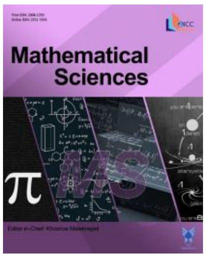

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On Recovering a Time-Dependent Inverse Problem in the Diffusion Model with a Dirichlet-Type Constraint and an Integral Overposed Condition: Theory and Simulation

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Abstract

Our aim in this paper is to recover a time-dependent source term, along with the solution of the diffusion equation, by using a new definition of the derivative, known as the conformable approach. Herein, the inverse problem is utilized subject to Dirichlet boundary constraint conditions and an integral overposed condition. An explicit solution in series form for the considered inverse source problem is obtained by employing the Fourier expansion scheme. The thoughtful mathematical structure, including the theoretical (existence, stability, and uniqueness) for the suggested regular solution, is settled and affirmed. Two time-conformable diffusion equation examples are considered to show the stability result. Various graphical plots and numerical tables are utilized to confirm the results discussed. The final remarks, highlights, and some focused references are given at the end.

Keywords: Diffusion equation; Fourier expansion method; Inverse source problem; Numerical simulation; Time-conformable derivative

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1. Introduction

In applied mathematics, a direct problem typically involves modeling certain physical fields, phenomena, or processes. The objective is to determine a function that describes the underlying physical process. Over the past decades, increasing attention has been devoted to the mathematical framework concerned with the reversal of measurements, referred to as an ISP.

These problems constitute a highly challenging area of research that lies at the intersection of mathematics and various scientific disciplines [1–5]. Many researchers are attentive to the study of an ISP for diffusion equations where the time derivative operators are different, for example, Riemann, Caputo, Fabrizio, Riesz's, or Atangana techniques [6-12]. Such time-fractional models of inverse problems can be applied to formulate various physical phenomena, including chemical engineering, thermoelasticity, heat conduction, and fluid flow in porous media [1-17].

The ISPs constitute a fundamental class of problems in theoretical and applied mathematics, dealing with the determination of an unknown source from observed effects subject to several constraints, like the over-determination constraint of integral type that is used in this essay.

Mathematically, given a mapping or operator that describes a forward process from parameters to data, the ISP seeks to recover the parameters when only the output data are obtainable.

Those problems are typically ill-posed in the sense of Hadamard, meaning they may lack existence, uniqueness, or stability of solutions. So, regularization schemes are therefore essential to impose constraints and obtain stable physically substantive simulations ISPs arise widely in several areas, including diffusion, heat, and wave equations, and the Hermite nabla difference equation. More details can be found in [1-11].

The CD provides an adaptable mathematical approach that enlarges the traditional calculus by incorporating a fractional order parameter $0 < \alpha \leq 1$. This allows for the modeling of complex models like the ISP with memory effects, hereditary properties, and nonlinear dynamics that cannot be characterized by integer-order derivatives alone. Herein, employing the CD can enhance the physical meaningfulness of the ISP by accounting for intermediate behaviors, anomalous diffusion, or time-dependent responses often observed in physical, biological, and engineering models but disregarded by classical differential operators.

Recently, conformable calculus has appeared in various areas of pure analysis, chemistry, engineering, biology, and mathematical physics as a powerful mathematical tool for various applications and processes [18-24]. More results on other fractional approaches can be found in [25-33].

Although classical fractional derivatives are non-local operators that incorporate memory effects and hereditary behaviors, the CD used in this work is a local operator that naturally generalizes the classical derivative. It does not involve non-locality or memory kernels, yet it retains many useful properties of standard differentiation and allows a continuous interpolation between integer orders. Compared to non-local fractional derivatives, which model complex processes with historical dependence, the CD provides simpler analytical and numerical treatment while remaining a flexible and effective tool for modeling diffusion-type processes and formulating the ISP considered here.

It was utilized to subedit different kinds of nonlinear time-conformable partial differential models through vantage given to supply a more global demonstration of chaos, dynamic process, and the modality of state variation over time.

This derivative depends on the limit definition of the derivative of a function. So, this derivative seems to be a natural extension of the ordinary derivative. The employment of the CD has acquired distinguished amendment and awareness in several domains of engineering and sciences. This work contributes to giving a solution set of an ISP for the diffusion equation involving a CD in time from an integral over-specified condition, together with Dirichlet BCCs. Hither, we are attentive to the following global 2-D TCDE:

$$\begin{cases} D_t^{\alpha}\wp(x,t) = \wp_{xx}(x,t) + f(x)a(t), & (x,t) \in [0,1]\times[0,T], \\ \wp(0,t) = \wp(1,t) = 0, & t \in [0,T], \\ \wp(x,0) = \phi(x), & x \in [0,1], \end{cases} \tag{1}$$

where $T > 0$, D_t^{α} stands for the time-CD of order α with $0 < \alpha \leq 1$, and $f(x)a(t)$ is the source term of x and t. Right after that, $\wp: [0,1]\times[0,T] \to \mathbb{R}$, $f: [0,1] \to \mathbb{R}$, $a: [0,T] \to \mathbb{R}$, and $\phi(x): [0,1] \to \mathbb{R}$.

The ISP here finesses the recovery of a time-dependent source term $a(t)$ together with the unknown suggested $\wp(x,t)$ solution in (1). Thus, it is indispensable to suggest some supplementary input on the so-called over-determination constraint of integral type as

$$\int_0^1 \wp(x,t)\,dx = \Pi(t), \quad t \in [0,T], \tag{2}$$

where $\Pi(t)$ is a fully absolutely continuous function in which the exporter control $a(t)$ needs to be specified by the exporter of thermal energy $\Pi(t)$.

Here, the time-CD and the conformable integral of order $0 < \alpha \leq 1$ are as follows:

$$D_t^{\alpha}\wp(x,t) = \lim_{\varepsilon\to 0}\frac{\wp(x,t+\varepsilon t^{1-\alpha}) - \wp(x,t)}{\varepsilon}. \tag{3}$$

$$\mathcal{J}_0^{\alpha}\wp(x,t) = \int_0^t \frac{\wp(x,\xi)}{\xi^{1-\alpha}}\,d\xi. \tag{4}$$

In conclusion, in this research, we need some results, such as if $\wp_1, \wp_2: [0,\infty[\to \mathbb{R}$ be two functions and $t > 0$, then, for $0 < \alpha \leq 1$, we have [18,19]

1. $D_t^{\alpha}(\wp_1(t)) = t^{1-\alpha}\wp_1'(t)$ if $\wp_1$ is differentiable.
2. $\mathcal{J}_0^{\alpha}D_t^{\alpha}\wp_1(t) = \wp_1(t) - \wp_1(0)$ if $D_t^{\alpha}\wp_1(t)$ is continuous.
3. $D_t^{\alpha}\mathcal{J}_0^{\alpha}\wp_1(t) = \wp_1(t)$ if $\wp_1$ is continuous.
4. $D_t^{\alpha}(a\wp_1 + b\wp_2) = aD_t^{\alpha}(\wp_1) + bD_t^{\alpha}(\wp_2)$ if $\wp_1, \wp_2$ are α-differentiable with $a, b \in \mathbb{R}$.

A function $\wp_1$ is α-differentiable at some point t if $D_t^{\alpha}(\wp_1)(t)$ exists. Indeed, a function could be α-differentiable for some value of α at a point but not differentiable. For example, the function $\wp_1(t) = 2\sqrt{t}$, $D_t^{0.5}(\wp_1)(0) = 1$, but $D_t^{1}(\wp_1)(0)$ does not exist.

However, what we will utilize and discuss in the research are as follows:

- Part 1: Presentation: figuration of the ISP and its significance.
- Part 2: Series formation: eigenvalues & eigenfunctions.
- Part 3: Existence and uniqueness: unique regular solution and consistency condition.
- Part 4: Continuously dependent on the data: stability analysis for a and $\wp$.
- Part 5: Numerical simulation: algorithm, applications, and analyses.

- Part 6: Outline: notes, future, and highlights.

2. Series representation of ISP

This section aims to extend the solution of the ISP for the TCDE in the context of Dirichlet BCCs and an integral overposed condition. The proposed approach is formulated using FEM.

Let us make clear what a solution is to the direct problem (1). First, let us consider the following spectral BCC problem:

$$\begin{cases} \mathcal{S}''(x) + \mu\mathcal{S}(x) = 0, & 0 \le x \le 1, \\ \mathcal{S}(0) = \mathcal{S}(1) = 0, \end{cases} \tag{5}$$

Herein, (5) is a classical Sturm-Liouville problem. The eigenvalues $\{\mu_n\}_n$ and eigenfunctions $\{\mathcal{S}_n\}_n$ form a complete orthogonal system, allowing the solution of the direct problem (1) to be represented as a series expansion in terms of $\{\mathcal{S}_n\}_n$, as commonly done in spectral methods for boundary value problems [34-40]. The eigenvalues and corresponding eigenfunctions are explicitly given by

$$\begin{cases} \mu_n = (n\pi)^2, & n = 1,2,\cdots, \\ \mathcal{S}_n(x) = \frac{1}{\sqrt{2}}\sin(\sqrt{\mu_n}x), & n = 1,2,\cdots. \end{cases} \tag{6}$$

During that, the set of eigenfunctions $\{\mathcal{S}_n\}_{n=1}^{\infty}$ forms an orthogonal basis for $L^2[0,1]$ space. However, by applying the procedure of FEM, we derive the solution $\wp(x,t)$ of the direct problem (1) as follows:

$$\wp(x,t) = \sum_{n=1}^{\infty} Y_n(t)\mathcal{S}_n(x), \tag{7}$$

where the unknown $Y_n(t)$ is the solution of the attached TCDE:

$$D_t^\alpha Y_n(t) = -\mu_n Y_n(t) + f_n a(t). \tag{8}$$

By using the appropriate constraint conditions in (1) and the Laplace technique, one can gain

$$Y_n(t) = \phi_n e^{-\mu_n\frac{t^\alpha}{\alpha}} + f_n\int_0^t s^{\alpha-1}a(s)e^{-\mu_n\frac{t^\alpha - s^\alpha}{\alpha}}ds, \tag{9}$$

where the coefficients ϕ_n and f_n are given, simultaneously, as

$$\begin{cases} \phi_n = \int_0^1 \phi(x)\mathcal{S}_n(x)dx, \\ f_n = \int_0^1 f(x)\mathcal{S}_n(x)dx. \end{cases} \tag{10}$$

Hence, the solution to the direct problem (1) is given by

$$\wp(x,t) = \sum_{n=1}^{\infty}\left\{\phi_n e^{-\mu_n\frac{t^\alpha}{\alpha}} + f_n\int_0^t s^{\alpha-1}a(s)e^{-\mu_n\frac{t^\alpha - s^\alpha}{\alpha}}ds\right\}\mathcal{S}_n(x). \tag{11}$$

Finally, let us determine the solution of the ISP from the over-specified condition (2) as

$$\int_0^1 D_t^\alpha \wp(x,t)dx = D_t^\alpha \Pi(t). \tag{12}$$

One can deduce the formal expression from the equation of the direct problem (1) as

$$a(t) = \left[\int_0^1 f(x)dx\right]^{-1}\left[D_t^\alpha\Pi(t) + \sum_{n=1}^{\infty}\mu_n\left\{\phi_n e^{-\mu_n\frac{t^\alpha}{\alpha}} + f_n\int_0^t s^{\alpha-1}a(s)e^{-\mu_n\frac{t^\alpha - s^\alpha}{\alpha}}ds\right\}N_n\right], \tag{13}$$

where $N_n = \int_0^1 \mathcal{S}_n(x)dx$.

For simplicity, put the following abbreviations in (11):

$$F(t) = \left[\int_0^1 f(x)dx\right]^{-1}\left[D_t^\alpha\Pi(t) + \sum_{n=1}^{\infty}\mu_n\phi_n e^{-\mu_n\frac{t^\alpha}{\alpha}}N_n\right]. \tag{14}$$

$$Q(t,s) = \left[\int_0^1 f(x)dx\right]^{-1}\sum_{n=1}^{\infty}\mu_n f_n s^{\alpha-1}e^{-\mu_n\frac{t^\alpha - s^\alpha}{\alpha}}N_n. \tag{15}$$

Consequently, we obtain the following Volterra integral equation for $a(t)$:

$$a(t) = F(t) + \int_0^t Q(t,s)a(s)ds. \tag{16}$$

The next lemma is useful in the derivations of the next results, which are included in the running sections. Herein,

$C^5[0,1] = \{\mathfrak{H}: [0,1] \to \mathbb{R}: \mathfrak{H} \in C[0,1], \mathfrak{H}^{(k)} \in C(0,1), \text{and } \mathfrak{H}^{(k)}(0) = \mathfrak{H}^{(k)}(1) = 0 \text{ for } k = 0,1,2,3,4,5\}$.
Indeed, $\|\mathfrak{H}\|_{C^5} = \max_{x\in[0,1]}\{|\mathfrak{H}^{(k)}(x)|: k = 0,1,2,3,4,5\}$.

Lemma 1. If $\mathfrak{H}(x) \in C^5[0,1]$ satisfies $\mathfrak{H}(0) = \mathfrak{H}(1) = 0$ and $\mathfrak{H}''(0) = \mathfrak{H}''(1) = 0$, then the following inequality holds:

$$\sum_{n=1}^{\infty} \mu_n |\mathfrak{H}_n| \leq c\|\mathfrak{H}\|_{C^5}. \tag{17}$$

Proof. One can remember that

$$\begin{aligned}\mathfrak{H}_n &= \frac{1}{\mu_n}\mu_n\langle \mathfrak{H}(x), \mathcal{S}_n(x)\rangle_{L^2} \\ &= \frac{1}{\mu_n}\langle \mathfrak{H}''(x), \mathcal{S}_n(x)\rangle_{L^2}.\end{aligned} \tag{18}$$

Applying Bessel and Schwarz inequalities, we obtain

$$\sum_{n=1}^{\infty} |\mathfrak{H}_n| \leq c\|\mathfrak{H}''\|_{L^2} \leq c\|\mathfrak{H}\|_{C^5}. \tag{19}$$

Finally, one can write

$$\sum_{n=1}^{\infty} \mu_n |\mathfrak{H}_n| \leq c\|\mathfrak{H}\|_{C^5}. \tag{20}$$

3. Existence and uniqueness results

This section aims to establish the uniqueness of the solution to the ISP for the TCDE. The proof is based on the contraction mapping principle and the Banach fixed-point theorem.

A solution set $\{a(t), \wp(x,t)\}$ is called a regular solution of the ISP (1) and (2) if $\wp(x,t)$, $\wp_{xx}(x,t)$, and $D_t^\alpha \wp(x,t)$ are in $C([0,1]\times[0,1])$ and $a(t) \in C[0,1]$.

Theorem 1. The ISP (1) and (2) processes a unique regular solution provided that the following are met:

1. $\phi \in C^3[0,1]$ such that $\phi(0) = \phi(1) = 0$ and $f \in C^5[0,1]$ such that $f(0) = f(1) = 0$,
2. $\int_0^1 f(x) \neq 0$ and $0 < \frac{1}{M_0} \leq \left|\int_0^1 f(x)dx\right|$, where $M_0 > 0$,
3. $\Pi(t) \in C^1[0,T]$ and satisfies the consistency condition $\int_0^1 \phi(x)dx = \Pi(0)$.

Proof. Firstly, consider $\{a(t), \wp(x,t)\}$ and let us prove the unique existence of solutions. Firstly, according to (13), we define the mapping $P: C[0,1] \to C[0,1]$ which satisfies

$$P\big(a(t)\big) = a(t), \tag{21}$$

such that

$$P\big(a(t)\big) = F(t) + \int_0^t Q(t,s)a(s)ds, \tag{22}$$

where $F(t)$ and $Q(t,s)$ are as given in (14) and (15), simultaneously. Now, we will prove that for $a(t) \in C[0,1]$ the image $P(a(t))$ represents a continuous function. Since,

$$\begin{aligned}|Q(t,s)| &= \left|\frac{\left[\int_0^1 f(x)dx\right]^{-1}}{\sum_{n=1}^{\infty} \mu_n f_n s^{\alpha-1} e^{-\mu_n \frac{t^\alpha - s^\alpha}{\alpha}} N_n}\right| \\ &\leq M_0 T^{\alpha-1} \sum_{n=1}^{\infty} \mu_n |f_n|.\end{aligned} \tag{23}$$

By using that $ze^{-bz} \leq \frac{1}{be}$ with $z \geq 0$ and $b > 0$, we get

$$\begin{aligned}|F(t)| &= \left|\frac{\left[\int_0^1 f(x)dx\right]^{-1}}{\left[D_t^\alpha \Pi(t) + \sum_{n=1}^{\infty} \mu_n \phi_n e^{-\mu_n \frac{t^\alpha}{\alpha}} N_n\right]}\right| \\ &\leq M_0\left\{M_1|\Pi(t)| + \frac{\alpha}{t^\alpha e}\sum_{n=1}^{\infty}\phi_n\right\}.\end{aligned} \tag{24}$$

Here, we note that the constant M_1 is well-defined. Since $\Pi(t) \in C^1[0,T]$, its CD satisfies

$$D_t^\alpha \Pi(t) = t^{1-\alpha}\Pi'(t), \tag{25}$$

which is continuous on the interval $[0,T]$. Consequently, $D_t^\alpha \Pi(t)$ is bounded, and we may define

$$M_1 = \max_t |D_t^\alpha \Pi(t)| < \infty. \tag{26}$$

The convergence in (23) and (24) is inferred using the continuity of $\phi(x)$ together with $f(x)$ and Lemma 1. Hence $P\big(a(t)\big)$ is well-defined.

Now, let us show that the mapping $P\big(a(t)\big) = a(t)$ in $C[0,1]$ is a contraction mapping. We consider $T \leq T_0$, where $T_0^\alpha = \frac{1}{\|f\|_{C^5}M_1}$ and $M_2 = M_0 T^{\alpha-1}\|f\|_{C^5}$. We have the following estimates:

$$\begin{aligned}\left|P\big(a(t)\big) - P\big(b(t)\big)\right| &\leq \int_0^t |Q(t,s)||a(s) - b(s)|ds \\ &\leq TM_2|a(s) - b(s)|.\end{aligned} \tag{27}$$

Consequently,

$$\begin{aligned}\|P(a) - P(b)\| &= \max_t \left|P\big(a(t)\big) - P\big(b(t)\big)\right| \\ &\leq TM_2\|a - b\|.\end{aligned} \tag{28}$$

Under the explicit condition $TM_2 < 1$, the mapping P is a contraction. Therefore, by applying the Banach fixed-point theorem, one can obtain a unique $a(t) \in C[0,T]$. Under the condition of Theorem 1 and the Schwartz inequality, the series (11) and the series representation of $\wp_{xx}$ are uniformly convergent in

$[0,1]\times[0,1]$. Subsequently, the convergence of the series representation of T_t^{α} is deduced from (1).

Suppose that $(a,\wp)$ and (b,v) are two solutions of (1) and (2). Then from (11) and (21), we get

$$\wp(x,t)-v(x,t) = \sum_{n=1}^{\infty}\left\{f_n\int_0^t s^{\alpha-1}\big(a(s) - b(s)\big)e^{-(n\pi)^2\frac{t^\alpha-s^\alpha}{\alpha}}ds\right\}S_n(x). \quad (29)$$

$$a(t)-b(t)=P[a(t)-b(t)].$$

Thus, $a(t)=b(t)$ and substituting $a(t)=b(t)$ in (29), one gets $\wp(x,t)=v(x,t)$.

4.Continuously dependent on the data

This section addresses the continuous dependence of the solution set on the given data. Specifically, we present a stability analysis for ISP (1) and (2), with emphasis on the role of the time-CD. First, considering a solution set $\{a(t),\wp(x,t)\}$, where $\wp(x,t)$, $\wp_{xx}(x,t)$, and $D_t^{\alpha}\wp(x,t)$ are in $C([0,1]\times[0,1])$ and $a(t)\in C[0,1]$.

Theorem 2. The solution set $\{a(t),\wp(x,t)\}$ of the ISP (1) and (2) depends continuously on the given data $\{\phi(x),f(x),\Pi(t)\}$ such that

$$\|f\|_{C^5}\le M_3,\quad \|\Pi\|_C\le M_4,\quad \|\phi\|_{C^3}\le M_5, \quad (30)$$

for some positive constants M_i with $i=3,4,5$.

Proof. Let $\{\wp(x,t),a(t)\}$ and $\{\tilde{\wp}(x,t),\tilde{a}(t)\}$ be two solution sets of ISP (1) and (2) corresponding to the data $\{f,\phi,\Pi\}$ and $\{\tilde{f},\tilde{\phi},\tilde{\Pi}\}$, simultaneously. According to (16), we have

$$a(t)=F(t)+\int_0^t Q(t,s)a(s)ds. \quad (31)$$

$$\tilde{a}(t)=\tilde{F}(t)+\int_0^t \tilde{Q}(t,s)\tilde{a}(s)ds. \quad (32)$$

Herein, we estimate the difference $a-\tilde{a}$ as

$$a(t)-\tilde{a}(t)=F(t)-\tilde{F}(t)+\int_0^t \tilde{Q}(t,s)\big(a(s)-\tilde{a}(s)\big) + a(t)\Big(Q(t,s) - \tilde{Q}(t,s)\Big)ds. \quad (33)$$

From (23) and (24) using $\Pi(t)\in C^1[0,1]$ and the Schwarz inequality, it is obvious that

$$\|F\|_C\le M_6,\quad \|Q(t,s)\|\le M_7,\quad \|a\|_C\le M_8, \quad (34)$$

where $M_6=M_0(M_4M_1+\frac{\alpha}{t^{\alpha}e}c_1M_5)$, $M_7=M_0T^{\alpha-1}c_2M_3$, and $M_8=\frac{M_6}{1-TM_7}$. The constants c_1 and c_2 appearing in the estimates of the Fourier coefficients ϕ_n and f_n follow directly from Lemma 1. Anyhow, from (14) and (15), we get

$$\|F-\tilde{F}\|_C\le M_1M_0\|\Pi-\tilde{\Pi}\|_C + \frac{\alpha}{t^{\alpha}e}c_1M_5M_0\|\phi-\tilde{\phi}\|_{C^3} + M_0^2(M_7+M_4)\|f-\tilde{f}\|_C. \quad (35)$$

$$\|Q-\tilde{Q}\|_C\le M_0T^{\alpha-1}c_2\|f-\tilde{f}\|_C. \quad (36)$$

Then, one can write

$$(1-TM_7)\|a-\tilde{a}\|_C \le M_9\|\Pi-\tilde{\Pi}\|_C + M_{10}\|\phi-\tilde{\phi}\|_{C^3} + M_{11}\|f-\tilde{f}\|_C, \quad (37)$$

where $M_9=M_1M_0$, $M_{10}=\frac{\alpha}{t^{\alpha}e}c_1M_5M_0$, and $M_{11}=M_0M_8T^{\alpha}\,c_2$.

Hence, after refinement, one gets the following.

$$\|a-\tilde{a}\|_C\le M_{12}\left(\|\Pi-\tilde{\Pi}\|_C+\|\phi-\tilde{\phi}\|_{C^3} + \|f-\tilde{f}\|_C\right), \quad (38)$$

where $M_{12}=\frac{1}{(1-TM_7)}\max\{M_9,M_{10},M_{11}\}$.

This proves a is continuously dependent on the input data. Similarly, using the same procedure, one can see that $\wp(x,t)$ is continuously dependent on the input data too.

5. Numerical simulation

To demonstrate the theoretical results presented in the preceding sections, we consider two applications. Through these applications, we show that $\{a(t),\wp(x,t)\}$ depends continuously on the input data. All numerical computations for the tables and the figures were performed using Mathematica 11.

Here, we utilize several measured formulas considering the random noise as:

$$f^{\delta}(\cdot)=\delta\,\mathrm{rand}(\cdot)+f(\cdot),\quad E^{\delta}(\cdot)=\delta\,\mathrm{rand}(\cdot)+E(\cdot),\quad \phi^{\delta}(\cdot)=\left(1+\frac{\delta\,\mathrm{rand}(\cdot)}{\|\phi\|_2}\right)\phi(\cdot), \quad (39)$$

providing $\mathrm{rand}(\cdot)$ is a certain generated random value.

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The numerical procedure for the presented TCDE of ISP subject to Dirichlet BCCs and an integral over-posed condition is summarized in the following algorithm:

Algorithm 1. Procedure of checking that $\{a(t), \wp(x,t)\}$ are continuously dependent on the given information data.

Phase 1. Put $t_i = i\Delta t$ for $i = 0,1,\cdots,N$.

Phase 2. Put $x_j = j\Delta x$ for $j = 0,1,\cdots,N$.

Phase 3. Put $\Delta x = \Delta t = \frac{1}{N}$ and fixed $N = 20$.

Phase 4. Numerical solution of (16): evaluate $F(t_i)$ and $\sum_{j=0}^{i-1} Q(t_i,t_j)$ using (14) and (15), with the series truncated at $K = 100$ terms. Then compute $a(t_i)$ using the quadrature scheme

$$a(t_i) \approx F(t_i) + \Delta t \sum_{j=0}^{i-1} Q(t_i,t_j)\, a(t_j), \tag{40}$$

starting from the initial value $a(0) = F(0)$.

Phase 5. Construct column vectors that collect all discrete values of $\{a(t_i), \wp(x_j,t_i)\}$ together with $\{a^\delta(t_i), \wp^\delta(x_j,t_i)\}$.

Phase 6. Estimated the attached error as

$$\mathrm{Err}_a = \frac{\sqrt{\sum_{i=0}^{N}|a^\delta(t_i) - a(t_i)|^2}}{\sqrt{\sum_{i=0}^{N}|a(t_i)|^2}},\qquad \mathrm{Err}_\wp = \frac{\sqrt{\sum_{i,j=0}^{N}\left|\wp^\delta(x_j,t_i) - \wp(x_j,t_i)\right|^2}}{\sqrt{\sum_{i,j=0}^{N}\left|\wp(x_j,t_i)\right|^2}}. \tag{41}$$

Example 1. Consider the ISP for the TCDE in the domain $[0,1]\times[0,1]$:

$$\begin{cases} D_t^\alpha \wp(x,t) = \wp_{xx}(x,t) + (1+\pi^2)\sin(\pi x)\, a(t), \\ \wp(0,t) = \wp(1,t) = 0, \\ \wp(x,0) = \sin(\pi x), \end{cases} \tag{42}$$

with an overposed condition

$$\int_0^1 \wp(x,t)\,dx = \frac{2}{\pi} e^{-\frac{t^\alpha}{\alpha}}. \tag{43}$$

During that, with order $0 < \alpha \le 1$, the set $\{a(t), \wp(x,t)\}$ can be constructed using (11) and (13) as

$$\begin{cases} a(t) = e^{-\frac{t^\alpha}{\alpha}}, \\ \wp(x,t) = e^{-\frac{t^\alpha}{\alpha}} \sin(\pi x). \end{cases} \tag{44}$$

An important outcome of this application is that the assumptions of Theorem 1 are verified.

We shall now come to the analytical side. Anyhow, to summarize the most important thing that we have done in the previous section, next, some numerical analyses and discussions based on Algorithm 1 are presented in the form of tables.

Table 1 represents Err_a in between $a(t)$ and its perturbed one $a^\delta(t)$ when $\delta \in \{0.0001, 0.002, 0.004, 0.01\}$ at $\alpha = 0.2$, $\alpha = 0.4$, and $\alpha = 0.7$ for Example 1.

Table 2 illustrates $\mathrm{Err}_\wp$ in between $\wp(x,t)$ and its perturbed one $\wp^d(x,t)$ when $\delta \in \{0.0001, 0.002, 0.004, 0.01\}$ at $\alpha = 0.2$, $\alpha = 0.4$, and $\alpha = 0.7$ for Example 1.

Next, we summarize the most important thing that we have done in the previous sections. Anyhow, some numerical analyses and discussions based on Algorithm 1 are utilized in the form of 1-D plots.

Fig. 1 explains the comparison of in-between $a(t)$ and its perturbed one $a(t)$ for $\delta = 0.005$ and $\delta = 0.05$ in the case of $\alpha = 0.2$ and $\alpha = 0.3$ for Example 1.

Now, some numerical analyses and discussions based on Algorithm 1 are utilized in the form of 2-D plots.

Figures 2 and 3 explain the behavior solution $\wp(x,t)$ and its perturbed one $\wp^\delta(x,t)$ for $\delta = 0.05$ in the case of $\alpha = 0.2$ and $\alpha = 0.4$ for Example 1.

From Tables 1 and 2 and Figures 2 and 3, we show numerically that any tiny modification in the given input information data leads to a tiny modification in the set $\{a(t), \wp(x,t)\}$.

Table 1. The output of Err_a for $a(t)$ and $a^\delta(t)$ for various α and δ in Example 1

α	$\delta_1 = 0.0001$	$\delta_2 = 0.002$	$\delta_3 = 0.004$	$\delta_4 = 0.01$
0.2	0.0000247870	0.000495736	0.000991464	0.00247859
0.4	0.0000180124	0.000360245	0.000720483	0.00180116
0.7	0.0000101923	0.000203844	0.000407685	0.00101918

Table 2. The output of $\mathrm{Err}_\wp$ for $\wp(x,t)$ and $\wp^d(x,t)$ for various α and δ in Example 1

α	$\delta_1 = 0.0001$	$\delta_2 = 0.002$	$\delta_3 = 0.004$	$\delta_4 = 0.01$
0.2	0.0003083710	0.001677420	0.01233480	0.03083710
0.4	0.0000286051	0.000572103	0.00114410	0.00286051
0.7	0.0000110591	0.000221182	0.00044235	0.00110591

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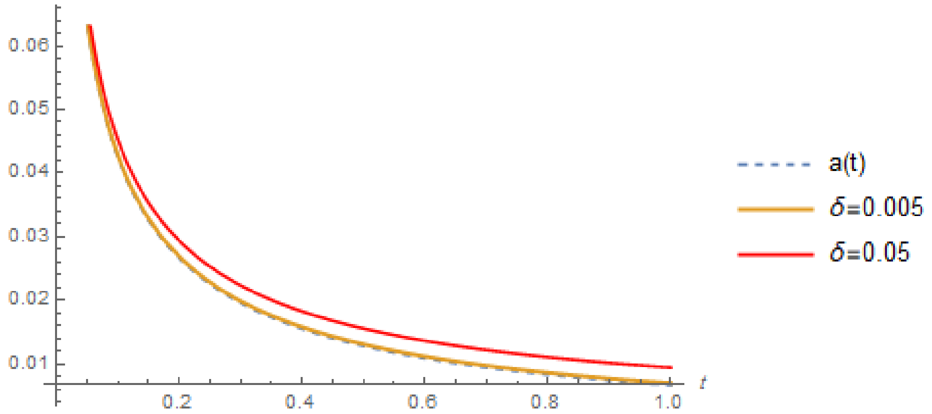
a) $\alpha = 0.2$



a) $\alpha = 0.3$

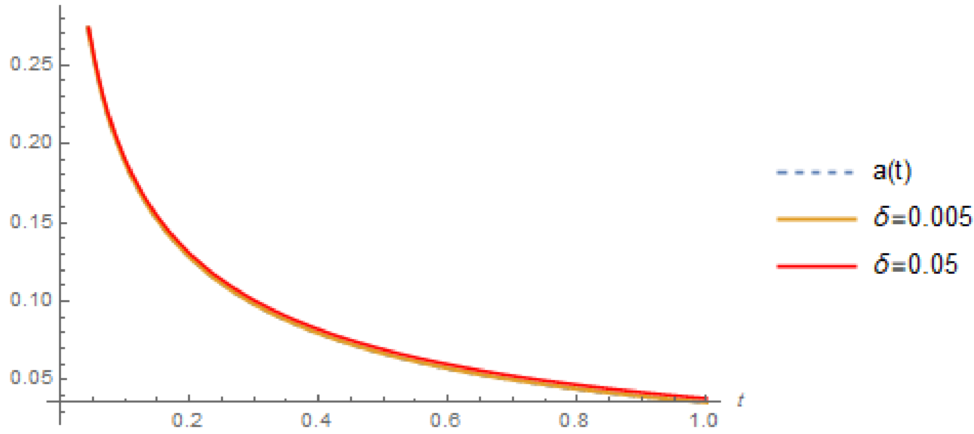
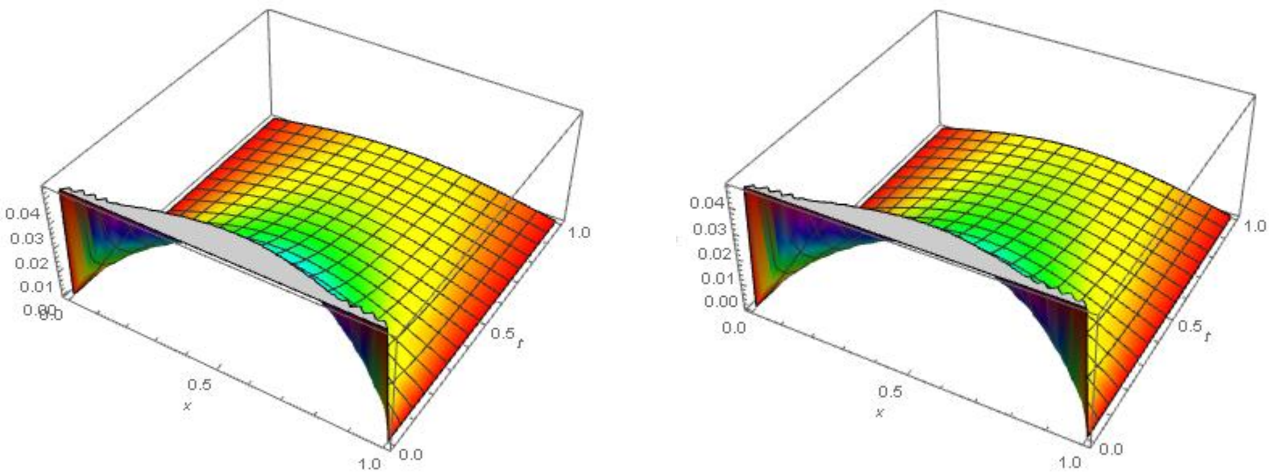


Figure 1. Comparison of in-between $a(t)$ and its perturbed one $a^{\delta}(t)$ in Example 1



a) $\wp(x, t)$ b) $\wp^{\delta}(x, t)$ for $\delta = 0.05$

Figure 2. Comparison of in-between $\wp(x, t)$ and its perturbed one $\wp^{\delta}(x, t)$ for $\delta = 0.05$ and $\alpha = 0.2$ in Example 1

Table 3. The output of Err_a for $a(t)$ and $a^{\delta}(t)$ for various α and δ in Example 2

α	$\delta_1 = 0.0001$	$\delta_2 = 0.002$	$\delta_3 = 0.004$	$\delta_4 = 0.01$
0.2	0.0000438023	0000876488	0.00176530	0.00434511
0.4	0.0000576114	0.001150440	0.00229714	0.00571496
0.7	0.0000738597	0.001474910	0.00294501	0.00732676

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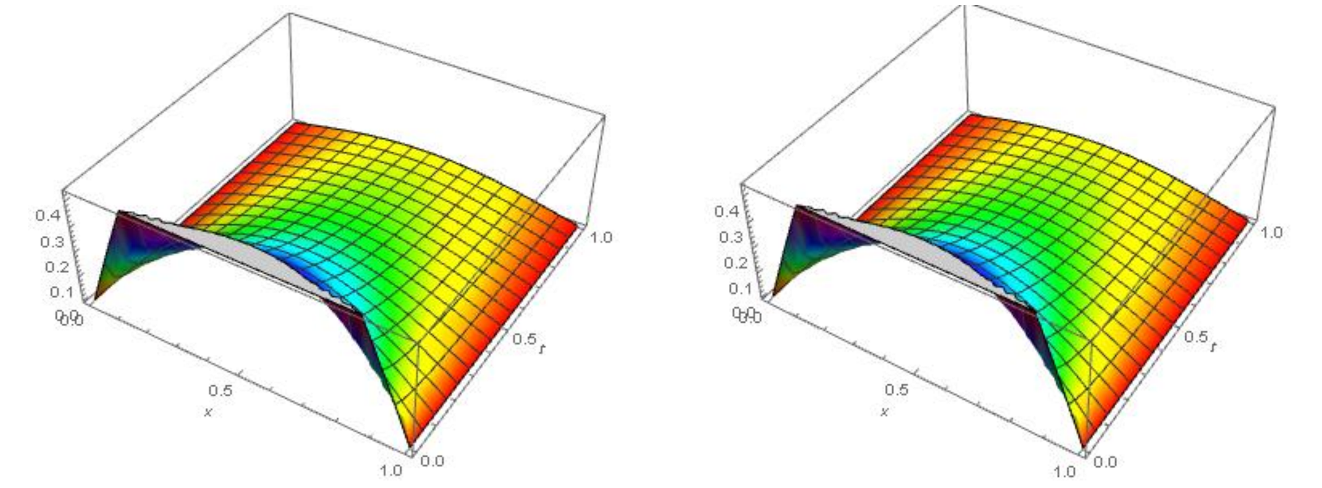

a) $\wp(x,t)$ b) $\wp^{\delta}(x,t)$ for $\delta = 0.05$

Figure 3. Comparison of in-between $\wp(x,t)$ and its perturbed one $\wp^{\delta}(x,t)$ for $\delta = 0.05$ and $\alpha = 0.4$ in Example 1

Table 4. The output of $\mathrm{Err}_{\wp}$ for $\wp(x,t)$ and $\wp^{d}(x,t)$ for various $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ in Example 2

α	$\delta_1 = 0.0001$	$\delta_2 = 0.002$	$\delta_3 = 0.004$	$\delta_4 = 0.01$
0.2	0.000116756	0.00233511	0.00467023	0.0116756
0.4	0.000133663	0.00267326	0.00534651	0.0133663
0.7	0.000160235	0.00320470	0.00640941	0.0160235

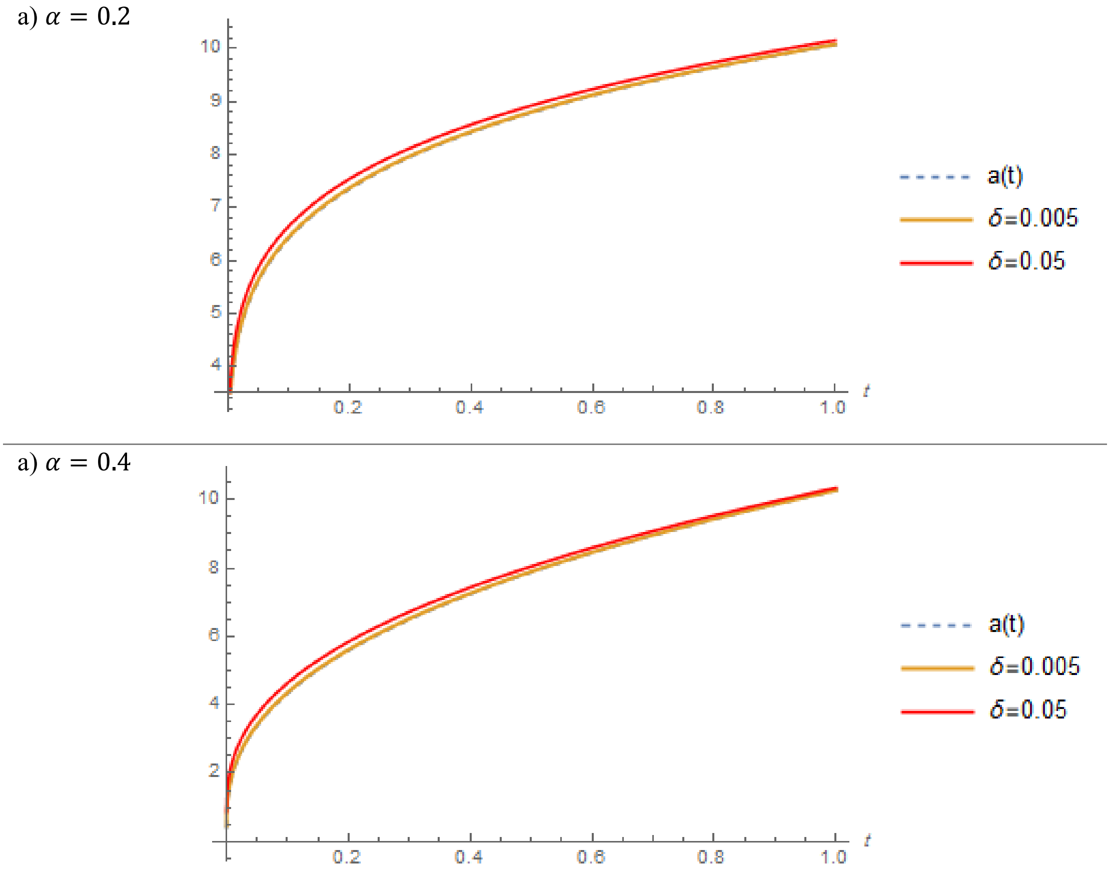

Figure 4. Comparison of in-between $a(t)$ and its regularized one $a^{\delta}(t)$ in Example 2

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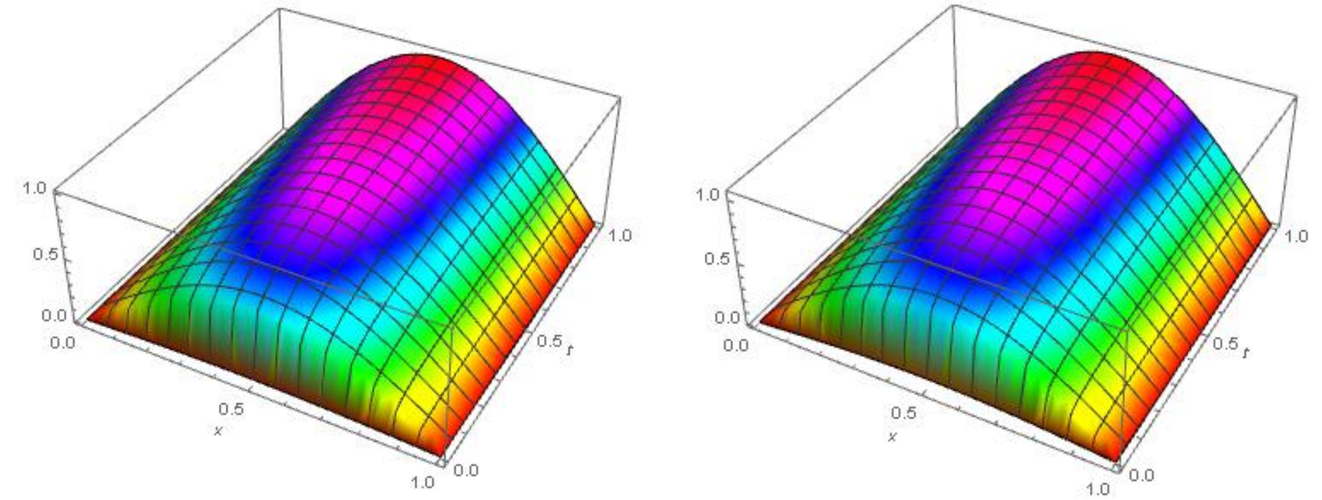
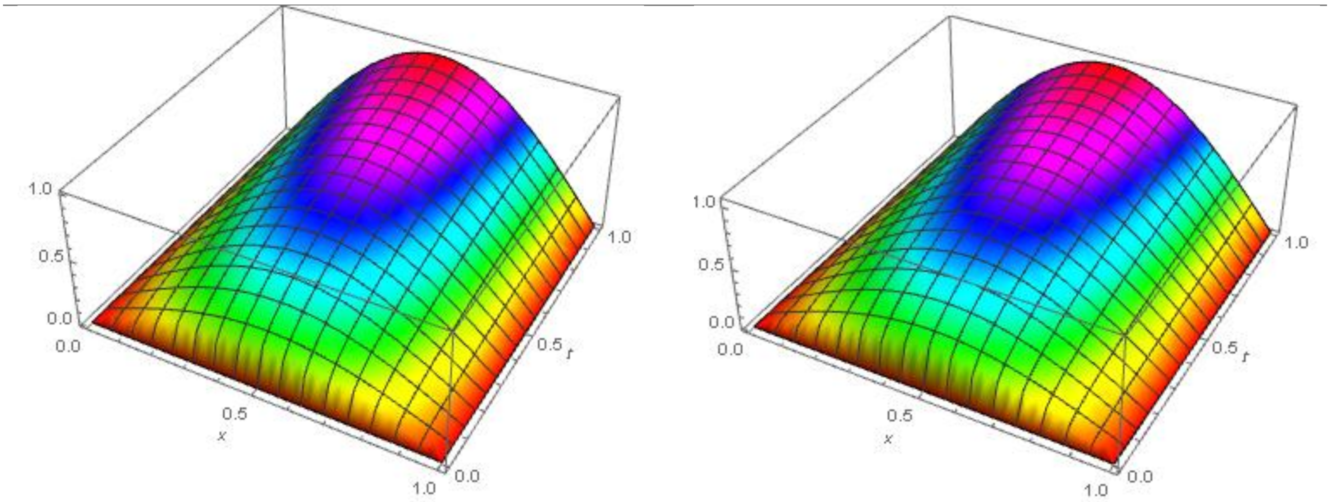


a) $\wp(x,t)$	b) $\wp^{\delta}(x,t)$ for $\delta = 0.05$

Figure 5. Comparison of in-between $\wp(x,t)$ and its perturbed one $\wp^{\delta}(x,t)$ for $\alpha = 0.2$ in Example 2



a) $\wp(x,t)$	b) $\wp^{\delta}(x,t)$ for $\delta = 0.05$

Figure 6. Comparison of in-between $\wp(x,t)$ and its perturbed one $\wp^{\delta}(x,t)$ for $\alpha = 0.4$ in Example 2

Example 2. Consider the ISP for the TCDE in the domain $[0,1] \times [0,1]$:

$$\begin{cases} D_t^{\alpha}\wp(x,t) = \wp_{xx}(x,t) + \sin(\pi x)\, a(t), \\ \wp(0,t) = \wp(1,t) = 0, \\ \wp(x,0) = 0, \end{cases} \tag{45}$$

with an overposed condition

$$\int_0^1 \wp(x,t)\,dx = \frac{2}{\pi} t^{\alpha}. \tag{46}$$

During that, with order $0 < \alpha \le 1$, the set $\{a(t), \wp(x,t)\}$ can be constructed using (11) and (13) as

$$\begin{cases} a(t) = (\alpha + \pi^2 t^{\alpha}), \\ \wp(x,t) = t^{\alpha}\sin(\pi x). \end{cases} \tag{47}$$

Again, as an important application result, one can check that the as Now, some numerical analyses and discussions based on Algorithm 1 are utilized in the form of 2-D plots. Figures 5 and 6 illustrate the solution $\wp(x,t)$ and the perturbed solution $\wp^{\delta}(x,t)$ for $\delta = 0.05$ in the case of $\alpha = 0.2$ and $\alpha = 0.4$ for Example 2. sumptions of Theorem 1 are satisfied.

We will now come to the analytical side. Anyhow, to summarize the most important thing that we have done in the previous sections, next, some numerical analyses and discussions based on Algorithm 1 are presented in the form of tables. Table 3 represents Err_a in between $a(t)$ and its perturbed one $a^{\delta}(t)$ when $\delta \in \{0.0001, 0.002, 0.004, 0.01\}$ at $\alpha = 0.2$, $\alpha = 0.4$, and $\alpha = 0.7$ for Example 2.

Table 2 illustrates $\mathrm{Err}_{\wp}$ in between $\wp(x,t)$ and its perturbed one $\wp^{d}(x,t)$ when $\delta \in \{0.0001, 0.002, 0.004, 0.01\}$ at $\alpha = 0.2$, $\alpha = 0.4$, and $\alpha = 0.7$ for Example 2.Next, we summarize the most important thing that we have done in the previous sections. Anyhow, some numerical analyses and

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discussions based on Algorithm 1 are utilized in the form of 1-D plots. Fig. 4 explains the comparison of in-between $a(t)$ and its perturbed one $a^{\delta}(t)$ for $\delta = 0.005$ and $\delta = 0.05$ in the case of $\alpha = 0.2$ and $\alpha = 0.4$ for Example 2.

Example 2. Consider the ISP for the TCDE in the domain $[0,1] \times [0,1]$:

$$\begin{cases} D_t^{\alpha} \wp(x,t) = \wp_{xx}(x,t) + \sin(\pi x)\, a(t), \\ \wp(0,t) = \wp(1,t) = 0, \\ \wp(x,0) = 0, \end{cases} \tag{45}$$

with an overposed condition

$$\int_0^1 \wp(x,t)\, dx = \frac{2}{\pi} t^{\alpha}. \tag{46}$$

During that, with order $0 < \alpha \leq 1$, the set $\{a(t), \wp(x,t)\}$ can be constructed using (11) and (13) as

$$\begin{cases} a(t) = (\alpha + \pi^2 t^{\alpha}), \\ \wp(x,t) = t^{\alpha} \sin(\pi x). \end{cases} \tag{47}$$

Again, as an important application result, one can check that the as Now, some numerical analyses and discussions based on Algorithm 1 are utilized in the form of 2-D plots.

Figures 5 and 6 illustrate the solution $\wp(x,t)$ and the perturbed solution $\wp^{\delta}(x,t)$ for $\delta = 0.05$ in the case of $\alpha = 0.2$ and $\alpha = 0.4$ for Example 2.sumptions of Theorem 1 are satisfied.

We will now come to the analytical side. Anyhow, to summarize the most important thing that we have done in the previous sections, next, some numerical analyses and discussions based on Algorithm 1 are presented in the form of tables. Table 3 represents Err_a in between $a(t)$ and its perturbed one $a^{\delta}(t)$ when $\delta \in \{0.0001, 0.002, 0.004, 0.01\}$ at $\alpha = 0.2$, $\alpha = 0.4$, and $\alpha = 0.7$ for Example 2. Table 2 illustrates $\mathrm{Err}_{\wp}$ in between $\wp(x,t)$ and its perturbed one $\wp^{d}(x,t)$ when $\delta \in \{0.0001, 0.002, 0.004, 0.01\}$ at $\alpha = 0.2$, $\alpha = 0.4$, and $\alpha = 0.7$ for Example 2. Next, we summarize the most important thing that we have done in the previous sections. Anyhow, some numerical analyses and discussions based on Algorithm 1 are utilized in the form of 1-D plots. Fig. 4 explains the comparison of in-between $a(t)$ and its perturbed one $a^{\delta}(t)$ for $\delta = 0.005$ and $\delta = 0.05$ in the case of $\alpha = 0.2$ and $\alpha = 0.4$ for Example 2. From Tables 3 and 4 and Figures 5 and 6, we show numerically that any tiny modification in the given input information data leads to a tiny modification in the set $\{a(t), \wp(x,t)\}$.

6. Conclusion and highlight

In this study, the problem of identifying a time-dependent source term for TCDE under the time-CD subject to Dirichlet BCCs and an integral over-posed condition has been considered. First, by using the FEM, we construct a series solution, and then the problem is proved to have a unique classical solution set by analyzing the convergence of the resulting infinite series and the Banach fixed-point theorem. Second, uniqueness and stability results are obtained. Finally, graphical plots and numerical tables are utilized to confirm the results discussed using the presented algorithm. Our future work will generalize the ISP for the TCDE subject to Robin BCCs and an integral over-posed condition in three independent variables.

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Conflicts of Interest
The authors declare that they have no conflicts of interest.

Data Source
All figures and tables were created by the authors.

Data Availability Statement
No datasets are associated with this manuscript. The datasets used for generating the plots and results during the current study can be directly obtained from the numerical simulation of the related mathematical equations in the manuscript.

Consent to Participate and Publish
The authors declare that they participated in this paper willingly, and they declare consent to the publication of this paper.

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This article does not contain any studies with human participants or animals performed by any of the authors.

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Smina Djennadi. Data Curation, Investigation, Software, Methodology, Validation, Roles/Writing - Original Draft, Writing - Review & Editing.
Omar Abu Arqub. Conceptualization, Formal Analysis, Investigation, Project Administration, Software, Writing - Review & Editing.
Marwan Abukhaled. Formal Analysis, Investigation, Methodology, Software, Writing - Review & Editing.

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