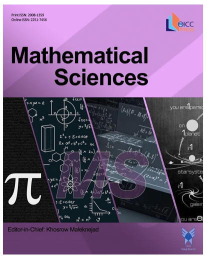

Math. Sci. 2025; 19(2) : 1-14

Mathematical Sciences (MS)

https://doi.org/10.57647/mathsci.2025.1902.08

A Novel Numeric-analytic Method for Time-fractional Swift–Hohenberg (S–H) Equation with Modified Caputo-Fabrizio Derivative

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Original Research

Received:
23 April 2025
Revised:
01 June 2025
Accepted:
21 June 2025

Abstract:

This work investigates utilizing Adomian decomposition technique (A-ADM) coupled with Aboodh transform as a hybrid approach the solution of Caputo-Fabrizio (CF) fractional order Swift-Hohenberg problem. Effective framework for modeling memory effects and complex system dynamics is provided by CF fractional derivative defined by non-singular exponential kernel. Classical Swift-Hohenberg equation is expanded to account for long-term memory effects, which are crucial in understanding pattern development and chaotic behavior in many physical systems by including CF fractional derivatives. With the suggested Aboodh-Adomian decomposition technique (A-ADM), an approximate analytical solution is provided showing its efficiency and accuracy in solving fractional nonlinear problem. Furthermore shown are convergence and originality of the solution. Graphical depictions of the acquired solutions make use of MATLAB package software. Theoretical results are confirmed by numerical simulations, which further expose the important influence of fractional order on the dynamic features of the system.

Keywords: Time-fractional-derivative; Swift-Hohenberg; Aboodh-Transform; Adomian-decomposition

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Cite this article: Şahin Y., Merdan M., A Novel Numeric-analytic Method for Time-fractional Swift–Hohenberg (S–H) Equation with Modified Caputo-Fabrizio Derivative. Math. Sci 2025; *19*(2) : 1-14 https://doi.org/10.57647/mathsci.2025.1902.08

1. Introduction

Fractional calculus (FC) is a branch of calculus that deals with derivatives and integrals of arbitrary real or complex orders. FC has achieved significant popularity over the past thirty years owing to its demonstrated applicability across several domains of research and engineering. Numerous mathematicians across several disciplines have conducted research on fractional differential equations (FDEs). Veeramani et al. [1] propose the "neutrosophic goal programming" method that takes into account uncertainties to solve the multi-objective fractional transportation problem. The model allows decision makers to optimize multiple objectives. Yao and Kumar [2] suggest a novel model for the Brownian motion of particles that is based on fractional differential equations. Analytical methods are employed to obtain the solution, which offers an alternative to classical models. Kumar [3] proposed the residual power series approach for fractional Burgers type problem solution. The approach offers quick fixes for nonlinear differential equations. Merdan and Merdan [4] employ the multi-step differential transformation approach to simulate the behavior of fractional-order economic systems. This technique uses mathematics to study economic growth and development processes. Merdan [5] used the modified Riemann-Liouville derivative to derive analytical solutions to fractional Riccati differential equations. Merdan [6] examines the resolution of time-fractional reaction-diffusion equations via the modified Riemann-Liouville derivative. The findings facilitate the modeling of anomalous diffusion processes in physical systems. Merdan [7] employed the variational iteration approach for fractional convection-diffusion equations. The approach effectively yields approximate analytical solutions. Merdan [8] proposes a method utilizing a modified Riemann–Liouville derivative to address fractional Klein-Gordon equations. The approach aids in the modeling of nonlinear physical systems.

The Caputo-Fabrizio time fractional Swift-Hohenberg (S-H) equation was resolved in this paper.

The Swift-Hohenberg (S-H) equation was developed by Jack Swift and Pierre Hohenberg as a universal model for Rayleigh-Bénard convective instability in fluids with temperature fluctuations [9]. This model establishes a correlation between thermal convection and temperature in the context of fluid dynamics. The (S-H) equation is crucial in the theory of pattern formation in fluid layers confined between horizontal, highly conductive boundaries, particularly regarding the amplitude mechanism of the optical electric field within the cavity and the patterning in thin vibrating granular layers [10]. The suggested issue exemplifies the multitude of localized and non-localized patterns present in many biological systems[11, 12, 13]. The description of the (S-H) equation is crucial in several scientific processes, including laser applications, hydrodynamics, liquid crystals, flame dynamics, and statistical mechanics[14, 15, 16].

The (S-H) equation's general time fractional form is [17].

$$\begin{aligned} {}_{0}^{CF}D_t^{\alpha}\Psi(x,t) &= (\mu-1)\,\Psi(x,t) - \frac{\partial^4\Psi(x,t)}{\partial x^4} \\ -2\frac{\partial^2\Psi(x,t)}{\partial x^2} &- \Psi^3(x,t)\,, 0<\alpha\le 1, t>0, \text{x}\in\text{R}. \end{aligned} \tag{1}$$

Subject to the initial conditions:

$$\Psi(x,0) = \frac{1}{10}\sin\left(\frac{\pi x}{L}\right)$$

where μ is real bifurcation parameter. The parameter L determines the length or width of the sinusoidal mode (regular and repeating wave structure) in the initial condition. The value of L indicates how large or small the waveform structure will be.

The (S-H) equation has been resolved using a variety of techniques, including the homotopy perturbation transform method (HPTM) [18], homotopy analysis method (HAM) [19], differential transform method (DTM) and homotopy perturbation method (HPM) [20], residual power series method (RPSM) [21], and variational iteration technique [22, 23]. In recent years, the (S-H) equation has been solved with new methods and various analyses have been made [24, 25, 26, 27].

Considering the most recent technical breakthroughs and comparing the outcomes with a more current technique, this work suggests to examine the usage of A-ADM to numerically solve the fractional Swift-Hohenberg (S-H) equation. The initial goal of this paper is to provide the Aboodh-Adomian decomposition method solving a fractional Swift-Hohenberg (S-H) equation for the first time with the recently developed hybrid technique is the second aim. Many of the responses from the literature that have not been taken into account thus far are found and their whole graphical characteristics are shown. Using the Aboodh transform method [28, 29, 30, 31], one solves linear equations incorporating the Laplace transform. To solve nonlinear equations, meanwhile, the Adomian decomposition method [32] needs be supplemented with a numerical method like the transform method. This study integrates Adomian decomposition technique with the Aboodh transform. A-ADM is a new approach to develop fresh numerical solutions for the fractional Swift-Hohenberg (S-H) problem presented in this work. [9]: This research examines the hydrodynamic fluctuations that arise during convective instabilities. The study introduces a theoretical framework designed to elucidate the impact of these instabilities on physical systems. [33]: This study conducts simulations utilizing the PFC (Phase-Field Crystal) methodology and the Swift–Hohenberg equation with a non-conservative potential to predict superconducting phases. The study seeks to enhance comprehension of phase transitions in superconductivity via computational techniques. [34] : This study examines three-dimensional morphological phase transitions in polymers and block and diblock copolymers using the integration of the Cahn-Hilliard and Swift-Hohenberg equations. This research introduces a novel methodology utilizing numerical simulations to model intricate material structures.

In this paper, we study the Cauchy-Dirichlet problem for the Swift-Hohenberg equation on the interval (0, L). It should be noted that in addition to μ, the length L is also an important parameter. Therefore, we consider the problem by writing the (S-H) equation in a more conventional form.

Other functions can be used instead of the sine function in the initial condition. This usually depends on the characteristics of the system and the needs of the solution. For example, cosine, exponential or random functions can be used. However, which function to use should take into account the characteristics of the solutions and the impact of the initial condition on the system. The sine is usually preferred for harmonic and symmetric structures, while other functions can create different time or space structures.

This study extends the Aboodh–Adomian decomposition method (A-ADM) to the Swift–Hohenberg (S–H) equation utilizing the Caputo–Fabrizio (CF) time-fractional derivative. It presents an explicit iterative A-ADM algorithm tailored to the CF kernel, which minimizes transform algebra relative to conventional Caputo formulations. Additionally, it includes a convergence analysis under lenient operator/Lipschitz conditions that quantifies the solution's radius of convergence within a Banach framework. Furthermore, a systematic numerical benchmarking is conducted, employing closed-form test problems and relevant parameter sets, demonstrating enhanced efficiency and comparable or superior accuracy against various baseline methods, including classical ADM, Laplace–Adomian ADM, homotopy perturbation method, and a spectral time-stepping scheme. Collectively, these contributions demonstrate the appropriateness of A-ADM+CF for medium-stiff nonlinear fractional PDEs, whereby the CF exponential kernel facilitates transform inversion and diminishes numerical evaluation expenses.

The present work is set up as follows. Section 2 addresses notations and simple definitions. The Aboodh Adomian transform method is investigated in Section 3 for components of the Caputo-Fabrizio temporal fractional order Swift-Hohenberg (S-H) issue. Analysis of numerical applications of the method to the fractional order Swift-Hohenberg (S-H) issue is given in Section 4. Section 5 comes with the conclusion.

2. Preliminaries

Here we present some basic definitions of Caputo-Fabrizio that are important to our research.

Definition 2.1 *Caputo-Fabrizio fractional derivative [35] of order $h(t)$*

$$ {}_{a}^{CF}D_t^{\alpha}h\left(t\right)=\frac{P\left(\alpha\right)}{1-\alpha}\int_a^t h'(\tau)e^{\left[\frac{-\alpha(t-\tau)}{1-\alpha}\right]}d\tau,0<\alpha\leq 1 \quad (2) $$

$h'\ \epsilon\ H'\ \ (a,b)\ ,\ b>0$ and $P(\alpha)$ is the normalization constant depending on a where

$$ P(0)=P(1)=1. $$

Definition 2.2 *Caputo fractional derivative of $h(t)$, where $\alpha>0$ is the order of the Caputo fractional derivative [35, 36, 37]:*

$$ {}^{c}D_t^{\alpha}h\left(t\right)=\begin{cases}\frac{1}{\Gamma(p-\alpha)}\int_0^t\left(t-s\right)^{p-\alpha-1}h^{(p)}\left(s\right)ds, p-1<\alpha\leq p\\ \\ \frac{\partial^n}{\partial t^n}h\left(t\right), & \alpha=n\epsilon\mathbb{N}.\end{cases} \quad (3) $$

It is defined as.

Definition 2.3 *The functions in the set $\mathcal{A}$ described as follows are taken into consideration by a new transformation for exponential functions known as the Aboodh transformation [28, 29, 30, 31].*

$$ \mathcal{A}=\left\{h\left(t\right):\ \exists\, M,\ k_1\ ,\ k_2>0\ ,\ \left|\,h\left(t\right)\right|<Me^{-st}\right\}. \quad (4) $$

Aboodh transform of function $h(t)$

$$ \mathcal{A}\left\{h(t)\right\}=\mathcal{A}\left\{h\left(s\right)\right\}=\frac{1}{s}\int_0^{\infty}h\left(t\right)e^{-st}dt, t\geq 0, $$
$$ k_1\leq s\leq k_2. \quad (5) $$

It is defined as.

Definition 2.4 *Laplace Transform of Caputo-Fabrizio fractional derivative [38, 39, 40]:*

$$ L\left\{{}_{0}^{CF}D_t^{\alpha+n}h(t)\right\} \quad (6) $$
$$ =\frac{s^{n+1}\ L\left\{h(s)\right\}-h\left(0\right)s^n-\ h'\left(0\right)s^{n-1}-\ldots-\ h^{(n)}(0)}{s+\alpha(1-s)}, $$
$$ 0<\alpha\leq 1 $$

for $n=0$

$$ L\left\{{}_{0}^{CF}D_t^{\alpha}h(t)\right\}=\frac{s\,L\left\{h\left(s\right)\right\}-h\left(0\right)}{s+\alpha(1-s)}. \quad (7) $$

Theorem 2.5 *The Aboodh transform of the Caputo-Fabrizio fractional derivative of order α, denoted as $\mathcal{A}\left\{h\left(s\right)\right\}$, where $h(t)$ is the function in question.*

$$ \mathcal{A}\left\{{}_{0}^{CF}D_t^{\alpha}h\left(t\right)\right\}=\frac{s\ \mathcal{A}\left\{h\left(s\right)\right\}-\frac{h(0)}{s}}{\alpha+s\left(1-\alpha\right)}. \quad (8) $$

3. Methods and metarials

In order to demonstrate the Aboodh decomposition method, we examine the more general nonhomogeneous nonlinear fractional partial differential equation.

$$ {}_{0}^{CF}D_t^{\alpha}\Psi\left(x,t\right)=-R\Psi\left(x,t\right)-N\Psi\left(x,t\right)+g\left(x,t\right) \quad (9) $$

with the initial condition

$$ \Psi\left(x,\ 0\right)=f\left(x\right) \quad (10) $$

where R represents the residual linear operator, N denotes the nonlinear operator, and $g(x,t)$ signifies the nonhomogeneous term.

Let's take the Aboodh transformation of both sides of Eq (9)

$$ \mathcal{A}\left\{{}_{0}^{CF}D_t^{\alpha}\Psi\left(x,t\right)\right\} = \mathcal{A}\left\{-R\Psi\left(x,t\right)\right. \quad (11) $$
$$ \left.-N\Psi\left(x,t\right)+g(x,t)\right\} $$

$$ \frac{s\mathcal{A}\left\{\Psi\left(x,t\right)\right\}-\frac{\Psi(x,0)}{s}}{\alpha+s\left(1-\alpha\right)} = \mathcal{A}\left\{-R\Psi\left(x,t\right)\right. \quad (12) $$
$$ \left.-N\Psi\left(x,t\right)+g(x,t)\right\} $$

$$ \mathcal{A}\left\{\Psi\left(x,t\right)\right\} = \frac{\Psi\left(x,0\right)}{s^2}+\left(\frac{\alpha}{s}+1-\alpha\right) \quad (13) $$
$$ \mathcal{A}\left\{-R\Psi\left(x,t\right)-N\Psi\left(x,t\right)+g\left(x,t\right)\right\} $$

Now, the unknown function $\Psi(x,t)$is replaced by an infinite series of Ψ_n's, i.e.,

$$ \Psi\left(x,t\right)=\sum_{n=0}^{\infty}\Psi_n\left(x,t\right)=\Psi_0+\Psi_1+\Psi_2+ \quad (14) $$

and the nonlinear term is represented by an infinite series of the Adomian polynomial B_n, which is given by

$$ N\Psi\left(x,t\right)=\sum_{n=0}^{\infty}B_n\left(\Psi_0,\Psi_1,\Psi_2,\ldots\right), n=0,1,2,\ldots \quad (15) $$

Where

$$ B_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty}\lambda^i\Psi_i\right)\right]_{\lambda=0}, n=0,1,2,\ldots \quad (16) $$

If we substitute (14) and (15) into (13) and take the inverse aboodh transformation of Eq (13);

$$ \sum_{n=0}^{\infty}\Psi_n(x,t) = \Psi(x,0) \quad (17) $$
$$ +\mathcal{A}^{-1}\left\{\left(\frac{\alpha}{s}+1-\alpha\right)\left\{\mathcal{A}\sum_{m=0}^{\infty}R\Psi_m(x,t)\right.\right. $$
$$ \left.\left.-\mathcal{A}\left\{\sum_{n=0}^{\infty}B_mt(x,t)\right\}+\mathcal{A}g(x,t)\right\}\right\} $$

equation is obtained.

https://doi.org/10.57647/mathsci.2025.1902.08

If the terms of the same order in Eq (17) are equated,

$$\Psi_0(x,t) = \Psi(x,0) = f(x)$$

$$\begin{aligned}\Psi_1(x,t) &= \mathcal{A}^{-1}\left\{\left(\frac{\alpha}{s}+1-\alpha\right)\left\{\mathcal{A}\sum_{m=0}^{\infty} R\Psi_0(x,t)\right.\right.\\ &\qquad \left.\left.-\mathcal{A}\left\{\sum_{n=0}^{\infty} B_0(x,t)\right\}\right\}\right\}\end{aligned}$$

$$\begin{aligned}\Psi_2(x,t) &= \mathcal{A}^{-1}\left\{\left(\frac{\alpha}{s}+1-\alpha\right)\left\{\mathcal{A}\sum_{m=0}^{\infty} R\Psi_1(x,t)\right.\right.\\ &\qquad \left.\left.-\mathcal{A}\left\{\sum_{n=0}^{\infty} B_1(x,t)\right\}\right\}\right\} \qquad (18)\end{aligned}$$

$$\begin{aligned}\Psi_3(x,t) &= \mathcal{A}^{-1}\left\{\left(\frac{\alpha}{s}+1-\alpha\right)\left\{\mathcal{A}\sum_{m=0}^{\infty} R\Psi_2(x,t)\right.\right.\\ &\qquad \left.\left.-\mathcal{A}\left\{\sum_{n=0}^{\infty} B_2(x,t)\right\}\right\}\right\}\end{aligned}$$

$$\vdots$$

Consequently, the Aboodh-Adomian decomposition approach yields the recursive relation for the solution of the ordinary differential equation (18) as follows:

$$\Psi_0(x,t) = \Psi(x,0) = f(x)$$

$$\begin{aligned}\Psi_{n+1}(x,t) &= \mathcal{A}^{-1}\left\{\left(\frac{\alpha}{s}+1-\alpha\right)\left\{\mathcal{A}\sum_{m=0}^{\infty} R\Psi_m(x,t)\right.\right.\\ &\qquad \left.\left.-\mathcal{A}\left\{\sum_{n=0}^{\infty} B_m(x,t)\right\}+\mathcal{A}g(x,t)\right\}\right\}, \qquad (19)\\ &\qquad n=0,1,\ldots\end{aligned}$$

4. Convergence analysis

Define the Banach space $C[0,T]$ consisting of all continuous functions on the interval $[0,T]$ with a supremum norm. In this part, we examine the functions $\Psi(x,t)$ and $\Psi_n(x,t) \in C[0,T]$.

Theorem 4.1 *(Theorem of Uniqueness [41, 42]) The unique solution for the nonlinear fractional differential Eq (1) derived by A-ADM is valid for* $0<\gamma<1$.

Proof. The resolution of nonlinear FPDEs in Eq (1) is delineated as follows.

$$\begin{cases} n=0;\ \Psi_0(x,t)=\Psi(x,0)=\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)\\ n\geq 0;\ \Psi_{n+1}(x,t)=\mathcal{A}^{-1}\left\{\left(\frac{\alpha}{s}+1-\alpha\right)\mathcal{A}\left\{(\mu-1)\sum_{n=0}^{\infty}\Psi_n\right.\right.\\ \qquad \left.\left.-\sum_{n=0}^{\infty}\Psi_{n_{xxxx}}-2\sum_{n=0}^{\infty}\Psi_{n_{xx}}-\Psi^3\right\}\right\} \end{cases} \qquad (20)$$

Assuming that Ψ and Φ are two distinct solutions of Eq (20), we may derive the following using the previously described equation.

$$\begin{aligned}|\Psi-\Phi| &= \left|\mathcal{A}^{-1}\left\{\left(\frac{\alpha}{s}+1-\alpha\right)\mathcal{A}\{(\mu-1)(\Psi-\Phi)\right.\right.\\ &\quad \left.\left.-\frac{\partial^4}{\partial x^4}(\Psi-\Phi)-2\frac{\partial^2}{\partial x^2}(\Psi-\Phi)-\left(\Psi^3-\Phi^3\right)\right\}\right\}\right|. \qquad (21)\end{aligned}$$

Utilizing the convolution theory for the Aboodh transform, it is derived as

$$\begin{aligned}|\Psi-\Phi| &\leq \int_0^t \left(|(\mu-1)(\Psi-\Phi)|-\left|\frac{\partial^4}{\partial x^4}(\Psi-\Phi)\right|\right.\\ &\quad \left.-\left|2\frac{\partial^2}{\partial x^2}(\Psi-\Phi)\right|+\left|\left(\Psi^3-\Phi^3\right)\right|\right)\frac{(t-\tau)^{\beta}}{\Gamma(1+\beta)}d\tau\\ &\leq \int_0^t \left(|(\mu-1)\|(\Psi-\Phi)|+\frac{\partial^4}{\partial x^4}|(\Psi-\Phi)|+2\frac{\partial^2}{\partial x^2}|(\Psi-\Phi)|\right.\\ &\quad \left.+\left|(\Psi-\Phi)(\Psi^2+\Psi\Phi+\Phi^2)\right|\right)\frac{(t-\tau)^{\beta}}{\Gamma(1+\beta)}d\tau\\ &\leq \int_0^t \left((\mu-1)|(\Psi-\Phi)|+\delta^4|(\Psi-\Phi)|+2\delta^2|(\Psi-\Phi)|\right.\\ &\quad \left.+\left|(\Psi-\Phi)(\Psi^2+\Psi\Phi+\Phi^2)\right|\right)\frac{(t-\tau)^{\beta}}{\Gamma(1+\beta)}d\tau \qquad (22)\end{aligned}$$

where $\delta^4=\frac{\partial^4}{\partial x^4}$ and$\delta^2=\frac{\partial^2}{\partial x^2}$. Subsequently, applying the integral mean-value theorem, it results in

$$\begin{aligned}|\Psi-\Phi| &\leq \left((\mu-1)|(\Psi-\Phi)|+\delta^4|(\Psi-\Phi)|\right.\\ &\quad \left.+2\delta^2|(\Psi-\Phi)|+\left|(\Psi-\Phi)\left(\Psi^2+\Psi\Phi+\Phi^2\right)\right|\right)T\end{aligned}$$

$$\gamma=\left((\mu-1)+\delta^4+2\delta^2+\left|\left(\Psi^2+\Psi\Phi+\Phi^2\right)\right|\right)T$$

$$|\Psi-\Phi|\leq\gamma|(\Psi-\Phi)|\ ,\ (1-\gamma)|\Psi-\Phi|\leq 0.$$

Therefore, $|\Psi-\Phi|=0$, as $0<\gamma<1$. Consequently, $\Psi=\Phi$. This demonstrates the solution's distinctiveness.

Theorem 4.2 *(Convergence Theorem [43, 44]) Let* X *be a Banach space and* $H: X\rightarrow X$ *be a nonlinear mapping. Should the inequality*

$$\|H(\Psi)-H(\Phi)\|\leq\gamma\|\Psi-\Phi\|, \forall\Psi,\Phi\in X. \qquad (23)$$

If it exists, then H possesses a fixed point according to Banach's fixed point theorem [44]. Moreover, for the arbitrary selection of Ψ_0 and Ψ_1 in X, the sequence generated by the A-ADM converges to a fixed point of H.

$$\|\Psi_m-\Psi_n\|\leq\frac{\gamma^n}{1-\gamma}\|\Psi_1-\Psi_0\|, \forall\Psi,\Phi\in X. \qquad (24)$$

Proof. Consider a Banach space $(C[J],\|.\|)$ comprising all continuous functions on J, with the norm defined as $\|h(\xi)\|=\max_{\xi\in J}|h(\xi)|$.

We now illustrate that the sequence $\{\Psi_n\}$ constitutes a Cauchy sequence within the Banach space:

$$\|\Psi_m-\Psi_n\|=\max_{\xi\in J}|\Psi_m-\Psi_n|$$

https://doi.org/10.57647/mathsci.2025.1902.08

$$\begin{aligned}
&= \max_{\xi\in J}\Big|\mathcal{A}^{-1}\Big\{\Big(\frac{\alpha}{s}+1-\alpha\Big)\mathcal{A}\{(\mu-1)\\
&(\Psi_{m-1}-\Psi_{n-1})-\frac{\partial^4}{\partial x^4}(\Psi_{m-1}-\Psi_{n-1})\\
&-2\frac{\partial^2}{\partial x^2}(\Psi_{m-1}-\Psi_{n-1})-\Big(\Psi^3_{m-1}-\Psi^3_{n-1}\Big)\Big\}\Big\}\Big|
\end{aligned}$$

$$\begin{aligned}
&\le \max_{\xi\in J}\Big[\mathcal{A}^{-1}\Big\{\Big(\frac{\alpha}{s}+1-\alpha\Big)\mathcal{A}\,\{(\mu-1)\,|\Psi_{m-1}\\
&-\Psi_{n-1}\,|+\Big|\frac{\partial^4}{\partial x^4}(\Psi_{m-1}-\Psi_{n-1})\Big|\\
&+2\Big|\frac{\partial^2}{\partial x^2}(\Psi_{m-1}-\Psi_{n-1})\Big|+\Big|\Psi^3_{m-1}-\Psi^3_{n-1}\Big|\Big\}\Big\}\Big].
\end{aligned}$$

Currently, the convolution theorem for Laplace transform is employed to provide the following:

$$\begin{aligned}
&\|\Psi_m-\Psi_n\|\le\max_{\xi\in J}\Big[\int_0^{\xi}\,[(\mu-1)\,|\Psi_{m-1}-\Psi_{n-1}|\\
&+\Big|\frac{\partial^4}{\partial x^4}(\Psi_{m-1}-\Psi_{n-1})\Big|+2\Big|\frac{\partial^2}{\partial x^2}(\Psi_{m-1}-\Psi_{n-1})\Big|\\
&+|\Psi^3_{m-1}-\Psi^3_{n-1}|]\,\frac{(\xi-\tau)^\beta}{\Gamma(1+\beta)}d\tau\Big]
\end{aligned}$$

$$\begin{aligned}
&\le\max_{\xi\in J}\Big[\int_0^{\xi}((\mu-1)|(\Psi_{m-1}-\Psi_{n-1})|\\
&+\delta^4|(\Psi_{m-1}-\Psi_{n-1})|+2\delta^2|(\Psi_{m-1}-\Psi_{n-1})|\\
&+|(\Psi_{m-1}-\Psi_{n-1})(\Psi^2+\Psi\Phi+\Phi^2)|)\frac{(\xi-\tau)^\beta}{\Gamma(1+\beta)}d\tau\Big].
\end{aligned}$$

Next, we obtain the result by employing the integral mean value theorem [44].

$$\begin{aligned}
&\|\Psi_m-\Psi_n\|\le\max_{\xi\in J}[((\mu-1)|(\Psi_{m-1}-\Psi_{n-1})|\\
&+\delta^4|(\Psi_{m-1}-\Psi_{n-1})|+2\delta^2|(\Psi_{m-1}-\Psi_{n-1})|\\
&+|(\Psi_{m-1}-\Psi_{n-1})(\Psi^2+\Psi\Phi+\Phi^2)|)T]
\end{aligned}$$

$$\|\Psi_m-\Psi_n\|\le\gamma\|\Psi_{m-1}-\Psi n-1\|.$$

Let $m = n + 1$, then we have

$$\begin{aligned}
|\Psi_{n+1}-\Psi_n\| &\le \gamma\|\Psi_n-\Psi_{n-1}\|\\
&\le \gamma^2\|\Psi_{n-1}-\Psi_{n-2}\|\\
&\ \ \vdots\\
&\le \gamma^n\|\Psi_1-\Psi_0\|.
\end{aligned}$$

Utilizing the triangular inequality, it generates the following:

$$\begin{aligned}
\|\Psi_m-\Psi_n\| &\le \|\Psi_{n+1}-\Psi_n\|+\|\Psi_{n+2}-\Psi_{n+1}\|\\
&\qquad+\cdots+\|\Psi_m-\Psi_{m-1}\|\\
&\le \left[\gamma^n+\gamma^{n+1}+\cdots+\gamma^{m-1}\right]\|\Psi_1-\Psi_0\|\\
&\le\gamma^n\left[1+\gamma+\gamma^2+\cdots+\gamma^{m-n-1}\right]\|\Psi_1-\Psi_0\|\\
&\le \gamma^n\left[\frac{1-\gamma^{m-n-1}}{1-\gamma}\right]\|\,\Psi_1-\Psi_0\|\,.
\end{aligned}$$

Because $\gamma\in(0,1)$, so $1-\gamma^{m-n-1}<1$, then we have

$$\|\Psi_m-\Psi_n\|\le\frac{\gamma^n}{1-\gamma}\|\Psi_1-\Psi_0\|.$$

Although $\|\Psi_1 - \Psi_0\| < \infty$, as m approaches infinity,$\|\Psi_m-\Psi_n\|$approaches zero. Consequently, the sequence $\{\Psi_n\}$ is a Cauchy sequence in $C[J]$, and is therefore convergent.

5. Applications of A-ADM

In this section, we apply the Aboodh transform and the Adomian decomposition method to the multidimensional fractional (S-H) equation. In particular, nonlinearity is controlled using Adomian polynomials.

Example 5.1 *We consider the Caputo-Fabrizio fractional S-H equation*

$$\begin{aligned}
{}^{CF}_{\ 0}D^{\alpha}_t\Psi(x,t) &= (\mu-1)\,\Psi-\Psi_{xxxx}\\
&\quad -2\Psi_{xx}-\Psi^3\ ,\quad 0<\alpha\le 1
\end{aligned}\tag{25}$$

$$\Psi\,(x,0)=\frac{1}{10}\sin\Big(\frac{\pi x}{L}\Big).\tag{26}$$

Determine the starting condition via the Aboodh-Adomian decomposition (A-ADM) approach.

Solution.(25) If the Aboodh transformation is applied to both sides of the equation;

$$\mathcal{A}\left\{{}^{CF}_{0}D^{\alpha}_t\Psi(x,t)\right\}=\mathcal{A}\left\{(\mu-1)\Psi-\Psi_{xxxx}-2\Psi_{xx}-\Psi^3\right\}$$

$$\begin{aligned}
\frac{s\mathcal{A}\{\Psi(x,t)\}-\frac{\Psi(x,0)}{s}}{\alpha+s\,(1-\alpha)} &= \mathcal{A}\{(\mu-1)\,\Psi\\
&\qquad-\Psi_{xxxx}-2\Psi_{xx}-\Psi^3\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}\{\Psi(x,t)\} &= \frac{\Psi(x,0)}{s^2}+\Big(\frac{\alpha}{s}+1-\alpha\Big)\mathcal{A}\{(\mu-1)\\
&\qquad\Psi-\Psi_{xxxx}-2\Psi_{xx}-\Psi^3\}
\end{aligned}\tag{27}$$

$$\begin{cases}
\Psi\,(x,t)=\sum\limits_{n=0}^{\infty}\Psi_n\,(x,t),\ \ \Psi_{xxxx}=\sum\limits_{n=0}^{\infty}\Psi_{n_{xxxx}},\\[2ex]
\Psi_{xx}=\sum\limits_{n=0}^{\infty}\Psi_{n_{xx}},\ \ \Psi=\sum\limits_{n=0}^{\infty}\Psi_n\\[2ex]
\Psi^3=\sum\limits_{n=0}^{\infty}B_n;\ \ B_0=\Psi_0^3,\ \ B_1=3{\Psi_0}^2\Psi_1.
\end{cases}\tag{28}$$

B_n Adomian polinoms.
Let's substitute the definitions in(28) in (27) ;

$$\mathcal{A}\left\{\sum_{n=0}^{\infty}\Psi_n(x,t)\right\}=\frac{\Psi(x,0)}{s^2}+(\frac{\alpha}{s}+1-\alpha)\tag{29}$$

$$\mathcal{A}\left\{(\mu-1)\sum_{n=0}^{\infty}\Psi_n-\sum_{n=0}^{\infty}\Psi_{n_{xxxx}}-2\sum_{n=0}^{\infty}\Psi_{n_{xx}}-\sum_{n=0}^{\infty}B_n\right\}.$$

https://doi.org/10.57647/mathsci.2025.1902.08

(29) If the equation is subjected to the inverse Aboodh transformation

$$
\begin{aligned}
\sum_{n=0}^{\infty} \varPsi_n\,(x,t) &= \Psi(x,0) \qquad (30)\\
&+\mathcal{A}^{-1}\Bigg\{(\frac{\alpha}{s}+1-\alpha)\mathcal{A}\Bigg\{(\mu-1)\sum_{n=0}^{\infty}\Psi_n\\
&-\sum_{n=0}^{\infty}\Psi_{n_{xxxx}}-2\sum_{n=0}^{\infty}\Psi_{n_{xx}}-\sum_{n=0}^{\infty}B_n\Bigg\}\Bigg\}
\end{aligned}
$$

$$
\begin{cases}
n=0;\ \varPsi_0(x,t)=\varPsi(x,0)=\frac{1}{10}\sin(\frac{\pi x}{L})\\[2ex]
n\geq 0;\ \varPsi_{n+1}(x,t)=\\
\mathcal{A}^{-1}\Bigg\{\left(\frac{\alpha}{s}+1-\alpha\right)\mathcal{A}\Bigg\{(\mu-1)\sum_{n=0}^{\infty}\varPsi_n\\
\quad -\sum_{n=0}^{\infty}\varPsi_{n_{xxxx}}-2\sum_{n=0}^{\infty}\varPsi_{n_{xx}}-\sum_{n=0}^{\infty}B_n\Bigg\}\Bigg\}.
\end{cases} \qquad (31)
$$

(31) in the equation $n=0,1,2,\dots$ using the values;

$$
\begin{aligned}
\Psi_1(x,t) &= \mathcal{A}^{-1}\Big\{(\frac{\alpha}{s}+1-\alpha\mathcal{A}\{(\mu-1)\Psi_0\\
&\qquad -\Psi_{0_{xxxx}}-2\varPsi_{0_{xx}}-B_0\}\}
\end{aligned} \qquad (32)
$$

$\varPsi_{0_{xxxx}} = \frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)$, $\varPsi_{0_{xx}} = \frac{-1}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)$
and Adomian polinom

$$
B_0=(\frac{1}{10})^3\sin^3\left(\frac{\pi x}{L}\right) \qquad (33)
$$

Let’s substitute the expressions (33) in (32);

$$
\begin{aligned}
&\Psi_1(x,t)=\mathcal{A}^{-1}\Big\{(\frac{\alpha}{s}+1-\alpha)\mathcal{A}\Big\{(\mu-1)\frac{1}{10}\sin(\frac{\pi x}{L})\\
&-\frac{1}{10}(\frac{\pi}{L})^4\sin(\frac{\pi x}{L})+\frac{2}{10}(\frac{\pi}{L})^2\sin(\frac{\pi x}{L})\\
&-(\frac{1}{10})^3\sin^3(\frac{\pi x}{L})\Big\}\Big\}\\
&=\Big\{(\mu-1)\frac{1}{10}\sin(\frac{\pi x}{L})-\frac{1}{10}(\frac{\pi}{L}^4\sin(\frac{\pi x}{L})\\
&+\frac{2}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&-\left(\frac{1}{10}\right)^3\sin^3(\frac{\pi x}{L})\Big\}\mathcal{A}^{-1}\Big\{(\frac{\alpha}{s}+1-\alpha)\frac{1}{s^2}\Big\}\\
&=\Big\{(\mu-1)\frac{1}{10}\sin(\frac{\pi x}{L})-\frac{1}{10}(\frac{\pi}{L})^4\sin(\frac{\pi x}{L})\\
&+\frac{2}{10}(\frac{\pi}{L})^2\sin(\frac{\pi x}{L})-(\frac{1}{10})^3\sin^3(\frac{\pi x}{L})\Big\}\mathcal{A}^{-1}\Big\{\frac{\alpha}{s^3}+\frac{1-\alpha}{s^2}\Big\}\\
&=\Big\{(\mu-1)\frac{1}{10}\sin(\frac{\pi x}{L})-\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin(\frac{\pi x}{L})\\
&+\frac{2}{10}(\frac{\pi}{L})^2\sin(\frac{\pi x}{L})-\left(\frac{1}{10}\right)^3\sin^3(\frac{\pi x}{L})\Big\}(\alpha t+1-\alpha)
\end{aligned}
$$

$$
\begin{aligned}
\Psi_1(x,t)=&\Big\{(\mu-1)\frac{1}{10}\sin(\frac{\pi x}{L})-\frac{1}{10}(\frac{\pi}{L})^4\sin(\frac{\pi x}{L})\\
&+\frac{2}{10}(\frac{\pi}{L})^2\sin(\frac{\pi x}{L})-(\frac{1}{10})^3\sin^3(\frac{\pi x}{L})\Big\}(\alpha t+1-\alpha)
\end{aligned}
$$

is obtained.

$$
\begin{aligned}
\Psi_2(x,t) &= \mathcal{A}^{-1}\Big\{(\frac{\alpha}{s}+1-\alpha)\mathcal{A}\{(\mu-1)\Psi_1\\
&\qquad -\Psi_{1_{xxxx}}-2\varPsi_{1_{xx}}-B_1\}\}
\end{aligned} \qquad (34)
$$

$$
\begin{aligned}
\Psi_{1_{xxxx}}=&\Big\{(\mu-1)\frac{1}{10}(\frac{\pi}{L})^4\sin(\frac{\pi x}{L})-\frac{1}{10}(\frac{\pi}{L})^8\sin(\frac{\pi x}{L})\\
&+\frac{2}{10}(\frac{\pi}{L})^6\sin(\frac{\pi x}{L})+12(\frac{1}{10})^3(\frac{\pi}{L})^4\sin(\frac{2\pi x}{L})\\
&+6(\frac{1}{10})^3(\frac{\pi}{L}^4\cos(\frac{2\pi x}{L})\cos(\frac{\pi x}{L})\\
&-3(\frac{1}{10})^3(\frac{\pi}{L})^4\sin(\frac{2\pi x}{L})\sin(\frac{\pi x}{L}) \qquad (35)\\
&-9(\frac{1}{10})^3(\frac{\pi}{L})^4\sin^2(\frac{\pi x}{L})\cos(\frac{\pi x}{L})\Big\}(\alpha t+1-\alpha)
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{1_{xx}}=&\Big\{(1-\mu)\frac{1}{10}(\frac{\pi}{L})^2\sin(\frac{\pi x}{L})+\frac{1}{10}(\frac{\pi}{L})^6\sin(\frac{\pi x}{L})\\
&-\frac{2}{10}(\frac{\pi}{L})^4\sin(\frac{\pi x}{L})-3(\frac{1}{10})^3(\frac{\pi}{L})^2\sin(\frac{2\pi x}{L})\\
&+3(\frac{1}{10})^3(\frac{\pi}{L})^2\sin^2(\frac{\pi x}{L})\cos(\frac{\pi x}{L})\Big\}(\alpha t+1-\alpha). \qquad (36)
\end{aligned}
$$

Adomian polinom,

$$
\begin{aligned}
B_1 &= 3\Psi_0^2\Psi_1 \qquad (37)\\
&= 3(\frac{1}{10})^2\sin^2(\frac{\pi x}{L})\Big\{(\mu-1)\frac{1}{10}\sin(\frac{\pi x}{L})\\
&\quad -\frac{1}{10}(\frac{\pi}{L})^4\sin(\frac{\pi x}{L})+\frac{2}{10}(\frac{\pi}{L})^2\sin(\frac{\pi x}{L})\\
&\quad -(\frac{1}{10})^3\sin^3(\frac{\pi x}{L})\Big\}(\alpha t+1-\alpha).
\end{aligned}
$$

Let’s substitute the expressions (35),(36) and (37) in (34);

$$
\begin{aligned}
&\Psi_2(x,t)\\
&=\mathcal{A}^{-1}\Big\{(\frac{\alpha}{s}+1-\alpha)\mathcal{A}\Big\{((\mu-1)((\mu-1)\frac{1}{10}\sin(\frac{\pi x}{L})\\
&-\frac{1}{10}(\frac{\pi}{L})^4\sin(\frac{\pi x}{L})+\frac{2}{10}(\frac{\pi}{L})^2\sin(\frac{\pi x}{L})-(\frac{1}{10})^3\sin^3(\frac{\pi x}{L}))\\
&\quad -((\mu-1)\frac{1}{10}(\frac{\pi}{L}^4\sin(\frac{\pi x}{L})-\frac{1}{10}(\frac{\pi}{L})^8\sin(\frac{\pi x}{L})\\
&\quad +\frac{2}{10}(\frac{\pi}{L})^6\sin(\frac{\pi x}{L})+12(\frac{1}{10}^3(\frac{\pi}{L})^4\sin(\frac{2\pi x}{L})\\
&\quad +6(\frac{1}{10})^3(\frac{\pi}{L})^4\cos(\frac{2\pi x}{L})\cos\left(\frac{\pi x}{L}\right)\\
&\quad -3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)\\
&\quad -9\left(\frac{1}{10}\right)^3(\frac{\pi}{L})^4\sin^2(\frac{\pi x}{L})\cos(\frac{\pi x}{L}))\\
&\quad -2((1-\mu)\frac{1}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&\quad +\frac{1}{10}\left(\frac{\pi}{L}\right)^6\sin\left(\frac{\pi x}{L}\right)-\frac{2}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)\\
&\quad -3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^2\sin\left(\frac{2\pi x}{L}\right)
\end{aligned}
$$

https://doi.org/10.57647/mathsci.2025.1902.08

$$\begin{aligned}
&+3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^2\sin^2\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)\\
&-3\left(\frac{1}{10}\right)^2\sin^2\left(\frac{\pi x}{L}\right)((\mu-1)\,\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&-\left(\frac{1}{10}\right)^3\sin^3(\frac{\pi x}{L})))\,(\alpha t+1-\alpha)\}\}
\end{aligned}$$

$$\begin{aligned}
\Psi_2(x,t)&=((\mu-1)((\mu-1)\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&-\left(\frac{1}{10}\right)^3\sin^3(\frac{\pi x}{L}))-((\mu-1)\,\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^8\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^6\sin\left(\frac{\pi x}{L}\right)\\
&+12\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin\left(\frac{2\pi x}{L}\right)\\
&+6\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\cos\left(\frac{2\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)\\
&-3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)\\
&-9\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin^2\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right))\\
&-2((1-\mu)\,\frac{1}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&+\frac{1}{10}\left(\frac{\pi}{L}\right)^6\sin\left(\frac{\pi x}{L}\right)-\frac{2}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)\\
&-3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^2\sin\left(\frac{2\pi x}{L}\right)\\
&+3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^2\sin^2\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right))\\
&-3\left(\frac{1}{10}\right)^2\sin^2\left(\frac{\pi x}{L}\right)((\mu-1)\,\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&-\left(\frac{1}{10}\right)^3\sin^3\left(\frac{\pi x}{L}\right)\Bigg)\Bigg)\Bigg)\,\mathcal{A}^{-1}\left\{\left(\frac{\alpha}{s}+1-\alpha\right)\left(\frac{\alpha}{s^3}+\frac{1-\alpha}{s^2}\right)\right\}
\end{aligned}$$

$$\begin{aligned}
\Psi_2\,(x,t)&=\{(\mu-1)\,((\mu-1)\,\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&-\left(\frac{1}{10}\right)^3\sin^3(\frac{\pi x}{L}))-((\mu-1)\,\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^8\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^6\sin\left(\frac{\pi x}{L}\right)\\
&+12\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin\left(\frac{2\pi x}{L}\right)\\
&+6\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\cos\left(\frac{2\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)\\
&-3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)\\
&-9\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin^2\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right))\\
&-2((1-\mu)\,\frac{1}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&+\frac{1}{10}\left(\frac{\pi}{L}\right)^6\sin\left(\frac{\pi x}{L}\right)-\frac{2}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)\\
&-3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^2\sin\left(\frac{2\pi x}{L}\right)\\
&+3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^2\sin^2\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right))\\
&-3\left(\frac{1}{10}\right)^2\sin^2\left(\frac{\pi x}{L}\right)((\mu-1)\,\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&-\left(\frac{1}{10}\right)^3\sin^3\left(\frac{\pi x}{L}\right))\}\left(\frac{\alpha^2t^2}{2!}+2\alpha\,(1-\alpha)\,t+(1-\alpha)^2\right)
\end{aligned}$$

is obtained.

$$\begin{aligned}
\mathit{\Psi}(x,t)&=\sum_{n=0}^{\infty}\mathit{\Psi}_n(x,t)\\
&=\mathit{\Psi}_0+\mathit{\Psi}_1+\mathit{\Psi}_2+\mathit{\Psi}_3+\ldots
\end{aligned}\qquad(38)$$

$$\begin{aligned}
\Psi(x,t)&=\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)+\{(\mu-1)\,\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)\qquad(39)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&-\left(\frac{1}{10}\right)^3\sin^3\left(\frac{\pi x}{L}\right)\}\,(\alpha t+1-\alpha)\\
&+\{(\mu-1)\,((\mu-1)\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&-\left(\frac{1}{10}\right)^3\sin^3(\frac{\pi x}{L}))-((\mu-1)\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^8\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^6\sin\left(\frac{\pi x}{L}\right)\\
&+12\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin\left(\frac{2\pi x}{L}\right)\\
&+6\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\cos\left(\frac{2\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)\\
&-3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)\\
&-9\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^4\sin^2\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right))
\end{aligned}$$

$$\begin{aligned}
&-2\left((1-\mu)\frac{1}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\right.\\
&+\frac{1}{10}\left(\frac{\pi}{L}\right)^6\sin\left(\frac{\pi x}{L}\right)-\frac{2}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)\\
&-3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^2\sin\left(\frac{2\pi x}{L}\right)\\
&\left.+3\left(\frac{1}{10}\right)^3\left(\frac{\pi}{L}\right)^2\sin^2\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)\right)\\
&-3\left(\frac{1}{10}\right)^2\sin^2\left(\frac{\pi x}{L}\right)\left((\mu-1)\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)\right.\\
&-\frac{1}{10}\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi x}{L}\right)+\frac{2}{10}\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)\\
&\left.-\left(\frac{1}{10}\right)^3\sin^3\left(\frac{\pi x}{L}\right)\right)\Bigg\}\left(\frac{\alpha^2t^2}{2!}+2\alpha\left(1-\alpha\right)t+\left(1-\alpha\right)^2\right)\\
&+\ldots
\end{aligned}$$

approximate solution is obtained.

If we examine the convergence by choosing $t = 1, x = 1, \alpha = 1,\ L = 9,\ \mu = 0.5$

$$r_0 = \frac{\|\Psi_1\|}{\|\Psi_0\|} = \frac{\|-0.0093\,\|}{\|0.0342\|} = 0.2723$$

$$r_1 = \frac{\|\Psi_2\|}{\|\Psi_1\|} = \frac{\|0.00027715\|}{\|-0.0093\|} = 0.0298$$

$$\vdots$$

values are perceived as being less than 1 [45]. In other words, the exact solution for this problem is obtained by A_ADM, as the series solution $\sum_{i=0}^{\infty}\Psi_i$ converges [46, 47].

6. Results and discussions

Figure 1 shows the three-dimensional graphs of the A-ADM solution for $\alpha = 1,\ 0.8,\ 0.6$ and 0.4. The contour plots of the approximate solutions for $\alpha = 0.7$ and 0.9 are given in Figure 2 , Figure 3 shows the behavior of the $\Psi(x, t)$ solution for $\alpha = 1,\ \mu = 0.3, t = 1, 4, 8, 12$ with (a) $L = 4$, (b) $L = 6$, (c) $L = 9$ and (d) $L = 12$ graphs. Figure 4 shows the solution behavior of $\Psi(x, t)$ for the values of $\alpha = 0.5$, μ=0.5, $t = 1, 4, 8, 12$ with (a) $L = 4$, (b) $L = 6$, (c) $L = 9$ and (d) $L = 12$ graphs. Figure 5 shows the solution behavior of $\Psi(x, t)$ for the values of $\alpha = 1$, μ=0.8 , $t = 1, 4, 8, 12$ with (a) $L = 4$, (b) $L = 6$, (c) $L = 9$ and (d) $L = 12$ graphs. Figure 6 shows the solution behavior of $\Psi(x, t)$ for the values of $\alpha = 0.5$, μ=0.8, $t = 1, 4, 8, 12$ with (a) $L = 4$, (b) $L = 6$, (c) $L = 9$ and (d) $L = 12$ graphs.

Contour diagrams of approximate solutions for $\alpha = 0.9, 0.7, x \in [0, 1]$and $t \in [0, 1]$ values are depicted in Figure 2 The graphs show that the results are extremely similar.

7. Conclusions

This paper investigates the efficacy of Time-Fractional Swift-Hohenberg (S-H) when A-ADM is implemented. Therefore, it is crucial to demonstrate the impact of the Caputo-Fabrizio fractional operator that is incorporated into the model being investigated. Additionally, MATLAB software was implemented to produce both 2B and 3B models. MATLAB software was employed to generate plots that illustrate the solutions of equation (1) for a variety of α values. In addition, the general structure of the 3B surface diagrams generated for the solutions of the equations was analyzed to determine the variety. Furthermore, the MATLAB program was employed to generate graphs of the numerical solutions. Swift-Hohenberg (S-H) numerical solutions were efficiently and effectively derived. Consequently, it can be inferred that A-ADM is highly efficient and resilient in its ability to compute numerical data for the purpose of solving a variety of fractional nonlinear partial differential equations. We anticipate that the A-ADM approach will be quite beneficial in the resolution of partial differential equations that include a variety of fractional derivatives in future research. The manuscript provides the mathematical answer but fails to relate it to the physical meaning of the Swift–Hohenberg equation. An analysis of how the solution embodies the pattern creation or dynamic behavior represented by the S-H equation would enhance the practical significance of the study.

This method has the potential to be implemented in a diverse array of scientific and engineering applications. The suggested approach is not confined to the particular fractional Swift-Hohenberg (S-H) problem examined; it may, in theory, be generalized to a wide array of nonlinear fractional partial differential equations (PDEs). The application hinges on the feasibility of an appropriate decomposition of nonlinear terms (e.g., by Adomian polynomials) and the proper definition of the fractional derivative operator (such as Caputo-Fabrizio) for the specified beginning and boundary conditions. When these assumptions are fulfilled, the approach remains applicable and may be modified for different nonlinear fractional models.

The results highlight the improved ability of the CF-based model to capture complex behaviors, making it a valuable tool for studying complex phenomena in fluid dynamics, nonlinear optics, and related fields.

Acknowledgments

The authors would like to express their gratitude to the reviewer for his insightful comments and recommendations, which have significantly enhanced the original manuscript in its current state.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

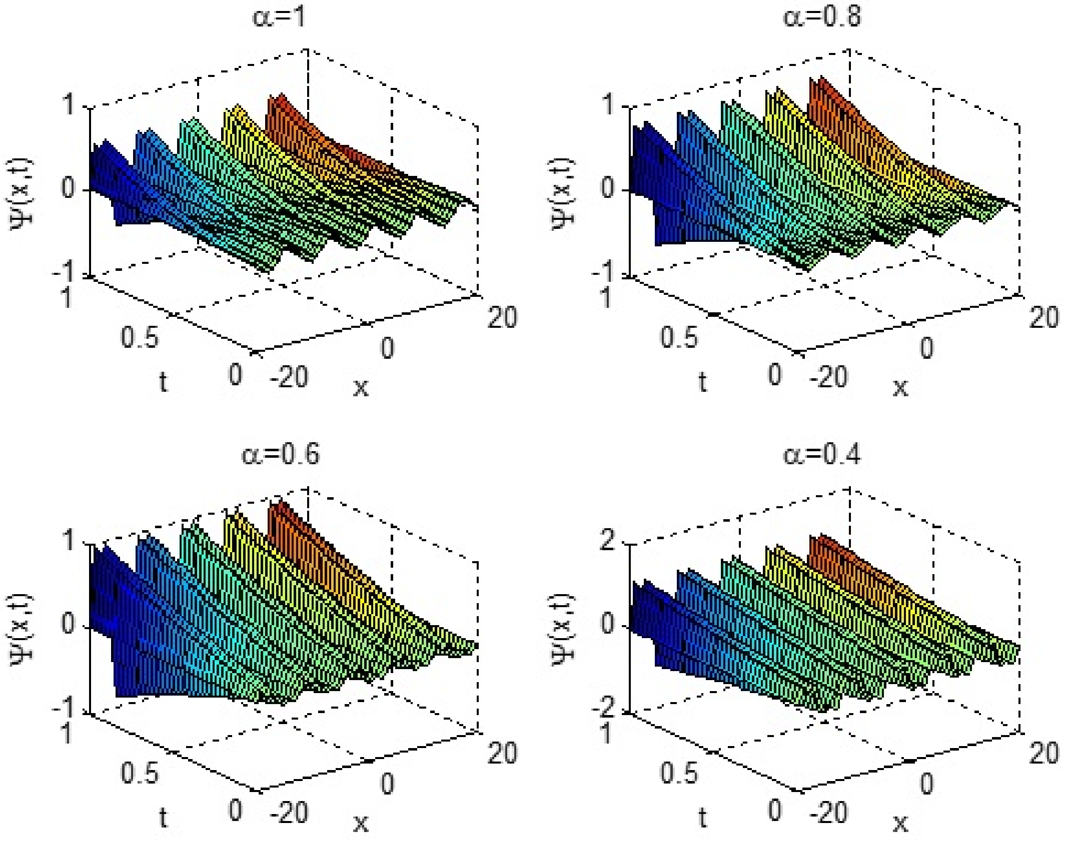

Figure 1. 3B graphical representation for $\Psi(x,\ t)$, the approximate solution when $\alpha = 1, 0.8, 0.6\ and\ 0.4$ of Example 5.1

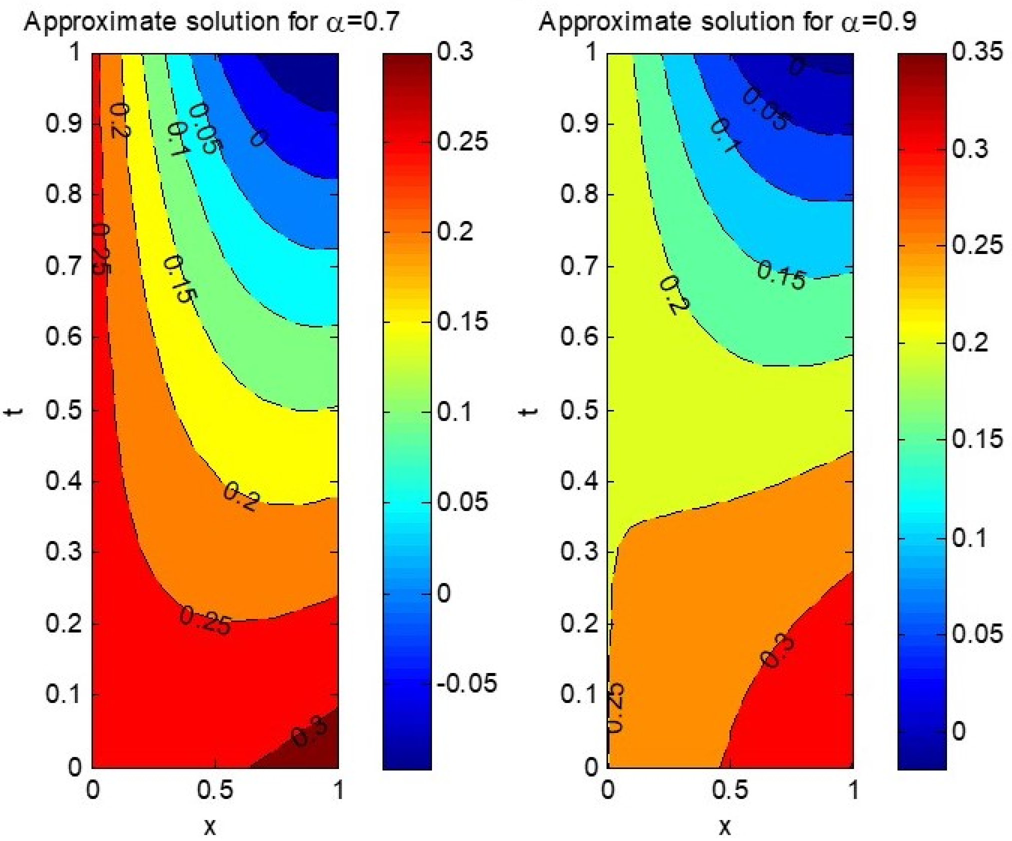

Figure 2. Contour plot of representation for $\Psi\ (x, t)$, the approximate solutions for Example 5.1 at $\alpha = 0.9, 0.7 x \in [0, 1]$ and $t \in [0, 1]$

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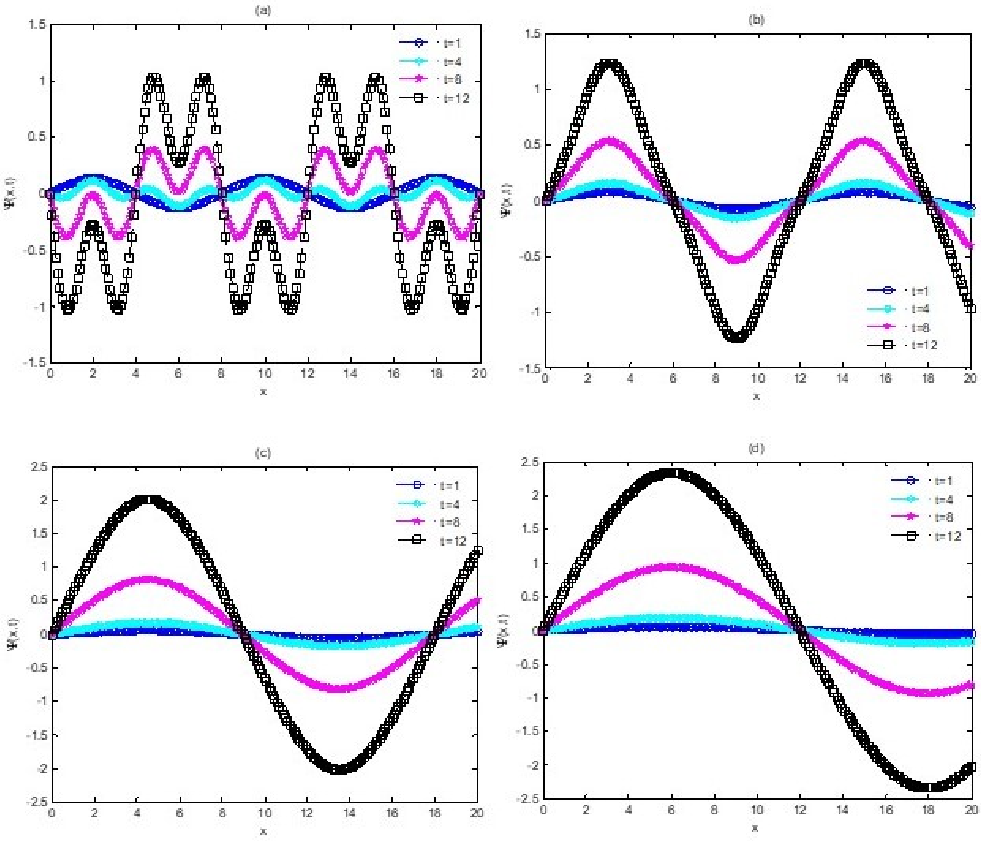

Figure 3. Plots of $\Psi(x,\ t)$ v.s at $\alpha = 1$, $\mu = 0, 3$ for (a) $L = 4$, (b) $L = 6$, (c) $L = 9$ and (d) $L = 12$ of Example 5.1

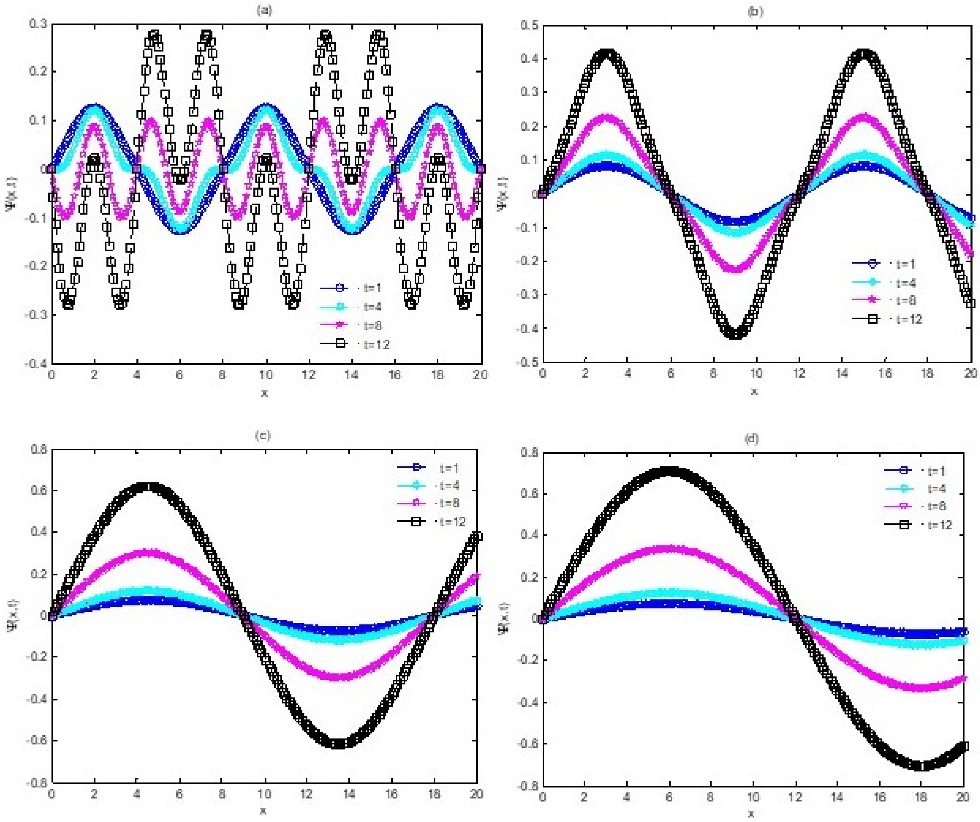

Figure 4. Plots of $\Psi(x,\ t)$ v.s at $\alpha = 0.5$, $\mu = 0.3$ for (a) $L = 4$, (b) $L = 6$, (c) $L = 9$ and (d) $L = 12$ of Example 5.1

https://doi.org/10.57647/mathsci.2025.1902.08

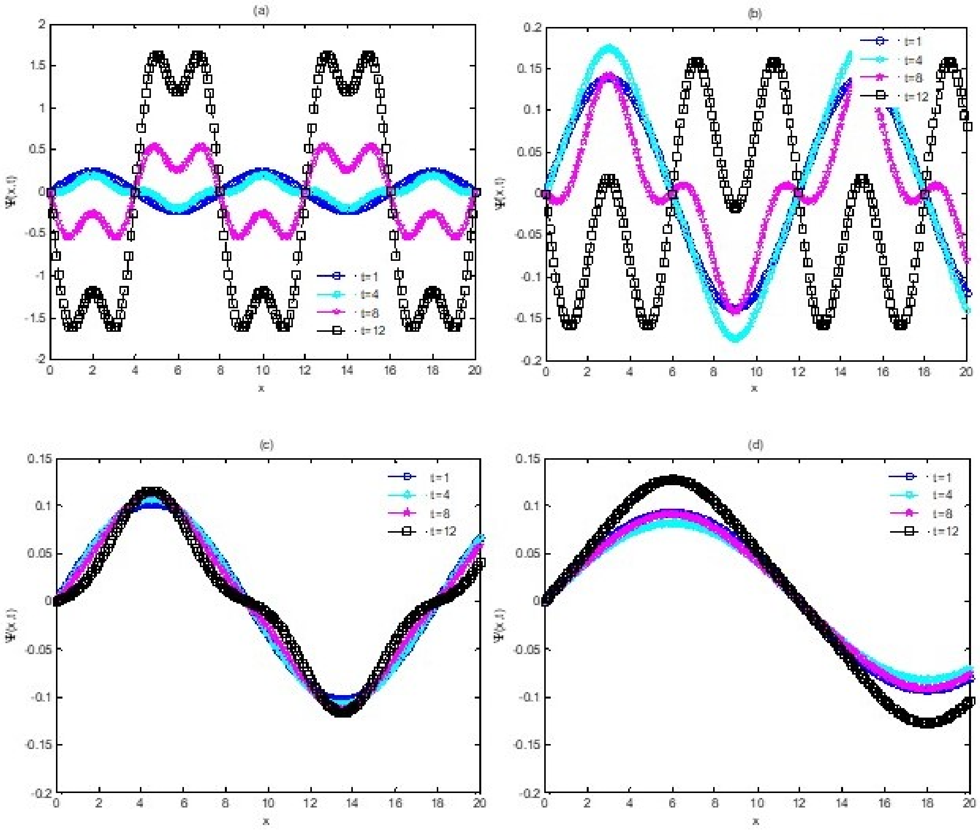

Figure 5. Plots of $\Psi(x,\ t)$ v.s at $\alpha = 1$, $\mu = 0.8$ for (a) $L = 4$, (b) $L = 6$, (c) $L = 9$ and (d) $L = 12$ of Example 5.1

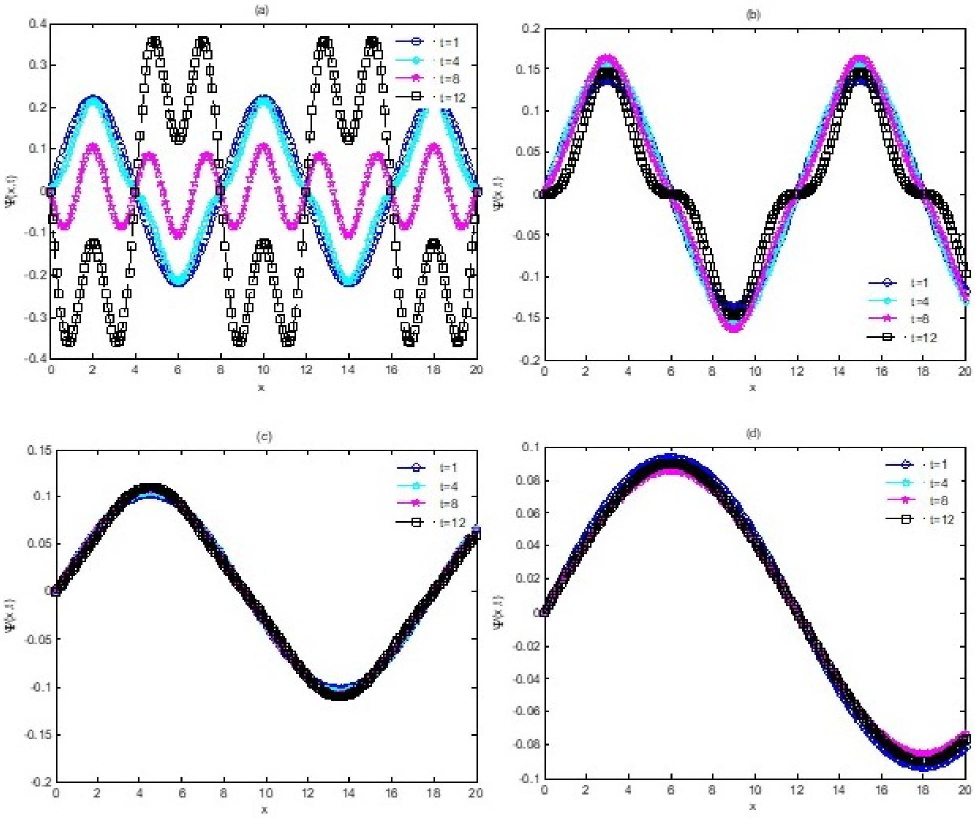

Figure 6. Plots of $\Psi(x,\ t)$ v.s at $\alpha = 0.5$, $\mu = 0.8$ for (a) $L = 4$, (b) $L = 6$, (c) $L = 9$ and (d) $L = 12$ of Example 5.1

Authors contributions
All authors have read and approved the final version of the manuscript for publication. All authors of this article have been contributed equally.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The authors declare no conflict of interest.

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