(Y|\lambda)\sim Poisson(\lambda)$ & $(Y|\lambda)\sim RLE(\eta,\vartheta)$

Now utilizing the PDF of RLE distribution (1) and probability mass function (PMF) of the Poi model.

$$P(Y=y,\eta,\vartheta)=\int_0^\infty \frac{e^{-\lambda}\lambda^y}{y!}\left(\frac{\eta}{\vartheta^2(\vartheta-1)}\right)(\vartheta^2-2\vartheta + \eta\lambda)e^{-\left(\frac{\eta}{\vartheta}\right)\lambda}\,d\lambda,$$

where y is a nonnegative integer (so $\Gamma(y+1)=y!$) and $\vartheta \neq 1$. Now, combine the exponentials and splitlynomial factor: ,

$$P(Y=y,\eta,\vartheta)=\frac{\eta}{\vartheta^2(\vartheta-1)y!}\left[\int_0^\infty \lambda^y(\vartheta^2 - 2\vartheta)e^{-\left(\frac{\eta}{\vartheta}+1\right)\lambda}\,d\lambda + \eta\int_0^\infty \lambda^{y+1}e^{-\left(\frac{\eta}{\vartheta}+1\right)\lambda}\,d\lambda\right].$$

Now utilizing the gamma integral $\int_0^\infty \lambda^k e^{-\left(\frac{\eta}{\vartheta}+1\right)\lambda}\,d\lambda=\frac{\Gamma(k+1)}{\left(\frac{\eta}{\vartheta}+1\right)^{k+1}}$. Thus

$$P(Y=y,\eta,\vartheta)=\frac{\eta}{\vartheta^2(\vartheta-1)y!}\left[\frac{(\vartheta^2-2\vartheta)\Gamma(y+1)}{\left(\frac{\eta}{\vartheta}+1\right)^{y+1}} + \frac{\eta\Gamma(y+2)}{\left(\frac{\eta}{\vartheta}+1\right)^{y+2}}\right].$$

Since $\Gamma(k+1)=k!$ and $\Gamma(k+2)=(k+1)k!$ this simplifies to

$$P(Y=y,\eta,\vartheta)=\frac{\eta}{\vartheta^2(\vartheta-1)}\left[\frac{\vartheta^2-2\vartheta}{\left(\frac{\eta}{\vartheta}+1\right)^{y+1}} + \frac{\eta(y+1)}{\left(\frac{\eta}{\vartheta}+1\right)^{y+2}}\right].$$

After simplification

$$P(Y=y,\eta,\vartheta)=\frac{\eta\big((\vartheta-2)(\eta+\vartheta)\vartheta^y+\eta\vartheta^y(y+1)\big)}{(\vartheta-1)(\eta+\vartheta)^{y+2}}\quad y=0,1,2,\ldots \tag{3}$$

Which is the required PMF of the PRLE distribution. Now, first we compute the limiting behaviour of the PMF at the lower and upper limits of the density function. The behaviour of the PMF at the lower limit $y\to 0$ is

$$\lim_{y\to 0}P(y)=\frac{\eta(\vartheta^2-2\vartheta+\eta\vartheta-\eta)}{(\vartheta-1)(\eta+\vartheta)^2},$$

and the limiting behaviour at the upper limit $y\to\infty$ is

$$\lim_{y\to\infty}P(y)=0.$$

We plot the PMF of various options of parameters and were illustrated in Fig. 1. The PRLE distribution is very flexible, as depicted in the plots. It gives broad and skewed shapes towards the right with larger dispersion when η and ϑ are smaller. The higher the value of either parameter, the more peaked and concentrated the distribution on fewer counts, and the less dispersed it is. In general, the PRLE model could depict over-dispersed and almost equi-dispersed count data where the shape and spread are governed by η andϑ .

The CDF of the PRLE distribution is obtained by applying the general definition as $P(Y\le y)$.

$$F(y)=\frac{\eta}{\vartheta^2(\vartheta-1)}\sum_{y=0}^{y}\left(\frac{\vartheta^2-2\vartheta}{\left(\frac{\eta}{\vartheta}+1\right)^{y+1}}+\frac{\eta(y+1)}{\left(\frac{\eta}{\vartheta}+1\right)^{y+2}}\right).$$

Using the geometric series formula for $|r|<1$

$$F(y)=\frac{\eta}{\vartheta^2(\vartheta-1)}\left[\frac{\vartheta^2(\vartheta-2)}{\eta}\left(1-\left(\frac{\vartheta}{\eta+\vartheta}\right)^{y+1}\right)+\frac{\vartheta^2}{\eta}\left(-\frac{\vartheta\left(\frac{\vartheta}{\eta+\vartheta}\right)^{y}\big((y+2)\eta+\vartheta\big)}{(\eta+\vartheta)^2}\right)\right].$$

After Simplification, we obtain the more compact form of CDF, and provided as

$$F(y)=1-\frac{\vartheta^{y+1}(y\eta+\vartheta\eta+\vartheta^2-\vartheta)}{(\vartheta-1)(\eta+\vartheta)^{y+2}},\quad y=0,1,2,\ldots \tag{4}$$

3. Statistical properties

This section presents the key statistical characteristics of the proposed distribution, attention being paid to moments, identifiability based on CDF and PMF, recurrence relation, survival function, reliability characteristics, Mills ratio, odd function, generating function and order statistics.

3.1. Identifiability using CDF

Assume the model parameter region where the CDF (4) is well-defined: $\eta+\vartheta>0, \eta\neq 0$ and $\vartheta\neq 1$. Now we show the identifiability of the PREL distribution in each parameter separately.

3.1.1. *Identifiability in* η $(for\ fixed\ \vartheta)$

Assume$F(y;\eta_1,\vartheta)=F(y;\eta_2,\vartheta)$ for all integers $y\ge 0$. Then, for all $y\ge 0$,

$$\frac{\vartheta^{y+1}\big(y\eta_1+\vartheta(\eta_1+\vartheta-1)\big)}{(\vartheta-1)(\eta_1+\vartheta)^{y+2}}=\frac{\vartheta^{y+1}\big(y\eta_2+\vartheta(\eta_2+\vartheta-1)\big)}{(\vartheta-1)(\eta_2+\vartheta)^{y+2}}.$$

$$\vartheta^{y+1}\big(y\eta_1+\vartheta(\eta_1+\vartheta-1)\big)(\eta_2+\vartheta)^{y+2}=\vartheta^{y+1}\big(y\eta_2+\vartheta(\eta_2+\vartheta-1)\big)(\eta_1+\vartheta)^{y+2}. \tag{5}$$

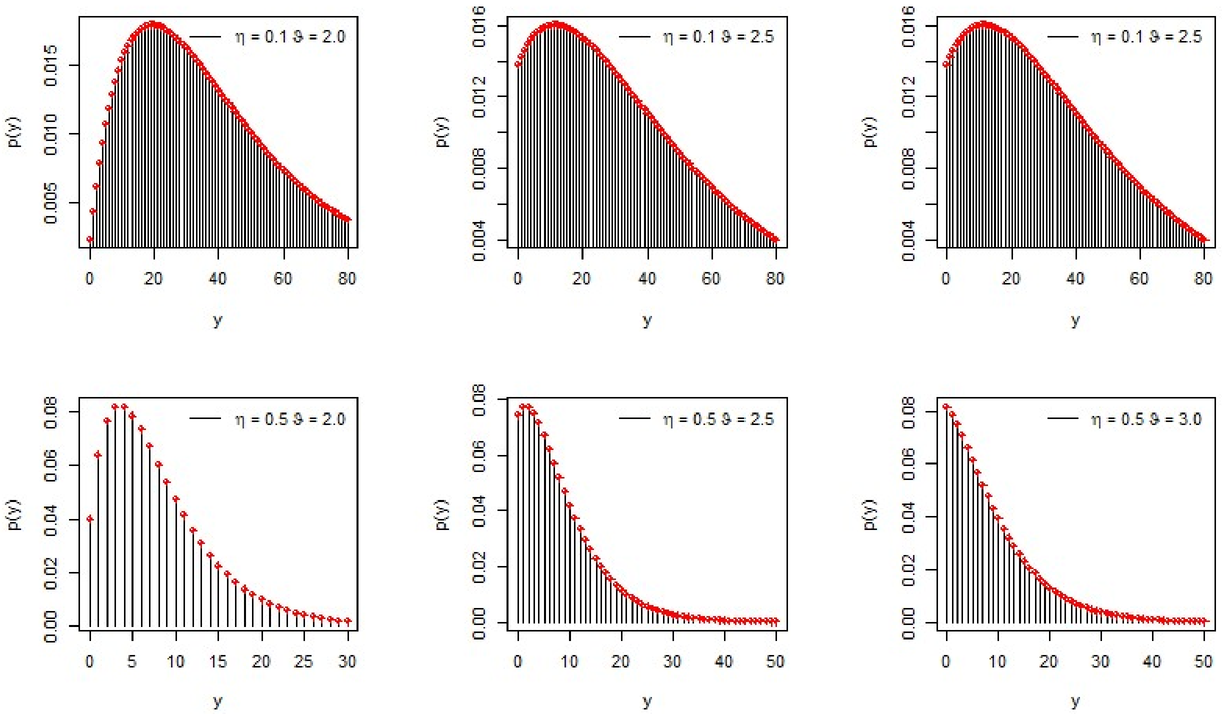

Figure 1. Plots of PRLE distribution for various parameter choices

Divide both sides of equ. (4) by $y(\eta_1+\vartheta)^{y+2}(\eta_2+\vartheta)^{y+2}$ and rearrange to isolate exponential ratios. For large values of the random variable, in terms of linear in y dominate the constant terms, hence asymptotically.

$$\eta_1(\eta_2+\vartheta)^{y+2} \sim \eta_2(\eta_1+\vartheta)^{y+2}, \qquad y \to \infty.$$

Thus

$$\left(\frac{\eta_2+\vartheta}{\eta_1+\vartheta}\right)^{y+2} \sim \frac{\eta_2}{\eta_1}, \qquad y \to \infty.$$

If $\eta_2+\vartheta \neq \eta_1+\vartheta$, the left-hand side vanishes as $y \to \infty$, a contradiction to the finite, nonzero right-hand side. Hence

$$\frac{\eta_2+\vartheta}{\eta_1+\vartheta} = 1 \qquad \Rightarrow \qquad \eta_1 = \eta_2.$$

This completes the proof of identifiability in η.

3.1.2. *Identifiability in* ϑ $(for\ fixed\ \eta)$

Assume $F(y;\eta,\vartheta_1) = F(y;\eta,\vartheta_2)$ for all integers $y \geq 0$. Equivalently, for all $y \geq 0$,

$$\frac{\vartheta_1^{y+1}\left(y\eta+\vartheta_1(\eta+\vartheta_1-1)\right)}{(\vartheta_1-1)(\eta_1+\vartheta_1)^{y+2}} = \frac{\vartheta_2^{y+1}\left(y\eta+\vartheta_2(\eta+\vartheta_2-1)\right)}{(\vartheta_2-1)(\eta+\vartheta_2)^{y+2}}.$$

By rearranging terms and comparing the dominant exponential rates as $y \to \infty$, equality of the CDFs implies

$$\frac{\vartheta_1}{\eta+\vartheta_1} = \frac{\vartheta_2}{\eta+\vartheta_2}.$$

Cross multiplies:

$$\vartheta_1(\eta+\vartheta_2) = \vartheta_2(\eta+\vartheta_1),$$
$$\vartheta_1\eta = \vartheta_2\eta,$$
$$\vartheta_1 = \vartheta_2, \quad (Since\ \eta \neq 0).$$

This completes the proof of identifiability in ϑ.

3.3. Identifiability using PMF (summary)

The identifiability of the PRLE distribution can also be established by utilizing the density function by considering the likelihood ratio of two parameter pairs (η_1,ϑ_1) and (η_2,ϑ_2). If the PMFs coincide for all $y \geq 0$, the logarithmic derivative of the likelihood ratio must vanish for all y, taking limits as $y \to \infty$ isolates the dominant exponential terms, leading to the condition

$$\vartheta_1(\eta_2+\vartheta_2) = \vartheta_2(\eta_1+\vartheta_1).$$

Additional evaluation at finite values of y then implies $\vartheta_1 = \vartheta_2$ and then $\eta_1 = \eta_2$.

Therefore

$$P(Y=y;\eta_1,\vartheta_1) = P(Y=y;\eta_2,\vartheta_2)\ \forall\, y \geq 0 \qquad \Rightarrow \qquad (\eta_1,\vartheta_1) = (\eta_2,\vartheta_2).$$

The PMF of PRLE distribution is identifiable: distinct parameter pairs (η,ϑ) produce distinct models. Equivalently, equality of the PMF for all integer values $y \geq 0$ implies equality of the parameters.

3.4. Recurrence relation

In this section, we derive the recursive relation of the PMF that supports us in identifying the mass at the subsequent point in terms of the mass at a previous point. Now using the representation of the PMF in equ. (3)

$$P_y = P(Y = y) = \frac{\eta\vartheta^y}{(\vartheta - 1)(\eta + \vartheta)^{y+2}}[(\vartheta - 2)(\eta + \vartheta) + \eta(y + 1)].$$

Now

$$P_{y+1} = \frac{\eta\vartheta^{y+1}}{(\vartheta - 1)(\eta + \vartheta)^{y+3}}[(\vartheta - 2)(\eta + \vartheta) + \eta(y + 2)].$$

Form the ratio $\frac{P_{y+1}}{P_y}$

$$\frac{P_{y+1}}{P_y} = \frac{\vartheta^{y+1}[(\vartheta - 2)(\eta + \vartheta) + \eta(y + 2)](\eta + \vartheta)^{y+2}}{\vartheta^y[(\vartheta - 2)(\eta + \vartheta) + \eta(y + 1)](\eta + \vartheta)^{y+3}}.$$

Simplify

$$\frac{P_{y+1}}{P_y} = \frac{\vartheta}{\eta + \vartheta}\frac{(\vartheta - 2)(\eta + \vartheta) + \eta(y + 2)}{(\vartheta - 2)(\eta + \vartheta) + \eta(y + 1)}.$$

Recurrence relation for P_{y+1}

$$P(Y = y + 1) = \frac{\vartheta}{\eta + \vartheta} \times \frac{(\vartheta - 2)(\eta + \vartheta) + \eta(y + 2)}{(\vartheta - 2)(\eta + \vartheta) + \eta(y + 1)} \times P(Y = y).$$

3.5. Survival function

The survival function (SF), which gives the probability that the system or component survives beyond time y, is given by

$$S(y; \eta, \vartheta) = P(Y > y) = 1 - F(y; \eta, \vartheta),$$

where $F(y; \eta, \vartheta)$ is the CDF. Now using the CDF equ. (4), the SF of the PRLE distribution is obtained as

$$S(y; \eta; \vartheta) = \frac{\vartheta^{y+1}(y\eta + \vartheta\eta + \vartheta^2 - \vartheta)}{(\vartheta - 1)(\eta + \vartheta)^{y+2}}, \quad y = 0,1,2, \ldots.. \qquad (6)$$

Since $S(y + 1; \eta; \vartheta) < S(y; \eta; \vartheta)$ for all $y \geq 0$, the survival function of the PRLE distribution is a strictly decreasing function of the random variable y.

The analytical behaviour can be derived for a better understanding of the trend. For this purpose, consider the forward ratio:

$$\frac{S(y + 1)}{S(y)} = \frac{(\eta\vartheta y + \eta\vartheta) + (\vartheta^2\eta + \vartheta^3 - \vartheta^2)}{(\eta + \vartheta)[y\eta + \vartheta\eta + \vartheta^2 - \vartheta]}.$$

This ratio is always less than one, proving that $S(y)$ is monotonically decreasing as $y \longrightarrow \infty$,

$$\frac{S(y + 1)}{S(y)} = \left(\frac{\vartheta}{\eta + \vartheta}\right),$$

indicating a geometric-type tail behaviour.

3.6. Hazard function

The hazard function, representing the conditional probability of failure at time y given survival up to that point, is obtained as

$$h(y; \eta, \vartheta) = \frac{\eta\big((\vartheta - 2)(\eta + \vartheta) + \eta(y + 1)\big)}{\vartheta\big(y\eta + \vartheta(\eta + \vartheta - 1)\big)}, y = 0,1,2, \ldots \qquad (7)$$

The behaviour of the hazard function at distribution limiting values is obtained as follows:

$$\lim_{y\to 0} h(y) = \frac{\eta^2 + \eta(\eta + \vartheta)(\vartheta - 2)}{(\eta + \vartheta - 1)\vartheta^2}, \text{ and } \lim_{y\to\infty} h(y) = \frac{\eta}{\vartheta}.$$

Thus, the $h(y)$ increases from a finite positive value at $y = 0$ and approaches an upper bound $\left(\frac{\eta}{\vartheta}\right)$ as $y \to \infty$. Now to investigate the trend of the $h(y; \eta, \vartheta)$, consider the forward difference:

$$\Delta h(y; \eta, \vartheta) = \frac{\eta^2(\eta + \vartheta)}{\vartheta[\eta\vartheta + \eta y + \vartheta^2 - \vartheta][\eta\vartheta + \eta y + \eta + \vartheta^2 - \vartheta]}.$$

Since $\vartheta > 1$ and $\eta > 0$, the denominator is positive for all $y \geq 0$. Consequently,

$$\Delta h(y; \eta, \vartheta) > 0. \quad \forall y \geq 0,$$

Which implies that $h(y; \eta, \vartheta)$ is strictly increasing in variable y. Therefore, the PRLE distribution exhibits an increasing failure rate (IFR) property.

To gain further insights into the dynamics of the proposed model's behaviour, we analyse the change in the hazard function based on various parameter values. We calculate the hazard function for various values of the parameter settings to present the resulting plots.

This will indicate the influence of the parameter settings on the risk evolution over time by clearly demonstrating the differences in the hazard function plots. This is presented in Fig. 2 below.

The hazard rate plots of the PRLE distribution show that $h(y)$ increases monotonically with y, indicating an increasing failure rate pattern.

This suggests that the likelihood of an event occurring grows as time or count progresses. For smaller values of η, the hazard rate increases slowly and remains relatively low, reflecting a gradual aging process.

As η increases, the hazard function rises more sharply, reaching its upper limit faster; this implies a system with faster wear-out behaviour. Similarly, higher ϑ values smooth the curve, reducing the rate of increase. Hence, the PRLE distribution exhibits flexible hazard behaviour, accommodating both gradual and rapidly increasing failure tendencies depending on parameter values.

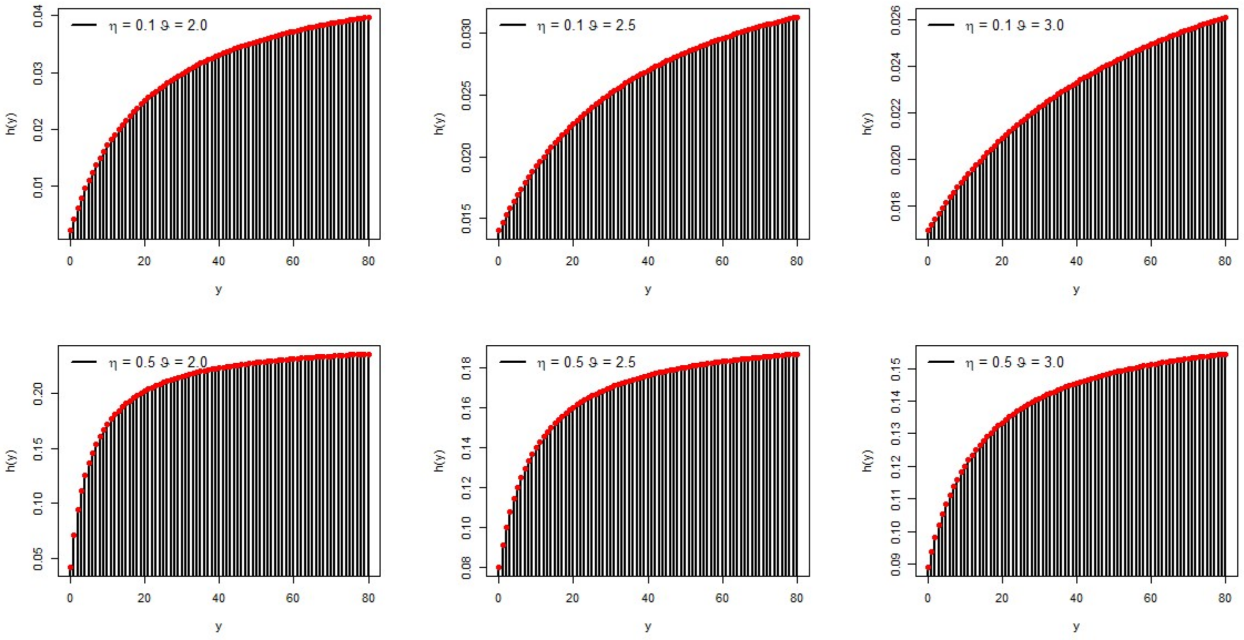

Figure 2. Hazard rate plots for various parameter values

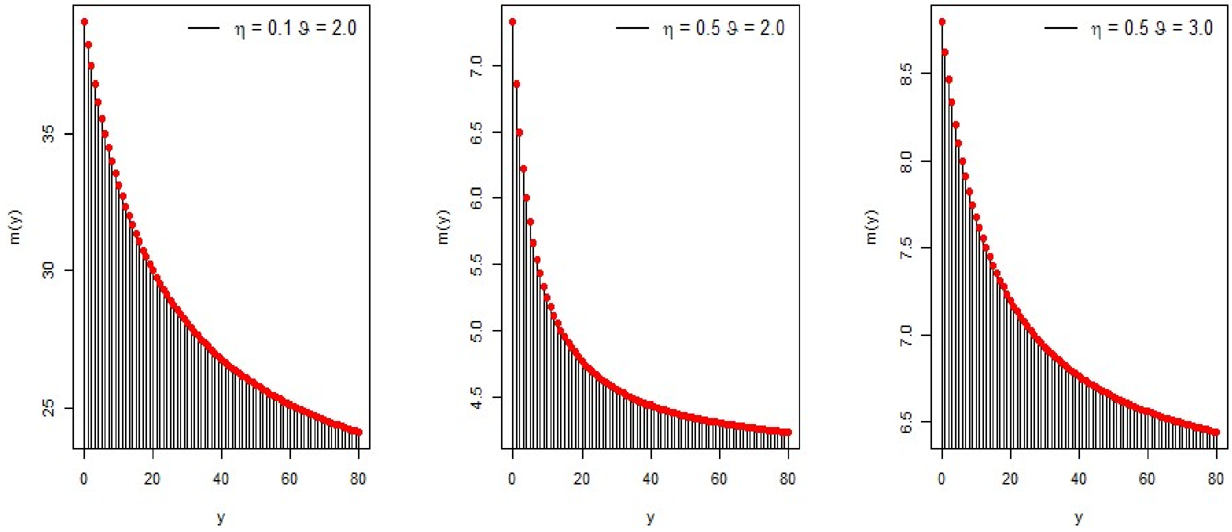

Figure 3. MRL plots for various parameter values

3.7. Mean residual life function

The discrete mean residual life (MRL) function is derived as

$$m(y) = E(Y - y \mid Y \geq y) = \sum_{k=1}^{\infty} \frac{S(y+k)}{S(y)}, \quad y = 0,1,2,\ldots$$

Now using the survival function (8), a closed-form expression for the MRL is

$$m(y) = \frac{\vartheta[\eta\vartheta + \eta y + \eta + \vartheta^2]}{\eta[\eta\vartheta + \eta y + \vartheta^2 - \vartheta]}, \quad y = 0,1,2,\ldots \tag{8}$$

As $y \to \infty$, $\lim_{y\to\infty} m(y) = \frac{\vartheta}{\eta}$.

Additionally, the forward difference of $m(y)$ satisfies

$$m(y+1) - m(y) = -\frac{\vartheta(\eta+\vartheta)}{(\eta\vartheta + \eta y + \vartheta^2 - \vartheta)(\eta\vartheta + \eta y + \eta + \vartheta^2 - \vartheta)} < 0.$$

Therefore, $m(y)$ is strictly decreasing in y, confirming that the probability model belongs to the decreasing MRL (DMRL) class.

3.8. Reverse and cumulative hazard rates

The reverse hazard rate, which measures the probability of failure at time y given that failure occurs at or before y, is expressed as

$$RH(y) = \frac{f(y;\eta,\vartheta)}{F(y;\eta,\vartheta)} = \frac{\eta\vartheta^{y}\big((\vartheta-2)(\eta+\vartheta) + \eta(y+1)\big)}{(\vartheta-1)(\eta+\vartheta)^{y+2} - \vartheta^{y+1}(y\eta + \vartheta\eta + \vartheta^2 - \vartheta)}.$$

10.57647/mathsci.2025.1903.16

The cumulative hazard function, representing the total accumulated risk up to time y, is defined by

$$H(y) = -ln\big(R(y;\eta,\vartheta)\big)$$
$$= -ln\left(\frac{\vartheta^{y+1}\big(y\eta + \vartheta(\eta + \vartheta - 1)\big)}{(\vartheta - 1)(\eta + \vartheta)^{y+2}}\right).$$

This measure is useful for understanding the overall reliability performance over time.

3.9. Mills ratio and odd function

The Mills ratio, defined as the ratio of the survival probability to the probability mass at y, is given by

$$MR(y) = \frac{R(y;\eta,\vartheta)}{P(y;\eta,\vartheta)} = \frac{\vartheta(y\eta + \vartheta\eta + \vartheta^2 - \vartheta)}{\eta\big((\vartheta - 2)(\eta + \vartheta) + \eta(y + 1)\big)}.$$

The Odds Function denoted as $O(y)$ is defined as the ratio of the cumulative probability (failure probability) to the survival probability and given as follows.

$$O(y) = \frac{F(y;\eta,\vartheta)}{1 - F(y;\eta,\vartheta)}$$
$$= \frac{(\vartheta - 1)(\eta + \vartheta)^{y+2} - \vartheta^{y+1}(y\eta + \vartheta\eta + \vartheta^2 - \vartheta)}{\vartheta^{y+1}(y\eta + \vartheta\eta + \vartheta^2 - \vartheta)}.$$

3.10. Moments

Let Y be a random variable following the PRLE distribution. The r^{th} factorial moment of Y is defined as

$$\mu_{[r]} = E\big[E\big(y^{(r)}|\theta\big)\big],$$

where $Y \mid \theta \sim Poisson(\theta)$. The conditional expectation inside the integral represents the r^{th}factorial moment of the Poisson distribution. Hence, the unconditional factorial moment can be expressed as

$$\mu_{[r]} = \int_0^\infty \left(\sum_{x=0}^{\infty} y^{(r)} \frac{e^{-\theta}\theta^y}{y!}\right)\left(\frac{\eta}{\vartheta^2(\vartheta - 1)}\right)(\vartheta^2 - 2\vartheta$$
$$+ \eta\theta)e^{-\left(\frac{\eta}{\vartheta}\right)\theta}\, d\theta.$$

The expression under the bracket in the above is the *rth* moment about the origin of the Poi distribution. So,

$$\mu_{[r]} = \int_0^\infty \theta^r \left(\frac{\eta}{\vartheta^2(\vartheta - 1)}\right)(\vartheta^2 - 2\vartheta + \eta\theta)e^{-\left(\frac{\eta}{\vartheta}\right)\theta}\, d\theta.$$

Expanding and simplifying the above integral gives

$$\mu_{[r]} = \left(\frac{\eta}{\vartheta^2(\vartheta - 1)}\right)\Bigg(\vartheta^2 \int_0^\infty \theta^r e^{-\left(\frac{\eta}{\vartheta}\right)\theta}\, d\theta$$
$$- 2\vartheta \int_0^\infty \theta^r e^{-\left(\frac{\eta}{\vartheta}\right)\theta}\, d\theta$$
$$+ \eta \int_0^\infty \theta^{r+1} e^{-\left(\frac{\eta}{\vartheta}\right)\theta}\Bigg).$$

Using the well-known gamma integral identity$\int_0^\infty y^{n-1}e^{-\eta x} = \frac{\Gamma(n)}{(\eta)^n}$, it leads to the compact form

$$\mu_{[r]} = \left(\frac{(\vartheta + r - 1)\vartheta^r r!}{(\vartheta - 1)\eta^r}\right).$$

By substituting $r = 1,2,3,4$, we obtain the first four factorial moments as

$$\mu_{[1]} = \frac{\vartheta^2}{\eta(\vartheta - 1)},$$
$$\mu_{[2]} = \frac{2\vartheta^3 + 2\vartheta^2}{\eta^2\vartheta - \eta^2},$$
$$\mu_{[3]} = \frac{6\vartheta^4 + 12\vartheta^3}{\eta^3\vartheta - \eta^3},$$
$$\mu_{[4]} = \frac{24\vartheta^5 + 72\vartheta^4}{\eta^4\vartheta - \eta^4}.$$

To calculate the moments about the origin, we use their intrinsic relationship with the factorial moments of the PRLE distribution. Accordingly, the first four moments about the origin for the PRLE distribution are obtained as follows.

$$\mu'_2 = \frac{\vartheta^2(\eta + 2\vartheta + 2)}{\eta^2(\vartheta - 1)},$$
$$\mu'_3 = \frac{6\vartheta^4 + 12\vartheta^3 + 6\eta\vartheta^3 + 6\eta\vartheta^2 + \eta^2\vartheta^2}{\vartheta\eta^3 - \eta^3},$$
$$\mu'_4$$
$$= \frac{\vartheta^2(\eta^3 + 14\eta^2 + 14\vartheta\eta^2 + 72\eta\vartheta + 36\eta\vartheta^2 + 24\vartheta^3 + 72\vartheta^2)}{\vartheta\eta^4 - \eta^4}.$$

The central moments of the PRLE distribution are derived from the moments about the origin using the standard relationships.

$$\mu_2 = \frac{\vartheta^2(\eta\vartheta - \eta + \vartheta^2 - 2)}{\eta^2(\vartheta - 1)^2},$$
$$\mu_3$$
$$= \frac{\vartheta^2\big(12\vartheta + (\eta(\vartheta - 1) + \vartheta^2 - 6)(\eta(\vartheta - 1) + 2\vartheta^2)\big)}{\eta^3(\vartheta - 1)^3}.$$

Similarly, the fourth central moment can be written as

$$\mu_4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4$$

Furthermore, using the relationship between moments of origin and moments about Mean, the first four mean moments can be determined as follows:

$$DI = \frac{(\eta\vartheta - \eta + \vartheta^2 - 2)}{\eta(\vartheta - 1)},$$
$$Var(Y) = \mu + \frac{\vartheta^2(\vartheta^2 + 2)}{\eta^2(\vartheta - 1)^2},$$
$$SK$$
$$= \frac{\vartheta^2(12\vartheta + \eta(\vartheta - 1) + \vartheta^2 - 6)(\eta(\vartheta - 1) + 2\vartheta^2))}{\big(\vartheta^2(\eta\vartheta - \eta + \vartheta^2 - 2)\big)^{3/2}},$$
$$K = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{(\sigma^2)^2}.$$

Table 1. Descriptive analysis of PRLE distribution for different choices of parameters

Parameters		Descriptive							
ϑ	η	μ	μ_2	μ_3	μ_4	Variance	DI	Skewness	Kurtosis
2.0	0.5	8.0000	104.00	1832.0	40616.0	40.000	5.0000	1.4230	6.025
	1.0	4.0000	28.000	268.00	3244.00	12.000	3.0000	1.4434	6.0833
	1.5	2.6667	13.333	91.556	797.926	6.2222	2.3333	1.4699	6.1607
	2.5	1.6000	5.4400	25.408	151.360	2.8800	1.8000	1.5321	6.3472
	3.0	1.3333	4.0000	16.444	86.3704	2.2222	1.6667	1.5652	6.4500
2.5	0.5	8.3333	125.00	2608.3	69325.0	55.556	6.6667	1.5474	6.4788
	1.0	4.1667	33.333	372.92	5333.33	15.972	3.8333	1.5811	6.6091
	1.5	2.7778	15.741	125.00	1272.53	8.0247	2.8889	1.6142	6.7360
	2.5	1.6667	6.3333	33.667	230.330	3.5556	2.1333	1.6794	6.9844
	3.0	1.3889	4.6296	21.528	129.010	2.7006	1.9444	1.7116	7.1070
3.0	0.5	9.0000	153.00	3681.0	113769.0	72.000	8.0000	1.6499	6.9306
	1.0	4.5000	40.500	517.50	8518.50	20.250	4.5000	1.6790	7.0576
	1.5	3.0000	19.000	171.00	1987.00	10.000	3.3333	1.7076	7.1800
	2.5	1.8000	7.5600	45.000	346.939	4.3200	2.4000	1.7641	7.4167
	3.0	1.5000	5.5000	28.500	191.500	3.2500	2.1667	1.7921	7.5325
3.5	0.5	9.8000	186.20	5066.6	178232.6	90.160	9.2000	1.7226	7.2885
	1.0	4.9000	49.000	703.15	13073.2	24.990	5.1000	1.7462	7.3995
	1.5	3.2667	22.867	229.76	2996.26	12.196	3.7333	1.7699	7.5073
	2.5	1.9600	9.0160	59.349	508.390	5.1744	2.6400	1.8176	7.7178
	3.0	1.6333	6.5333	37.294	277.301	3.8656	2.3667	1.8415	7.8215

The results of the descriptive statistics of various combinations of the parameter values are shown in Table 1. The findings indicate that the PRLE distribution is positively skewed and leptokurtic, which is consistent with the trends in the graphical representation of the distribution. Furthermore, the distribution is over-dispersed, which proves that the PRLE model is effective in the dispersion properties under investigation.

3.11. Over-dispersion

The PRLE distribution is always over-dispersed $\left(Mean(X) < Var(X)\right)$. We have

$$Var(y)-\mu=\frac{\vartheta^2(\eta+2\vartheta+2)}{\eta^2(\vartheta-1)}-\frac{\vartheta^2}{\eta(\vartheta-1)}$$

We may write $\frac{\vartheta^2}{\eta(\vartheta-1)}=\frac{\vartheta^2\eta(\vartheta-1)}{\eta^2(\vartheta-1)^2}$

$$Var(y)-\mu=\frac{\vartheta^2(\eta+2\vartheta+2)}{\eta^2(\vartheta-1)}-\frac{\vartheta^2\eta(\vartheta-1)}{\eta^2(\vartheta-1)^2}$$

After simplification, we can express it as

$$Var(y)=\mu+\frac{\vartheta^2(\vartheta^2-2)}{\eta^2(\vartheta-1)^2}=\mu\left(1+\frac{(\vartheta^2-2)}{\eta(\vartheta-1)}\right)$$

This makes clear that $Var > \mu$ exactly when the extra term is positive.

3.12. Generating functions

The moment generating function (MGF) of the Poisson Ramos-Louzada Exponential (PRLE) distribution is derived as follows.

$$M_z(t)=E(e^{tz})=\frac{\eta}{\vartheta^2(\vartheta-1)}\left((\vartheta^2-2\vartheta)\sum_{y=0}^{\infty}\frac{e^{ty}}{\left(\frac{\eta}{\vartheta}+1\right)^{y+1}}+\eta\sum_{y=0}^{\infty}\frac{e^{tz}(y+1)}{\left(\frac{\eta}{\vartheta}+1\right)^{y+2}}\right)$$

After simplification, we obtain

$$M_z(t) = \frac{\eta}{\vartheta^2(\vartheta-1)}\left((\vartheta^2-2\vartheta)\sum_{y=0}^{\infty}\frac{e^{ty}}{\left(\frac{\eta}{\vartheta}+1\right)^{y+1}} + \eta\sum_{z=0}^{\infty}\frac{ye^{tz}}{\left(\frac{\eta}{\vartheta}+1\right)^{y+2}} + \eta\sum_{z=0}^{\infty}\frac{e^{ty}}{\left(\frac{\eta}{\vartheta}+1\right)^{y+2}}\right)$$

Evaluating these series and simplifying gives the closed-form expression.

$$M_y(t) = \frac{\eta}{\vartheta^2(\vartheta-1)\left(\frac{\eta}{\vartheta}+1\right)}\left((\vartheta^2-2\vartheta)\left(\frac{\eta+\vartheta}{\eta+\vartheta-e^t\vartheta}\right) + \left(\frac{\eta\vartheta^2e^t}{(\eta+\vartheta-e^t\vartheta)^2}\right) + \left(\frac{\eta\vartheta}{\eta+\vartheta-e^t\vartheta}\right)\right)$$

After simplification,

$$M_y(t) = \frac{\eta\vartheta}{\vartheta^2(\vartheta-1)(\eta+\vartheta)}\left(\frac{\vartheta(\eta+\vartheta)(\eta\vartheta+(e^t-1)(\vartheta-2)\vartheta-\eta)}{(\eta+\vartheta-e^t\vartheta)^2}\right)$$

Now, the characteristic function and probability generating function can simply be obtained by replacing e^t by $e^{it}$and t^y respectively,

$$\phi_z(it) = \frac{\eta\vartheta}{\vartheta^2(\vartheta-1)(\eta+\vartheta)}\left(\frac{\vartheta(\eta+\vartheta)(\eta\vartheta+(e^{it}-1)(\vartheta-2)\vartheta-\eta)}{(\eta+\vartheta-e^{it}\vartheta)^2}\right).$$

and

$$P_z(t) = \frac{\eta\vartheta}{\vartheta^2(\vartheta-1)(\eta+\vartheta)}\left(\frac{\vartheta(\eta+\vartheta)(\eta\vartheta+(t^y-1)(\vartheta-2)\vartheta-\eta)}{(\eta+\vartheta-t^y\vartheta)^2}\right).$$

3.13. Order statistics

Let $Y_{(1)}, Y_{(2)}, Y_{(3)}, \ldots, Y_{(n)}$be the ordered statistics of the random sample $Y_1, Y_2, Y_3, \ldots, Y_n$drawn from the discrete distribution with cumulative distribution function $F(y;\eta,\vartheta)$and probability mass function$P(y;\eta,\vartheta)$, then the Cumulative distribution function of i^{th} order statistics$Y_i$is given by

$$F_{y_{|i}} = \sum_{k=i}^{n}\binom{n}{i}\,[\,F(y;\eta,\vartheta)]^k[1-F(y;\eta,\vartheta]^{n-k}$$

Using equ. (4), the Cumulative distribution function of i^{th} order statistics of the PRLE distribution is given by:

$$F_{y_{|j}} = \sum_{i=j}^{n}\binom{n}{i}\left[1-\frac{\vartheta^{y+1}(y\eta+\vartheta\eta+\vartheta^2-\vartheta)}{(\vartheta-1)(\eta+\vartheta)^{y+2}}\right]^{i}$$

$$\left[\frac{\vartheta^{y+1}(y\eta+\vartheta\eta+\vartheta^2-\vartheta)}{(\vartheta-1)(\eta+\vartheta)^{y+2}}\right]^{n-i}$$

The PMF of the i^{th}order statistics of a discrete distribution can be expressed as:

$$f_{y_{|j}} = n.\binom{n-1}{j-1}\,[\,F(y;\eta,\vartheta)]^{i-1}[1 - Fy;\eta,\vartheta]^{n-i}\,P(y;\eta,\vartheta).$$

Using equations (3) and (4), we can obtain the PMF of i^{th}order statistics PRLE distribution is follows as

$$f_{y_{|1}} = n\binom{n-1}{j-1}\left[1-\frac{\vartheta^{y+1}(y\eta+\vartheta\eta+\vartheta^2-\vartheta)}{(\vartheta-1)(\eta+\vartheta)^{y+2}}\right]^{i-1}$$

$$\left[\frac{\vartheta^{y+1}(y\eta+\vartheta\eta+\vartheta^2-\vartheta)}{(\vartheta-1)(\eta+\vartheta)^{y+2}}\right]^{n-i}$$

$$\frac{\eta\big((\vartheta-2)(\eta+\vartheta)\vartheta^y+\eta\vartheta^y(y+1)\big)}{(\vartheta-1)(\eta+\vartheta)^{y+2}}$$

Then, the PMF of the first order Y_1PRLE distribution is given by

$$f_{y_{|1}} = n\left[\frac{\vartheta^{y+1}(y\eta+\vartheta\eta+\vartheta^2-\vartheta)}{(\vartheta-1)(\eta+\vartheta)^{y+2}}\right]^{n-1}$$

$$\frac{\eta\big((\vartheta-2)(\eta+\vartheta)\vartheta^y+\eta\vartheta^y(y+1)\big)}{(\vartheta-1)(\eta+\vartheta)^{y+2}}$$

Similarly, the PMF of n^{th}order Y_nPRLE distribution is given as

$$f_{y_{|n}} = n\left[\,1-\frac{\vartheta^{y+1}(y\eta+\vartheta\eta+\vartheta^2-\vartheta)}{(\vartheta-1)(\eta+\vartheta)^{y+2}}\right]^{n-1}$$

$$\left[\frac{\delta^2\big((\delta+1)^2+(z+1)\big)}{(\delta^2+\delta+1)(\delta+1)^{z+2}}\right].$$

5. Parameter estimation and simulation study

In this section, we will derive the maximum likelihood estimators (MLEs) for the parameters of the PRLE distribution. The parameters η and ϑ are estimated by maximizing the likelihood function based on a random sample $Y_1, Y_2, Y_3, \ldots, Y_n$ drawn from the PRLE distribution with the probability mass function provided in equ. (3). The likelihood function for the observed sample is expressed as

$$L(x) = \prod_{i=1}^{n} P(Y=y_i)$$

$$= \frac{\eta^n\vartheta^{\sum_{i=1}^{n}y_i}}{(\vartheta-1)^n(\eta+\vartheta)^{\sum_{i=1}^{n}(y_i+2)}}\prod_{i=1}^{n}\big(((\vartheta - 2)(\eta+\vartheta)+\eta(y_i+1)\big)\big)$$

10.57647/mathsci.2025.1903.16

Table 2. Monte Carlo Simulation analysis for the PRLE distribution

Parameters	n	$\hat{\eta}$				$\hat{\vartheta}$			
		$E(\hat{\eta})$	$Bias(\hat{\eta})$	$MRE(\hat{\eta})$	$RMSE(\hat{\eta})$	$E(\hat{\vartheta})$	$Bias(\hat{\vartheta})$	$MRE(\hat{\vartheta})$	$RMSE(\hat{\vartheta})$
$\eta = 0.3$ $\vartheta = 2.0$	50	0.3196	0.0196	0.0653	0.2017	2.0261	0.0261	0.0130	2.4712
	100	0.3034	0.0034	0.0113	0.0245	1.9971	0.0029	0.0014	0.1480
	200	0.3011	0.0011	0.0037	0.0173	2.0018	0.0018	0.0009	0.0860
	300	0.3010	0.0010	0.0033	0.0141	2.0025	0.0025	0.0012	0.0656
	500	0.3005	0.0005	0.0016	0.0100	2.0025	0.0025	0.0013	0.0480
	800	0.3005	0.0005	0.0015	0.0100	2.0013	0.0013	0.0007	0.0374
	1000	0.3002	0.0002	0.0005	0.0100	2.0018	0.0018	0.0009	0.0332
$\eta = 0.3$ $\vartheta = 2.5$	50	0.5233	0.2233	0.7443	1.2311	5.4177	2.9177	1.1671	16.4733
	100	0.3742	0.0742	0.2474	0.6286	3.4815	0.9815	0.3926	8.4048
	200	0.3085	0.0085	0.0284	0.1183	2.6007	0.1007	0.0403	1.5794
	300	0.3039	0.0039	0.0131	0.0224	2.5460	0.0460	0.0184	0.2968
	500	0.3020	0.0020	0.0066	0.0141	2.5209	0.0209	0.0084	0.1997
	800	0.3010	0.0010	0.0035	0.0100	2.5118	0.0118	0.0047	0.1456
	1000	0.3013	0.0013	0.0043	0.0100	2.5125	0.0125	0.0050	0.1304
$\eta = 0.3$ $\vartheta = 3.0$	50	0.8876	0.5876	1.9588	2.0052	11.406	8.4056	2.8019	28.638
	100	0.5853	0.2853	0.9511	1.1845	7.1726	4.1726	1.3909	17.397
	200	0.3941	0.0941	0.3136	0.5753	4.3912	1.3912	0.4637	8.6882
	300	0.3351	0.0351	0.1169	0.2798	3.5113	0.5113	0.1704	4.1566
	500	0.3091	0.0091	0.0305	0.0781	3.1284	0.1284	0.0428	1.1848
	800	0.3044	0.0044	0.0147	0.0300	3.0587	0.0587	0.0196	0.4853
	1000	0.3029	0.0029	0.0096	0.0200	3.0376	0.0376	0.0125	0.3010
$\eta = 0.5$ $\vartheta = 2.0$	50	0.5456	0.0456	0.0911	0.5287	2.1579	0.1579	0.0790	4.1359
	100	0.5126	0.0126	0.0252	0.2421	2.0538	0.0538	0.0269	1.8334
	200	0.5031	0.0031	0.0062	0.0283	2.0044	0.0044	0.0022	0.0970
	300	0.5020	0.0020	0.0039	0.0224	2.0033	0.0033	0.0017	0.0755
	500	0.5013	0.0013	0.0026	0.0173	2.0032	0.0032	0.0016	0.0592
	800	0.5007	0.0007	0.0014	0.0141	2.0030	0.0030	0.0015	0.0469
	1000	0.5004	0.0004	0.0007	0.0141	2.0020	0.0020	0.0010	0.0412
$\eta = 0.5$ $\vartheta = 2.5$	50	0.9372	0.4372	0.8743	2.2628	5.9625	3.4625	1.3850	18.303
	100	0.6431	0.1431	0.2863	1.1462	3.6381	1.1381	0.4552	9.3754
	200	0.5301	0.0301	0.0602	0.4192	2.7342	0.2342	0.0937	3.4947
	300	0.5088	0.0088	0.0177	0.0458	2.5648	0.0648	0.0259	0.3825
	500	0.5038	0.0038	0.0076	0.0283	2.5246	0.0246	0.0098	0.2177
	800	0.5029	0.0029	0.0059	0.0200	2.5162	0.0162	0.0065	0.1612
	1000	0.5018	0.0018	0.0036	0.0173	2.5111	0.0111	0.0044	0.1411
$\eta = 0.5$ $\vartheta = 3.0$	50	1.6879	1.1879	2.3757	3.8723	13.215	10.215	3.4050	33.019
	100	1.0561	0.5561	1.1122	2.2934	7.8390	4.8390	1.6130	19.977
	200	0.6693	0.1693	0.3386	0.9986	4.5028	1.5028	0.5009	8.9565
	300	0.5857	0.0857	0.1714	0.6280	3.7482	0.7482	0.2494	5.6077
	500	0.5231	0.0231	0.0461	0.2377	3.2068	0.2068	0.0689	2.2106
	800	0.5071	0.0071	0.0143	0.0412	3.0603	0.0603	0.0201	0.4037
	1000	0.5061	0.0061	0.0122	0.0346	3.0538	0.0538	0.0179	0.3457

10.57647/mathsci.2025.1903.16

Table 3. Monte Carlo Simulation analysis for the PRLE distribution

Parameters	n	$\hat{\eta}$				$\hat{\vartheta}$			
		$E(\hat{\eta})$	$Bias(\hat{\eta})$	$MRE(\hat{\eta})$	$RMSE(\hat{\eta})$	$E(\hat{\vartheta})$	$Bias(\hat{\vartheta})$	$MRE(\hat{\vartheta})$	$RMSE(\hat{\vartheta})$
$\eta = 1.0$ $\vartheta = 2.0$	50	1.1892	0.1892	0.1892	2.3827	2.6447	0.6447	0.3223	8.6356
	100	1.0319	0.0319	0.0319	0.5280	2.0943	0.0943	0.0471	2.1060
	200	1.0092	0.0092	0.0092	0.0648	2.0185	0.0185	0.0093	0.1476
	300	1.0064	0.0064	0.0064	0.0520	2.0078	0.0078	0.0039	0.1109
	500	1.0023	0.0023	0.0023	0.0387	2.0048	0.0048	0.0024	0.0825
	800	1.0018	0.0018	0.0018	0.0316	2.0017	0.0017	0.0009	0.0640
	1000	1.0013	0.0013	0.0013	0.0283	2.0026	0.0026	0.0013	0.0574
$\eta = 1.0$ $\vartheta = 2.0$	50	2.3485	1.3485	1.3485	5.8909	7.8261	5.3261	2.1305	23.638
	100	1.5168	0.5168	0.5168	3.2324	4.5648	2.0648	0.8259	13.209
	200	1.1148	0.1148	0.1148	1.2577	2.9503	0.4503	0.1801	5.2230
	300	1.0282	0.0282	0.0282	0.2953	2.5995	0.0995	0.0398	1.2281
	500	1.0129	0.0129	0.0129	0.1175	2.5429	0.0429	0.0172	0.5085
	800	1.0066	0.0066	0.0066	0.0480	2.5243	0.0243	0.0097	0.2066
	1000	1.0057	0.0057	0.0057	0.0412	2.5219	0.0219	0.0088	0.1764
$\eta = 1.0$ $\vartheta = 3.0$	50	3.5992	2.5992	2.5992	8.0711	14.223	11.223	3.7409	35.013
	100	2.5869	1.5869	1.5869	5.9179	9.8930	6.8930	2.2977	25.751
	200	1.5974	0.5974	0.5974	3.0048	5.6441	2.6441	0.8814	13.482
	300	1.2910	0.2910	0.2910	1.8367	4.2859	1.2859	0.4286	8.2495
	500	1.0625	0.0625	0.0625	0.5273	3.2709	0.2709	0.0903	2.3906
	800	1.0246	0.0246	0.0246	0.1658	3.1052	0.1052	0.0351	0.7813
	1000	1.0149	0.0149	0.0149	0.0843	3.0636	0.0636	0.0212	0.4156
$\eta = 1.5$ $\vartheta = 2.0$	50	2.0644	0.5644	0.3763	4.7952	3.3483	1.3483	0.6742	11.834
	100	1.5828	0.0828	0.0552	1.2509	2.1833	0.1833	0.0917	3.3227
	200	1.5153	0.0153	0.0102	0.1054	2.0219	0.0219	0.0109	0.1892
	300	1.5108	0.0108	0.0072	0.0837	2.0121	0.0121	0.0060	0.1442
	500	1.5073	0.0073	0.0049	0.0656	2.0083	0.0083	0.0042	0.1082
	800	1.5038	0.0038	0.0025	0.0500	2.0088	0.0088	0.0044	0.0849
	1000	1.5039	0.0039	0.0026	0.0447	2.0049	0.0049	0.0024	0.0735
$\eta = 1.5$ $\vartheta = 2.5$	50	4.4874	2.9874	1.9916	11.243	10.4013	7.9013	3.1605	29.794
	100	2.7721	1.2721	0.8481	6.6186	5.9406	3.4406	1.3763	18.239
	200	1.8357	0.3357	0.2238	2.9868	3.3946	0.8946	0.3578	8.3659
	300	1.5825	0.0825	0.0550	1.0295	2.7024	0.2024	0.0810	2.8119
	500	1.5291	0.0291	0.0194	0.2114	2.5719	0.0719	0.0287	0.5923
	800	1.5132	0.0132	0.0088	0.0800	2.5268	0.0268	0.0107	0.2415
	1000	1.5083	0.0083	0.0055	0.0678	2.5198	0.0198	0.0079	0.2066
$\eta = 1.5$ $\vartheta = 3.0$	50	6.5605	5.0605	3.3737	14.469	17.445	14.445	4.8150	41.277
	100	4.4271	2.9271	1.9514	9.7922	11.515	8.5148	2.8383	28.515
	200	2.8190	1.3190	0.8793	5.7016	6.8864	3.8864	1.2955	16.826
	300	2.1346	0.6346	0.4231	3.5135	4.8678	1.8678	0.6226	10.493
	500	1.6849	0.1849	0.1232	1.5145	3.5420	0.5420	0.1807	4.5258
	800	1.5754	0.0754	0.0503	0.7273	3.2192	0.2192	0.0731	2.2202
	1000	1.5380	0.0380	0.0253	0.2285	3.1104	0.1104	0.0368	0.7338

Taking the natural logarithm of the likelihood function, the log-likelihood function becomes

$$l(y) = nln(\eta) + ln(\vartheta)\sum_{i=1}^{n} y_i - nln(\vartheta - 1) - ln(\eta + \vartheta)\sum_{i=1}^{n}(y_i + 2) + \sum_{i=1}^{n} ln\big((\vartheta - 2)(\eta + \vartheta) + \eta(y_i + 1)\big)$$

To obtain the MLEs of η and ϑ, we differentiate the log-likelihood with respect to both parameters and equate the results to zero.

$$\frac{\partial l}{\partial \eta} = \frac{n}{\eta} - \frac{\sum_{i=1}^{n}(y_i + 2)}{\eta + \vartheta} + \sum_{i=1}^{n}\frac{\vartheta + y_i - 1}{(\vartheta - 2)(\eta + \vartheta) + \eta(y_i + 1)} = 0,$$

$$\frac{\partial l}{\partial \vartheta} = \frac{\sum_{i=1}^{n} y_i}{\vartheta} - \frac{n}{(\vartheta - 1)} - \frac{\sum_{i=1}^{n}(y_i + 2)}{(\eta + \vartheta)} + \sum_{i=1}^{n}\frac{2\vartheta + \eta - 2}{(\vartheta - 2)(\eta + \vartheta) + \eta(y_i + 1)} = 0.$$

These two nonlinear equations are solved simultaneously to obtain the estimates of η and ϑ. Since analytical solutions are generally not possible, numerical optimization methods such as the Newton–Raphson or Fisher scoring algorithms are recommended to compute the MLEs.

In this section, we also performed a detailed Monte Carlo simulation study to explore the behaviour of the above-derived ML estimators based on different sample sizes and different choices of parameters. The random samples are generated from the PRLE distribution based on the following parameter settings: $(\eta = 0.3, \vartheta = 2.0)$, $(\eta = 0.3, \vartheta = 2.5)$, $(\eta = 0.3, \vartheta = 3.0)$, $(\eta = 0.5, \vartheta = 2.0)$, $(\eta = 0.3, \vartheta = 2.5)$, $(\eta = 0.3, \vartheta = 3.0)$, $(\eta = 1.0, \vartheta = 2.0)$, $(\eta = 1.0, \vartheta = 2.5)$, $(\eta = 1.0, \vartheta = 3.0)$, $(\eta = 1.5, \vartheta = 2.0)$, $(\eta = 1.5, \vartheta = 2.5)$, $(\eta = 1.5, \vartheta = 3.0)$. For each sample size, ranging from n=50, 100, 200, 300, 500, 800, and 1000, a total of N =10,000 samples are generated from the PRLE distribution. To access the estimation accuracy, renowned indicators are utilized, such as Bias, mean relative error (MRE), and root mean square error (RMSE). The computation for each indicator is as follows.

$$Bias(\hat{\Omega}) = \frac{1}{N}\sum_{i=1}^{N}\left|\hat{\Omega} - \Omega\right|,$$

$$MRE(\hat{\Omega}) = \frac{1}{N}\sum_{i=1}^{N}\frac{\left|\hat{\Omega} - \Omega\right|}{\Omega},$$

and

$$RMSE(\hat{\omega}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\Omega} - \Omega)^2}.$$

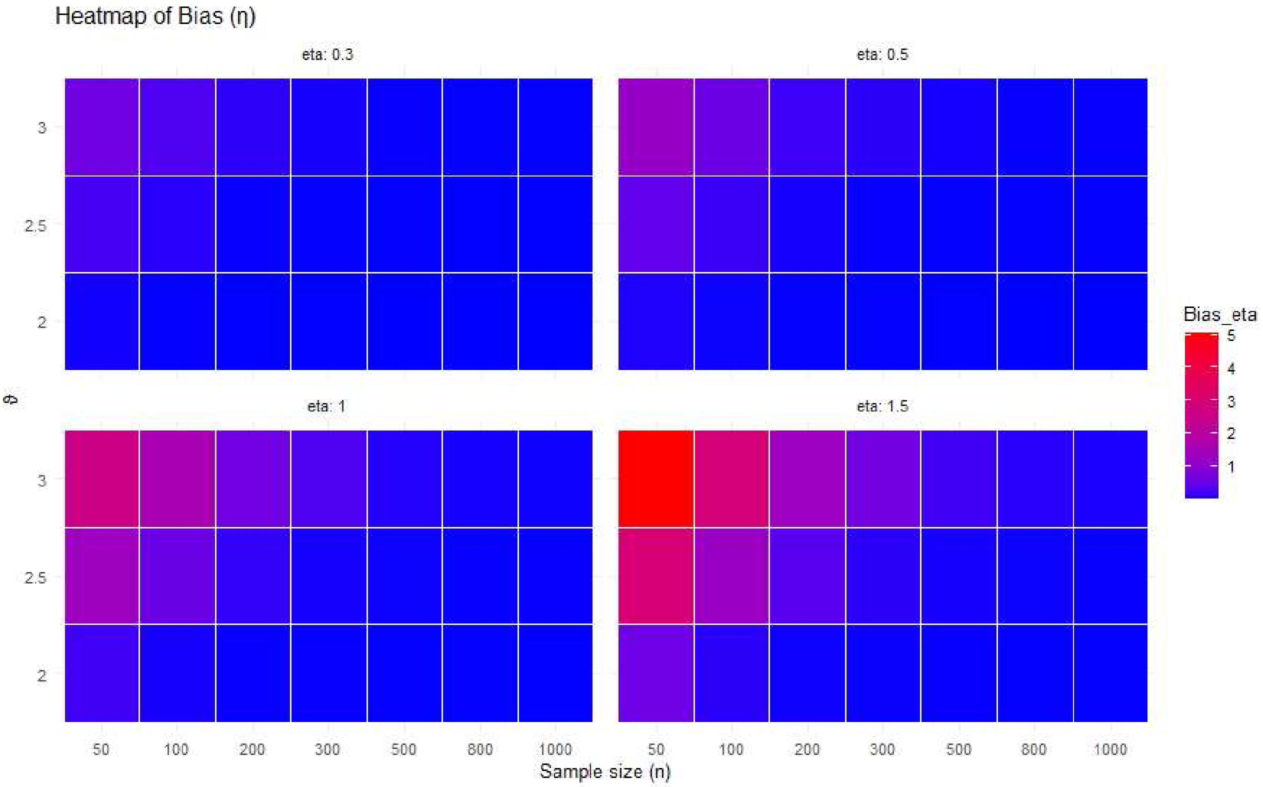

Figure 4. Heatmap plot of Bias for the parameter η

The simulated study in Tables 2 and 3 suggests that both parameters can be associated with a consistent estimation as the sample size increases, and η exhibited more favourable behaviour and lower bias than ϑ. The dispersion parameter ϑ necessitates bigger samples to make credible estimation, especially when the counts of observations are skewed. These trends verify the consistency, asymptotic unbiasedness, and increasing precision of the MLEs of the PRLE model.

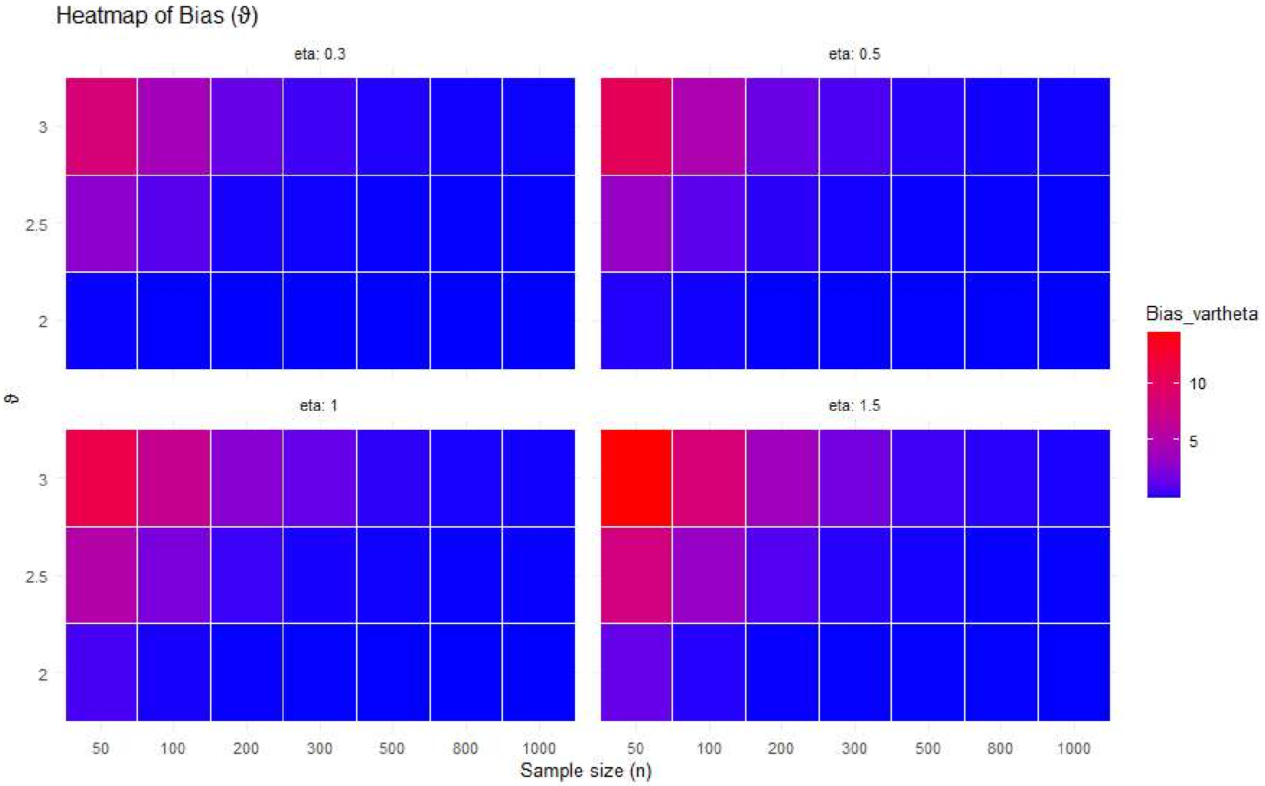

Figure 5. Heatmap plot of Bias for the parameter ϑ

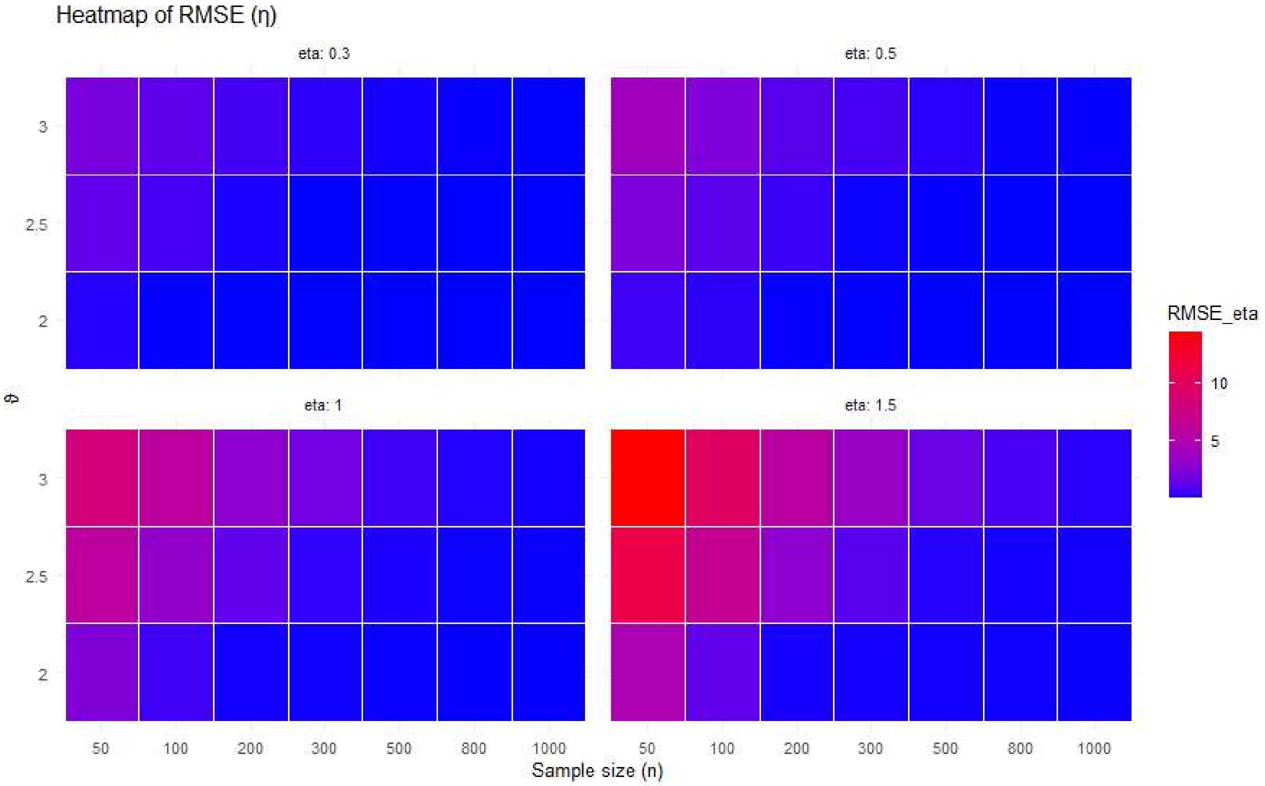

Figure 6. Heatmap plot of RMSE for the parameter η

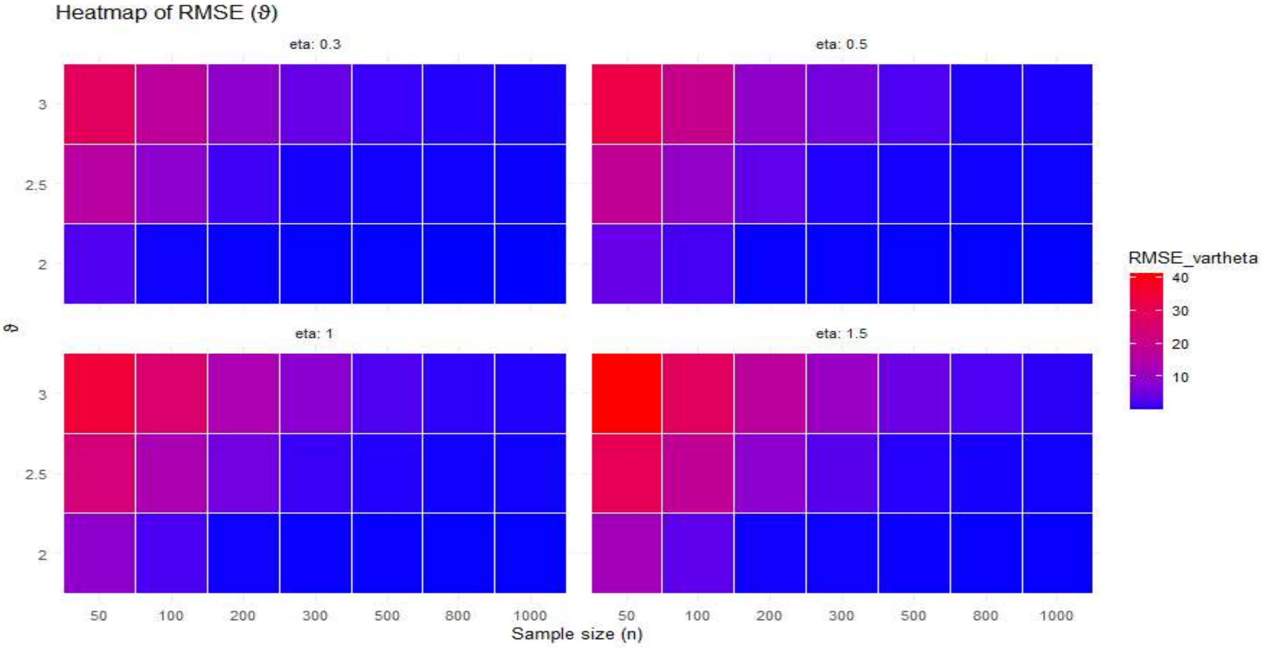

Figure 7. Heatmap plot of RMSE for the parameter ϑ

4. Construction of PRLE regression model

As the earlier part of this work indicates that the PRLE distribution may be applied to examine data sets that are over-dispersed, a critical point to note, as in real life, most data are over-dispersed.

In this part, the PRLE distribution proposed a new count regression to test over-dispersed data sets.

4.1. Model derivation

Suppose that Y follows a PRLE distribution. To begin with, let's consider the following re-parameterization

$$\vartheta = \frac{\eta\mu \pm \sqrt{\eta^2\mu^2 - 4\eta\mu}}{2}$$

By using this approach, we can obtain the PMF of PRLE distribution in terms of mean $E(Y) = \mu > 0$ and $\vartheta > 0$. Then the corresponding PMF is obtained as

$$P(Y = y, \eta, \vartheta) = \frac{\frac{\vartheta^2}{\mu(\vartheta - 1)}\left((\vartheta - 2)\left(\frac{\eta\mu + \sqrt{\eta^2\mu^2 - 4\eta\mu}}{2} + \vartheta\right)\vartheta^y + \frac{\eta\mu + \sqrt{\eta^2\mu^2 - 4\eta\mu}}{2}\vartheta^y(y + 1)\right)}{(\vartheta - 1)\left(\frac{\eta\mu + \sqrt{\eta^2\mu^2 - 4\eta\mu}}{2} + \vartheta\right)^{y+2}} \tag{9}$$

The above PMF is represented as $PRLE(\eta, \vartheta)$, thus we write $Y \sim PRLE(\eta, \vartheta)$. Moreover, the mean response variable is associated with the set of covariates through a log-linear link function expressed as

$$\mu_i = E(Y_i|\eta) = \exp(x_i^T\eta), \quad i = 1,2,\dots,n \tag{10}$$

where x_i represents the vector of explanatory variables corresponding to the i^{th} observation, and $\eta = (\eta_0, \eta_1, \eta_2, \dots, \eta_n)^T$ is the vector of regression coefficients. Now, substituting equ. (10) into equ. (9), the PMF of $(Y_i|x_i^T) \sim PRLE(\mu_i, \eta)$ can be expressed in a regression form as follows:

$$P\left({}^{y_i}/_{x_i^T}, \eta\right) = \left[\frac{\frac{\vartheta^2}{\mu(\vartheta - 1)} \times \left(\begin{array}{c}(\vartheta - 2)\left(\frac{\eta\exp(x_i^T\eta) + \sqrt{\eta^2(\exp(x_i^T\eta))^2 - 4\eta\exp(x_i^T\eta)}}{2} + \vartheta\right) \times \vartheta^y \\ + \frac{\eta\exp(x_i^T\eta) + \sqrt{\eta^2(\exp(x_i^T\eta))^2 - 4\eta\exp(x_i^T\eta)}}{2}\vartheta^y(y + 1)\end{array}\right)}{(\vartheta - 1)\left(\frac{\eta\exp(x_i^T\eta) + \sqrt{\eta^2(\exp(x_i^T\eta))^2 - 4\eta\exp(x_i^T\eta)}}{2} + \vartheta\right)^{y+2}}\right] \tag{11}$$

10.57647/mathsci.2025.1903.16

4.2. Model estimation

The regression parameter vector η and the model parameter η are estimated utilizing the MLE technique. For the proposed regression, the log-likelihood function for a random sample $(y_1, x_1), (y_2, x_2), \dots, (y_n, x_n)$ is given by

$$l(\Phi) = \sum_{i=1}^{n} \left[\frac{\dfrac{\vartheta^2}{\mu(\vartheta-1)} \times \left(\begin{array}{c} (\vartheta-2)\left(\dfrac{\eta \exp(x_i^T\eta) + \sqrt{\eta^2(\exp(x_i^T\eta))^2 - 4\eta \exp(x_i^T\eta)}}{2} + \vartheta \right) \times \vartheta^{y_i} \\ + \dfrac{\eta \exp(x_i^T\eta) + \sqrt{\eta^2(\exp(x_i^T\eta))^2 - 4\eta \exp(x_i^T\eta)}}{2} \vartheta^{y_i}(y_i+1) \end{array} \right)}{(\vartheta-1)\left(\dfrac{\eta \exp(x_i^T\eta) + \sqrt{\eta^2(\exp(x_i^T\eta))^2 - 4\eta \exp(x_i^T\eta)}}{2} + \vartheta \right)^{y_i+2}} \right]$$

.

The MLE of the parameter vector $\Phi = (\eta, \eta)$ is obtained by maximizing the above log-likelihood function. The numerical maximization is performed utilizing the *optim* function of the R software.

6. Application of PRLE distribution

In this section, we demonstrate the application of the PRLE distribution by fitting it to two real-world datasets from different fields. This helps illustrate that the proposed distribution provides a better fit compared to several other distributions.

The comparisons are made against Poisson Xrani (PXR) distribution, Poisson Ram Awadh (PRA) distribution (Shukla et al., 2022), Poisson weighted entropy-based exponential (PEWE) distribution, Poisson Ramose-Louzada (PRL) distribution (Alkhairy, 2023), discrete inverted top Leone (DITL) distribution (Eldeeb et al., 2021), discrete Bilal (DBI) distribution (Altun et al., 2020), and classical Poisson distribution. The competing distributions were selected to give a thorough comparison in terms of their applicability to the modelling of overdispersed or heterogeneous count data. This choice will provide an evaluation of a broad spectrum of modelling methods of count data, which will bring out the practical benefits of the PRLE distribution.

6.1. Performance metrics

To identify the best-fitting distribution, we use several evaluation criteria to assess the distribution flexibility and applicability.

- **Log-Likelihood:** The log-likelihood calculates how well the distribution explains the observed data. The definition of log-likelihood is expressed as:

  $$l(\Phi) = \log\left(\prod_{i=1}^{n} P(Y = y_i; \Phi) \right).$$

  The higher the value of l indicate the better the fitting.

- **Akaike Information Criterion (AIC):** The AIC is utilized to assess the relative merits of various statistical distributions. The AIC is defined by the formula:

  $$AIC = 2p - 2l(\Phi)$$

  where m is the number of parameters and n number of observations, directly affecting the $l(\Phi)$ which measures the distribution's ability to accurately describe the observed data.

- **Bayesian Information Criterion (BIC):** The BIC model selection is like AIC. The BIC is defined by the formula:

  $$AIC = m \ln(n) - 2l(\Phi)$$

6.1. Radiation data

The initial set of data is the number of dibasic centric chromosomes in the peripheral blood following a dose of 1480 MeV oxygen ions at 1.600 Gy. Earlier, researchers (Oliveira et al., 2016; Puig & Barquinero, 2011) have also explored this radiation information. We plot the non-parametric plots to explore the nature of the dataset and present it in Fig. 4. The first dataset is highly skewed. The scatter plot and strip chart highlight the clustering of low counts and sparse higher values. The boxplot and violin plots confirm a low median, a small interquartile range, and the presence of outliers at larger values. The Q-Q plot also supports these findings, which are skewed and deviated from normality. Table 4 shows observed and expected frequencies of the dicentric chromosome data as well as estimates of parameters and goodness-of-fit (GOF) measures of some of the competing models. The findings indicate that the PRLE distribution is the most optimal and has the smallest log-likelihood ($-l$ = 227.0341), AIC (458.0683), and BIC (464.3749) values. It also contains the least chi-square value ($\chi^2 =$ 1.3890), and the largest p-value (0.4993), which means that there is an excellent agreement between observed and expected frequencies.

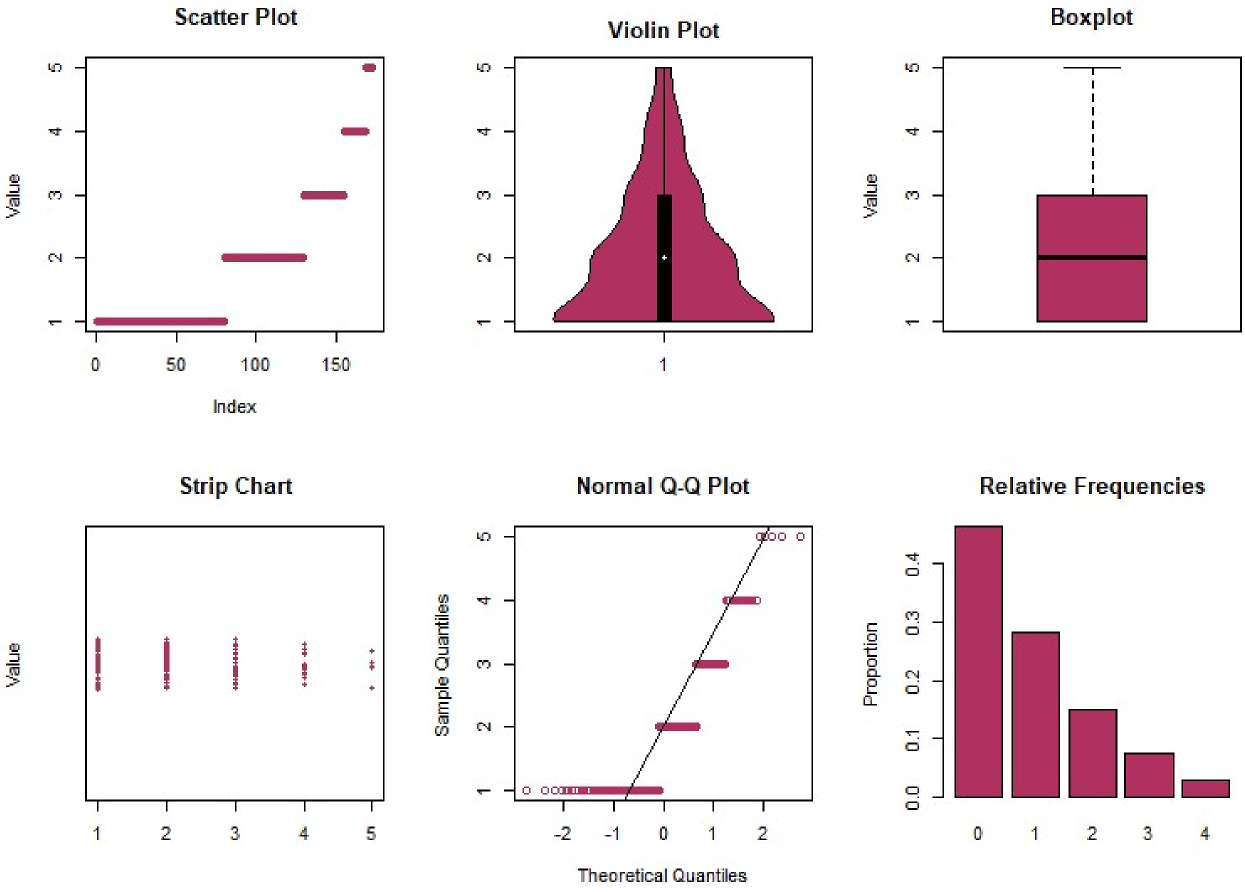

Figure 4. Non-parameter plots for the radiation dataset

Table 4. MLEs and model goodness of fit measures for the radiation dataset

Y	Observed frequency	Expected frequencies							
		PRLE	PXR	PRA	PEWE	PRL	DITL	DBI	Poisson
0	80	79.0367	94.4006	93.6724	41.3816	26.5072	91.5822	71.9069	68.6068
1	49	53.1521	38.39141	36.5745	48.4700	25.0014	44.3277	61.5992	63.4544
2	26	24.7801	18.4145	18.9113	35.0905	22.3219	17.2545	25.8529	29.3445
3	13	10.0851	9.9055	10.9864	21.6982	19.2274	7.9340	9.1737	9.0469
4	5	5.9458	11.8879	12.8555	26.3597	79.9421	11.9015	4.4673	2.5474
Total	$n = 173$								
ML Est.	$\hat{\vartheta}$	1.8638	1.9447	2.5491	1.2348	2.6395	2.6198	1.7060	1.4833
	$\hat{\eta}$	4.3483				-			
GOF Measures	$-l$	227.0341	234.8651	235.1868	258.4307	338.5122	236.2535	228.6398	229.7038
	AIC	458.0683	471.7301	472.3735	518.8613	679.0245	474.5069	459.2797	461.4076
	BIC	464.3749	474.8834	475.5268	522.0146	682.1778	477.6600	462.4330	464.5609
	χ^2	1.3890	13.2104	14.0434	59.1953	203.8647	13.6267	5.1482	9.6545
	Degree of Freedom	2	3	3	3	3	3	3	3
	p-value	0.4993	0.0042	0.0028	0.0000	0.0000	0.0034	0.1612	0.0217

10.57647/mathsci.2025.1903.16

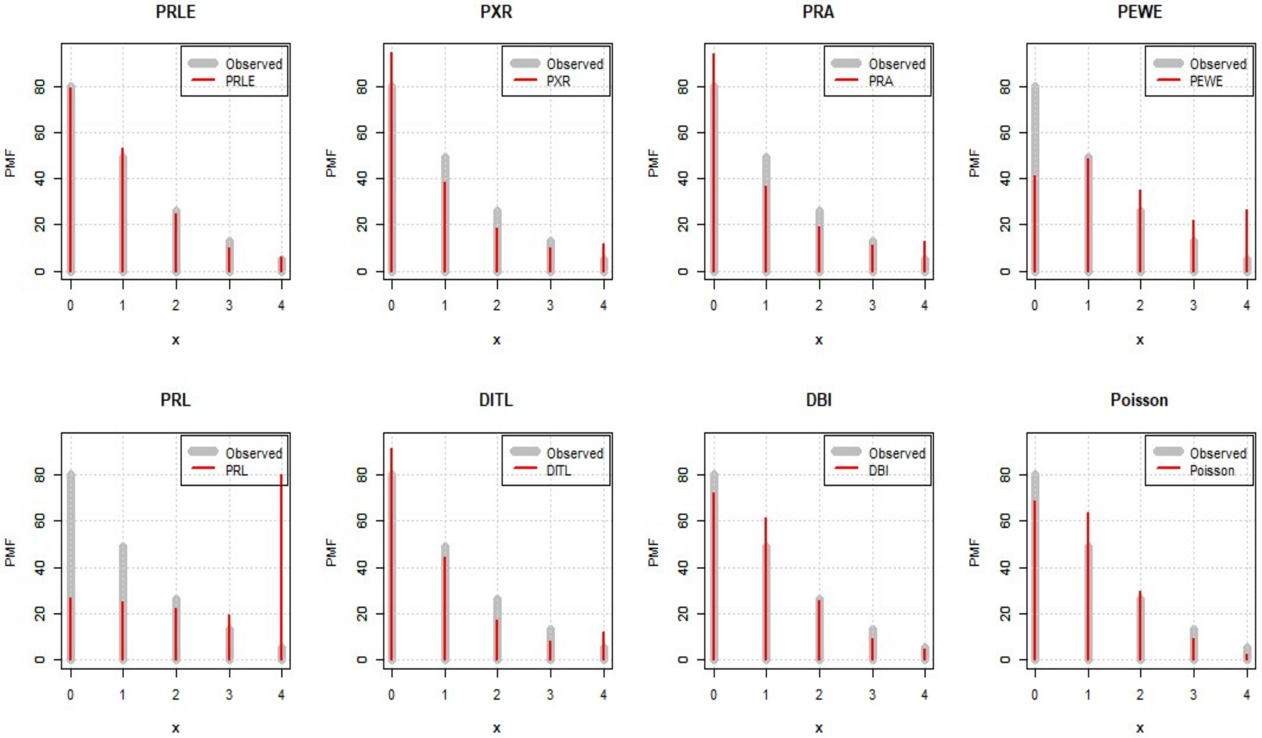

Figure 5. Fitted PMFs for radiation data

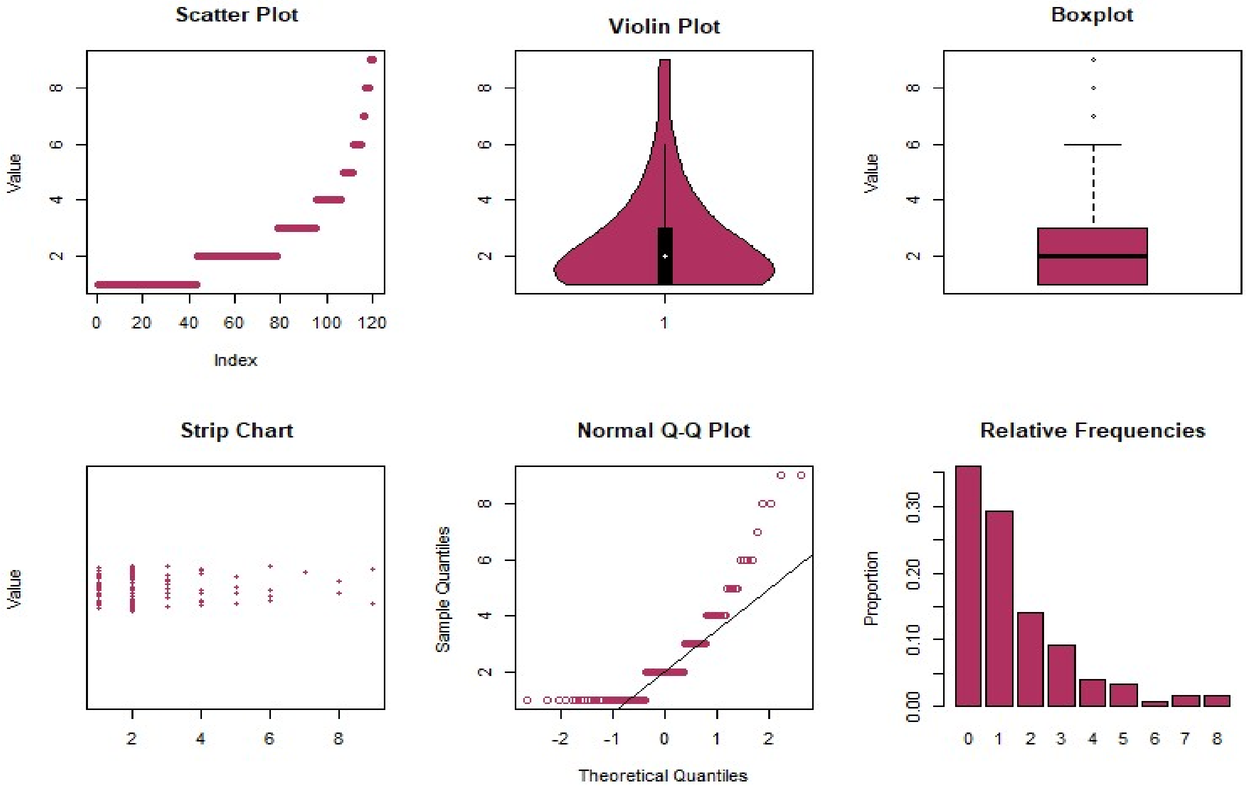

Figure 6. Non-parameter plot for European corn borer dataset

Conversely, the other models like PXrani, PRA, PEWE, and PRL present greater values of chi-square and significantly small p-values, implying poor fits. The DITL, DBI, and Poisson models prove to be unimpressive, though not yet at par with PRLE. Therefore, it is known that the PRLE distribution obviously gives better results than all other competitive models and gives the most acceptable expression of the dicentric chromosome data.In addition, the values of the estimated parameters,

$\hat{\vartheta} = 1.8638$ and $\hat{\eta} = 4.3483$,

signify a moderate value of the mean count with high dispersion and tail heaviness, which explains why the suggested model fits.

Table 5. MLEs and model goodness of fit measures for the corn borer dataset

Y	Observed frequency	Expected frequencies							
		PRLE	PXR	PRA	PEWE	PRL	DITL	DBI	Poisson
0	43	44.6150	50.5569	49.5250	29.5926	18.0502	52.1886	32.73939	27.2265
1	35	30.4592	25.6348	25.0011	32.0156	17.4679	30.4244	39.5880	40.3851
2	17	19.0659	15.8936	16.7041	23.5134	15.7629	14.1124	24.2756	29.9516
3	11	11.3365	10.6241	11.6089	15.0096	13.6192	7.4663	12.5053	14.8091
4	5	6.5152	7.0031	7.5526	8.9052	11.4229	4.3900	5.9683	5.4916
5	4	3.6548	4.4104	4.5283	5.0507	9.3766	2.7906	2.7363	1.6291
6	1	2.3942	2.6418	2.5236	2.7784	7.5718	1.8811	1.2258	0.4028
7	2	1.0937	1.5115	1.3230	1.4950	6.0364	1.3271	0.5415	0.0853
8	2	1.2459	1.7234	1.2331	1.6391	20.6917	5.4195	0.4194	0.0189
Total	$n = 120$								
ML	$\hat{\vartheta}$	2.7548	1.7019	2.25030	1.1332	2.5243	1.9840	2.3767	1.4833
Est.	$\hat{\eta}$	2.9151	-	-	-	-	-	-	-
GOF Measures	$-l$	200.4152	202.156	202.3913	205.2743	244.501	205.1517	204.6753	219.1879
	AIC	404.8303	406.3139	406.7826	412.5486	491.002	412.3033	411.3505	440.3759
	BIC	410.4053	409.1014	409.5701	415.3361	493.790	415.0908	414.138	443.1634
	χ^2	1.4444	5.3752	5.7969	11.300	84.667	6.9766	9.6427	38.5362
	DF	7				7			
	P-value	0.6951	0.2509	0.2148	0.0457	0.0000	0.1371	0.0468	0.0000

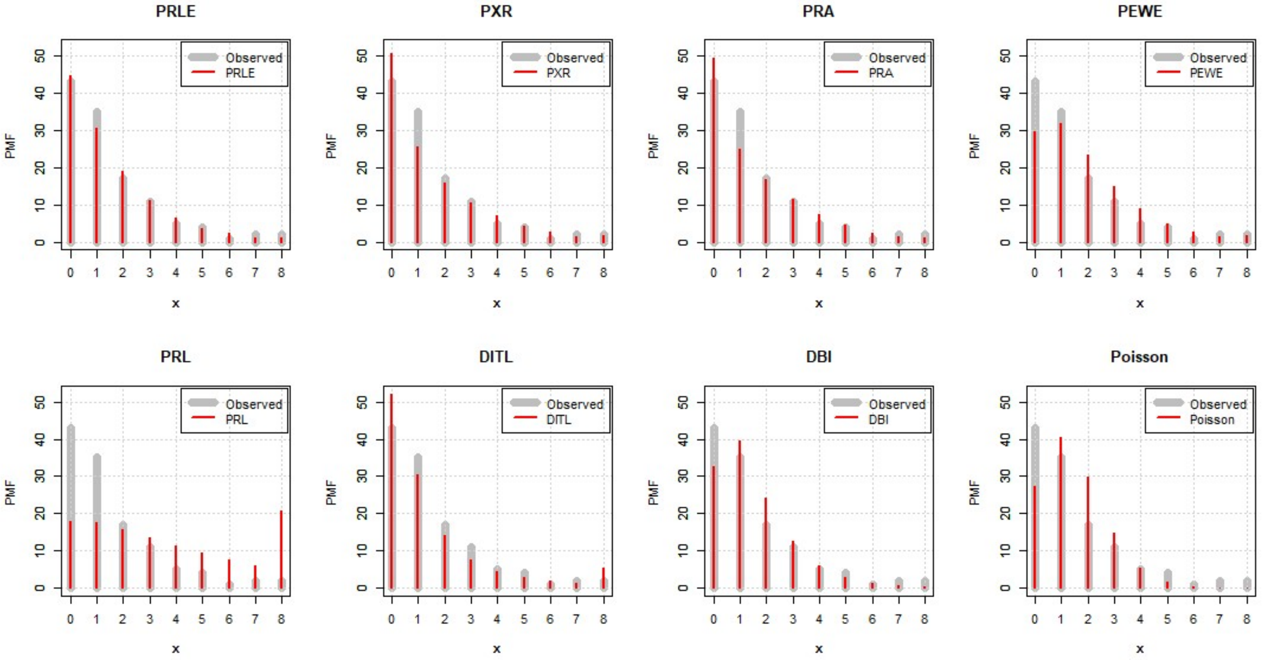

Figure 7. Fitted PMFs for European corn borer data

6.2. European Corn Borer

The second data is based on a biological experiment, the European corn borer, as given in Table 5. In the experiment, the researcher counted the number of borers per hill of corn on 15 random replications with 8 hills each(Beall, 1940). This dataset is very informative concerning the distribution of borers and could be used to analyze the operation of the suggested model. The values of the estimated parameters, $\hat{\vartheta} = 2.7548$ and $\hat{\eta} = 2.9151$, suggest a higher mean count with moderate dispersion, supporting the flexibility and applied relevance of the PRLE distribution. The observed and expected frequencies of the second dataset are summarized in Table 5, and parameter estimates and goodness-of-fit statistics are provided for a variety of candidate models. These findings also indicate clearly that the PRLE distribution is the most suitable of all models. It is the one with the minimum log-likelihood ($-l = 200.4152$), AIC (404.8303), BIC (410.4053), and the lowest chi-square ($\chi^2 = 1.4444$) and the highest p-value (0.6951). These results prove the high level of correspondence between the measured and predicted frequencies under the PRLE model. The PRLE p-value is lower and the chi-square greater than the PRLE, making other models like PXr, PRA, and DITL offer moderate fits to the data. Other models, such as PEWE, PRL, DBI, and Poisson, have significantly poorer performance, where the p-values are very low, which implies poor fitting. Overall, such findings support the idea that PRLE distribution is the best model to be used to describe the second dataset and illustrate its versatility and strength over rival models.

7. Conclusion

In this study, we introduced the Poisson Ramos Louzada-Exponential compound distribution as a flexible and parsimonious probability distribution for over-dispersed count observations. The statistical properties of the new model are derived, including moments, identifiability based on CDF and PMF, recurrence relation, survival function, hazard function, mean residual life function, reversed hazard function, cumulative hazard rates, Mills ratio, odd function, generating function, and order statistics. The parameter estimation of the PRLE distribution is performed by utilizing the renowned maximum likelihood estimation approach. Comprehensive Monte Carlo simulation results demonstrate desirable estimation properties, small bias, decreasing RMSE with higher values of sample size, and stable convergence behavior, confirming the robustness of the derived estimators. Applications to two empirical datasets, radiation and European corn borer, indicate the proposed count probability distribution consistently achieves competitive or superior values of AIC, BIC, and Chi-Square goodness-of-fit measures when compared to considered models. These model selection measures highlight the applicability and effectiveness of the PRLE distribution as a tractable model alternative for analysis of over-dispersed count data. Future research directions include the detailed study of regression models, zero-inflated and hurdle generalizations, survival analysis based on censored data, or time-dependent count data.

Funding
This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [KFU254763].

Conflict of Interest
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability
The data supporting the findings of this study are included within the article.

References

Adetunji, A. A., & Sabri, S. R. (2023). On the Poisson transmuted Ailamujia distribution with applications to dispersed and skewed count data. *J. Stat. Manage. Syst.*, *26*, 929–943.Afify, A. Z.,

Ahsan-ul-Haq, M., Aljohani, H. M., Alghamdi, A. S., Babar, A., & Gómez, H. W. (2022). A new one-parameter discrete exponential distribution: Properties, inference, and applications to COVID-19 data. *Journal of King Saud University - Science*, *34*(6), 102199. https://doi.org/10.1016/j.jksus.2022.102199

Ahsan-ul-Haq, M. (2022). On Poisson Moment Exponential Distribution with Applications. *Annals of Data Science*. https://doi.org/10.1007/s40745-022-00400-0

Ahsan-ul-Haq, M., Al-Bossly, A., El-Morshedy, M., & Eliwa, M. S. (2022). Poisson XLindley Distribution for Count Data: Statistical and Reliability Properties with Estimation Techniques and Inference. *Computational Intelligence and Neuroscience*, *2022*. https://doi.org/10.1155/2022/6503670

Ahsan-ul-Haq, M., Memon, A. Z., & Chand, S. (2022). An Analysis of Generalized Exponential Uniform Distribution. *Mathematical Problems in Engineering*, 1–15. https://doi.org/10.1155/2022/6040943

Alghamdi, A. S., Ahsan-ul-Haq, M., Babar, A., Aljohani, H. M., & Afify, A. Z. (2022). The discrete power-Ailamujia distribution: properties, inference, and applications. *AIMS Mathematics*, *7*(5), 8344–8360.

Alghamdi, F. M., Ahsan-ul-Haq, M., Hussain, M. N. S., Hussam, E., Almetwally, E. M., Aljohani, H. M., Mustafa, M. S., Alshawarbeh, E., & Yusuf, M. (2024). Discrete Poisson Quasi-XLindley distribution with mathematical properties, regression model, and data analysis. *Journal of Radiation Research and Applied Sciences*, *17*(2), 100874. https://doi.org/ 10.1016/j.jrras.2024.100874

Aljohani, H. M., Zaghdoun, F. M., Meraou, M. A., Alharthi, A. S., Almohri, W. A. J., Kalantan, Z. I., EL-Helbawy, A. A., Hussam, E., & Muse, A. H. (2025). A novel extension of the exponential distribution with application in modeling complex lifetime and environmental data. In *Scientific Reports* (Vol. 15, Issue 1). https://doi.org/10.1038/s41598-025-18711-6

Alkhairy, I. (2023). Classical and Bayesian inference for the discrete Poisson Ramos-Louzada distribution with application to COVID-19 data. *Mathematical Biosciences and Engineering*, *20*(8), 14061–14080. https://doi.org/10.3934/mbe.2023628

Alomair, A., & Ahsan-ul-Haq, M. (2023). A new extension of Poisson distribution for asymmetric count data: theory, classical and Bayesian estimation with application to lifetime data. *PeerJ Computer Science*, *9*, e1748. https://doi.org/10.7717/peerj-cs.1748

Alomair, A. M., & Ahsan-ul-Haq, M. (2025a). A new mixed Poisson Komal distribution with application on radiation, agricultural and medical sciences data. *Journal of Radiation Research and Applied Sciences*, *18*(2), 101500.

Alomair, A. M., & Ahsan-ul-Haq, M. (2025b). Analysis of radiation and corn borer data using discrete Poisson Xrama distribution. *Journal of Radiation Research and Applied Sciences*, *18*(2), 101388.

https://doi.org/10.1016/j.jrras.2025.101388

Alomair, A. M., Ayyaz, F., Tariq, S., & Ahsan-ul-Haq, M. (2025). Discrete Poisson Haq distribution with mathematical properties and count data modeling. *Scientific Reports*, *15*(1), 1–18. https://doi.org/10.1038/s41598-025-07223-y

Alomair, A. M., Nasir, M., Hussain, S., Javed, A., & Ahsan-ul-haq, M. (2025). A new discrete Burr III distribution for modeling radiation and biological data. *Journal of Radiation Research and Applied Sciences*, *18*(2), 101366. https://doi.org/10.1016/j.jrras.2025.101366

Alomair, M. A., Alomair, A. M., Ahsan-ul-Haq, M., & Alotaibi, E. S. (2025). Poisson length biased Chris-Jerry distribution for analyzing radiation, ecology, agricultural, and health data. *Journal of Radiation Research and Applied Sciences*, *18*(3), 101753. https://doi.org/10.1016/j.jrras.2025.101753

Alrumayh, A., & Khogeer, H. A. (2023). A New Two-Parameter Discrete Distribution for Overdispersed and Asymmetric Data: Its Properties, Estimation, Regression Model, and Applications. *Symmetry*, *15*(6), 1289. https://doi.org/10.3390/sym15061289

Altun, E., El-Morshedy, M., & Eliwa, M. S. (2020). A study on discrete Bilal distribution with properties and applications on integer-valued autoregressive process. *Revstat. Stat. J*, *18*, 70–99.

Beall, G. (1940). The fit and significance of contagious distributions when applied to observations on larval insects. *Ecology*, *21*(4), 460–474.

Borbye, S., Nasiru, S., & Ajongba, K. K. (2024). Poisson XRani Distribution: An Alternative Discrete Distribution for Overdispersed Count Data. *International Journal of Mathematics and Mathematical Sciences*, *2024*. https://doi.org/10.1155/2024/555494

Eldeeb, A. S., Ahsan-ul-Haq, M., & Babar, A. (2021). A Discrete Analog of Inverted Topp-Leone Distribution: Properties, Estimation and Applications. *International Journal of Analysis and Applications*, *19*(5), 695–708.

Hassan, H., Dar, S. A., & Ahmad, P. B. (2019). Poisson Ishita distribution: A new compounding probability model. *IOSR Journal of Engineering (IOSRJEN)*, *9*(2), 38–46.

Mahnashi, A. M., & Zaagan, A. A. (2025). Poisson copoun distribution : An alternative discrete model for count data analysis. *Alexandria Engineering Journal*, *128*(May), 571–584. https://doi.org/10.1016/j.aej.2025.05.085

Naz, T., Sallam, G., Abdalla, S., Shafiq, A., & Abushal, T. A. (2025). A new statistical framework for overdispersed count data : Applications in public health , radiation dosimetry and finance. *Journal of Radiation Research and Applied Sciences*, *18*(4), 101987. https://doi.org/10.1016/j.jrras.2025.10198

Oliveira, M., Einbeck, J., Higueras, M., Ainsbury, E., Puig, P., & Rothkamm, K. (2016). Zero-inflated regression models for radiation-induced chromosome aberration data: A comparative study. *Biometrical Journal*, *58*(2), 259–279

Puig, P., & Barquinero, J. F. (2011). An application of compound Poisson modelling to biological dosimetry. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *467*(2127), 897–910.

Shukla, K. K., Shanker, R., & Tiwari, M. K. (2022). A new one parameter discrete distribution and its applications. *Journal of Statistics and Management Systems*, *25*(1), 269–283.

Sindhu, T. N., Shafiq, A., Atangana, A., Abushal, T. A., & Alkhathami, A. A. (2025). Control Charts for Overdispersed Count Data: Exploring the Poisson Chris-Jerry Distribution in Agriculture and Medicine. *Quality and Reliability Engineering International.*