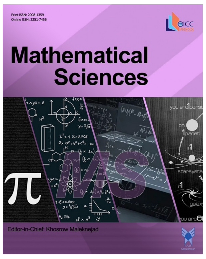

Math. Sci. 2026; 20 (1): 1-18

Mathematical Sciences (MS)

https://doi.org/10.57647/mathsci.2025.1904.18008

Original Research

Transmuted One-Parameter Sarhan-Tadj-Hamilton Distribution: Properties and Applications

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Received:
06 September 2025
Revised:
21 October 2025
Accepted:
27 October 2025

Published in Issue:
31 March 2026

Abstract:

Transmuted distributions have gained attention in statistical modeling due to their flexibility and ability to enhance the performance of baseline distributions. In this article, we introduce the transmuted one-parameter Sarhan-Tadj-Hamilton distribution. Various structural properties of the proposed distribution, such as explicit expressions, stochastic orders, moments, and order statistics are derived. Six parameter estimation methods are examined, with their relative performance compared through Monte Carlo simulations and ranked across different sample sizes. The proposed model is further validated using multiple real data sets, demonstrating its practical flexibility.

Keywords: Transmuted distribution; Skewed distribution; Estimation; Simulation; Goodness of fit

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Cite this article: Shiha. F.A., Aldawsari A.M.A., Alotaibi. R., Magar. A.M., Transmuted One-Parameter Sarhan-Tadj-Hamilton Distribution: Properties and Applications. Math. Sci. 2026; *20* (1): 1-18 https://doi.org/10.57647/mathsci.2025.1904.18008

1. Introduction

Probability distributions are essential in statistics for modeling real-world data in many fields. While many standard distributions are widely used, they often cannot capture the complexity of real data. This is why generalized forms of these distributions are needed. Generalized distributions are created by adding extra parameter(s) to the baseline probability models, making them more flexible and better suited to real life data. Over time, researchers have developed several families of generalized distributions based on different modification methods. Among these, Marshall and Olkin [1] introduced a simple method of adding a single parameter into a family of distributions. Azzalini [2] introduced a class of skew normal distributions by adding asymmetry parameter. Exponentiated families of distributions have been proposed (see for example, Gupta and Kundu [3], Mudholkar and Srivastava [4], and Mudholkar and Hutson [5]). Balakrishnan and Ristić [6] introduced gamma-generated family of distributions. Barakat [7] proposed a method for adding two parameters to a family of distributions to obtain a new family of distributions whose skewness and kurtosis can be controlled. Eugene et al. [8] defined a general family of distributions generated from the logit of the beta random variable. Ahsan-ul-Haq [9] proposed the generalized exponential uniform distribution.

In recent years, the use of mathematical and statistical techniques in many applied areas such as lifetime analysis, finance and insurance have grown significantly. To address the skewness and kurtosis often observed in such applications, Shaw and Buckley [10] introduced a new approach to generalize distributions. Their approach introduced the concept of a quadratic rank transmutation map, which combines the cumulative distribution function (cdf) of one distribution with the inverse cumulative distribution (quantile) function of another. Using this approach various generalized distributions were generated and studied. For example, Ahsan-ul-Haq [11] introduced transmuted modified Lehmann-type II distribution, Merovci [12] developed the transmuted Rayleigh distribution. Khan and King [13] introduced transmuted modified Weibull distribution. Ahsan-ul-Haq [14] proposed transmuted exponentiated inverse Rayleigh distribution, Hussian [15] introduced the transmuted exponentiated gamma. Dey et al.[16] provided an extensive review of nearly thirty transmuted distributions.

In this paper, we employ the quadratic rank transmutation map proposed by Shaw and Buckley [10] to introduce a new continuous distribution, referred to as the transmuted one-parameter Sarhan-Tadj-Hamilton distribution TSTH-I. This distribution generalizes the one-parameter Sarhan-Tadj-Hamilton distribution introduced by Sarhan et al. [17]. By extending the bias distribution, the proposed TSTH-I distribution offers increased flexibility for modeling lifetime data in real-world applications.

The remainder of this paper is organized as follows: In Section 2, the main characteristics of the TSTH-I distribution are introduced, including its reliability functions,stochastic orders, quantiles, and order statistics. Section 3 derives the statistical properties of the TSTH-I distribution such as the moment generating function, mean, variance, skewness, and kurtosis in general form. These measures are then computed numerically for selected values of the parameters to investigate the behavior of the distribution under different settings. Section 4 presents a simulation study designed to compare the performance of six estimation methods for the proposed distribution parameters, using Monte Carlo simulations implemented in R. Separate rankings were conducted based on accuracy (using the mean squared error MSE and the mean absolute error MAE) and robustness (using the failure rate FR), allowing for a clearer evaluation of each method in terms of both precision and reliability. Section 5 demonstrates the applicability of the TSTH-I distribution through analyses of multiple real lifetime data sets, and Section 6 concludes the paper.

2. Transmuted STH-I distribution

A random variable Y is said to have a one-parameter Sarhan-Tadj-Hamilton distribution (STH-I) with parameter ω if its cumulative distribution function (cdf) and probability density function (pdf) are of the form [17]:

$$G(y;\omega) = 1 - \frac{1}{1+\omega}\left[\omega + (1+\omega y)e^{-\omega y}\right] e^{-\omega y}, \quad (1)$$

and

$$g(y;\omega) = \frac{\omega}{1+\omega}\left[\omega + (1+2\omega y)e^{-\omega y}\right] e^{-\omega y}, \quad (2)$$

where $y \geq 0$, $\omega > 0$.

It is worth noting that, the pdf (2) is a mixture of three components: $\mathrm{Exp}(\omega)$, $\mathrm{Exp}(2\omega)$, and $\mathrm{G}(2, 2\omega)$ distributions with mixture proportions $p_1 = \frac{\omega}{1+\omega}$ and $p_2 = p_3 = \frac{1}{2(1+\omega)}$ as follows:

$$g(y;\omega) = p_1\mathrm{Exp}(\omega) + p_2\mathrm{Exp}(2\omega) + p_3\mathrm{G}(2, 2\omega). \quad (3)$$

Definition 2.1 *A continuous random variable Y is said to have transmuted distribution [10] if its cdf and pdf are given by*

$$F(y) = G(y)\left[1 + \eta\,(1 - G(y))\right] \quad ,|\eta| \leq 1, \quad (4)$$

and

$$f(y) = g(y)\ \left[1 + \eta - 2\eta\, G(y)\right] \quad ,|\eta| \leq 1, \quad (5)$$

where $G(y)$ and $g(y)$ are the cdf and pdf of the base distribution. If $\eta = 0$, $F(y)$ and $f(y)$ coincide with $G(y)$ and $g(y)$, respectively.

Following Definition 2.1 and applying equations (1) and (2), a random variable Y follows a TSTH-I distribution if its cdf and pdf are given by

$$\begin{aligned} F(y;\omega,\eta) =& 1 - \frac{\left[\omega + (1+\omega y)e^{-\omega y}\right] e^{-\omega y}}{(1+\omega)} \\ &\times \left(1 - \eta + \frac{\eta\left[\omega + (1+\omega y)e^{-\omega y}\right] e^{-\omega y}}{(1+\omega)}\right), \quad (6)\end{aligned}$$

and

$$\begin{aligned} f(y;\omega,\eta) =& \frac{\omega\ \left[\omega + (1+2\omega y)e^{-\omega y}\right]\ e^{-\omega y}}{1+\omega} \\ &\times \left(1 - \eta + \frac{2\eta\ \left[\omega + (1+\omega y)e^{-\omega y}\right] e^{-\omega y}}{1+\omega}\right), \\ & \quad (7)\end{aligned}$$

where $y \geq 0$, $\omega > 0$ and $-1 \leq \eta \leq 1$.

The graphical representations of the TSTH-I distribution's pdf and cdf, corresponding to certain parameter values, appear in Figs. 1 and 2.

Fig. 1 shows that the TSTH-I distribution is positively skewed with its shape varying significantly across different parameter values. We varied these parameters systematically to observe their effects on the distribution's shape. The parameter η modifies skewness, kurtosis, and peak behavior of the distribution, positive η creating heavier right tails and sharper peak. So η affects the shape (especially asymmetry and heaviness of tails), then it's acting as a shape parameter. The scale parameter ω controls decay rate (tail-shape) of the distribution. Higher ω leads to faster decay (shorter tails), compressing values closer to 0. Lower ω tends to slow decay creating heavier tails. Thus, the TSTH-I distribution flexibility allows it to handle various data patterns, including those that are unimodal, positively skewed, or exhibit heavy tails. Consistent with probability theory, the cdf in Fig. 2 approaches 1 as $y \to \infty$ and 0 as $y \to 0$.

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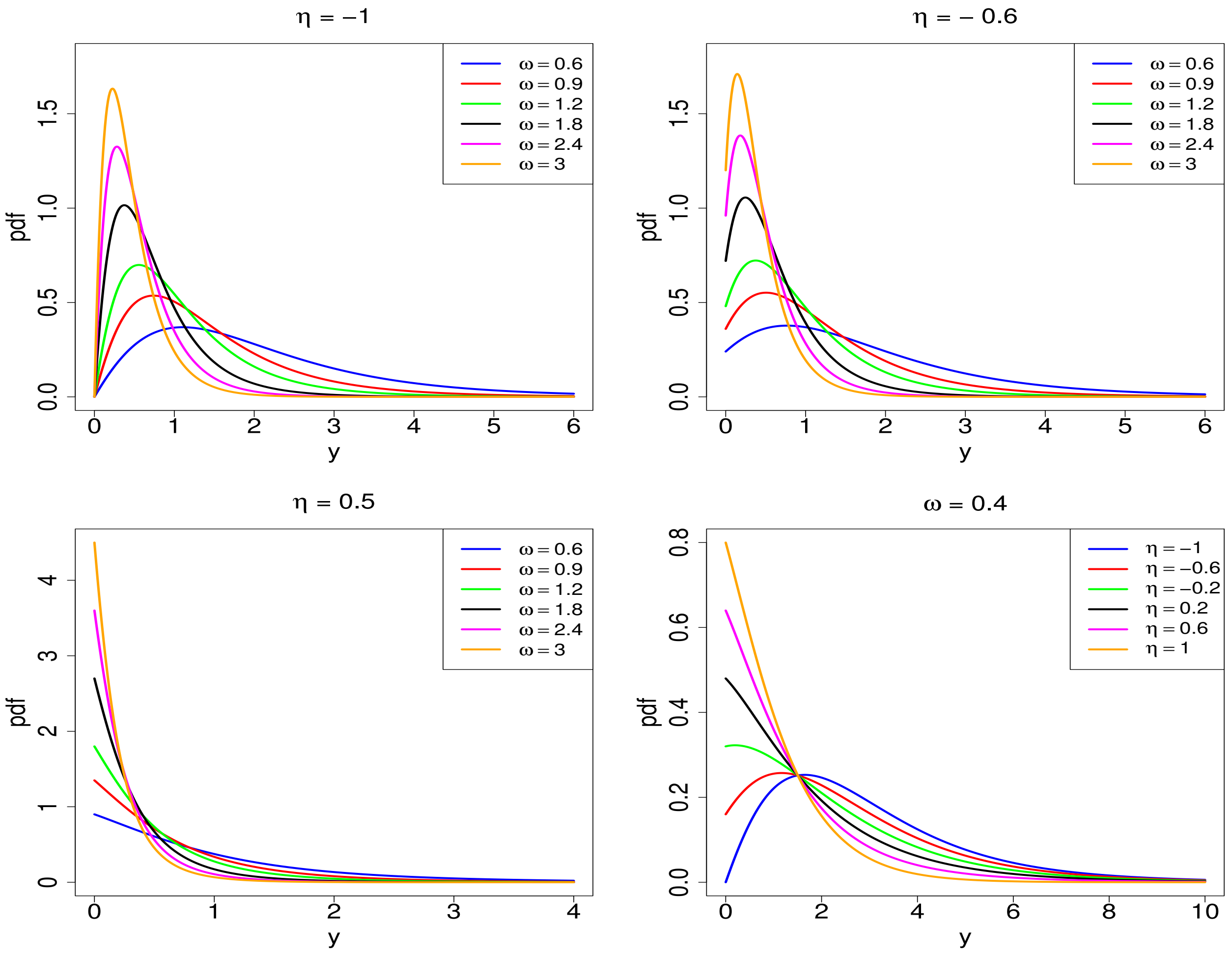

Figure 1. The TSTH-I's pdf across different parameter settings.

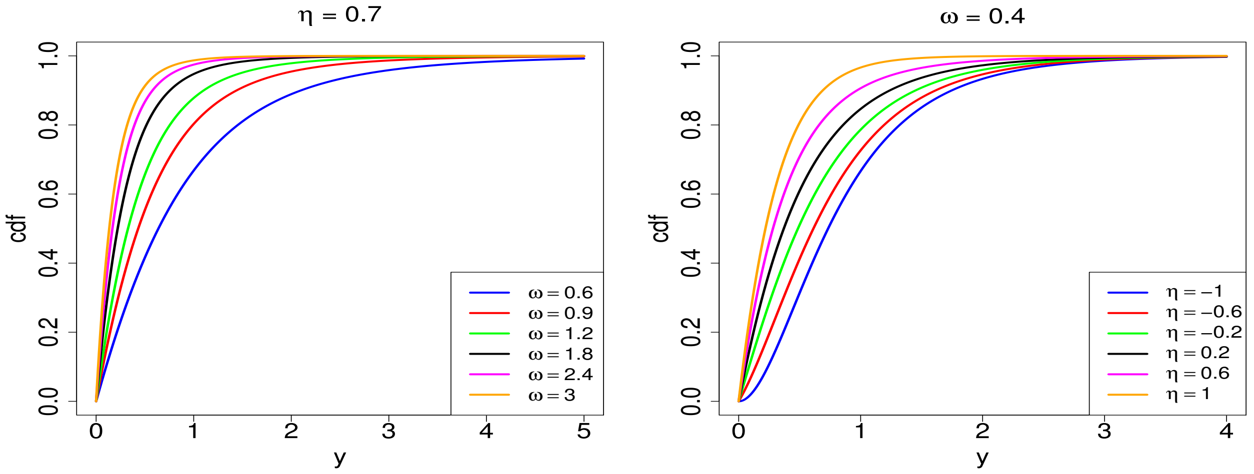

Figure 2. The TSTH-I's cdf across different parameter settings.

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2.1 Reliability

In the current subsection, we present the theoretical derivation of both the survival function (reliability function) and the hazard rate function (failure rate function). The survival function $S(t) = S(t;\omega,\eta)$ represents the probability that a subject survives beyond time t: $S(t;\omega,\eta) = P(Y > t) = 1 - F(t;\omega,\eta)$. Using the cdf of the TSTH-I distribution from (6), the survival function for the TSTH-I is derived as:

$$\begin{aligned} S(t;\omega,\eta) =& \frac{\left[\omega + (1+\omega t)e^{-\omega t}\right] e^{-\omega t}}{(1+\omega)} \\ &\times \left(1 - \eta + \frac{\eta\left[\omega + (1+\omega t)e^{-\omega t}\right] e^{-\omega t}}{(1+\omega)}\right). \end{aligned}$$

Fig. 3 displays the survival function plots of the TSTH-I distribution for various parameter combinations. The parameter values were systematically selected across their admissible ranges to demonstrate their effects on the function's behavior and shape characteristics. The plots confirm the TSTH-I's survival function satisfies fundamental reliability requirements: Begins at unity (certain survival) at time zero, decreases monotonically with increasing time, and complete failure certainty as $t \to \infty$.

The hazard (or failure rate) function $h(t) = h(t,\omega,\eta)$ describes the likelihood of failure in an infinitesimally small time interval $[t, t+\Delta t]$, given that the component has survived until time t. It is derived as the ratio of the pdf to the survival function:

$$\begin{aligned} h(t;\omega,\eta) =& \frac{\omega\left(\omega + e^{-\omega t} + 2\omega t e^{-\omega t}\right)}{\omega + e^{-\omega t} + \omega t e^{-\omega t}} \\ &\times \frac{(1+\omega)(1-\eta) + 2\eta\left(\omega e^{-\omega t} + (1+\omega t)e^{-2\omega t}\right)}{(1+\omega)(1-\eta) + \eta\left(\omega e^{-\omega t} + (1+\omega t)e^{-2\omega t}\right)}, \end{aligned}$$

$$h(0;\omega,\eta) = \omega(1+\eta), \qquad \lim_{t\to\infty} h(t;\omega,\eta) = \begin{cases} \omega, & \eta \neq 1, \\ 2\omega, & \eta = 1. \end{cases}$$

The hazard function is increasing from values below ω to ω when $\eta < 0$, it equals ω at both the origin and infinity when $\eta = 0$, it decreases from values above ω towards ω when $0 < \eta < 1$, and it starts and ends at 2ω when $\eta = 1$. Fig. 4 shows the hazard for some different choices of ω and η which demonstrates the flexibility of the hazard function of the TSTH-I distribution.

2.2 Stochastic orders

In this subsection, we focus on stochastic orders, which provide a systematic way to compare random variables in terms of their "location" or overall "magnitude". Let X and Y be two random variables. We say that X is stochastically smaller than Y (denoted $X \leq_{st} Y$) if $F_X(y) \geq F_Y(y)$ for all y. Similarly, X is said to be smaller than Y in the hazard rate order (denoted $X \leq_{hr} Y$) if $h_X(y) \geq h_Y(y)$ for all y. Moreover, X is smaller than Y in the likelihood ratio order (denoted $X \leq_{lr} Y$) if the ratio $f_X(y)/f_Y(y)$ is decreasing in y. It is well known that these orderings are related as follows:

$$X \leq_{lr} Y \;\Rightarrow\; X \leq_{hr} Y \;\Rightarrow\; X \leq_{st} Y.$$

For further details, see Shaked and Shanthikumar [18].

Theorem 2.2 *Let Y_1 and Y_2 be two random variables following the TSTH-I distribution with parameters (ω_1, η_1) and (ω_2, η_2), respectively. If $\omega_1 = \omega_2$ and $\eta_1 < \eta_2$, then $Y_2 \leq_{lr} Y_1$ and hence $Y_2 \leq_{hr} Y_1$ and $Y_2 \leq_{st} Y_1$.*

Proof Recall that, the pdf of the TSTH-I distributed random variable Y is given by

$$\begin{aligned} f(y;\omega,\eta) =& \frac{\omega\left[\omega + (1+2\omega y)e^{-\omega y}\right] e^{-\omega y}}{1+\omega} \\ &\times \left(1 - \eta + \frac{2\eta\left[\omega + (1+\omega y)e^{-\omega y}\right] e^{-\omega y}}{1+\omega}\right). \end{aligned}$$

Define the functions

$$A(y;\omega) = \frac{\omega(\omega + (1+2\omega y)e^{-\omega y})e^{-\omega y}}{1+\omega},$$

and

$$D(y;\omega) = \frac{2(\omega + (1+\omega y)e^{-\omega y})e^{-\omega y}}{1+\omega},$$
$$B(y;\omega,\eta) = 1 - \eta + \eta D(y;\omega).$$

Then the pdf can be expressed as

$$f(y;\omega,\eta) = A(y;\omega)\, B(y;\omega,\eta).$$

When $\omega_1 = \omega_2 = \omega$ and $\eta_1 \neq \eta_2$. The likelihood ratio reduces to

$$R(y) = \frac{f(y;\omega,\eta_1)}{f(y;\omega,\eta_2)} = \frac{B(y;\omega,\eta_1)}{B(y;\omega,\eta_2)} = \frac{1-\eta_1+\eta_1 D(y;\omega)}{1-\eta_2+\eta_2 D(y;\omega)}.$$

Differentiating, we obtain

$$\frac{d}{dy}\log R(y) = \frac{D'(y;\omega)(\eta_1-\eta_2)}{(1-\eta_1+\eta_1 D(y;\omega))\,(1-\eta_2+\eta_2 D(y;\omega))},$$

where

$$D'(y;\omega) = -\frac{2\omega}{1+\omega}\left(\omega e^{-\omega y} + (1+2\omega y)e^{-2\omega y}\right).$$

Thus

$$\begin{aligned} &\frac{d}{dy}\log R(y) = \\ &\frac{-2(\eta_1-\eta_2)[\omega^2 e^{-\omega y} + \omega(1+2\omega y)e^{-2\omega y}]}{(1+\omega)(1-\eta_1+\eta_1 D(y;\omega))\,(1-\eta_2+\eta_2 D(y;\omega))}. \end{aligned}$$

Consequently, if $\eta_1 < \eta_2$, then $\frac{d}{dy}\log R(y) > 0$, which implies that $R(y)$ increasing in y. Hence $Y_2 \leq_{lr} Y_1$. If $\eta_1 = \eta_2$, then $R(y)$ is constant and no ordering holds. □

Next case where $\eta_1 = \eta_2 = \eta$ but $\omega_1 \neq \omega_2$. The likelihood ratio becomes

$$R(y) = \frac{f(y;\omega_1,\eta)}{f(y;\omega_2,\eta)} = \frac{A(y;\omega_1)}{A(y;\omega_2)} \cdot \frac{B(y;\omega_1,\eta)}{B(y;\omega_2,\eta)}.$$

Differentiating, we obtain

$$\begin{aligned} \frac{d}{dy}\log R(y) =& \frac{d}{dy}\left[\log A(y,\omega_1) + \log B(y,\omega_1,\eta)\right. \\ &\left. - \log A(y,\omega_2) - \log B(y,\omega_2,\eta)\right]. \end{aligned} \tag{8}$$

Straightforward calculations give

$$\frac{d}{dy}\log A(y,\omega) = \frac{-\omega^2\left(e^{-\omega y} + 4y e^{-2\omega y}\right)}{\omega e^{-\omega y} + (1+2\omega y)e^{-2\omega y}},$$

$$\frac{d}{dy}\log B(y,\omega,\eta) = \frac{\eta D'(y;\omega)}{1-\eta+\eta D(y;\omega)}.$$

Substituting these expressions into (8) results in a complicated form. Therefore, the sign of $\frac{d}{dy}\log R(y)$ is investigated numerically using R for different pairs of ω values while keeping η fixed. The results, illustrated in Fig. 5, show that for $\omega_1 < \omega_2$, the derivative is positive and hence $Y_2 \leq_{lr} Y_1$. Conversely, for $\omega_2 < \omega_1$, the derivative is negative and $Y_1 \leq_{lr} Y_2$.

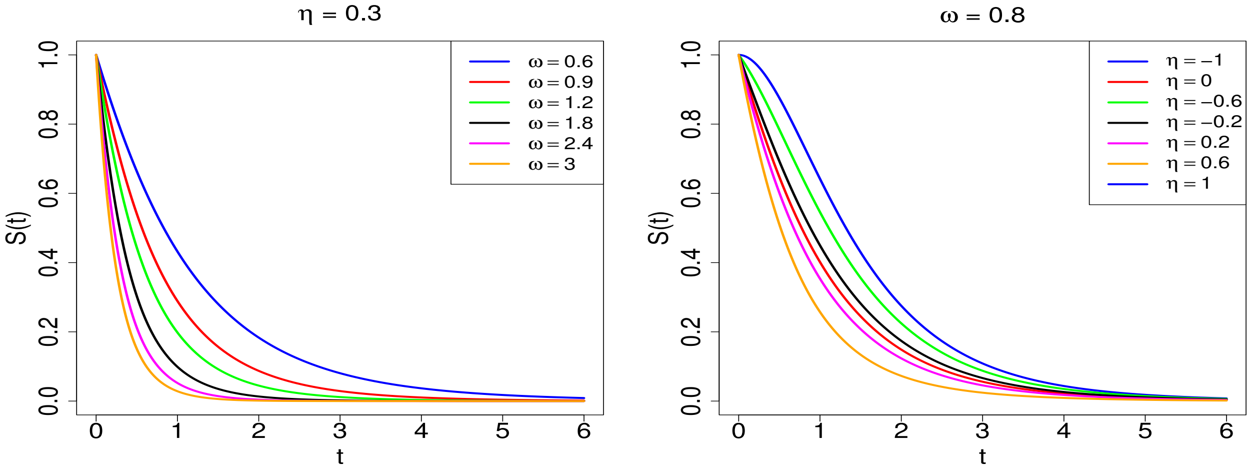

Figure 3. The Survival function of the TSTH-I distribution at different parameter values.

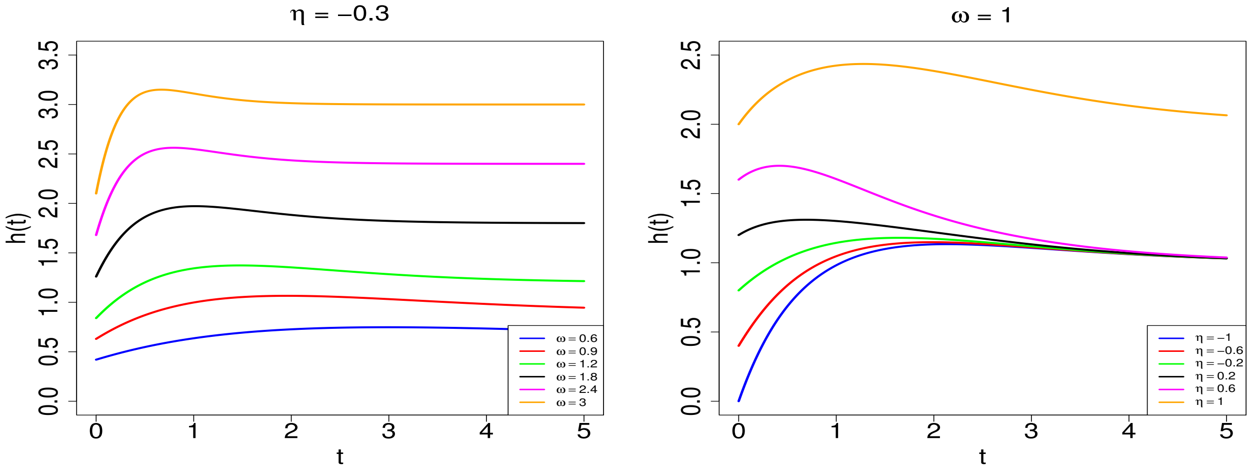

Figure 4. The hazard function of the TSTH-I distribution for some parameter values.

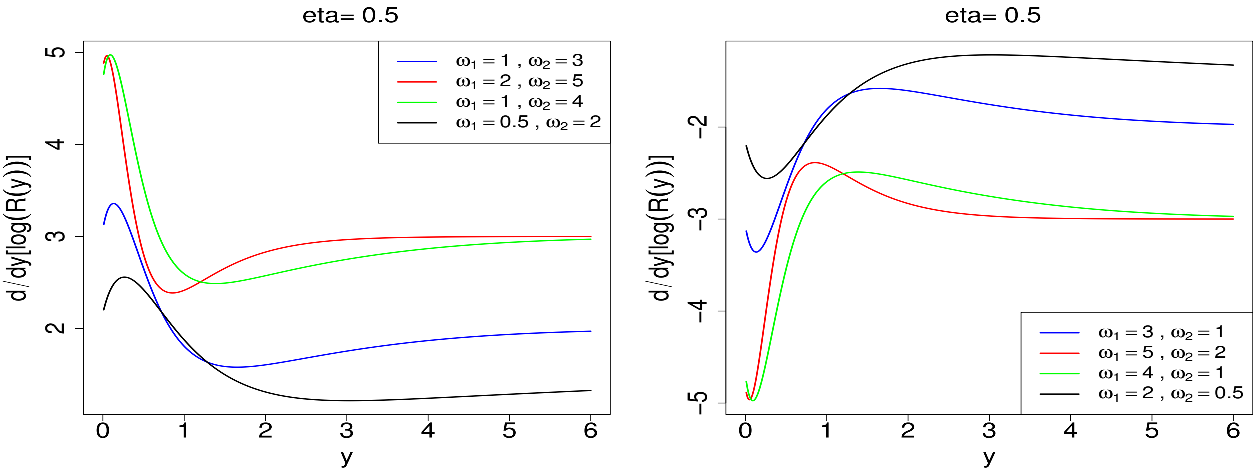

Figure 5. Behavior of $\frac{d}{dy} \log R(y)$ for different parameter values ($\eta = 0.5$).

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2.3 Quantiles

For a real-valued continuous random variable Y with distribution function $F(y)$, the quantile function $Q(u)$ is defined as the inverse of $F(y)$:

$$Q(u) = F^{-1}(u), \quad 0 < u < 1. \tag{9}$$

Common examples include the first quartile $Q(0.25)$, the median $Q(0.5)$, and the third quartile $Q(0.75)$.

The cdf of TSTH-I distribution can be rewritten as

$$F(y;\omega,\eta) = 1 - B\,(1-\eta+\eta B)\,, \tag{10}$$

where

$$B = \frac{A(y)}{1+\omega}\;, \quad A(y) = \left[\omega + (1+\omega y)e^{-\omega y}\right]e^{-\omega y}.$$

Let $u \in (0,1)$, setting $F(y;\omega,\eta) = u$ gives

$$1-u = B\,(1-\eta+\eta B) \quad \Rightarrow \quad \eta B^2 + (1-\eta)B - (1-u) = 0. \tag{11}$$

This quadratic equation in B has solution

$$B = \begin{cases} 1-u, & \eta = 0, \\ \dfrac{-(1-\eta)+\sqrt{(1-\eta)^2+4\eta(1-u)}}{2\eta}, & \eta \neq 0, \end{cases}$$

where the root is chosen such that $B \in [0,1]$.

Thus, the inversion problem reduces to solving

$$\omega e^{-\omega y} + (1+\omega y)e^{-2\omega y} - (1+\omega)B = 0, \quad \text{for } y \geq 0. \tag{12}$$

This equation cannot be solved analytically. Therefore, the quantile function $Q(u;\omega,\eta)$ is obtained numerically by solving equation (12) with B given above for the chosen u, ω, η.

Fig. 6 displays the three quartiles Q_1, Q_2, and Q_3 of the proposed distribution for varying values of η (left), and ω (right). The interquartile range ($IQR = Q_3 - Q_1$) decreases as η grows, indicating lower dispersion. All quartiles decline monotonically with increasing ω, displaying a hyperbolic-like decay pattern. Larger ω values lead to a stronger concentration of the distribution near zero.

2.4 Order statistic

Consider a random sample $\{Y_i\}_{i=1}^{n}$ drawn from TSTH-I distribution with cdf $F(y;\omega,\eta)$ and pdf $f(y;\omega,\eta)$. Let $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}$ represent the corresponding order statistics obtained by arranging the sample in non-decreasing order. Then the pdf of the κ-th order statistic $Y_{\kappa:n}$, $1 \leq \kappa \leq n$ (see [19]) is given by

$$\begin{aligned} f_{\kappa:n}(y;\omega,\eta) = & \,\kappa\binom{n}{\kappa}\left[F(y;\omega,\eta)\right]^{\kappa-1} \\ & \times \left[1 - F(y;\omega,\eta)\right]^{n-\kappa} f(y;\omega,\eta). \end{aligned} \tag{13}$$

Using equation (6) and equation (7) in equation (13), we obtain the pdf of κ-th order statistic of the TSTH-I distribution.

$$\begin{aligned} f_{\kappa:n}(y;\omega,\eta) = & \\ & \kappa\binom{n}{\kappa}\left[1 - \frac{\left[\omega+(1+\omega y)e^{-\omega y}\right]e^{-\omega y}}{(1+\omega)} \right. \\ & \left. \times \left(1-\eta+\frac{\eta\left[\omega+(1+\omega y)e^{-\omega y}\right]e^{-\omega y}}{(1+\omega)}\right)\right]^{\kappa-1} \\ & \times \left[\frac{\left[\omega+(1+\omega y)e^{-\omega y}\right]e^{-\omega y}}{(1+\omega)} \right. \\ & \left. \times \left(1-\eta+\frac{\eta\left[\omega+(1+\omega y)e^{-\omega y}\right]e^{-\omega y}}{(1+\omega)}\right)\right]^{n-\kappa} \\ & \times \left[\frac{\omega}{1+\omega}\left[\omega+(1+2\omega y)e^{-\omega y}\right]e^{-\omega y} \right. \\ & \left. \left(1-\eta+\frac{2\eta}{1+\omega}\left[\omega+(1+\omega y)e^{-\omega y}\right]e^{-\omega y}\right)\right]. \end{aligned} \tag{14}$$

By setting $\kappa = 1$, $\frac{n+1}{2}$, and n, we obtain the pdf of minimum $Y_{1:n}$, the median (for odd n), and the maximum $Y_{n:n}$ order statistics of the TSTH-I distribution, respectively

$$f_{1:n}(y;\omega,\eta) = n[1 - F(y;\omega,\eta)]^{n-1} f(y;\omega,\eta), \tag{15}$$

$$\begin{aligned} f_{median}(y;\omega,\eta) = & \frac{n!}{(\frac{n-1}{2})!^2}\left[F(y;\omega,\eta)\right]^{\frac{n-1}{2}} \\ & \times \left[1 - F(y;\omega\eta)\right]^{\frac{n-1}{2}} f(y;\omega,\eta), \end{aligned} \tag{16}$$

$$f_{n:n}(y;\omega,\eta) = n[F(y;\omega,\eta)]^{n-1} f(y;\omega,\eta). \tag{17}$$

Fig. 7 illustrates the behaviour of minimum, median, and maximum order statistics for $n = 5$, across various values of η and ω. As shown in Fig. 7, increasing the shape parameter η from negative to positive increases the right-skewness and tail heaviness of the distribution, particularly in the behavior of the maximum order statistic. Increasing the scale parameter ω shifts the distribution toward zero, resulting in a more concentrated and peaked shape with lighter tails.

For modeling minima, choosing $\eta < 0$ helps reduce the occurrence of extreme small values, leading to light-tailed distributions suitable for low-risk or bounded-outcome scenarios (e.g., reliability engineering). Conversely, for modeling maxima, selecting $\eta > 0$ allows the model to capture heavy-tailed behavior, which is desirable in contexts dominated by rare but extreme events, such as in finance or insurance. Overall, the shape parameter η plays a central role in governing the tail behavior of the TNL distribution: as η approaches 1, the distribution becomes increasingly heavy tailed amplifying the probability of extreme values whereas as η approaches -1, the tails become lighter, making the distribution more appropriate for modeling safer or more constrained processes.

3. Statistical properties

Various structural and statistical characteristics of the proposed TSTH-I model will be investigated. These characteristics include the moment generating function, moments, variance, kurtosis, and skewness.

3.1 Moment generating function

The moment generating function (MGF) is a powerful analytical tool that systematically encodes all the moments of a probability distribution. Furthermore, skewness and kurtosis, which are

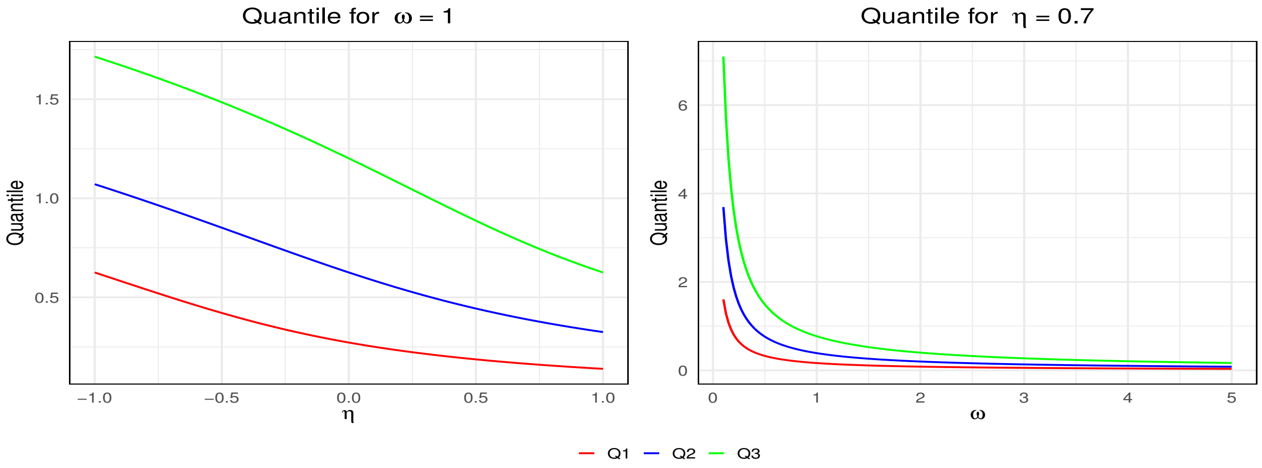

Figure 6. The quartiles of TSTH-I distribution.

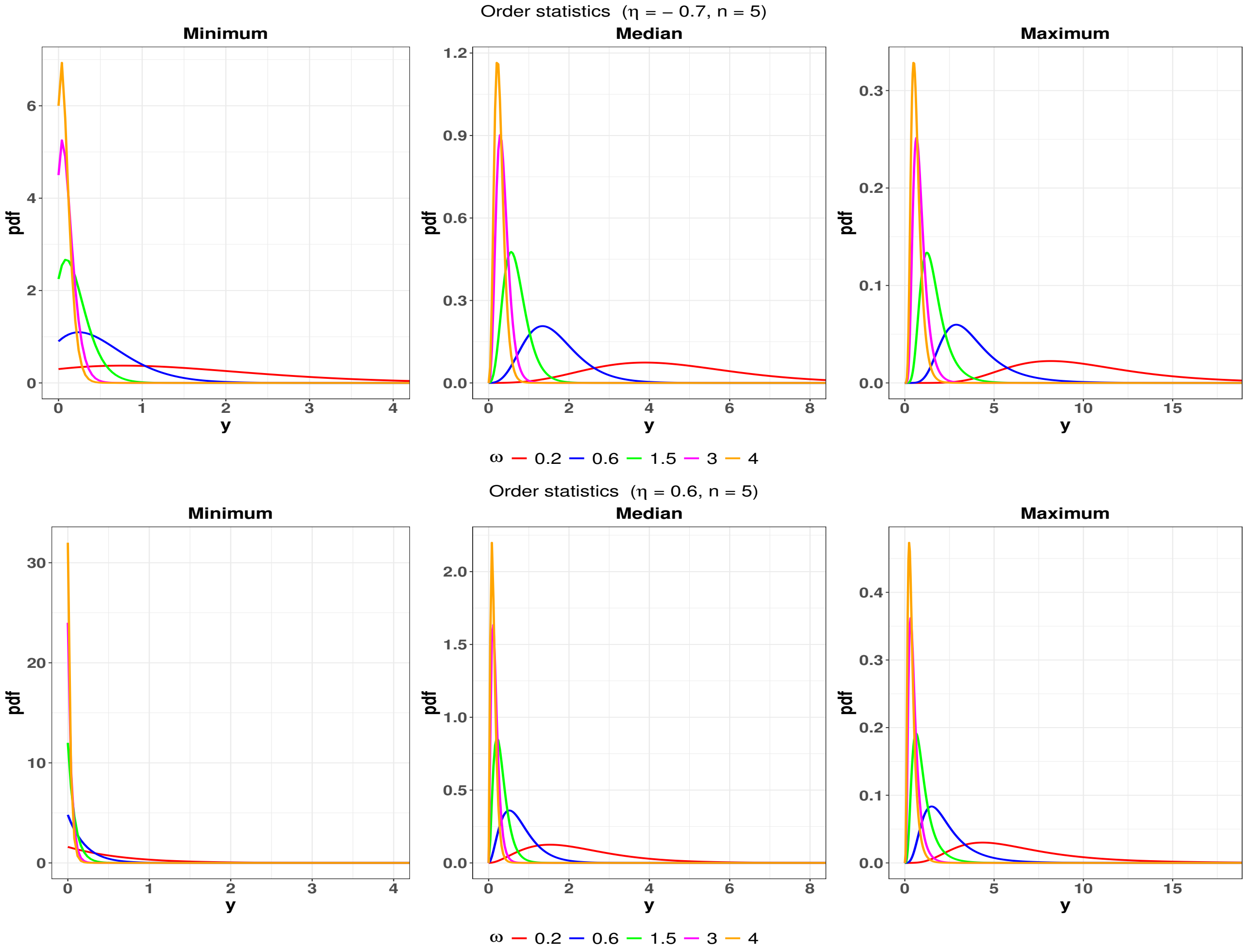

Figure 7. The pdf’s for the minimum, median, and maximum order statistics of TSTH-I distribution.

derived from the moment generating function, offer valuable insights into the asymmetry and tail behavior of the distribution.

Theorem 3.1 *The moment generating function of a TSTH-I distributed random variable is given by*

$$
\begin{aligned}
M_Y(t) = \sum_{\kappa=0}^{\infty} \frac{t^\kappa}{\kappa!} \cdot \frac{3^{-\kappa-1}\, 4^{-\kappa-2}\, \omega^{-\kappa}\, \Gamma(\kappa+1)}{(\omega+1)^2} \times \\
\Big[\eta\, 2^{2\kappa+5}\, (\kappa+3)\omega + 3^{\kappa+1}\Big(\eta\left(\kappa^2 + 2^{\kappa+4}\,\omega^2 + 9\kappa + 16\right) \\
- (\eta-1)2^{\kappa+3}\,(\omega+1)\left(2^{\kappa+1}\,\omega + \kappa + 2\right)\Big)\Big].
\end{aligned}
\tag{18}
$$

Proof

$$
\begin{aligned}
M_Y(t) &= E(e^{tY}) = \int_0^\infty e^{ty}\, f(y;\omega,\eta)dy \\
&= \int_0^\infty \sum_{\kappa=0}^{\infty} \frac{(ty)^\kappa}{\kappa!}\, \frac{\omega\,\left[\omega + (1+2\omega y)e^{-\omega y}\right] e^{-\omega y}}{1+\omega} \\
&\quad \times \left(1 - \eta + \frac{2\eta\,\left[\omega + (1+\omega y)e^{-\omega y}\right] e^{-\omega y}}{1+\omega}\right) dy \\
&= \sum_{\kappa=0}^{\infty} \frac{t^\kappa}{\kappa!} \int_0^\infty y^\kappa \frac{\omega\,\left[\omega + (1+2\omega y)e^{-\omega y}\right]\, e^{-\omega y}}{1+\omega} \\
&\quad \times \left(1 - \eta + \frac{2\eta\,\left[\omega + (1+\omega y)e^{-\omega y}\right] e^{-\omega y}}{1+\omega}\right) dy \\
&= \sum_{\kappa=0}^{\infty} \frac{t^\kappa}{\kappa!} \left[\int_0^\infty y^\kappa \frac{\omega\,\left[\omega + (1+2\omega y)e^{-\omega y}\right]\, e^{-\omega y}}{1+\omega} \right. \\
&\quad \left. \times \left(\frac{2\eta\,\left[\omega + (1+\omega y)e^{-\omega y}\right] e^{-\omega y}}{1+\omega}\right) dy\right] \\
&\quad + \sum_{\kappa=0}^{\infty} \frac{t^\kappa}{\kappa!} \left[\int_0^\infty y^\kappa (1-\eta) \right. \\
&\quad \left. \times \frac{\omega\,\left[\omega + (1+2\omega y)e^{-\omega y}\right]\, e^{-\omega y}}{1+\omega} dy\right] \\
&= \sum_{\kappa=0}^{\infty} \frac{t^\kappa}{\kappa!} \cdot \frac{3^{-\kappa-1}\, 4^{-\kappa-2}\, \omega^{-\kappa}\Gamma(\kappa+1)}{(\omega+1)^2} \times \\
&\quad \Big[\eta\, 2^{2\kappa+5}\, (\kappa+3)\omega \\
&\quad + 3^{\kappa+1}\Big(\eta\left(\kappa^2 + 2^{\kappa+4}\omega^2 + 9\kappa + 16\right) \\
&\quad - (\eta-1)2^{\kappa+3}(\omega+1)\left(2^{\kappa+1}\omega + \kappa + 2\right)\Big)\Big]. \qquad \square
\end{aligned}
$$

Differentiating the moment generating function κ times with respect to t and evaluating it at $t = 0$ yields the κth raw moment (about the origin) $\mu'_\kappa = E(Y^\kappa)$ as

$$
\begin{aligned}
\mu'_\kappa = \frac{3^{-\kappa-1}\, 4^{-\kappa-2}\, \omega^{-\kappa}\Gamma(\kappa+1)}{(\omega+1)^2} \times \\
\Big[\eta\, 2^{2\kappa+5}(\kappa+3)\omega + 3^{\kappa+1}\Big(\eta\left(\kappa^2 + 2^{\kappa+4}\omega^2 + 9\kappa + 16\right) \\
- (\eta-1)2^{\kappa+3}(\omega+1)\left(2^{\kappa+1}\omega + \kappa + 2\right)\Big)\Big].
\end{aligned}
\tag{19}
$$

By setting $\kappa = 1,2,3,4$, we derive the first four raw moments for the TSTH-I distribution:

$$
\mu'_1 = E(Y) = \frac{72\,(\omega+1)(4\,\omega+3) - \eta\,[8\,\omega(18\,\omega+31) + 99]}{288\,\omega(\omega+1)^2},
\tag{20}
$$

which is mean of the TSTH-I distribution.

$$
\mu'_2 = \frac{1728\,(\omega+1)(2\,\omega+1) - \eta\,\left[32\,\omega\,(81\,\omega+122) + 1215\right]}{1728\,\omega^2\,(\omega+1)^2}.
\tag{21}
$$

$$
\mu'_3 = \frac{432\,(\omega+1)(16\,\omega+5) - \eta\,\left[16\,\omega\,(378\,\omega+503) + 1809\right]}{1152\,\omega^3\,(\omega+1)^2}.
\tag{22}
$$

$$
\mu'_4 = \frac{768\,(\omega+1)(16\,\omega+3) - \frac{4\eta}{81}\,\left[64\,\omega\,(3645\,\omega+4393) + 42525\right]}{512\,\omega^4\,(\omega+1)^2}.
\tag{23}
$$

We note that, each raw moment μ'_κ $(\kappa = 1,2,2,4)$ can be written in the form

$$
\mu'_\kappa = \frac{A_\kappa(\omega) - \eta\,B_\kappa(\omega)}{C_\kappa(\omega)},
$$

with $A_\kappa(\omega), B_\kappa(\omega), C_\kappa(\omega) > 0$ for $\omega > 0$. Hence for fixed $\omega > 0$, we have

$$
\frac{\partial}{\partial\,\eta}\mu'_\kappa = -\frac{B_\kappa(\omega)}{C_\kappa(\omega)} < 0,
$$

so each μ'_κ is strictly decreasing in η. Therefore, on the interval $-1 \le \eta \le 1$, the moments attain their maximum at $\eta = -1$ and their minimum at $\eta = 1$,

$$
\begin{gathered}
\frac{144\omega^2 + 256\omega + 117}{288\,\omega(\omega+1)^2} \le \mu'_1 \le \frac{432\omega^2 + 752\omega + 315}{288\,\omega(\omega+1)^2}, \\
\frac{864\omega^2 + 1280\omega + 513}{1728\,\omega^2(\omega+1)^2} \le \mu'_2 \le \frac{6048\omega^2 + 9088\omega + 2943}{1728\,\omega^2(\omega+1)^2}, \\
\frac{864\omega^2 + 1024\omega + 351}{1152\,\omega^3(\omega+1)^2} \le \mu'_3 \le \frac{12960\omega^2 + 17120\omega + 3969}{1152\,\omega^3(\omega+1)^2}, \\
\frac{15552\omega^2 + 14336\omega + 4131}{10368\,\omega^4(\omega+1)^2} \le \mu'_4 \le \\
\frac{482112\omega^2 + 576640\omega + 89181}{10368\,\omega^4(\omega+1)^2}.
\end{gathered}
$$

As $\omega \longrightarrow 0^+$, both the boundaries of $\mu'_\kappa, \kappa = 1,2,3,4$ tend to ∞, and as $\omega \longrightarrow \infty$ they tend to 0^+. Therefore

$$
0 < \mu'_\kappa < \infty, \quad \kappa = 1,2,3,4.
$$

Numerical computations of the first four moments of the TSTH-I distribution for different values of the parameters η and ω are presented in Table 1. It is evident that the moments decrease as η and ω increase.

The variance of the TSTH-I distribution are derived as:

$$
\sigma^2 = E(Y^2) - (E(Y))^2 = \mu'_2 - (\mu'_1)^2.
$$

The measures of skewness (Sk) and kurtosis (Kur) for the TSTH-I distribution can be computed from the first four ordinary moments using

$$
\text{Sk} = \frac{\mu'_3 - 3\mu'_2\,\mu'_1 + 2(\mu'_1)^3}{[\mu'_2 - (\mu'_1)^2]^{3/2}},
$$

and

$$
\text{Kur} = \frac{\mu'_4 - 4\mu'_3\,\mu'_1 + 6\mu'_2\,(\mu'_1)^2 - 3{\mu'_1}^4}{[\mu'_2 - (\mu'_1)^2]^2}.
$$

https://doi.org/10.57647/mathsci.2025.1904.18008

Table 1. The first four raw moments of the TSTH-I distribution for different values of the model parameters.

η	ω	μ'_1	μ'_2	μ'_3	μ'_4	ω	μ'_1	μ'_2	μ'_3	μ'_4
-1.0	0.1	11.321	187.112	4168.530	120895.000	2.0	0.684	0.728	1.086	2.124
-0.8	0.1	10.602	171.507	3784.830	109261.000	2.0	0.639	0.666	0.984	1.918
-0.6	0.1	9.884	155.903	3401.120	97627.900	2.0	0.594	0.604	0.883	1.712
-0.4	0.1	9.165	140.299	3017.410	85994.400	2.0	0.549	0.541	0.781	1.506
-0.2	0.1	8.446	124.695	2633.710	74360.800	2.0	0.504	0.479	0.680	1.300
0.0	0.1	7.727	109.091	2250.000	62727.300	2.0	0.458	0.417	0.578	1.094
0.2	0.1	7.008	93.487	1866.290	51093.700	2.0	0.413	0.354	0.477	0.888
0.4	0.1	6.290	77.883	1482.590	39460.200	2.0	0.368	0.292	0.375	0.682
0.6	0.1	5.571	62.278	1098.880	27826.700	2.0	0.323	0.230	0.274	0.476
0.8	0.1	4.852	46.674	715.174	16193.100	2.0	0.278	0.167	0.172	0.270
1.0	0.1	4.133	31.070	331.468	4559.580	2.0	0.232	0.105	0.071	0.064
-1.0	0.5	2.466	9.258	48.670	341.584	3.0	0.467	0.340	0.346	0.458
-0.8	0.5	2.306	8.473	44.136	308.467	3.0	0.436	0.311	0.313	0.414
-0.6	0.5	2.146	7.688	39.602	275.350	3.0	0.405	0.282	0.281	0.369
-0.4	0.5	1.986	6.903	35.068	242.233	3.0	0.374	0.253	0.249	0.325
-0.2	0.5	1.827	6.118	30.534	209.117	3.0	0.343	0.224	0.216	0.281
0.0	0.5	1.667	5.333	26.000	176.000	3.0	0.312	0.194	0.184	0.236
0.2	0.5	1.507	4.548	21.466	142.883	3.0	0.282	0.165	0.152	0.192
0.4	0.5	1.347	3.763	16.932	109.767	3.0	0.251	0.136	0.119	0.147
0.6	0.5	1.187	2.978	12.398	76.650	3.0	0.220	0.107	0.087	0.103
0.8	0.5	1.027	2.193	7.864	43.533	3.0	0.189	0.078	0.055	0.058
1.0	0.5	0.867	1.408	3.330	10.416	3.0	0.158	0.049	0.023	0.014
-1.0	1.0	1.301	2.616	7.389	27.680	4.0	0.355	0.197	0.152	0.152
-0.8	1.0	1.216	2.392	6.699	24.994	4.0	0.332	0.180	0.138	0.138
-0.6	1.0	1.131	2.169	6.008	22.308	4.0	0.308	0.163	0.123	0.123
-0.4	1.0	1.045	1.946	5.318	19.622	4.0	0.285	0.146	0.109	0.108
-0.2	1.0	0.960	1.723	4.628	16.936	4.0	0.261	0.129	0.095	0.093
0.0	1.0	0.875	1.500	3.938	14.250	4.0	0.238	0.112	0.081	0.079
0.2	1.0	0.790	1.277	3.247	11.564	4.0	0.214	0.096	0.067	0.064
0.4	1.0	0.705	1.054	2.557	8.878	4.0	0.190	0.079	0.052	0.049
0.6	1.0	0.619	0.831	1.867	6.192	4.0	0.167	0.062	0.038	0.034
0.8	1.0	0.534	0.608	1.176	3.506	4.0	0.143	0.045	0.024	0.019
1.0	1.0	0.449	0.384	0.486	0.820	4.0	0.120	0.028	0.010	0.005
-1.0	1.5	0.894	1.242	2.420	6.215	5.0	0.287	0.128	0.080	0.064
-0.8	1.5	0.836	1.136	2.194	5.612	5.0	0.268	0.117	0.072	0.058
-0.6	1.5	0.777	1.030	1.968	5.009	5.0	0.249	0.106	0.065	0.052
-0.4	1.5	0.718	0.924	1.741	4.406	5.0	0.230	0.095	0.057	0.046
-0.2	1.5	0.659	0.817	1.515	3.803	5.0	0.211	0.084	0.050	0.039
0.0	1.5	0.600	0.711	1.289	3.200	5.0	0.192	0.073	0.042	0.033
0.2	1.5	0.541	0.605	1.063	2.597	5.0	0.173	0.062	0.035	0.027
0.4	1.5	0.482	0.499	0.836	1.994	5.0	0.154	0.051	0.028	0.021
0.6	1.5	0.423	0.393	0.610	1.391	5.0	0.135	0.040	0.020	0.014
0.8	1.5	0.364	0.286	0.384	0.788	5.0	0.115	0.029	0.013	0.008
1.0	1.5	0.306	0.180	0.158	0.185	5.0	0.096	0.018	0.005	0.002

Since closed form expressions for the variance, skewness, and kurtosis bounds cannot be derived easily, their patterns are investigated numerically to demonstrate their dependence on the parameters. Table 2 summarizes numerical values of median, σ^2, SK, and Kur of the TSTH-I distribution computed for various values of the parameters ω and η to investigate the distribution's behavior.

Table 2 illustrates how the main statistical measures of the TSTH-I distribution vary with different values of the shape parameter η and the scale parameter ω. The following key observations can be made:

1. For a fixed ω, the median, and variance decrease as η increases, indicating a shift in the distribution toward smaller values and reduced dispersion.
2. Skewness and kurtosis increase with increasing η, especially for $\eta > 0$, which reflects greater asymmetry and heavier right tails.
3. For a fixed η, the median and variance decrease as ω increases, whereas the skewness and kurtosis first increase and then decrease, indicating a more sharply peaked distribution with lighter tails.
4. A noticeable shift in tail behavior occurs at $\eta = 1$.

These results show the structural flexibility of the TSTH-I distribution in capturing diverse data behaviors. It can model positively skewed data with both light and heavy tails, making it well suited for applications in lifetime modeling, reliability analysis, and fields where skewness and tail behavior are essential.

4. Simulation

In this section, we assess the performance of six estimation methods using Monte Carlo simulation implemented in R: maximum likelihood estimation (MLE), Cramér–von Mises (CVM), Anderson–Darling (AD), maximum product of spacings (MPS), ordinary least squares (OLS), and weighted least squares (WLS). For further details on these methods, see [20, 21, 22]. Data were generated from the TSTH-I distribution with known parameter values using the inverse transform sampling method. The quantile function was computed numerically via root-finding algorithms. For each sample size $n = 30, 50, 100, 150, 250,$ and 500, a total of $N = 10000$ samples were simulated under four different parameter settings of ω and η.

The accuracy of each estimation method was evaluated using three standard error metrics: the absolute bias |Bias|, the mean squared error (MSE) and the mean absolute error (MAE), defined respectively as

$$|\text{Bias}| = \left| \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}_i - \theta \right|,$$

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\theta}_i - \theta\right)^2,$$

$$\text{MAE} = \frac{1}{N} \sum_{i=1}^{N} \left|\hat{\theta}_i - \theta\right|,$$

where $\hat{\theta}_i$ denotes the estimate obtained in the i-th simulation run, θ is the true parameter value, and N is the total number of simulations.

In addition to error based metrics, the failure rate (FR) was used to assess the robustness and reliability of each method. FR is defined as the proportion of simulation runs in which the estimation procedure failed to converge or produced invalid results:

$$\text{FR} = \frac{\text{Number of failed estimation attempts}}{\text{Total number of simulation runs}} \times 100\%.$$

While the |Bias|, MSE, and MAE reflect the accuracy of the estimates when the method succeeds, FR quantifies how frequently the method fails entirely. Hence, combining these metrics provides a more comprehensive evaluation of performance under varying sample sizes and parameter configurations.

To ensure a well-rounded comparison, the performance of each method was assessed using three separate criteria: accuracy, based on the ranks of |bias|, MSE, and MAE, and robustness, based on the FR. Lower ranks in each category indicate better performance.

Tables 3–6 present the simulation results for various combinations of η and ω, ranking the estimation methods (MLE, CVM, AD, MPS, OLS, WLS) based on their performance metrics |Bias|, MSE, MAE, and FR. The results clearly demonstrate that all estimators are consistent, as indicated by the systematic decrease in the values of |Bias|, MSE, and MAE with increasing sample size.

A comparative summary of the overall performance is presented in Table 7. The overall rank for each estimation method was calculated by summing its individual ranks based on accuracy (|Bias|, MSE, MAE) and robustness (failure rate). Lower total ranks indicate better overall performance. Among all evaluated methods, the maximum likelihood estimator (MLE) consistently outperformed the others, achieving the lowest overall ranks and exhibiting a zero failure rate across all scenarios (see Tables 3–6). These findings highlight the superior accuracy and robustness of the MLE.

Based on the simulation results, MLE is recommended for estimating the parameters of the TSTH-I distribution. Although the ordinary least squares (OLS) method ranked second in terms of accuracy, it was found to be unreliable due to its high failure rate. The maximum product of spacing (MPS) method, which ranked second in robustness, is therefore suggested as a secondary alternative for parameter estimation.

5. Applications

To evaluate the flexibility and practical applicability of the proposed distribution, we apply it to four real-world data sets from different fields, all exhibiting right skewness. The first data set, discussed in [23], consists of the waiting times (in minutes) before service for 100 bank customers. The second data set, presented in [24] and originally attributed to [25], presents the time between failures for 30 repairable item. The third data set, presented in [24] and originally attributed to [26], pertains to fatigue fracture tests conducted on Kevlar 373/epoxy specimens subjected to a constant stress level of 90% until complete failure occurred. The data set consists of 76 observations. The fourth data set, discussed by [27], consists of 34 measurements of Vinyl chloride concentrations (in mg/l) collected from clean up-gradient monitoring wells. The data sets are provided in Appendix A–D. For simplicity, we refer to these data as waiting times, failure times, fatigue fracture, and Vinyl levels data sets, respectively.

In our analysis, the proposed distribution is compared with the following existing models:

- The one-parameter Sarhan-Tadj-Hamilton distribution (STH-I) [17].
- The two-parameter Sarhan–Tadj–Hamilton distribution (STH-II), which is a power transformation of the STH-I

Table 2. The median, variance, skewness, and kurtosis of the TSTH-I distribution for different values of the model parameters.

η	ω	Med	σ^2	Sk	Kur	ω	Med	σ^2	Sk	Kur
-1.0	0.1	9.674	58.942	1.581	7.695	2.0	0.560	0.260	1.742	7.989
-0.8	0.1	8.954	59.096	1.570	7.592	2.0	0.515	0.258	1.753	8.041
-0.6	0.1	8.190	58.217	1.597	7.647	2.0	0.467	0.251	1.797	8.244
-0.4	0.1	7.395	56.305	1.656	7.860	2.0	0.418	0.240	1.871	8.613
-0.2	0.1	6.594	53.359	1.742	8.249	2.0	0.370	0.225	1.974	9.181
0.0	0.1	5.818	49.380	1.856	8.846	2.0	0.323	0.207	2.106	9.996
0.2	0.1	5.101	44.368	1.994	9.697	2.0	0.281	0.184	2.268	11.138
0.4	0.1	4.469	38.322	2.153	10.862	2.0	0.244	0.157	2.460	12.718
0.6	0.1	3.929	31.243	2.312	12.342	2.0	0.213	0.125	2.667	14.833
0.8	0.1	3.477	23.131	2.375	13.537	2.0	0.187	0.090	2.788	16.964
1.0	0.1	3.102	13.985	1.672	7.099	2.0	0.166	0.051	1.960	8.974
-1.0	0.5	2.053	3.177	1.796	8.756	3.0	0.382	0.122	1.711	7.750
-0.8	0.5	1.894	3.155	1.792	8.727	3.0	0.351	0.121	1.724	7.809
-0.6	0.5	1.727	3.082	1.825	8.866	3.0	0.318	0.118	1.771	8.016
-0.4	0.5	1.554	2.957	1.889	9.183	3.0	0.284	0.113	1.847	8.386
-0.2	0.5	1.381	2.782	1.982	9.705	3.0	0.251	0.106	1.951	8.949
0.0	0.5	1.214	2.556	2.103	10.475	3.0	0.219	0.097	2.085	9.757
0.2	0.5	1.060	2.278	2.254	11.563	3.0	0.190	0.086	2.248	10.886
0.4	0.5	0.926	1.949	2.430	13.064	3.0	0.165	0.073	2.442	12.451
0.6	0.5	0.811	1.569	2.613	15.026	3.0	0.144	0.059	2.652	14.549
0.8	0.5	0.716	1.138	2.695	16.798	3.0	0.126	0.042	2.780	16.693
1.0	0.5	0.637	0.656	1.825	8.179	3.0	0.112	0.024	1.978	9.042
-1.0	1.0	1.071	0.922	1.789	8.452	4.0	0.290	0.071	1.692	7.609
-0.8	1.0	0.986	0.914	1.794	8.475	4.0	0.266	0.070	1.706	7.671
-0.6	1.0	0.897	0.891	1.833	8.657	4.0	0.241	0.068	1.754	7.879
-0.4	1.0	0.805	0.853	1.903	9.014	4.0	0.216	0.065	1.831	8.247
-0.2	1.0	0.714	0.801	2.001	9.573	4.0	0.190	0.061	1.936	8.806
0.0	1.0	0.626	0.734	2.129	10.385	4.0	0.166	0.056	2.070	9.607
0.2	1.0	0.545	0.653	2.287	11.527	4.0	0.144	0.050	2.234	10.727
0.4	1.0	0.475	0.557	2.473	13.105	4.0	0.125	0.043	2.429	12.278
0.6	1.0	0.415	0.447	2.670	15.198	4.0	0.109	0.034	2.639	14.360
0.8	1.0	0.366	0.322	2.773	17.221	4.0	0.095	0.025	2.771	16.500
1.0	1.0	0.325	0.183	1.905	8.686	4.0	0.084	0.014	1.986	9.062
-1.0	1.5	0.733	0.442	1.763	8.181	5.0	0.234	0.046	1.678	7.516
-0.8	1.5	0.674	0.438	1.772	8.223	5.0	0.215	0.046	1.694	7.580
-0.6	1.5	0.613	0.426	1.815	8.420	5.0	0.195	0.044	1.742	7.788
-0.4	1.5	0.549	0.408	1.887	8.787	5.0	0.174	0.043	1.819	8.154
-0.2	1.5	0.486	0.383	1.989	9.354	5.0	0.153	0.040	1.925	8.711
0.0	1.5	0.425	0.351	2.119	10.171	5.0	0.134	0.037	2.059	9.506
0.2	1.5	0.370	0.312	2.280	11.317	5.0	0.116	0.033	2.224	10.618
0.4	1.5	0.322	0.266	2.470	12.903	5.0	0.100	0.028	2.419	12.159
0.6	1.5	0.281	0.213	2.673	15.020	5.0	0.087	0.022	2.630	14.228
0.8	1.5	0.247	0.154	2.788	17.123	5.0	0.077	0.016	2.763	16.363
1.0	1.5	0.219	0.087	1.941	8.884	5.0	0.068	0.009	1.990	9.066

Table 3. Simulated |Bias|, MSE, MAE, and FR values for $\eta = 0.5$ and $\omega = 1.4$.

	Methods	MLE	CVM	AD	MPS	OLS	WLS
$n = 30$	$\lvert\text{Bias}\rvert_\eta$	0.1354_1	0.3160_6	0.2799_4	0.1839_2	0.2960_5	0.2662_3
	$\lvert\text{Bias}\rvert_\omega$	0.0533_1	0.5048_6	0.3608_4	0.1939_2	0.4821_5	0.3484_3
	MSE_η	0.1289_1	0.3796_6	0.2686_4	0.1953_2	0.3617_5	0.2544_3
	MSE_ω	0.1971_1	0.9119_6	0.4832_4	0.3551_2	0.9055_5	0.4693_3
	MAE_η	0.3088_2	0.4409_6	0.3486_4	0.2973_1	0.4341_5	0.3379_3
	MAE_ω	0.3594_1	0.6976_6	0.4907_4	0.4171_2	0.6952_5	0.4833_3
	$\sum$ ranks	7_1	36_6	24_4	11_2	30_5	18_3
	FR	$0.00\%_1$	$0.04\%_3$	$0.06\%_4$	$0.03\%_2$	$0.32\%_5$	$0.38\%_6$
$n = 50$	$\lvert\text{Bias}\rvert_\eta$	0.1126_2	0.2987_6	0.2266_5	0.1338_3	0.0154_1	0.2223_4
	$\lvert\text{Bias}\rvert_\omega$	0.0615_1	0.4258_6	0.2895_5	0.1353_3	0.0967_2	0.2803_4
	MSE_η	0.1190_1	0.3055_6	0.1999_4	0.1524_2	0.1685_3	0.2002_5
	MSE_ω	0.1517_1	0.6094_6	0.3231_3	0.2324_2	0.3779_5	0.3236_4
	MAE_η	0.2951_2	0.3995_6	0.3051_3	0.2746_1	0.3077_5	0.3053_4
	MAE_ω	0.3223_1	0.5781_6	0.4042_4	0.3461_2	0.4472_5	0.4038_3
	$\sum$ ranks	8_1	36_6	$24_{4.5}$	13_2	21_3	$24_{4.5}$
	FR	$0.00\%_1$	$0.04\%_{2.5}$	$0.06\%_4$	$0.04\%_{2.5}$	$0.56\%_6$	$0.54\%_5$
$n = 100$	$\lvert\text{Bias}\rvert_\eta$	0.0937_2	0.2738_6	0.1841_5	0.0881_1	0.1351_3	0.1840_4
	$\lvert\text{Bias}\rvert_\omega$	0.0587_1	0.3583_6	0.2241_4	0.0893_3	0.0771_2	0.2250_5
	MSE_η	0.1003_1	0.2140_6	0.1446_4	0.1149_3	0.1026_2	0.1454_5
	MSE_ω	0.1106_1	0.3701_6	0.2030_5	0.1539_3	0.1188_2	0.2015_4
	MAE_η	0.2667_4	0.3375_6	0.2638_3	0.2519_1	0.2530_2	0.2673_5
	MAE_ω	0.2763_1	0.4495_6	0.3219_4	0.2894_3	0.2841_2	0.3238_5
	$\sum$ ranks	10_1	36_6	25_4	14_3	13_2	28_5
	FR	$0.00\%_{1.5}$	$0.00\%_{1.5}$	$0.08\%_3$	$0.09\%_4$	$0.83\%_6$	$0.43\%_5$
$n = 150$	$\lvert\text{Bias}\rvert_\eta$	0.0793_2	0.2613_6	0.1739_5	0.0584_1	0.1272_3	0.1658_4
	$\lvert\text{Bias}\rvert_\omega$	0.0527_1	0.3278_6	0.2087_5	0.0632_2	0.0814_3	0.1978_4
	MSE_η	0.0915_2	0.1751_6	0.1233_5	0.0949_3	0.0899_1	0.1198_4
	MSE_ω	0.0990_2	0.2757_6	0.1653_5	0.1196_3	0.0929_1	0.1599_4
	MAE_η	0.2541_5	0.3055_6	0.2493_4	0.2359_1	0.2361_2	0.2460_3
	MAE_ω	0.2621_3	0.3895_6	0.2944_5	0.2604_2	0.2520_1	0.2894_4
	$\sum$ ranks	15_3	36_6	29_5	12_2	11_1	23_4
	FR	$0.00\%_{1.5}$	$0.00\%_{1.5}$	$0.10\%_4$	$0.02\%_3$	$1.16\%_6$	$0.34\%_5$
$n = 250$	$\lvert\text{Bias}\rvert_\eta$	0.0662_2	0.2522_6	0.1567_5	0.0334_1	0.1195_3	0.1512_4
	$\lvert\text{Bias}\rvert_\omega$	0.0449_2	0.3025_6	0.1821_5	0.0364_1	0.0800_3	0.1762_4
	MSE_η	0.0785_3	0.1398_6	0.0977_5	0.0743_1	0.0778_2	0.0945_4
	MSE_ω	0.0813_2	0.2040_6	0.1218_5	0.0859_3	0.0761_1	0.1171_4
	MAE_η	0.2342_5	0.2777_6	0.2254_4	0.2149_1	0.2207_2	0.2210_3
	MAE_ω	0.2375_3	0.3353_6	0.2557_5	0.2272_1	0.2284_2	0.2505_4
	$\sum$ ranks	17_3	36_6	29_5	8_1	13_2	23_4
	FR	$0.00\%_{1.5}$	$0.00\%_{1.5}$	$0.15\%_4$	$0.07\%_3$	$0.96\%_6$	$0.60\%_5$
$n = 500$	$\lvert\text{Bias}\rvert_\eta$	0.0562_2	0.2287_6	0.1329_5	0.0132_1	0.0939_3	0.1261_4
	$\lvert\text{Bias}\rvert_\omega$	0.0431_2	0.2639_6	0.1520_5	0.0161_1	0.0659_3	0.1444_4
	MSE_η	0.0594_2	0.1023_6	0.0689_5	0.0556_1	0.0610_3	0.0680_4
	MSE_ω	0.0588_2	0.1388_6	0.0833_5	0.0603_3	0.0570_1	0.0811_4
	MAE_η	0.2022_5	0.2419_6	0.1910_2	0.1914_3	0.1960_4	0.1903_1
	MAE_ω	0.2011_3	0.2798_6	0.2116_5	0.1961_1	0.1969_2	0.2092_4
	$\sum$ ranks	$16_{2.5}$	36_6	27_5	10_1	$16_{2.5}$	21_4
	FR	$0.00\%_{1.5}$	$0.00\%_{1.5}$	$0.37\%_4$	$0.10\%_3$	$1.23\%_6$	$0.81\%_5$

Table 4. Simulated |Bias|, MSE, MAE, and FR values for $\eta = 0.7$ and $\omega = 0.4$.

	Methods	MLE	CVM	AD	MPS	OLS	WLS
$n = 30$	$\lvert\text{Bias}\rvert_\eta$	0.0470_3	0.0459_2	0.5150_6	0.4265_4	0.0400_1	0.5066_5
	$\lvert\text{Bias}\rvert_\omega$	0.0358_3	0.0120_1	0.1881_6	0.1363_4	0.0139_2	0.1843_5
	MSE_η	0.1070_3	0.0696_1	0.4630_6	0.3611_4	0.0737_2	0.4526_5
	MSE_ω	0.0195_3	0.0171_1	0.0738_6	0.0519_4	0.0174_2	0.0723_5
	MAE_η	0.2858_3	0.2546_1	0.5178_6	0.4303_4	0.2580_2	0.5097_5
	MAE_ω	0.1048_3	0.1037_2	0.2005_6	0.1609_4	0.1035_1	0.1977_5
	$\sum$ ranks	18_3	8_1	36_6	24_4	10_2	30_5
	FR	$0.00\%_1$	$0.94\%_4$	$0.46\%_3$	$0.21\%_2$	$2.60\%_6$	$1.60\%_5$
$n = 50$	$\lvert\text{Bias}\rvert_\eta$	0.0579_3	0.0332_2	0.4727_5	0.3835_4	0.0318_1	0.4746_6
	$\lvert\text{Bias}\rvert_\omega$	0.0338_3	0.0086_1	0.1653_5	0.1210_4	0.0103_2	0.1669_6
	MSE_η	0.1013_3	0.0677_2	0.3803_5	0.3006_4	0.0672_1	0.3861_6
	MSE_ω	0.0151_3	0.0123_1	0.0527_5	0.0397_4	0.0126_2	0.0542_6
	MAE_η	0.2732_3	0.2497_2	0.4769_5	0.3926_4	0.2483_1	0.4800_6
	MAE_ω	0.0939_3	0.0910_1	0.1733_5	0.1408_4	0.0921_2	0.1757_6
	$\sum$ ranks	18_3	13_2	30_4	32_5	9_1	36_6
	FR	$0.00\%_1$	$0.83\%_4$	$0.49\%_3$	$0.24\%_2$	$3.07\%_6$	$1.94\%_5$
$n = 100$	$\lvert\text{Bias}\rvert_\eta$	0.0805_3	0.0226_2	0.4459_6	0.3331_4	0.0205_1	0.4384_5
	$\lvert\text{Bias}\rvert_\omega$	0.0347_3	0.0067_1	0.1503_6	0.1045_4	0.0080_2	0.1493_5
	MSE_η	0.0953_3	$0.0633_{1.5}$	0.3183_6	0.2349_4	$0.0633_{1.5}$	0.3119_5
	MSE_ω	0.0118_3	0.0086_1	0.0395_6	0.0277_4	0.0087_2	0.0389_5
	MAE_η	0.2587_3	0.2388_2	0.4517_6	0.3499_4	0.2385_1	0.4445_5
	MAE_ω	0.0852_3	0.0791_1	0.1551_6	0.1190_4	0.0799_2	0.1541_5
	$\sum$ ranks	18_3	8.5_1	36_6	24_4	9.5_2	30_5
	FR	$0.00\%_1$	$0.78\%_4$	$0.69\%_3$	$0.21\%_2$	$3.05\%_6$	$2.54\%_5$
$n = 150$	$\lvert\text{Bias}\rvert_\eta$	0.0810_3	0.0121_2	0.4239_6	0.3092_4	0.0106_1	0.4230_5
	$\lvert\text{Bias}\rvert_\omega$	0.0316_3	0.0077_1	$0.1416_{5.5}$	0.0970_4	0.0087_2	$0.1416_{5.5}$
	MSE_η	0.0887_3	0.0612_2	0.2794_5	0.2037_4	0.0609_1	0.2808_6
	MSE_ω	0.0101_3	0.0073_2	0.0332_5	0.0232_4	0.0075_1	0.0335_6
	MAE_η	0.2471_3	0.2335_2	0.4296_6	0.3294_4	0.2321_1	0.4292_5
	MAE_ω	0.0788_3	0.0748_1	0.1450_5	0.1096_4	0.0752_2	0.1453_6
	$\sum$ ranks	18_3	10_2	32.5_5	24_4	8_1	33.5_6
	FR	$0.00\%_1$	$0.76\%_4$	$0.62\%_3$	$0.15\%_2$	$3.58\%_6$	$3.07\%_5$
$n = 250$	$\lvert\text{Bias}\rvert_\eta$	0.0917_3	0.0013_1	0.4117_5	0.2860_4	0.0019_2	0.4127_6
	$\lvert\text{Bias}\rvert_\omega$	0.0333_3	0.0092_1	0.1360_5	0.0902_4	0.0095_2	0.1361_6
	MSE_η	0.0816_3	0.0561_1	0.2475_5	0.1769_4	0.0564_2	0.2519_6
	MSE_ω	0.0089_3	$0.0061_{1.5}$	0.0283_5	0.0194_4	$0.0061_{1.5}$	0.0286_6
	MAE_η	0.2337_3	0.2199_1	0.4166_5	0.3104_4	0.2202_2	0.4181_6
	MAE_ω	0.0739_3	$0.0690_{1.5}$	0.1381_5	0.1015_4	$0.0690_{1.5}$	0.1385_6
	$\sum$ ranks	18_3	7_1	30_5	24_4	11_2	36_6
	FR	$0.00\%_1$	$0.76\%_4$	$0.66\%_3$	$0.15\%_2$	$4.12\%_6$	$3.43\%_5$
$n = 500$	$\lvert\text{Bias}\rvert_\eta$	0.0965_3	0.0092_1	0.4023_6	0.2480_4	0.0121_2	0.3930_5
	$\lvert\text{Bias}\rvert_\omega$	0.0326_3	0.0095_1	0.1311_6	0.0782_4	0.0103_2	0.1280_5
	MSE_η	0.0694_3	0.0493_1	0.2190_6	0.1375_4	0.0498_2	0.2145_5
	MSE_ω	0.0071_3	0.0048_1	0.0240_6	0.0144_4	0.0049_2	0.0236_5
	MAE_η	0.2103_3	0.2011_1	0.4045_6	0.2758_4	0.2024_2	0.3961_5
	MAE_ω	0.0653_3	0.0611_1	0.1320_6	0.0885_4	0.0617_2	0.1293_5
	$\sum$ ranks	18_3	6_1	36_6	24_4	12_2	30_5
	FR	$0.00\%_1$	$0.69\%_4$	$0.65\%_3$	$0.13\%_2$	$4.81\%_6$	$3.45\%_5$

Table 5. Simulated |Bias|, MSE, MAE, and FR values for $\eta = -0.6$ and $\omega = 0.8$.

	Methods	MLE	CVM	AD	MPS	OLS	WLS
$n = 30$	$\lvert\text{Bias}\rvert_\eta$	0.0190_1	0.0378_2	0.1779_4	0.2752_6	0.0434_3	0.1864_5
	$\lvert\text{Bias}\rvert_\omega$	0.0115_3	0.0016_1	0.0612_4	0.0995_6	0.0050_2	0.0637_5
	MSE_η	0.1650_1	0.1719_2	0.2311_4	0.3213_6	0.1790_3	0.2390_5
	MSE_ω	0.0375_1	0.0384_2	0.0398_5	0.0487_6	$0.0395_{3.5}$	$0.0395_{3.5}$
	MAE_η	0.3146_1	0.3249_2	0.3576_4	0.4141_6	0.3319_3	0.3638_5
	MAE_ω	0.1503_1	0.1554_2	0.1588_4	0.1751_6	0.1568_3	0.1591_5
	$\sum$ ranks	8_1	11_2	25_4	36_6	17.5_3	28.5_5
	FR	$0.00\%_1$	$0.26\%_3$	$0.43\%_4$	$0.11\%_2$	$2.51\%_6$	$1.95\%_5$
$n = 50$	$\lvert\text{Bias}\rvert_\eta$	0.0208_2	0.0192_1	0.1222_5	0.1946_6	0.0325_3	0.1138_4
	$\lvert\text{Bias}\rvert_\omega$	0.0045_3	0.0021_1	0.0439_5	0.0717_6	0.0043_2	0.0404_4
	MSE_η	0.1112_1	0.1197_2	0.1456_5	0.2141_6	0.1202_3	0.1438_4
	MSE_ω	0.0224_1	0.0252_5	$0.0248_{3.5}$	0.0313_6	$0.0248_{3.5}$	0.0246_2
	MAE_η	0.2508_1	0.2687_2	0.2764_5	0.3151_6	0.2689_3	0.2759_4
	MAE_ω	0.1171_1	0.1257_5	0.1231_3	0.1347_6	0.1250_4	0.1225_2
	$\sum$ ranks	9_1	16_2	26.5_5	36_6	18.5_3	20_4
	FR	$0.00\%_1$	$0.22\%_3$	$0.31\%_4$	$0.04\%_2$	$3.31\%_6$	$2.04\%_5$
$n = 100$	$\lvert\text{Bias}\rvert_\eta$	0.0056_2	0.0042_1	0.0583_4	0.1038_6	0.0097_3	0.0601_5
	$\lvert\text{Bias}\rvert_\omega$	0.0032_3	0.0012_2	0.0206_4	0.0411_6	0.0001_1	0.0215_5
	MSE_η	0.0541_1	0.0592_2	0.0599_3	0.0942_6	0.0627_5	0.0622_4
	MSE_ω	0.0113_1	0.0122_4	0.0114_2	0.0148_6	0.0127_5	0.0117_3
	MAE_η	0.1768_1	0.1901_4	0.1837_2	0.2067_6	0.1955_5	0.1861_3
	MAE_ω	0.0827_1	0.0874_4	0.0843_2	0.0911_6	0.0888_5	0.0849_3
	$\sum$ ranks	9_1	$17_{2.5}$	$17_{2.5}$	36_6	24_5	23_4
	FR	$0.00\%_1$	$0.18\%_3$	$0.28\%_4$	$0.17\%_2$	$4.71\%_6$	$2.39\%_5$
$n = 150$	$\lvert\text{Bias}\rvert_\eta$	$0.0010_{1.5}$	$0.0010_{1.5}$	0.0370_5	0.0651_6	0.0054_3	0.0349_4
	$\lvert\text{Bias}\rvert_\omega$	0.0048_3	0.0029_2	0.0142_5	0.0259_6	0.0003_1	0.0131_4
	MSE_η	0.0327_1	0.0399_4	0.0365_2	0.0471_6	0.0408_5	0.0367_3
	MSE_ω	0.0070_1	0.0082_4	0.0074_3	0.0083_5	0.0084_6	0.0073_2
	MAE_η	0.1402_1	0.1566_5	0.1467_3	0.1555_4	0.1584_6	0.1465_2
	MAE_ω	0.0658_1	0.0715_5	0.0678_3	0.0701_4	0.0726_6	0.0672_2
	$\sum$ ranks	8.5_1	21.5_4	21_3	31_6	27_5	17_2
	FR	$0.00\%_1$	$0.20\%_3$	$0.22\%_4$	$0.09\%_2$	$5.18\%_6$	$2.70\%_5$
$n = 250$	$\lvert\text{Bias}\rvert_\eta$	0.0015_3	0.0012_2	0.0232_5	0.0367_6	0.0009_1	0.0193_4
	$\lvert\text{Bias}\rvert_\omega$	0.0011_2	0.0006_1	0.0083_5	0.0158_6	0.0017_3	0.0072_4
	MSE_η	0.0185_1	0.0232_5	0.0210_3	0.0213_4	0.0236_6	0.0201_2
	MSE_ω	0.0041_1	0.0049_5	0.0044_4	0.0043_3	0.0050_6	0.0042_2
	MAE_η	0.1079_1	0.1204_5	0.1127_4	0.1124_3	0.1217_6	0.1109_2
	MAE_ω	0.0506_1	0.0554_5	0.0527_4	0.0521_3	0.0560_6	0.0517_2
	$\sum$ ranks	9_1	23_3	$25_{4.5}$	$25_{4.5}$	28_6	16_2
	FR	$0.00\%_1$	$0.18\%_3$	$0.23\%_4$	$0.10\%_2$	$5.90\%_6$	$2.48\%_5$
$n = 500$	$\lvert\text{Bias}\rvert_\eta$	0.0015_3	0.0007_2	0.0109_5	0.0177_6	0.0006_1	0.0074_4
	$\lvert\text{Bias}\rvert_\omega$	0.0005_2	0.0002_1	0.0039_5	0.0081_6	0.0007_3	0.0035_4
	MSE_η	0.0092_1	0.0116_5	0.0106_4	0.0098_2	0.0117_6	0.0100_3
	MSE_ω	0.0020_1	0.0024_5	0.0022_4	$0.0021_{2.5}$	0.0025_6	$0.0021_{2.5}$
	MAE_η	0.0764_1	0.0857_5	0.0805_4	0.0774_2	0.0860_6	0.0783_3
	MAE_ω	0.0355_1	0.0393_5	0.0371_4	0.0363_2	0.0397_6	0.0367_3
	$\sum$ ranks	9_1	23_4	26_5	20.5_3	28_6	19.5_2
	FR	$0.00\%_1$	$0.07\%_2$	$0.34\%_4$	$0.13\%_3$	$6.92\%_6$	$2.30\%_5$

Table 6. Simulated |Bias|, MSE, MAE, and FR values for $\eta = -0.3$ and $\omega = 1$.

	Methods	MLE	CVM	AD	MPS	OLS	WLS
$n = 30$	$\lvert\text{Bias}\rvert_\eta$	0.0213_3	0.0038_2	0.1694_5	0.2612_6	0.0023_1	0.1677_4
	$\lvert\text{Bias}\rvert_\omega$	0.0415_3	0.0262_1	0.0734_5	0.1329_6	0.0290_2	0.0710_4
	MSE_η	0.2089_1	0.2327_3	0.2609_5	0.3164_6	0.2321_2	0.2608_4
	MSE_ω	0.0854_3	0.0962_5	0.0801_2	0.0959_4	0.0977_6	0.0795_1
	MAE_η	0.3781_1	0.4105_3	0.4241_5	0.4621_6	0.4100_2	0.4234_4
	MAE_ω	0.2312_1	0.2481_4	0.2344_3	0.2582_6	0.2495_5	0.2317_2
	$\sum$ ranks	12_1	$18_{2.5}$	25_5	34_6	$18_{2.5}$	19_4
	FR	$0.00\%_1$	$0.09\%_3$	$0.11\%_4$	$0.07\%_2$	$0.97\%_5$	$0.99\%_6$
$n = 50$	$\lvert\text{Bias}\rvert_\eta$	0.0116_1	0.0166_3	0.1403_4	0.2335_6	0.0150_2	0.1441_5
	$\lvert\text{Bias}\rvert_\omega$	0.0141_3	0.0084_1	0.0632_4	0.1178_6	0.0097_2	0.0651_5
	MSE_η	0.1653_1	0.1790_3	0.2051_5	0.2722_6	0.1758_2	0.2002_4
	MSE_ω	0.0565_1	0.0644_5	0.0598_3	0.0737_6	0.0640_4	0.0584_2
	MAE_η	0.3224_1	0.3475_3	0.3612_5	0.4088_6	0.3436_2	0.3562_4
	MAE_ω	0.1880_1	0.2051_5	0.2000_3	0.2207_6	0.2041_4	0.1972_2
	$\sum$ ranks	8_1	20_3	24_5	36_6	16_2	22_4
	FR	$0.00\%_1$	$0.12\%_3$	$0.13\%_4$	$0.07\%_2$	$1.03\%_5$	$1.43\%_6$
$n = 100$	$\lvert\text{Bias}\rvert_\eta$	0.0209_1	0.0217_2	0.0879_4	0.1661_6	0.0285_3	0.0933_5
	$\lvert\text{Bias}\rvert_\omega$	0.0002_1	0.0025_2	0.0425_4	0.0832_6	0.0045_3	0.0444_5
	MSE_η	0.1051_1	0.1094_2	0.1159_4	0.1831_6	0.1097_3	0.1204_5
	MSE_ω	0.0324_1	0.0367_5	0.0340_2	0.0469_6	0.0365_4	0.0350_3
	MAE_η	0.2440_1	0.2595_3	0.2572_2	0.3078_6	0.2601_4	0.2632_5
	MAE_ω	0.1406_1	0.1528_4	0.1450_2	0.1666_6	0.1530_5	0.1477_3
	$\sum$ ranks	6_1	$18_{2.5}$	$18_{2.5}$	36_6	22_4	26_5
	FR	$0.00\%_1$	$0.13\%_3$	$0.18\%_4$	$0.11\%_2$	$1.33\%_5$	$1.76\%_6$
$n = 150$	$\lvert\text{Bias}\rvert_\eta$	0.0168_1	0.0241_2	0.0617_4	0.1243_6	0.0248_3	0.0678_5
	$\lvert\text{Bias}\rvert_\omega$	0.0001_1	0.0074_2	0.0303_4	0.0639_6	0.0077_3	0.0311_5
	MSE_η	0.0734_1	0.0766_2	0.0794_4	0.1315_6	0.0783_3	0.0823_5
	MSE_ω	0.0222_1	0.0252_4	0.0237_2	0.0336_6	0.0255_5	0.0241_3
	MAE_η	0.2001_1	0.2138_4	0.2099_2	0.2493_6	0.2153_5	0.2132_3
	MAE_ω	0.1147_1	0.1258_4	0.1201_3	0.1358_6	0.1260_5	0.1200_2
	$\sum$ ranks	6_1	18_2	19_3	36_6	24_5	23_4
	FR	$0.00\%_1$	$0.10\%_2$	$0.18\%_4$	$0.13\%_3$	$1.32\%_5$	$2.50\%_6$
$n = 250$	$\lvert\text{Bias}\rvert_\eta$	0.0111_1	0.0118_3	0.0383_5	0.0758_6	0.0115_2	0.0360_4
	$\lvert\text{Bias}\rvert_\omega$	0.0011_1	0.0036_2	0.0179_4	0.0409_6	0.0037_3	0.0188_5
	MSE_η	0.0425_1	0.0459_5	0.0443_3	0.0704_6	0.0446_4	0.0438_2
	MSE_ω	0.0130_1	0.0153_5	0.0137_3	0.0187_6	0.0150_4	0.0135_2
	MAE_η	0.1535_1	0.1650_5	0.1590_3	0.1780_6	0.1618_4	0.1575_2
	MAE_ω	0.0877_1	0.0973_5	0.0914_3	0.0988_6	0.0960_4	0.0902_2
	$\sum$ ranks	6_1	25_5	$21_{3.5}$	36_6	$21_{3.5}$	17_2
	FR	$0.00\%_1$	$0.06\%_2$	$0.27\%_4$	$0.09\%_3$	$1.71\%_5$	$3.02\%_6$
$n = 500$	$\lvert\text{Bias}\rvert_\eta$	0.0009_1	0.0071_3	0.0184_4	0.0330_6	0.0060_2	0.0223_5
	$\lvert\text{Bias}\rvert_\omega$	0.0014_1	0.0023_3	0.0084_4	0.0188_6	0.0018_2	0.0113_5
	MSE_η	0.0178_1	0.0217_5	0.0189_2	0.0218_6	0.0212_4	0.0197_3
	MSE_ω	0.0058_1	0.0076_6	0.0061_2	0.0066_4	0.0073_5	0.0064_3
	MAE_η	0.1030_1	0.1139_6	0.1074_2	0.1102_4	0.1132_5	0.1089_3
	MAE_ω	0.0597_1	0.0684_6	0.0621_2	0.0625_3	0.0669_5	0.0627_4
	$\sum$ ranks	6_1	$29_{5.5}$	16_2	$29_{5.5}$	$23_{3.5}$	$23_{3.5}$
	FR	$0.00\%_1$	$0.04\%_2$	$0.37\%_4$	$0.19\%_3$	$1.81\%_5$	$4.19\%_6$

distribution [17],

$$f_Y(y;\omega,\eta) = \frac{\omega\,\eta\,y^{\eta-1}}{1+\omega}\left[\omega + (1+2\omega y^{\eta})e^{-\omega y^{\eta}}\right]e^{-\omega y^{\eta}},$$

where $y \geq 0$, ω, $\eta > 0$.

https://doi.org/10.57647/mathsci.2025.1904.18008

Table 7. Performance ranking of estimation methods based on simulation results.

Parameters			MLE	CVM	AD	MPS	OLS	WLS
$\eta = 0.5$ and $\omega = 1.4$	sum of partial ranks	|bias|, MSE, and MAE	11.5	36	27.5	11	14.5	24.5
		FR	9.5	16	24	17.5	31	30
$\eta = 0.7$ and $\omega = 0.4$	sum of partial ranks	|bias|, MSE, and MAE	18	8	32	25	10	33
		FR	6	24	18	12	36	30
$\eta = -0.6$ and $\omega = 0.8$	sum of partial ranks	|bias|, MSE, and MAE	6	17.5	24	31.5	28	19
		FR	6	17	24	13	36	30
$\eta = -0.3$ and $\omega = 1$	sum of partial ranks	|bias|, MSE, and MAE	6	20.5	21	35.5	20.5	22.5
		FR	6	15	24	15	30	36
$\sum$ ranks (MSE, MAE)			41.5	82	104	104	73.5	98
Overall rank (MSE, MAE)			1	3	5.5	5.5	2	4
$\sum$ ranks (FR)			27.5	72	90	57.5	133	126
Overall rank (FR)			1	3	4	2	6	5

- The transmuted Aradhana distribution (TAD)[23]:

$$f_Y(y;\omega.\eta) = \frac{\omega^3}{\omega^2+2\omega+2}(1+y)^2 e^{-\omega y} \times \left[(1-\eta)+2\eta\left(1+\frac{\omega y(\omega y+2\omega+2)}{\omega^2+2\omega+2}\right)e^{-\omega y}\right],$$

  where $y \geq 0$, $\omega > 0$, $-1 \leq \eta \leq 1$.
- The xgamma distribution (XGD) [28]:

$$f(y;\omega) = \frac{\omega^2}{1+\omega}\left(1+\frac{\omega y^2}{2}\right)e^{-\omega y}, \quad y > 0,\ \omega > 0.$$

- The Lindley distribution (LD) [29]:

$$f(y;\omega) = \frac{\omega^2(1+y)}{1+\omega}e^{-\omega y}, \qquad y \geq 0,\ \omega > 0.$$

- The exponential distribution (EXP)

$$f(y;\omega) = \omega e^{-\omega y}, \qquad y \geq 0,\ \omega > 0.$$

To assess the quality of fit, we apply several evaluation tools, including the Kolmogorov–Smirnov (K–S) and Anderson–Darling (A-D) goodness-of-fit tests with their associated p-values, as well as model selection criteria such as the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). These tools not only measure the closeness between the model and the observed data but also account for model complexity. The best-fitting model is identified based on the smallest A-D and K-S test statistics and the highest corresponding p-values. Generally, a p-value below 0.05 indicates a poor fit, whereas higher values suggest better alignment between the fitted model and the empirical data.

Table 8 summarizes the parameter estimates, obtained using the maximum likelihood estimation (MLE) method, and the corresponding goodness-of-fit statistics for all models across the four data sets.

The results in Table 8 show that the proposed TSTH-I distribution demonstrates superior and consistent performance across several data sets. Although the EXP distribution attains the smallest information criteria for the vinyl levels data, it exhibits a poor fit for the waiting times and fatigue fracture data sets (p-value < 0.05).

For each data set, we present the empirical and fitted density functions, distribution functions, Q-Q plots, and P-P plots. Each plot includes a comparison among the candidate distributions: the proposed TSTH-I, STH-II, TAD, STH-I, XGD, LD, and EXP distributions. Figs. 8 through 11 correspond to the four data sets, respectively. We observe that the proposed TSTH-I distribution provides a good fit and successfully captures the empirical shape of the data. These visual results, together with the numerical evidence in Table 8, confirm the effectiveness of the proposed TSTH-I distribution as a versatile and powerful model for diverse types of lifetime data.

6. Conclusions

In this paper, we proposed a new two-parameter distribution, denoted by TSTH-I distribution , obtained by applying the quadratic rank transmutation map to the STH-I distribution. From the analysis of its density function, hazard rate, stochastic orders, quantiles, and order statistics, along with skewness and kurtosis measures, it is evident that the TSTH-I distribution offers substantial flexibility to model a variety of data patterns, including unimodal, positively skewed, and heavy- or light-tailed behaviors. We derived several statistical properties of the proposed distribution and employed six estimation methods to estimate its parameters. The results showed that the maximum likelihood estimator (MLE) generally provides the most reliable estimates. Applications to real data sets confirmed that the proposed distribution consistently provides a better fit compared to the STH-II, TA, STH-I, XG, LD, and EXP distributions, making it a practical choice for modeling data in different applications.

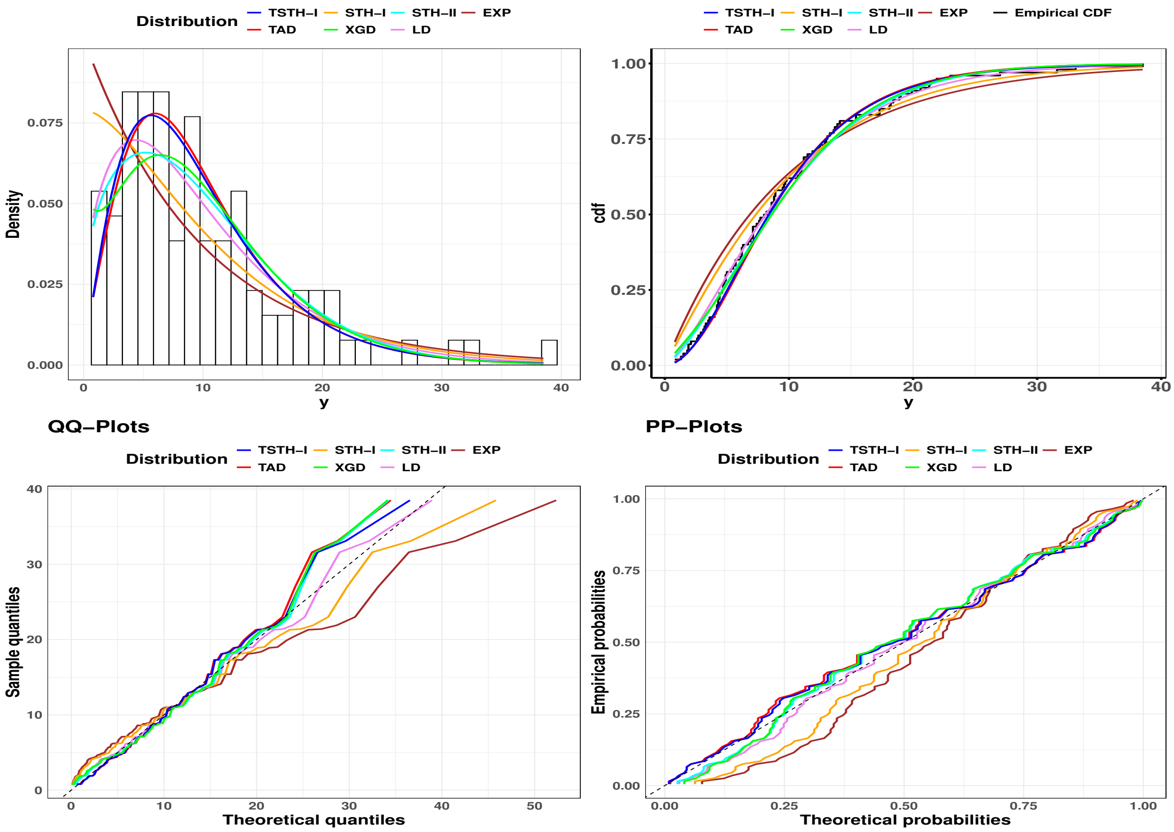

Figure 8. The fitted pdf, cdf, QQ, and PP for the waiting times data.

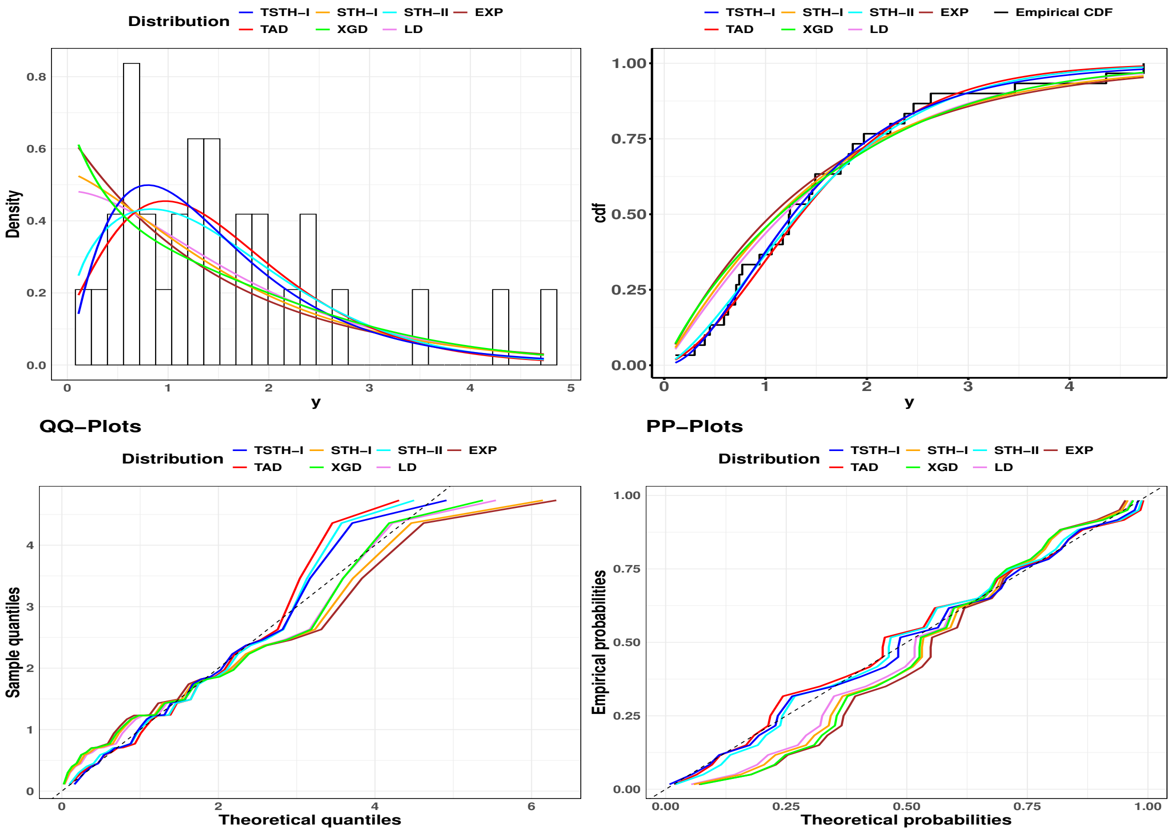

Figure 9. The fitted pdf, cdf, QQ, and PP for the failure times data.

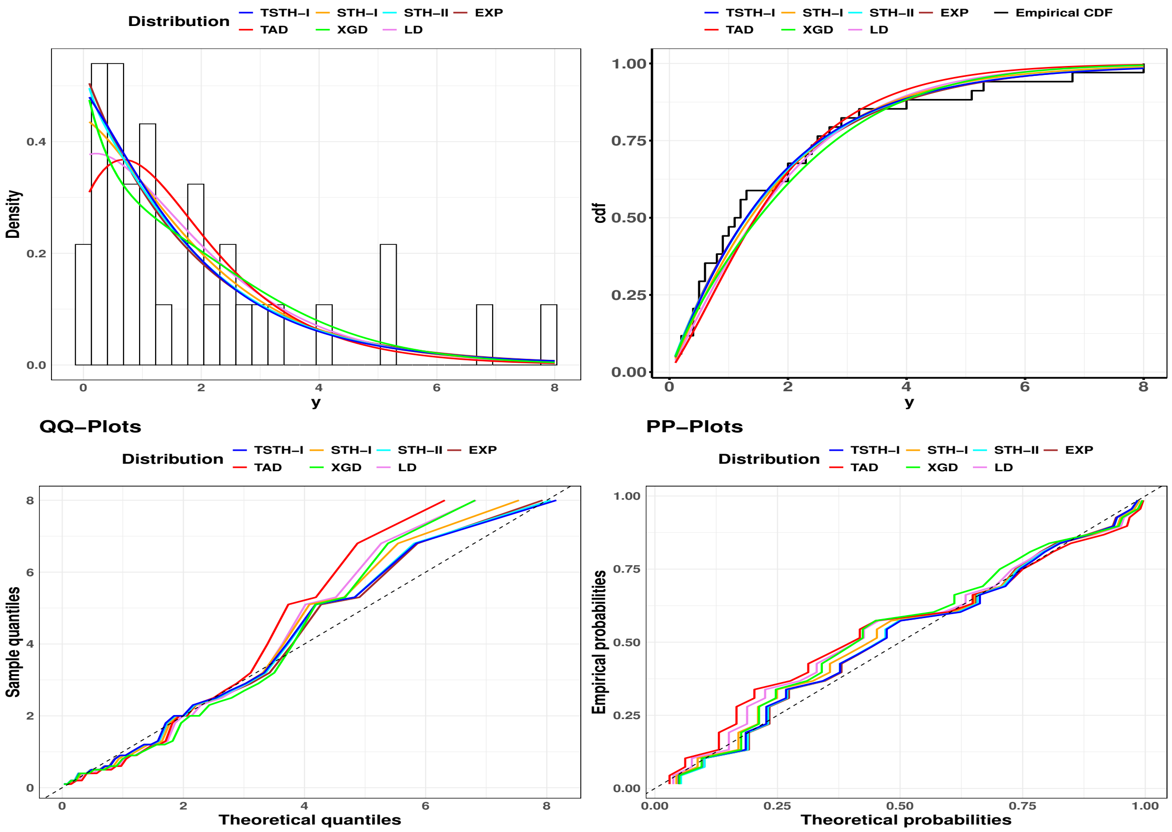

Figure 10. The fitted pdf, cdf, QQ, and PP for the fatigue fracture data.

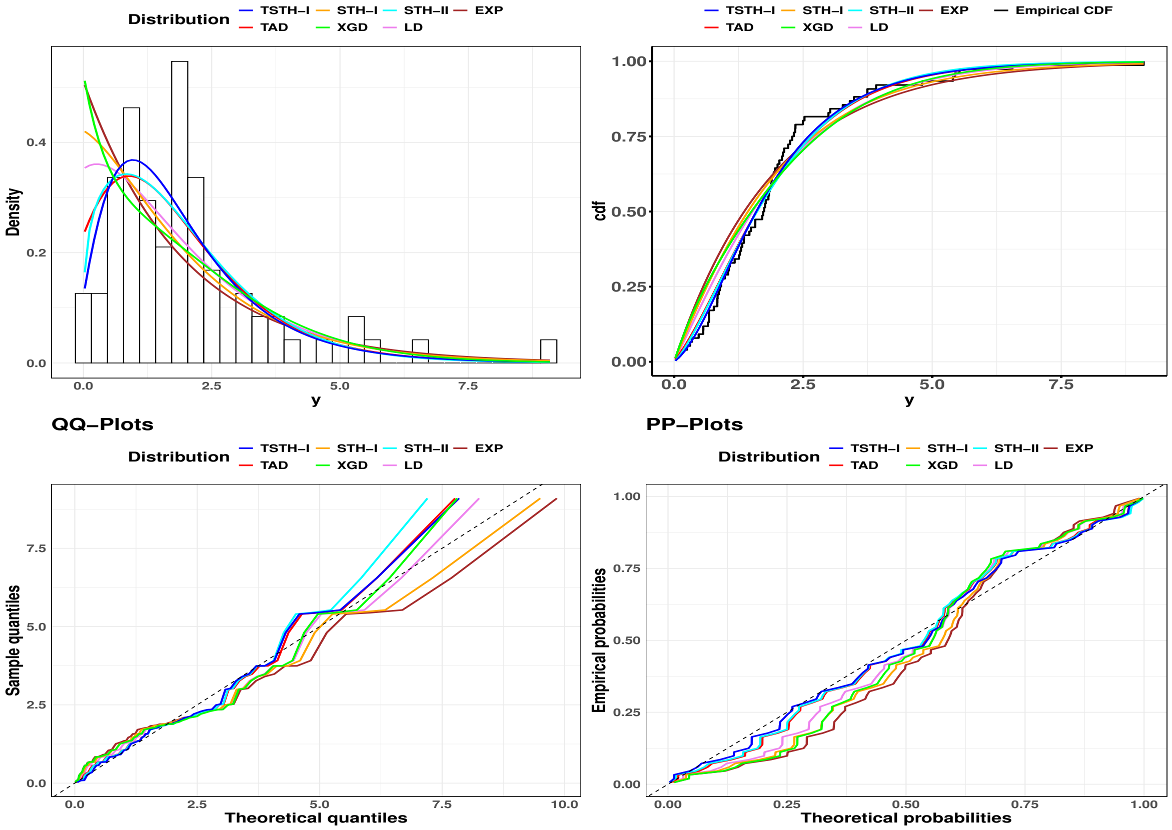

Figure 11. The fitted pdf, cdf, QQ, and PP for the vinyl levels data.

Table 8. Parameter estimates, and goodness-of-fit measures for the data sets.

Data	Model	Est. parameters	AIC	BIC	A-D(p-value)	K-S(p-value)
Waiting times	TSTH-I	$\hat{\omega} = 0.1157$, $\hat{\eta} = -0.9970$	**639.4585**	644.6688	**0.35046 (0.8958)**	0.0622 (0.8340)
	STH-II	$\hat{\omega} = 0.0355$, $\hat{\eta} = 1.2993$	642.7645	647.9749	0.4783 (0.7687)	**0.0592 (0.8747)**
	TAD	$\hat{\omega} = 0.2432$, $\hat{\eta} = 0.4098$	640.6321	645.8425	0.4814 (0.7655)	0.0704 (0.7045)
	STH-I	$\hat{\omega} = 0.0791$	651.3136	653.9188	2.5310 (0.0478)	0.1387 (0.0428)
	XGD	$\hat{\omega} = 0.2634$	644.0405	646.6457	0.6448 (0.6063)	0.0625 (0.8297)
	LD	$\hat{\omega} = 0.1866$	640.0748	**642.6800**	0.4863 (0.7604)	0.0677 (0.7494)
	EXP	$\hat{\omega} = 0.1012$	660.0418	662.6469	4.2293 (0.0068)	0.1730 (0.0050)
Failure times	TSTH-I	$\hat{\omega} = 0.8326$, $\hat{\eta} = -1.0000$	**83.2744**	**86.0768**	**0.1229 (0.9998)**	0.0709 (0.9982)
	STH-II	$\hat{\omega} = 0.4103$, $\hat{\eta} = 1.3671$	83.9784	86.7808	0.1928 (0.9923)	**0.0707 (0.9983)**
	TAD	$\hat{\omega} = 0.8880$, $\hat{\eta} = 1.0000$	84.8325	87.6349	0.2946 (0.9417)	0.0844 (0.9832)
	STH-I	$\hat{\omega} = 0.5373$	86.2563	87.6574	0.9419 (0.3884)	0.1577 (0.445)
	XGD	$\hat{\omega} = 1.2526$	87.3704	88.7716	1.1343 (0.2934)	0.1745 (0.3204)
	LD	$\hat{\omega} = 0.9762$	85.0946	86.4958	0.7266 (0.5357)	0.1407 (0.5928)
	EXP	$\hat{\omega} = 0.6482$	88.0108	89.4119	1.3298 (0.2228)	0.1845 (0.2590)
Fatigue fracture	TSTH-I	$\hat{\omega} = 0.6003$, $\hat{\eta} = -0.7976$	**245.7916**	**250.4530**	**0.4605 (0.7868)**	**0.0887 (0.5578)**
	STH-II	$\hat{\omega} = 0.3303$, $\hat{\eta} = 1.2497$	248.0627	252.7242	0.6860 (0.5701)	0.0999 (0.4076)
	TAD	$\hat{\omega} = 0.7569$, $\hat{\eta} = 0.8317$	246.9999	251.6613	0.68938 (0.5672)	0.0987 (0.4224)
	STH-I	$\hat{\omega} = 0.4213$	251.8741	254.2049	2.1086 (0.0803)	0.1403 (0.0912)
	XGD	$\hat{\omega} = 1.0332$	254.6521	256.9828	2.2325 (0.0689)	0.1474 (0.0662)
	LD	$\hat{\omega} = 0.7948$	249.3503	251.6810	1.4751 (0.1826)	0.1156 (0.2423)
	EXP	$\hat{\omega} = 0.5104$	256.2287	258.5594	2.9881 (0.0279)	0.1663 (0.0263)
Vinyl levels	TSTH-I	$\hat{\omega} = 0.3456$, $\hat{\eta} = 0.4325$	115.0427	118.0955	0.2725 (0.9570)	**0.0870 (0.9590)**
	STH-II	$\hat{\omega} = 0.4702$, $\hat{\eta} = 0.9420$	115.3607	118.4134	0.2880 (0.9466)	0.0903 (0.9445)
	TAD	$\hat{\omega} = 0.9936$, $\hat{\eta} = 0.4513$	118.4045	121.4572	1.0330 (0.3399)	0.1500 (0.4281)
	STH-I	$\hat{\omega} = 0.4425$	113.5707	115.0970	0.3971 (0.8506)	0.1064 (0.8362)
	XGD	$\hat{\omega} = 1.0313$	114.9701	116.4965	0.6292 (0.6196)	0.1384 (0.5330)
	LD	$\hat{\omega} = 0.8238$	114.6073	116.1336	0.6872 (0.5685)	0.1326 (0.5881)
	EXP	$\hat{\omega} = 0.5321$	**112.9052**	**114.4316**	**0.2720 (0.9574)**	0.0890 (0.9507)

Appendix A

The observations of the Waiting times data set:

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1
2.6	2.7	2.9	3.1	3.2	3.3	3.5	3.6
4.0	4.1	4.2	4.2	4.3	4.3	4.4	4.4
4.6	4.7	4.7	4.8	4.9	4.9	5.0	5.3
5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6
7.7	8.0	8.2	8.6	8.6	8.6	8.8	8.8
8.9	8.9	9.5	9.6	9.7	9.8	10.7	10.9
11.0	11.0	11.1	11.2	11.2	11.5	11.9	12.4
12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4
18.9	19.0	19.9	20.6	21.3	21.4	21.9	23.0
27.0	31.6	33.1	38.5				

Appendix B

The observations of the time between failures for repairable item data set are:

1.43	0.11	0.71	0.77	2.63	1.49	3.46
2.46	0.59	0.74	1.23	0.94	4.36	0.40
1.74	4.73	2.23	0.45	0.70	1.06	1.46
0.30	1.82	2.37	0.63	1.23	1.24	1.97
1.86	1.17					

Appendix C

The observations of the fatigue fracture data set are:

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451
0.4763	0.5650	0.5671	0.6566	0.6748	0.6751
0.6753	0.7696	0.8375	0.8391	0.8425	0.8645
0.8851	0.9113	0.9120	0.9836	1.0483	1.0596
1.0773	1.1733	1.2570	1.2766	1.2985	1.3211
1.3503	1.3551	1.4595	1.4880	1.5728	1.5733
1.7083	1.7263	1.7460	1.7630	1.7746	1.8275
1.8375	1.8503	1.8808	1.8878	1.8881	1.9316
1.9558	2.0048	2.0408	2.0903	2.1093	2.1330
2.2100	2.2460	2.2878	2.3203	2.3470	2.3513
2.4951	2.5260	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455	3.9143	4.8073	5.4005
5.4435	5.5295	6.5541	9.0960		

Appendix D

The observations of the Vinyl levels data set are:

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0
0.8	0.4	0.6	0.9	0.4	2.0	0.5	5.3	3.2
2.7	2.9	2.5	2.3	1.0	0.2	0.1	0.1	1.8
0.9	2.0	4.0	6.8	1.2	0.4	0.2		

Acknowledgment

This study is supported via funding Prince Sattam bin Abdulaziz University project number (PSAU/2026/R/1447).

Authors contributions

All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

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