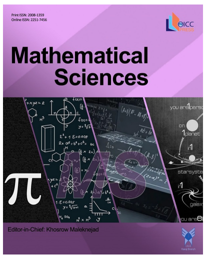

Math. Sci. 2025; 19 (4): 1-11

Mathematical Sciences (MS)

https://doi.org/10.57647/mathsci.2025.1904.19

Original Research

Approximate Solution for the Fractional Riccati/Logistic Differential Equations Employing the β-Khalouta Decomposition Technique

M. M. Khader[1], K. Lotfy[2], A. Alawfi[1], M. Adel[3,*],

[1] *Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt*
[3] *Department of Mathematics, Faculty of Science, Islamic University of Madinah, Medina, Saudi Arabia*

*Corresponding author: adel@sci.cu.edu.eg

Received:
16 November 2025
Revised:
28 November 2025
Accepted:
06 December 2025
Published in Issue:
31 December 2025

Abstract:

We provide an approximation solution to the Riccati/Logistic differential equations (RDE/LDE) with the Caputo-Katugampola fractional derivative. The proposed methodology relies on the β-Khalouta decompositionmethod (β-KDM). The proposed approach integrates the β-Khalouta transform method with a decomposition technique. The stability study encompasses the uniqueness, convergence, and error estimation of the proposed scheme. The residual error function is computed and utilized as a fundamental criterion for assessing the accuracy and efficiency of the specified numerical method. We employ the exact solution and the fourth-order Runge-Kutta method for comparison with the results obtained from the employed method. The results confirm that the used method is a straightforward and efficient instrument for the numerical simulation of these models. Illustrative models are provided to validate the efficacy and utility of the suggested approach.

Keywords: Riccati/Logistic differential equation; Caputo-Katugampola fractional derivative; Khalouta transform technique; Decomposition method; Convergence analysis

Cite this article: Khader. M.M., Lotfy. K., Alawfi. A., Adel. M., Approximate Solution for the Fractional Riccati/Logistic Differential Equations Employing the β-Khalouta Decomposition Technique. Math. Sci. 2025; *19* (4): 1-11 https://doi.org/10.57647/mathsci.2025.1904.19

1. Introduction

Financial mathematics is one of the most important branches of mathematics, which includes applications to problems of diffusion, random processes, and others. One of the most important of these problems is the Riccati differential equation [1], the basic theories of which were presented in [2]. This equation type has numerous applications in engineering sciences, alongside significant and conventional uses such as stochastic perception theory, network architecture, optimum control, and elastic stability. From this standpoint, many classical numerical techniques have been used, including, but not limited to, Euler's forward method, Runge-Kutta methods, HAM [3], variational iteration method [4], unconditional stable method [5], and others [6].

In 1838, the so-called logistic differential equation model was introduced by Pierre Verhulst [7]. These models have been employed to characterize chaotic behavior and periodic multiplication of some well-known dynamic systems. One of the most popular models of this type of equation is the population model, which contains many variables [8]. In medicine, the logistic curve is also commonly utilized to model tumor progression using the LDE, in which the tumor size at time t is expressed as $N(t)$. Finally, ecology is an important field in which this type of logistic differential equation appears.

Some studies demonstrate breakthroughs in the analysis and numerical treatment of complicated differential systems with memory effects, stochastic dynamics, singular perturbations, and temporal delays ([9]-[10]). The development of optimal- and higher-order numerical schemes for reaction-diffusion, convection-diffusion, and Robin-type parabolic problems with small perturbation parameters, boundary layers, and large delays shows a holistic approach to robustness ([11]-[12]).

Many papers illustrate a broad yet deeply connected advancement in modern scientific computing, unifying classical analytical perspectives with emerging computational paradigms ([13], [14]). The progression from traditional option-pricing algorithms to more efficient and robust computational strategies reflects the growing demand for scalable methods across complex application domains. Innovations such as modified iterative PINN algorithms for strongly coupled boundary-layer convection-diffusion-reaction systems ([15]-[16]).

In fact, fractional derivatives (FD) provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Many researchers have found that the fractional differential equations (FDEs) play important roles in many fields, such as physics, population dynamics, chemical technology, biotechnology, and economics [17]. The corresponding FDs were introduced in [18], which so-called Katugampola fractional operators, and the existence and uniqueness results for this derivative are given in [19]. This type of derivative generalizes two other fractional operators, by introducing a new parameter $\beta > 0$ in the definition. Indeed, if we take $\beta \longrightarrow 1$, we have the Riemann-Liouville fractional derivative. In addition, Khalouta transform is a generalization of many integral transforms having exponential type kernel, such as the Laplace-Carson transform, Elzaki transform, Sumudu transform, Aboodh transform, Shehu transform, and Natural transform.

In general, semi-approximate analysis techniques or traditional analysis techniques cannot provide exact closed-form or approximate solutions for nonlinear systems of the FDEs. Therefore, there is an urgent need for efficient numerical techniques capable of finding exact or accurate approximate solutions for these systems ([20], [21]). The main motivation of this paper is to propose a new methodology of β-KDM to tackle the solutions of the Caputo-Katugampola fractional derivative (CKFD), for the proposed models. The β-KDM approach combines the β-Khalouta transform method [22] and a new decomposition method [23]. The proposed method yields a closed-form solution expressed as an infinite series, which converges swiftly to the precise solution. Furthermore, it has been noted that the outcomes achieved surpass those of the approaches documented in the literature. This method provides approximate solutions for nonlinear systems of FDEs.

https://doi.org/10.57647/mathsci.2025.1904.19

2. Basic concepts

This section explains some basic formulas, concepts, and results for the fractional calculus and β-Khalouta transform of the CKFD and related formulas that will be relevant throughout the work.

2.1 Basic concepts on fractional calculus

Definition 2.1 *Let the function $\Theta : \mathbb{R}^+ \longrightarrow \mathbb{R}$, then the Katugampola fractional integral of order α, β is given as:*

$$\mathbb{I}^{\alpha,\beta}\Theta(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\beta - \tau^\beta}{\beta}\right)^{\alpha-1} \frac{\Theta(\tau)}{\tau^{1-\beta}} d\tau, \qquad \beta > 0,\ 0 < \alpha \leq 1, \tag{1}$$

where $\Gamma(.)$ is the gamma function.

Definition 2.2 *Let the function $\Theta : \mathbb{R}^+ \longrightarrow \mathbb{R}$, then the Caputo-Katugampola FD of order α, β is given as [24]:*

$$\begin{aligned}\mathbb{D}^{\alpha,\beta}\Theta(t) &= \mathbb{I}^{n-\alpha,\beta}\left[\eta^n\, \Theta(t)\right] \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \left(\frac{t^\beta - \tau^\beta}{\beta}\right)^{n-\alpha-1} \frac{\eta^n \Theta(\tau)}{\tau^{1-\beta}} d\tau,\end{aligned} \tag{2}$$

where the differential operator η is given by $\eta = t^{1-\beta}\frac{d}{dt}$ and $n-1 < \alpha \leq n, \beta > 0$.

For equations (1) and (2), we have the following relations:

$$\mathbb{D}^{\alpha,\beta}\mathbb{I}^{\alpha,\beta}\Theta(t) = \Theta(t), \tag{3}$$

$$\mathbb{I}^{\alpha,\beta}\mathbb{D}^{\alpha,\beta}\Theta(t) = \Theta(t) - \sum_{j=0}^{n} \frac{D^{\alpha-j,\beta}\Theta(0)}{\Gamma(\alpha-j+1)} \left(\frac{t^\beta}{\beta}\right)^{\alpha-j}. \tag{4}$$

For more details concerning the fractional calculus, see ([25]-[26]).

2.2 Basic concepts on β-Khalouta transform

Now, we present our main results concerning the β-Khalouta transform of the CKFD

Definition 2.3 *[22] The β-Khalouta transform of the function $\Theta(t)$ is defined as:*

$$\begin{aligned}\mathbb{KH}_\beta[\Theta(t)] &= \mathcal{K}_\beta(s,\gamma,\rho) \\ &= \frac{s}{\gamma\rho} \int_0^\infty \exp\left(-\frac{s}{\gamma\rho}\frac{t^\beta}{\beta}\right) \frac{\Theta(t)}{t^{1-\beta}} dt, \qquad \beta > 0,\end{aligned} \tag{5}$$

where $s > 0, \gamma > 0$, and $\rho > 0$ are the Khalouta transform variables.

Theorem 2.4 1. *For all real constants c_1 and c_2, we have:*

$$\begin{aligned}&\mathbb{KH}_\beta[c_1\Theta(t) \pm c_2\Psi(t)] \\ &= c_1\mathbb{KH}_\beta[\Theta(t)] \pm c_2\mathbb{KH}_\beta[\Psi(t)].\end{aligned} \tag{6}$$

2. *Let* a, b, *and* $c \in \mathbb{R}$ *and* $\beta > 0$, *then*

$$\begin{aligned}
&\mathbb{KH}_\beta[a] = a,\\
&\mathbb{KH}_\beta\left[t^b\right] = \left(\frac{\beta\gamma\rho}{s}\right)^{\frac{b}{\beta}} \Gamma\left(\frac{b}{\beta}+1\right),\\
&\mathbb{KH}_\beta\left[\frac{t^{n\beta}}{\beta^n}\right] = \left(\frac{\gamma\rho}{s}\right)^n \Gamma(n+1),\\
&\mathbb{KH}_\beta\left[\exp\left(c\,\frac{t^\alpha}{\alpha}\right)\right] = \frac{s}{s-c\gamma\rho}.
\end{aligned} \tag{7}$$

3. *Let* $\Theta \in C_\eta^{n-1}(\mathbb{R}^+)$, *then the* β*-Khalouta transform of* $\eta^n\Theta(t)$ *is defined by:*

$$\begin{aligned}
&\mathbb{KH}_\beta\left[\eta^n\,\Theta(t)\right]\\
&= \left(\frac{s}{\gamma\rho}\right)^n \mathbb{KH}_\beta[\Theta(t)] - \sum_{\ell=0}^{n-1}\left(\frac{s}{\gamma\rho}\right)^{n-\ell} \eta^\ell\Theta(0).
\end{aligned} \tag{8}$$

Proof See [22]. □

Theorem 2.5 *The* β*-Khalouta transform of the CKFD of order* $\alpha, \beta > 0$ *of the function* $\Theta(t)$ *is expressed as:*

$$\begin{aligned}
&\mathbb{KH}_\beta\left[\mathbb{D}^{\alpha,\beta}\Theta(t)\right]\\
&= \left(\frac{s}{\gamma\rho}\right)^\alpha \mathbb{KH}_\beta[\Theta(t)] - \sum_{\ell=0}^{n-1}\left(\frac{s}{\gamma\rho}\right)^{\alpha-\ell} \eta^\ell\,\Theta(0).
\end{aligned} \tag{9}$$

Proof The proof of this Theorem can be found in [27]. □

3. Description of β-Khalouta decomposition method

This section describes the new methodology of β-KDM to solve the following nonlinear FDE:

$$\mathbb{D}^{\alpha,\beta}(\phi(t)) + \mathbb{L}(\phi(t)) + \mathbb{N}(\phi(t)) = g(t), \tag{10}$$

with the initial condition (I.C):

$$\phi(0) = \phi^0, \tag{11}$$

where $0 < \alpha \leq 1$ and $\beta > 0$; $\mathbb{L}$ and $\mathbb{N}$ represent linear and nonlinear operators, respectively, and g is the non-homogeneous term. This will be achieved by following the following steps [28]:

1. To solve equation (10) with the I.C (11), we consider the following equation:

$$\begin{aligned}
&\mathbb{D}^{\alpha,\beta}\phi_p(t) = p\left[g(t) - \mathbb{L}\left(\phi_p(t)\right) - \mathbb{N}\left(\phi_p(t)\right)\right],\\
&\phi_p(0) = \phi_p^0, \quad p \in [0,1].
\end{aligned} \tag{12}$$

2. We suppose that the solution of (12) will be given in the following form:

$$\phi_p(t) = \sum_{k=0}^{\infty} p^k\,\phi_{p,k}(t). \tag{13}$$

3. Applying the β-Khalouta transform on equation (12), we get:

$$\begin{aligned}
&\mathbb{KH}_\beta\left[\mathbb{D}^{\alpha,\beta}\phi_p(t)\right]\\
&= p\,\mathbb{KH}_\beta\left[g(t) - \mathbb{L}\left(\phi_p(t)\right) - \mathbb{N}\left(\phi_p(t)\right)\right].
\end{aligned} \tag{14}$$

4. Using the formula (9), we have:

$$\begin{aligned}
&\mathbb{KH}_\beta\left[\phi_p(t)\right] = \phi_p(0)\\
&+ \left(\frac{\gamma\rho}{s}\right)^\alpha p\,\mathbb{KH}_\beta\left[g(t) - \mathbb{L}\left(\phi_p(t)\right) - \mathbb{N}\left(\phi_p(t)\right)\right].
\end{aligned} \tag{15}$$

5. Taking the inverse β-Khalouta transform of equation (15) to get:

$$\begin{aligned}
\phi_p(t) = \phi_p^0 + p\,\mathbb{KH}_\beta^{-1}\Bigg[\left(\frac{\gamma\rho}{s}\right)^\alpha\\
\mathbb{KH}_\beta\left[g(t) - \mathbb{L}\left(\phi_p(t)\right) - \mathbb{N}\left(\phi_p(t)\right)\right]\Bigg].
\end{aligned} \tag{16}$$

6. Substituting from (13) into (16), the following equation is obtained:

$$\begin{aligned}
&\sum_{k=0}^{\infty} p^k\,\phi_{p,k}(t) = \phi_p^0 + p\,\mathbb{KH}_\beta^{-1}\\
&\times\left[\left(\frac{\gamma\rho}{s}\right)^\alpha \mathbb{KH}_\beta\left[g(t) - \mathbb{L}\left(\sum_{k=0}^{\infty} p^k\phi_{p,k}(t)\right)\right.\right.\\
&\left.\left.-\mathbb{N}\left(\sum_{k=0}^{\infty} p^k\,\phi_{p,k}(t)\right)\right]\right].
\end{aligned} \tag{17}$$

7. Application of the new decomposition method [23] to equation (17) implies:

$$\begin{aligned}
&\sum_{k=0}^{\infty} p^k\,\phi_{p,k}(t) = \phi_p^0 + p\,\mathbb{KH}_\beta^{-1}\\
&\times\left[\left(\frac{\gamma\rho}{s}\right)^\alpha \mathbb{KH}_\beta\left[g - \mathbb{L}\left(\sum_{k=0}^{\infty} p^k\phi_{p,k}\right)\right.\right.\\
&\left.\left.-\sum_{k=0}^{\infty} p^k\mathbb{A}_{p,k}\right]\right],
\end{aligned} \tag{18}$$

where $\mathbb{A}_{p,\ell}$ are the Adomian's polynomials of $\phi_{p,0}, \phi_{p,1}, \ldots, \phi_{p,k}$ defined by:

$$\begin{aligned}
&\mathbb{A}_{p,\ell} = \frac{1}{\ell!}\frac{d^\ell}{d\lambda^\ell}\left[\mathbb{N}\left(\sum_{j=0}^{\ell}\lambda^j\phi_{p,j}\right)\right]_{\lambda=0},\\
&\ell = 0, 1, 2, \ldots.
\end{aligned} \tag{19}$$

8. By equating the identical powers of p in (18), we can get the following relations:

$$\begin{aligned}
\phi_{p,0}(t) =&\phi_p^0,\\
\phi_{p,1}(t) =&\mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^\alpha \mathbb{KH}_\beta\left[g(t)\right.\right.\\
&\left.\left.-\mathbb{L}\left(\phi_{p,0}(t)\right) - \mathbb{A}_{p,0}\right]\right],\\
\phi_{p,k}(t) =&\mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^\alpha \mathbb{KH}_\beta\left[-\mathbb{L}\left(\phi_{p,k-1}(t)\right)\right.\right.\\
&\left.\left.-\mathbb{A}_{p,k-1}\right]\right], \quad k = 2, 3, 4, \ldots.
\end{aligned} \tag{20}$$

9. Substituting the components of (20) into (13) gives the solution of equation (12). Now, according to (13), we get:

$$\phi(t) = \lim_{p\to 1} \phi_p(t) = \sum_{\ell=0}^{\infty} \phi_{p,\ell}(t). \tag{21}$$

Substituting (20) into (21), we get:

$$\begin{aligned}\phi(t) &= \sum_{\ell=0}^{\infty} \phi_{p,\ell}(t)\\ &=\phi_0(t) + \phi_1(t) + \sum_{\ell=2}^{\infty} \mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^{\alpha} \mathbb{KH}_\beta\right.\\ &\quad\left.\times\left[-\mathbb{L}\left(\phi_{\ell-1}\right) - \mathbb{A}_{\ell-1}\right]\right].\end{aligned} \tag{22}$$

4. Convergence analysis

Here, we state the convergence and uniqueness statements of the solutions.

Theorem 4.1 *(Uniqueness theorem) [27] The solution for the nonlinear FDE (10) in the domain $[0,T]$ obtained by β-KDM is unique for $0<\delta<1$, and $|\mathbb{L}(\phi-\bar{\phi})| < \xi_1|\phi-\bar{\phi}|$, $|\mathbb{N}(\phi-\bar{\phi})| < \xi_2|\phi-\bar{\phi}|$, for some constants $\xi_1, \xi_2 < 1$:*

$$\delta = (\xi_1+\xi_2)\kappa T, \qquad \kappa = \max_{0\le t\le T} \frac{(t-\tau)^{\alpha\beta}}{\Gamma(\alpha+1)\,\beta^{\alpha}}.$$

Theorem 4.2 *(Convergence theorem) Consider the Banach space $\mathcal{B} = ([0,1], \|.\|_{\mathcal{B}})$ of all continuous functions on $[0,T]$ with the norm expressed as $\|\phi(t)\|_{\mathcal{B}} = \max_{0\le t\le 1}|\phi(t)|$. Then, according to Banach's fixed-point theorem [29], the sequence obtained by β-KDM converges to the solution (fixed point) under the following condition:*

$$\left\|S_k - S_q\right\|_{\mathcal{B}} \le \frac{\delta^q}{1-\delta}\left\|S_1 - S_0\right\|_{\mathcal{B}}, \tag{23}$$

where δ is defined in Theorem 4.1, and $\{S_k\}_{k\ge 0}$ is the sequence of partial sums of the series described by $S_k = \sum_{j=0}^{k}\phi_j(t)$.

Proof We demonstrate that the sequence $\{S_k\}_{k\ge 0}$ is a Cauchy sequence (CS) in $\mathcal{B}$:

$$\begin{aligned}\left\|S_k - S_q\right\|_{\mathcal{B}} &= \max_{0\le t\le 1}\left|S_k - S_q\right|\\ &= \max_{0\le t\le 1}\left|\mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^{\alpha}\mathbb{KH}_\beta\right.\right.\\ &\quad\left.\left.\times\left[\mathbb{L}\left(S_{k-1}-S_{q-1}\right) + \mathbb{N}\left(S_{k-1}-S_{q-1}\right)\right]\right]\right|\\ &\le \max_{0\le t\le 1}\left|\mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^{\alpha}\mathbb{KH}_\beta\right.\right.\\ &\quad\left.\left.\times\left[\left|\mathbb{L}(S_{k-1}-S_{q-1})\right| + \left|\mathbb{N}(S_{k-1}-S_{q-1})\right|\right]\right]\right|\\ &\le \max_{0\le t\le 1}\left(\xi_1\left|S_{k-1}-S_{q-1}\right|\right.\\ &\quad\left.+\xi_2\left|S_{k-1}-S_{q-1}\right|\right)\mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^{\alpha}\right]\\ &\le \max_{0\le t\le 1}\left(\xi_1\left|S_{k-1}-S_{q-1}\right|\right.\\ &\quad\left.+\xi_2\left|S_{k-1}-S_{q-1}\right|\right)\frac{t^{\alpha\beta}}{\Gamma(\alpha+1)\beta^{\alpha}}\\ &\le (\xi_1+\xi_2)\left\|S_{k-1}-S_{q-1}\right\|_{\mathcal{B}}\frac{t^{\alpha\beta}}{\Gamma(\alpha+1)\,\beta^{\alpha}}.\end{aligned} \tag{24}$$

Now, by using the convolution theorem, we can obtain the following formula:

$$\begin{aligned}&\left\|S_k - S_q\right\|_{\mathcal{B}}\\ &\le \int_0^t (\xi_1+\xi_2)\left\|S_{k-1}-S_{q-1}\right\|_{\mathcal{B}}\frac{(t-\tau)^{\alpha\beta}}{\Gamma(\alpha+1)\,\beta^{\alpha}}d\tau.\end{aligned} \tag{25}$$

Using the mean value theorem (MVT) of integral calculus [30], leads us to:

$$\begin{aligned}\left\|S_{\ell,k} - S_{\ell,q}\right\|_{\mathcal{B}} &\le ((\xi_1+\xi_2)\kappa T)\left\|S_{\ell,k-1} - S_{\ell,(q-1)}\right\|_{\mathcal{B}}\\ &\le \delta\left\|S_{\ell,k-1} - S_{\ell,(q-1)}\right\|_{\mathcal{B}},\end{aligned} \tag{26}$$

where κ and δ are defined in Theorem 3. Choosing $k = q+1$, then we find:

$$\begin{aligned}\left\|S_{q+1} - S_q\right\|_{\mathcal{B}} &\le \delta\left\|S_q - S_{q-1}\right\|_{\mathcal{B}} \le \delta^2\left\|S_{q-1} - S_{q-2}\right\|_{\mathcal{B}}\\ &\le \ldots \le \delta^q\left\|S_1 - S_0\right\|_{\mathcal{B}}.\end{aligned} \tag{27}$$

Using the triangle inequality, we get:

$$\begin{aligned}\left\|S_k - S_q\right\|_{\mathcal{B}} &= \left\|S_{q+1} - S_q + S_{q+2} - S_{q+1}\right.\\ &\quad\left.+\ldots + S_k - S_{k-1}\right\|_{\mathcal{B}}\\ &\le \left\|S_{q+1} - S_q\right\|_{\mathcal{B}} + \left\|S_{q+2} - S_{q+1}\right\|_{\mathcal{B}}\\ &\quad+\ldots + \left\|S_k - S_{k-1}\right\|_{\mathcal{B}}\\ &\le \delta^q\left\|S_1 - S_0\right\|_{\mathcal{B}} + \delta^{q+1}\left\|S_1 - S_0\right\|_{\mathcal{B}}\\ &\quad+\ldots + \delta^{k-1}\left\|S_1 - S_0\right\|_{\mathcal{B}}\\ &= \delta^q\left(1+\delta+\ldots+\delta^{k-q-1}\right)\left\|S_1 - S_0\right\|_{\mathcal{B}}\\ &\le \delta^q\left(\frac{1-\delta^{k-q}}{1-\delta}\right)\left\|S_1 - S_0\right\|_{\mathcal{B}}.\end{aligned} \tag{28}$$

Now, since $0<\delta<1$, we have $1-\delta^{k-q} < 1$, thus we have:

$$\left\|S_k - S_q\right\|_{\mathcal{B}} \le \frac{\delta^q}{1-\delta}\left\|S_1 - S_0\right\|_{\mathcal{B}}. \tag{29}$$

https://doi.org/10.57647/mathsci.2025.1904.19

For $\|S_1 - S_0\|_{\mathcal{B}} < +\infty$, so as $q \to \infty$ then $\left\|S_k - S_q\right\|_{\mathcal{B}} \to 0$. Thus, the sequence $\{S_k\}_{k\geq 0}$ is the CS in $\mathcal{B}$, and so the convergence of the sequence is achieved. $\square$

5. Application the approximation method

5.1 Implementation β-KDM on LDE

In this example, we consider the following LDE ([31], [26]):

$$\mathbb{D}^{\alpha,\beta}\theta(t) = \xi\,\theta(t)(1-\theta(t)), \quad \theta(0) = \hat{\theta}, \quad \xi > 0. \tag{30}$$

Where $\theta(t) = \hat{\theta}\left((1-\hat{\theta})e^{-\xi t} + \hat{\theta}\right)^{-1}$ is the corresponding exact solution. The existence and uniqueness of (30) can be found in detail at [32].

To apply the β-KDM and obtain the required scheme for solving (30) inside the domain $[0,1]$, we follow the following steps (algorithm):

1. We assume the following model ($p \in [0,1]$):

   $$\mathbb{D}^{\alpha,\beta}\theta_p(t) = p\,\left[\xi\,\theta_p(t)(1-\theta_p(t))\right], \tag{31}$$

   and the solution of equation (31) will be expressed as:

   $$\theta_p(t) = \sum_{k=0}^{\infty} p^k\,\theta_{p,k}(t). \tag{32}$$

2. Applying the β-Khalouta transform on (31) and Theorem 4.2, we get:

   $$\mathbb{KH}_\beta\left[\theta_p(t)\right] = \hat{\theta} + \left(\frac{\gamma\rho}{s}\right)^{\alpha} p\,\mathbb{KH}_\beta\left[\xi\,\theta_p(t)(1-\theta_p(t))\right]. \tag{33}$$

3. Taking the inverse β-Khalouta transform on (33), we get:

   $$\theta_p(t) = \hat{\theta} + p\,\mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^{\alpha}\mathbb{KH}_\beta\left[\xi\,\theta_p(t) - \xi\,\theta_p^2(t)\right]\right]. \tag{34}$$

4. Application of the new decomposition method [23] implies:

   $$\sum_{i=0}^{\infty} p^i\,\theta_{p,i}(t) = \hat{\theta} + p\,\mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^{\alpha}\mathbb{KH}_\beta \times\left[\xi\sum_{i=0}^{\infty} p^i\,\theta_{p,i} - \xi\sum_{i=0}^{\infty} p^i\,\mathbb{A}_{p,i}\right]\right], \tag{35}$$

   where $\mathbb{A}_{p,i}$ are Adomian's polynomials that respectively represent the nonlinear term $\theta_{p,i}^2(t)$.

5. According to the relation (19), the first components of $\mathbb{A}_{p,i}$ are defined by:

   $$\begin{aligned}\mathbb{A}_{p,0} &= \theta_{p,0}^2,\\ \mathbb{A}_{p,1} &= 2\theta_{p,0}\theta_{p,1},\\ \mathbb{A}_{p,2} &= 2\theta_{p,0}\theta_{p,2} + \theta_{p,1}^2.\end{aligned} \tag{36}$$

6. By equating the identical powers of p in (35), we can obtain the following relations:

   $$\begin{aligned}\theta_{p,0}(t) &= \hat{\theta},\\ \theta_{p,1}(t) &= \mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^{\alpha}\mathbb{KH}_\beta\left[\xi\theta_{p,0} - \xi\mathbb{A}_{p,0}\right]\right],\end{aligned} \tag{37}$$

   $$\begin{aligned}&\theta_{p,i}(t)\\ &= \mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^{\alpha}\mathbb{KH}_\beta\left[\xi\theta_{p,i-1} - \xi\mathbb{A}_{p,i-1}\right]\right],\\ &i = 2,3,\ldots.\end{aligned} \tag{38}$$

7. Now, according to (32), we have:

   $$\theta(t) = \sum_{i=0}^{\infty}\theta_{p,i}(t). \tag{39}$$

   Thus, the following approximations are obtained successively:

   $$\theta_{p,0}(t) = \hat{\theta} = 0.25, \;\; \theta_{p,1}(t) = \frac{0.45}{\Gamma(\alpha+1)}\,\frac{t^{\alpha\beta}}{\beta^{\alpha}}, \;\ldots. \tag{40}$$

   The solution is finally expressed and approximated as follows:

   $$\theta_m(t) = \sum_{i=0}^{m}\theta_{p,i}(t). \tag{41}$$

   The closed solution of equation (30) can be given as follows:

   $$\theta(t) = \lim_{m\to\infty}\theta_m(t). \tag{42}$$

8. To conduct a comprehensive numerical analysis, we assume that the approximate solution ($\theta_m(t)$) for the presented issue may be provided. Consequently, we estimate the residual error function (REF) as follows:

   $$\begin{aligned}&\mathrm{REF}(t;\alpha,\beta,m)\\ &= \mathbb{D}^{\alpha,\beta}\theta_m(t) - \xi\theta_m(t)(1-\theta_m(t)) \simeq 0.\end{aligned} \tag{43}$$

   The minimal residual ($\mathrm{REF}(t;\alpha,\beta,m) \to 0$) indicates that the approximate solution roughly corresponds with the exact solution, indicating that the error approaches zero. This pattern of error is used when the exact solution is unknown, which is sometimes difficult in the case of fractional derivatives. Ultimately, the REF possesses various other forms; for further details, refer to [33].

Remark 5.1 *We have the following three cases:*

1. *Taking $\beta = 1$ in (42), the solution of equation (30) based on the CKFD can be obtained.*
2. *Taking $\alpha = 1$ in (42), the solution of equation (30) based on the CKFD can be obtained.*
3. *Taking $\alpha = \beta = 1$ in (42), the exact solution of equation (30) is available.*

5.2 Implementation β-KDM on RDE

We consider the RDE as follows ([25], [34]):

$$\mathbb{D}^{\alpha,\beta}\phi(t) = 1 - \phi^2(t), \qquad \phi(0) = \hat{\phi}. \tag{44}$$

The exact solution at $\alpha = \beta = 1$ is $\phi(t) = (\hat{\phi}e^{2t} - 1)(1 + \hat{\phi}e^{2t})^{-1}$.

To apply the β-KDM and obtain the required scheme for solving (44) inside the domain $[0, 2.5]$, we follow the following steps (algorithm):

1. We assume the following equation ($p \in [0, 1]$):

   $$\mathbb{D}^{\alpha,\beta}\phi_p(t) = p\left[1 - \phi_p^2(t)\right], \tag{45}$$

   and the solution of equation (45) will be expressed as:

   $$\phi_p(t) = \sum_{i=0}^{\infty} p^i\, \phi_{p,i}(t). \tag{46}$$

2. Applying the β-Khalouta transform on (45) and Theorem 2, we get:

   $$\mathbb{KH}_\beta\left[\phi_p(t)\right] = \hat{\phi} + \left(\frac{\gamma\rho}{s}\right)^\alpha p\, \mathbb{KH}_\beta\left[1 - \phi_p^2(t)\right]. \tag{47}$$

3. Taking the inverse β-Khalouta transform on (47), we get:

   $$\phi_p(t) = \hat{\phi} + p\, \mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^\alpha \mathbb{KH}_\beta\left[1 - \phi_p^2(t)\right]\right]. \tag{48}$$

4. Application of the new decomposition method [23] implies:

   $$\sum_{i=0}^{\infty} p^i\, \phi_{p,i}(t) = \hat{\phi} + p\, \mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^\alpha \mathbb{KH}_\beta\left[1 - \sum_{i=0}^{\infty} p^i\, \mathbb{A}_{p,i}\right]\right], \tag{49}$$

   where $\mathbb{A}_{p,i}$ are Adomian's polynomials which respectively represent the nonlinear term $\phi_{p,i}^2(t)$.

5. According to the relation (19), the first components of $\mathbb{A}_{p,i}$ are given by:

   $$\begin{aligned} \mathbb{A}_{p,0} &= \phi_{p,0}^2, \\ \mathbb{A}_{p,1} &= 2\phi_{p,0}\phi_{p,1}, \\ \mathbb{A}_{p,2} &= 2\phi_{p,0}\phi_{p,2} + \phi_{p,1}^2. \end{aligned} \tag{50}$$

6. By comparing the identical powers of p in (49), the following relations are obtained.

   $$\begin{aligned} \phi_{p,0}(t) &= \hat{\phi}, \\ \phi_{p,1}(t) &= \mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^\alpha \mathbb{KH}_\beta\left[1 - \mathbb{A}_{p,0}\right]\right], \end{aligned} \tag{51}$$

   $$\phi_{p,i}(t) = \mathbb{KH}_\beta^{-1}\left[\left(\frac{\gamma\rho}{s}\right)^\alpha \mathbb{KH}_\beta\left[1 - \mathbb{A}_{p,i-1}\right]\right], \quad i = 2, 3, \dots. \tag{52}$$

7. Now, according to (46), we get:

   $$\phi(t) = \phi_{p,0}(t) + \phi_{p,1}(t) + \sum_{i=2}^{\infty} \phi_{p,i}(t). \tag{53}$$

   Thus, the following approximations are obtained successively:

   $$\phi_{p,0}(t) = \hat{\phi} = 1, \quad \phi_{p,1}(t) = \frac{0.45}{\Gamma(\alpha+1)}\frac{t^{\alpha\beta}}{\beta^\alpha}, \dots. \tag{54}$$

   The solution is finally expressed and approximated as follows:

   $$\phi_m(t) = \sum_{i=0}^{m} \phi_{p,i}(t). \tag{55}$$

   The closed solution of equation (44) can be given by the following form:

   $$\phi(t) = \lim_{m\to\infty} \phi_m(t). \tag{56}$$

8. To conduct a comprehensive numerical analysis, we assume that the approximate solution ($\phi_m(t)$) for the issue under consideration may be provided. Consequently, we estimate the REF as follows:

   $$\mathrm{REF}(t; \alpha, \beta, m) = \mathbb{D}^{\alpha,\beta}\phi_m(t) - 1 + \phi_m^2(t)) \simeq 0. \tag{57}$$

6. Numerical results

6.1 For fractional LDE

The approximate solution of the equation (30) in $[0, 1]$ is presented in Figs. 1–5 with distinct values of initial solutions $\hat{\theta}$, m, ξ, α, β.

1. Fig. 1 presents the exact & approximate solutions using the β-KDM at $\hat{\theta} = 0.25$, with $m = 10$, $\xi = 0.5$ and $\alpha = \beta = 1$.
2. Fig. 2 plots the approximate solution at distinct quantities of $\alpha = \beta = 0.65, 0.75, 0.85, 0.95$, $\xi = 0.5$, $m = 9$, and $\hat{\theta} = 0.25$.
3. Fig. 3 plots the approximate solution at various quantities of $\hat{\theta} = 0.25, 0.5, 0.75, 1.0$ and $\xi = 0.5$, $m = 9$, and $\alpha = \beta = 0.85$.
4. Fig. 4 displays the impact of the parameter ξ on the approximate solution with distinct values of $\xi = 0.4, 0.8, 1.2, 1.4$, at $m = 10$, $\hat{\theta} = 0.5$, and $\alpha = \beta = 0.90$.
5. Fig. 5 presents the REF with $\xi = 0.75$, $\hat{\theta} = 0.5$, $m = 14$, and $\alpha = \beta = 0.93$.

These figures, and the excellent agreement between the given approximate solution and the exact solution, allow us to draw the conclusion that the approach can be implemented successfully to solve the given model. Also, the given approximate solutions depend on the changes in the initial solutions, m, α, β, and ξ.

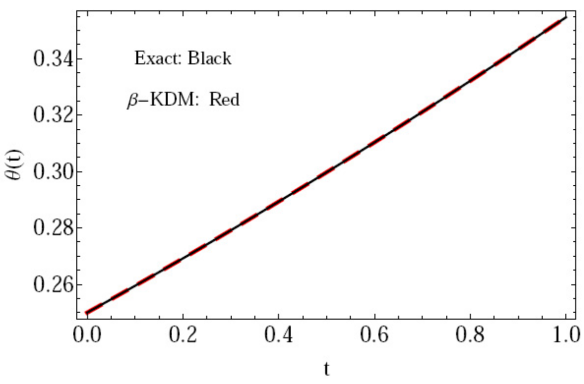

Figure 1. The approximate & exact solutions using β-KDM.

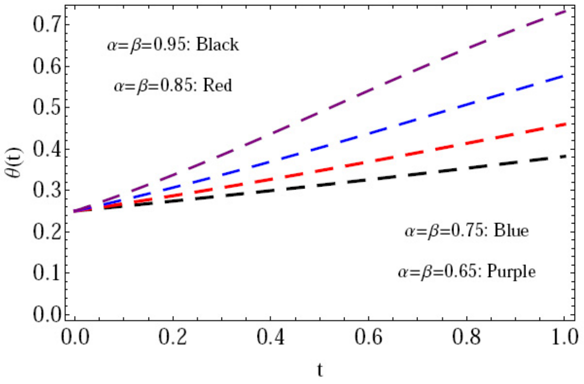

Figure 2. The numerical solution with distinct α, β.

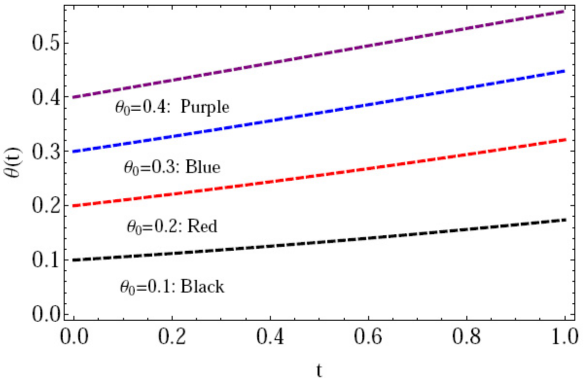

Figure 3. The numerical solution with distinct $\hat{\theta}$.

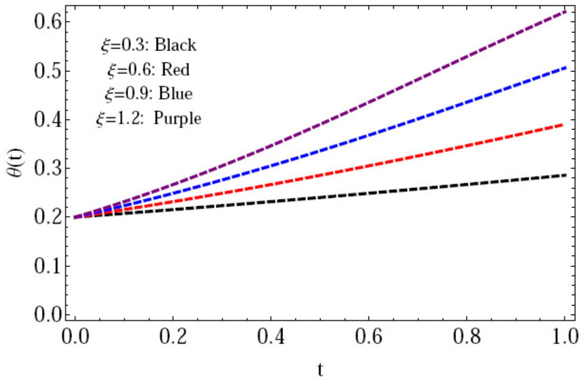

Figure 4. The numerical solution versus ξ.

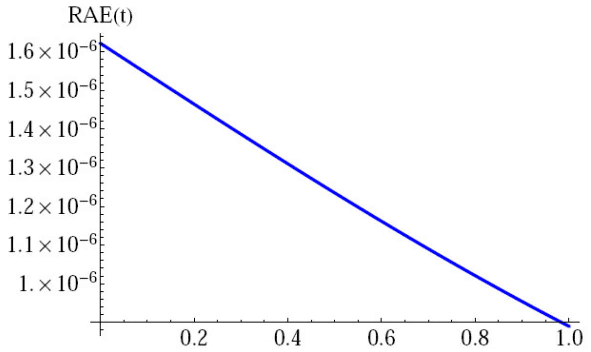

Figure 5. The RAF of the solution at $\xi = 0.75$.

To validate the numerical solutions of the LDE at $\hat{\theta} = 0.25$, $\alpha = \beta = 1.0$ and $\xi = 0.5$ with various values of the approximation order m, a comparison of the AE is presented in Table 1 for the β-KDM and RK4M for the same model. This comparison demonstrates how thorough the approach suggested in this article is a suitable for solving the model under investigation.

Furthermore, to reinforce the assumptions and proofs of the convergence theorems, the values of the convergence constants ξ, ξ_1, ξ_2, κ, δ, and the resulting error were estimated in Table 2. This table shows that when the assumptions are met, the conditions for convergence are fulfilled, but there are some cases where these conditions are not met. This confirms the strong agreement between the theoretical and numerical analysis of the problem under consideration.

6.2 For fractional RDE

Figs. 6–9 present the numerical solution of the equation (44) on $[0, 2.5]$, with varying values of α, β, $\hat{\phi}$, m.

1. Fig. 6 presents the exact & approximate solutions implementing the β-KDM at $\alpha = \beta = 1$, $m = 9$, and $\hat{\phi} = 0$.
2. Fig. 7 gives the approximate solution with different values of $\alpha = \beta = 0.65, 0.75, 0.85, 0.95$, $m = 9$, and $\hat{\phi} = 0$.
3. Fig. 8 gives the approximate solution with different values of initial solutions $\hat{\phi} = 0.2, 0.4, 0.6, 0.8$, $m = 10$ and $\alpha = \beta = 0.9$.
4. Fig. 9 presents the REF with $\hat{\phi} = 0.5$, $m = 11$, $\alpha = \beta = 0.92$.

These figures, and the excellent agreement between the given numerical solution and exact solution, allow us to draw the conclusion that the approach can be implemented successfully to solve the given model.

To validate the numerical solutions of the RDE at $\hat{\phi} = 0.5$ and $\alpha = \beta = 0.92$ with various values of the approximation order m, a comparison of the AE is presented in Table 3 for the β-KDM and the RK4M for the same model. This comparison demonstrates how thorough the approach suggested in this article is for solving the model under study.

Table 1. Comparison of the AE for approximate solutions of the LDE by the β-KDM and RK4M.

	AE of the present method		AE of the RK4M	
t	$m = 8$	$m = 16$	$h = 0.2$	$h = 0.1$
0.0	2.159753E-04	3.456123E-06	5.741025E-07	2.321045E-08
0.2	5.852014E-05	3.025874E-06	1.021345E-07	3.123456E-08
0.4	3.852014E-04	2.652413E-05	2.654123E-06	5.963258E-07
0.6	7.952147E-05	3.980021E-06	5.321412E-07	8.956542E-08
0.8	8.654123E-05	2.012365E-06	3.632584E-07	5.123054E-08
1.0	1.852014E-05	0.014785E-06	3.321004E-07	3.696325E-09

Table 2. Values of the convergence constants ξ, ξ_1, ξ_2, κ, δ and the resulting error.

ξ	ξ_1	ξ_2	T	κ	δ	error
0.2	0.2	0.4	0.5	0.573205	0.08598< 1	5.6425E-09
0.4	0.4	0.8	0.5	0.573205	0.25794< 1	5.9510E-07
0.6	0.6	1.2	0.5	0.573205	0.42990< 1	0.5247E-08
0.8	0.8	1.6	0.5	0.573205	0.60186< 1	4.2580E-09
1.0	1.0	2.0	0.5	0.573205	0.77382< 1	7.3542E-08
1.2	1.2	2.4	0.5	0.573205	1.03173> 1	1.9563E-02
1.4	1.4	2.8	0.5	0.573205	1.28970> 1	5.0145E-01

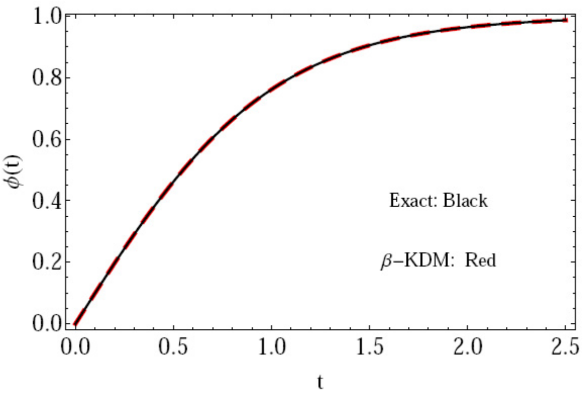

Figure 6. A comparison between the approximate & exact solutions using β-KDM.

Furthermore, to reinforce the assumptions and proofs of the convergence theorems, the values of the convergence constants ξ_1, ξ_2, κ, δ and the resulting error were estimated in Table 4. This table shows that when the assumptions are met, the conditions for convergence are fulfilled, but there are some cases where these conditions are not met. This confirms the strong agreement between the theoretical and numerical analysis of the problem under consideration.

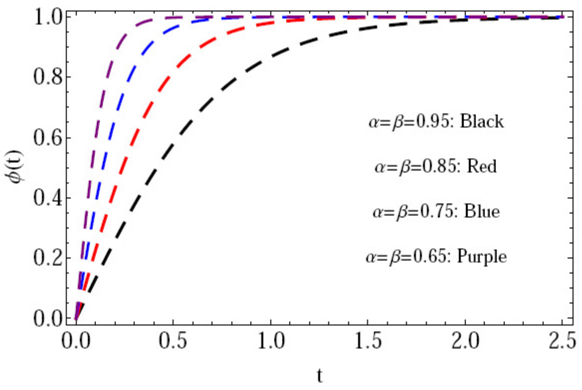

Figure 7. The approximate solution versus α, β.

7. Conclusions

This research employed the β-KDM to derive approximate solutions for two significant models: the fractional RDE and LDE, utilizing various initial conditions and parameters. Through the comparison of numerical solutions, exact solutions, and RK4M for each suggested model, we concluded that the approximate solutions derived from the specified technique align very well with the actual solution. The numerical findings demonstrate the efficacy of this technique in addressing the studied challenges, underscoring its validity and potential. This

Table 3. Comparison of the AE for numerical solutions of the RDE by the β-KDM and RK4M.

	AE of the present method		AE of the RK4M	
t	$m = 8$	$m = 16$	$h = 0.2$	$h = 0.1$
0.0	2.159753E-04	3.456123E-06	5.741025E-07	2.321045E-08
0.2	5.852014E-05	3.025874E-06	1.021345E-07	3.123456E-08
0.4	3.852014E-04	2.652413E-05	2.654123E-06	5.963258E-07
0.6	7.952147E-05	3.980021E-06	5.321412E-07	8.956542E-08
0.8	8.654123E-05	2.012365E-06	3.632584E-07	5.123054E-07
1.0	1.852014E-05	0.014785E-06	3.321004E-07	3.696325E-09

https://doi.org/10.57647/mathsci.2025.1904.19

Table 4. Values of the convergence constants ξ, ξ_1, ξ_2, κ, δ and the resulting error.

ξ	ξ_1	ξ_2	T	κ	δ	error
1.0	0.0	2.0	0.25	0.26065	0.130327< 1	3.2541E-07
1.0	0.0	2.0	0.35	0.36248	0.253736< 1	5.0147E-09
1.0	0.0	2.0	0.45	0.46372	0.417349< 1	1.2601E-08
1.0	0.0	2.0	0.55	0.56451	0.620963< 1	9.2954E-09
1.0	0.0	2.0	0.65	0.66493	0.864416< 1	0.2541E-08
1.0	0.0	2.0	0.75	0.76505	1.147583> 1	9.0248E-02
1.0	0.0	2.0	0.85	0.86490	1.470338> 1	8.2654E-01

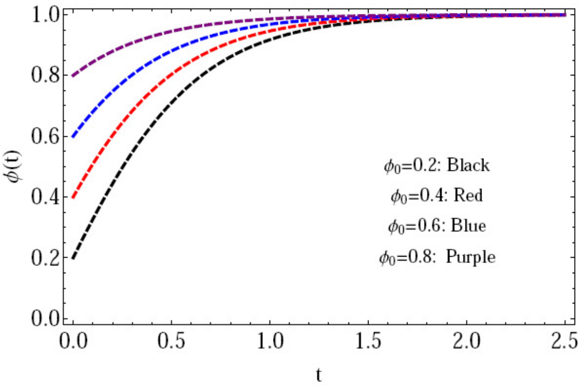

Figure 8. The numerical solution versus $\hat{\phi}$.

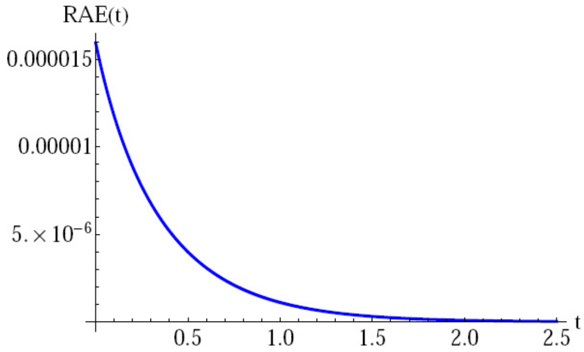

Figure 9. The REF of the solution at $\hat{\phi} = 0.5$.

perspective on analytical and numerical solutions of dynamic variables arises from their presence and significant impact on numerous models and domains of applied mathematics. We theoretically presented the proof of the uniqueness and convergence. The evidence obtained from the results of the examples illustrates not only the ability, effectiveness, and reliability of the proposed method but also opens some possibilities for further investigation. This study was carried out using several values of α, β, m, and the initial conditions to derive numerical solutions for the models under investigation. For future work, we can try to provide a theoretical study of the convergence order measurement, with an expanded focus on the study of convergence and stability of the given technique. Also, the technique will be applied to more complex models or systems of nonlinear ordinary or partial differential equations in order to expand the scope of application of the proposed technique.

Acknowledgment

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP-RP25).

Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data sets generated during the current study are available from the corresponding author upon reasonable request.
Conflict of interests
The authors declared that they have no conflict of interest.

References

1. Lasiecka I and Triggiani R. Differential and Algebraic Riccati Equations with Application to Boundary/Point Control Problems. Vol. 164. Lecture Notes in Control and Information Sciences. Berlin: Springer, 1991
2. Reid WT. Riccati Differential Equations. Vol. 86. Mathematics in Science and Engineering. New York: Academic Press, 1972
3. Tan Y and Abbasbandy S. Homotopy analysis method for quadratic Riccati differential equation. Communications in Nonlinear Science and Numerical Simulation 2008; 13:539–46
4. Jafari H and Tajadodi H. He's variational iteration method for solving fractional Riccati differential equation. International Journal of Differential Equations 2010; 2010:1–8

5. Dubois F and Saidi A. Unconditionally stable scheme for Riccati equation. ESAIM: Proceedings 2000; 8:39–52
6. Khader MM, Shloof A, and Ali H. Numerical investigation based on the Chebyshev-HPM for Riccati/Logistic differential equations. AIMS Mathematics 2025; 10:7906–19
7. Pastijn H. Chaotic Growth with the Logistic Model of P.-F. Verhulst. *Understanding Complex Systems: The Logistic Map and the Route to Chaos*. Springer, 2006 :3–11
8. Ausloos M. The Logistic Map and the Route to Chaos: From the Beginnings to Modern Applications. Understanding Complex Systems. Springer, 2006 :411
9. Anukiruthika K, Muthukumar P, and Das P. Semigroup-based theoretical prospects on optimal control of stochastic second-order Gurtin–Pipkin integro-differential equations. Optimization 2025; 5:1–18
10. Das P. Comparison of a priori and a posteriori meshes for singularly perturbed nonlinear parameterized problems. Journal of Computational and Applied Mathematics 2015; 290:16–25
11. Das P and Vigo-Aguiar J. Parameter uniform optimal order numerical approximation of a class of singularly perturbed system of reaction–diffusion problems involving a small perturbation parameter. Journal of Computational and Applied Mathematics 2015; 354:533–44
12. Das P and Natesan S. Optimal error estimate using mesh equidistribution technique for singularly perturbed system of reaction–diffusion boundary-value problems. Applied Mathematics and Computation 2014; 249:265–77
13. Sabawi SA, Santra S, Das P, and Mohammad NA. H^1 stability and convergence analysis of the L1/L2 Legendre spectral method for non-local weakly singular integro-PDEs. CALCOLO 2025; 62:42
14. Kumar S, Ishwariya R, and Das P. Impact of mixed boundary conditions and nonsmooth data on layer-originated nonpremixed combustion problems: Higher-order convergence analysis. CALCOLO 2024; 153:e12763
15. Patawari A and Das P. From traditional to computationally efficient scientific computing algorithms in option pricing: Current progresses with future directions. Archives of Computational Methods in Engineering 2025; 45:1–15
16. Santra S, Das P, and Jørgensen P. A unified multifaceted space–time wavelet framework and its singularity analysis for weakly singular non-local partial integro-differential equations in high dimensions. Acta Applicandae Mathematicae 2025; 200:230–50
17. Burton TA and Kirk CA. Fixed point theorem of Krasnoselskii–Schaefer type. Mathematische Nachrichten 1998; 189:23–31
18. Katugampola UN. A new approach to generalized fractional derivatives. Bulletin of Mathematical Analysis and Applications 2014; 6:1–15
19. Katugampola UN. Existence and uniqueness results for a class of generalized fractional differential equations. arXiv preprint. 2014. arXiv: 1411.5229
20. Khader MM, Shloof A, and Ali H. On the numerical simulation and convergence study for a system of non-linear fractional dynamical model of marriage. New Trends in Mathematical Sciences 2017; 5:130–41
21. Khader MM, Shloof A, and Ali H. Numerical simulation utilizing modified fractional Euler formula for the Ebola virus model and blood ethanol concentration system. Scientia Iranica 2025; 16:1–18
22. Khalouta A. Conformable fractional Khalouta transform and its applications to fractional differential equations. Nonlinear Dynamics and Systems Theory 2024; 24:495–504
23. Khalouta A. A new decomposition transform method for solving nonlinear fractional logistic differential equations. The Journal of Supercomputing 2024; 80:8179–201
24. Jarad F, Abdeljawad T, and Baleanu D. On the generalized fractional derivatives and their Caputo modification. Journal of Nonlinear Sciences and Applications 2017; 10:2607–19
25. Khader MM. Numerical treatment for solving the fractional Riccati differential equation. Journal of the Egyptian Mathematical Society 2013; 21:32–7
26. Khader MM. Numerical treatment for solving the fractional logistic differential equation. Differential Equations and Dynamical Systems 2016; 24:99–107
27. Khalouta A and Efendiev RF. New methodology for a system of nonlinear time-fractional partial differential equations based on the Khalouta–Caputo–Katugampola fractional derivative. Transactions of the National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences: Mathematics 2025; 45:1–18
28. Khalouta A. New approaches for solving Caputo time-fractional nonlinear system of equations describing the unsteady flow of a polytropic gas. International Journal of Nonlinear Analysis and Applications 2023; 14:33–46
29. Granas A and Dugundji J. Fixed Point Theory. New York: Springer-Verlag, 2003
30. Diethelm K. The mean value theorems and a Nagumo-type uniqueness theorem for Caputo's fractional calculus. Fractional Calculus and Applied Analysis 2012; 15:304–13

https://doi.org/10.57647/mathsci.2025.1904.19

31. Khader MM and Adel A. Numerical approach for solving the Riccati and Logistic equations via QLM-rational Legendre collocation method. Computational and Applied Mathematics 2020; 39:1–9
32. El-Sayed AMA, El-Mesiry AEM, and El-Saka HAA. On the fractional-order logistic equation. Applied Mathematics Letters 2007; 20:817–23
33. Parand K and Delkhosh M. Operational matrices to solve nonlinear Volterra–Fredholm integro-differential equations of multi-arbitrary order. Gazi University Journal of Science 2016; 29:895–907
34. Aminikhah H and Hemmatnezhad M. An efficient method for quadratic Riccati differential equation. Communications in Nonlinear Science and Numerical Simulation 2010; 15:835–9
35. Das P. An a posteriori based convergence analysis for a nonlinear singularly perturbed system of delay differential equations on an adaptive mesh. Numerical Algorithms 2019; 81:465–87
36. Saini S, Das P, and Kumar S. Parameter uniform higher order numerical treatment for singularly perturbed Robin type parabolic reaction–diffusion multiple scale problems with large delay in time. Applied Numerical Mathematics 2024; 126:1–21
37. Patawari A, Kumar S, and Das P. A modified iterative PINN algorithm for strongly coupled systems of boundary layer originated convection–diffusion–reaction problems in MHD flows with analysis. Networks and Heterogeneous Media 2025; 20:1026–60
38. Khader MM and Adel A. Modeling and numerical simulation for covering the fractional COVID-19 model using spectral collocation-optimization algorithms. Fractal and Fractional 2022; 6:1–19
39. Srivastava HM, Nain AK, Vats RK, and Das P. A theoretical study of the fractional-order p-Laplacian nonlinear Hadamard type turbulent flow models having the Ulam–Hyers stability. Revista Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2023; 117:160
40. Das P, Rana S, and Ramos H. A perturbation-based approach for solving fractional-order Volterra–Fredholm integro-differential equations and its convergence analysis. International Journal of Computer Mathematics 2019; 97:1994–2014
41. Das P. Investigations of a class of Liouville–Caputo fractional order Pennes bioheat flow partial differential equations through orthogonal polynomials on collocation points. Bulletin des Sciences Mathématiques 2025; 201:103637
42. Santra S, Mohapatra J, Das P, and Choudhuri D. Higher order approximations for fractional order integro-parabolic partial differential equations on an adaptive mesh with error analysis. Computers and Mathematics with Applications 2023; 150:87–101