


A Two-Parameter Ridge Estimators Approach to Mitigate Multicollinearity: Simulation and Application Results

Muhammad Haseeb ^{1,*} , Muhammad Yousaf Shad¹, Ali Rashash R Alzahrani ², Asma Ahmad Alzahrani³

¹Department of Statistics Quaid-i-Azam University Islamabad, Pakistan

²Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah, Saudi Arabia

³Department of Mathematics, Faculty of Science, Al-Baha University

*Corresponding author: mhaseeb@stat.qau.edu.pk

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Multicollinearity in regression analysis arises when predictor variables are highly correlated, making it difficult to accurately estimate regression coefficients. This issue can distort model interpretations and inflate coefficient variances, making estimates sensitive to small data changes. To address this issue, several ridge estimators have been developed in the past to reduce the effect of multicollinearity and improve the model stability. To overcome the negative affect of multicollinearity, we introduce three newly proposed two-parameter ridge estimators, named HITPR1, HITPR2, and HITPR3, which dynamically adjust the ridge parameter for different multicollinearity structures. We evaluate the performance of these proposed estimators through a comprehensive simulation study and employing Mean Squared Error (MSE) criterion. The numerical results show that HITPR1 estimator performs better with higher efficiency and lower MSE, outperforming the other estimators in different settings. To further investigate the performance and applicability of the newly proposed estimators, two real-world datasets, have been utilized.

Keywords: Regression analysis; Multicollinearity issues; Ridge regression methods; Mean squared error evaluation; Monte Carlo simulation studies

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1. Introduction

Regression analysis is a statistical method used to model the relationship between a dependent variable (Y) and one or more independent variables (X). The most common estimation technique is the Ordinary Least Squares (OLS), which minimizes the sum of squared residuals between observed and predicted values. The equation for the linear regression model is represented as:

$$Y = X\beta + \epsilon. \tag{1}$$

Here, $Y \in \mathbb{R}^{n \times 1}$ represents the vector of observed responses, $X \in \mathbb{R}^{n \times p}$ represents the design matrix comprising the predictor variables, $\beta \in \mathbb{R}^{p \times 1}$ is the vector of unknown regression coefficients, and $\epsilon \in \mathbb{R}^{n \times 1}$ is the vector of random errors. The OLS estimator for the

regression coefficients is given as:

$$\hat{\beta}_{OLS} = (X'X)^{-1} X'Y, \tag{2}$$

where, $X'X$ is the matrix of predictor variables. However, OLS assumes no multicollinearity among the predictors. When multicollinearity exists i.e., when predictor variables are highly correlated, the OLS estimators become unstable and unreliable. The condition number (CN) are used to assess multicollinearity in data. A large CN (greater than 30) often indicates high multicollinearity in data. The CN is ratio of the largest to the smallest eigenvalues $CN = \lambda_{\max}/\lambda_{\min}$.

A high condition number (> 30) indicate severe multicollinearity, OLS estimates unreliable and large error variances [1].

To address this hazardous behavior of the OLS es-

timators and to get reliable estimates, [2] introduced ridge regression, demonstrating its ability to stabilize estimators when predictor variables are highly correlated. The ridge estimator is given below:

$$\widehat{\beta}_{\text{Ridge}} = (X'X + kI)^{-1} X'Y, \quad k > 0, \quad (3)$$

where, k is the shrink parameter and I is the identity matrix. This shrinkage technique mitigates the effects of multicollinearity by penalizing large coefficient values. Improvements in selection of penalty parameter (k) through cross-validation were discussed in [3]. The selection of the penalty parameter (k) is a very crucial aspect in mitigating multicollinearity in data. Numerous researchers have sought to enhance ridge regression performance through modifications and to reduce the affect of severe multicollinearity. Over time, researchers have introduced various shrinkage parameters to address this issue. As a result, a substantial body of literature has emerged on basic ridge penalty estimators designed specifically to handle multicollinearity [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

The authors [20] introduced two-parameter ridge estimator, (k and q) for the coefficients of the regression model which is defined as:

$$\widehat{\beta}_{(q,k)} = q(X'X + kI)^{-1} X'Y, \quad (4)$$

where, q represents scaling factor defined in Eq. (5) below.

$$\widehat{q} = \frac{(X'y)' (X'X + kI)^{-1} X'y}{(X'y)' (X'X + kI)^{-1} X'X (X'X + kI)^{-1} X'y}. \quad (5)$$

This two-parameter ridge estimators approach enhances flexibility in model tuning, making it more suitable for high-dimensional datasets with severe levels of multicollinearity.

Eq. (4) represents a generalized form of Eq. (2) and (3) for $q = 1$ and $q = k$ or $k = 0$.

Several researchers have extended the basic ridge regression model to address multicollinearity more effectively. [21]-[22] introduced and modified two-parameter ridge estimators to encounter multicollinearity. Recently [23] modified few ridge estimators to handle severe multicollinearity in dataset.

It is evident from the old literature that the existing single-parameter and two-parameter ridge estimators do not effectively address the issue of severe multicollinearity. As a result, over time, variations in the ridge penalty have been introduced to mitigate the effects of multicollinearity. The choice of regularization parameter often depends on the specific characteristics of the data, such as the level of multicollinearity or the number of predictors. Moreover, the regularization effect of the basic ridge parameter does not always adequately capture the complexity of multicollinearity. To address this gap, this study introduces three newly improved two-parameter ridge estimators, namely, HITPR1, HITPR2,

and HITPR3 which are designed to tackle severe multicollinearity in datasets. These newly estimators adjust the ridge parameter based upon data properties, such as the multicollinearity structure, offering a more flexible and effective approach compared to existing ridge estimators and providing more reliable and interpretable results

The proposed estimators are optimized using the ridge parameters k and q , and their performances are assessed through a Monte Carlo simulation study, as a theoretical comparison of the new estimators is challenging due to the involvement of two parameters. Furthermore, two real-world datasets are analyzed to illustrate the practical application and effectiveness of the proposed estimators. The newly proposed estimators encounter the issue of multicollinearity which is a core problem in regression theory. The rest of the paper is organized as follows. Section 2 discusses the statistical methodology, including an overview of existing and proposed ridge estimators. Section 3 presents the simulation study, while Section 4 applies the estimators to real-world datasets. Section 5 concludes the research paper.

2. Material and Methods

The orthogonal form of Eq. (1) is given below:

$$y = Z\alpha + \epsilon, \quad (6)$$

Eq (6) can be written as $Z = XD$, with $\alpha = D^T \beta$, where $D^T D = I_p$, and I_p represents the identity matrix. D is an orthogonal matrix containing the eigenvectors of the $X^T X$ matrix. Furthermore, $\Lambda = D^T X^T X D$, where, Λ is a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_p$, arrange in ascending order.

The transformations form of Eq. (2),(4) & (5) expressed in the canonical forms as follows:

$$\widehat{\alpha} = \Lambda^{-1} Z^T y, \quad (7)$$

$$\widehat{\alpha}_k = (\Lambda + kI_p)^{-1} Z^T y, \quad (8)$$

$$\widehat{\alpha}_{(q,k)} = q(\Lambda + kI_p)^{-1} Z^T y. \quad (9)$$

Eq. (9) describes the two-parameter ridge estimator, which enhances ridge estimator by including an additional parameter (q).

2.1 Existing Basic Ridge Estimators

Few of the well-known existing two-parameter ridge estimators are discussed below.

- i. The initial ridge estimator was introduced by [2] denoted by (HK), and mathematically expressed as:

$$\widehat{k}_{HK} = \frac{\widehat{\sigma}^2}{\widehat{\alpha}_{max}^2}. \quad (10)$$

- ii. [24] modified the HK estimator, denoted by HKB and is defined as:

$$\widehat{k}_{HKB} = \frac{p\widehat{\sigma}^2}{\sum_{i=1}^p \widehat{\alpha}_i^2}. \quad (11)$$

iii. Kibria [12] proposed three new modified ridge estimators based upon averages, given as follows:

$$\hat{k}_{KAM} = \frac{1}{p} \sum_{i=1}^p \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}, \tag{12}$$

$$\hat{k}_{GM} = \frac{\hat{\sigma}^2}{\left(\prod_{i=1}^p \hat{\alpha}_i^2\right)^{\frac{1}{p}}}, \tag{13}$$

$$\hat{k}_{Med} = \text{Med} \left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} \right). \tag{14}$$

iv. [21] modified usual ridge estimator to mitigate the multicollinearity. The modified estimator is represented by KMS. Mathematically, KMS is defined as:

$$\hat{k}_{KMS} = \lambda_{\max} \frac{\sum_{i=1}^p |\hat{\alpha}_i|}{\left\{ \frac{\hat{\sigma}^2}{\hat{\alpha}_{\max}^2} \right\}}, \tag{15}$$

which combines the largest eigenvalue with parameter magnitudes.

2.1.1 Two-Parameter Ridge Estimator

Two-parameters ridge estimators approach adds flexibility with an additional parameter q , along with k to handle complex multicollinearity.

- i. [20] introduced the two-parameter ridge estimator (TPR) to counter the multicollinearity affect in data.
- ii. [25] developed two-parameter ridge estimator, denoted by TK, in which the optimal values for q and k are obtained using Eqs. (16) and (17), respectively.

$$\hat{q}_{opt} = \frac{\sum_{i=1}^p \hat{\alpha}_i^2 \frac{\lambda_i}{\lambda_i+k}}{\sum_{i=1}^p \frac{\hat{\sigma}^2 \lambda_i + \hat{\alpha}_i^2 \lambda_i^2}{(\lambda_i+k)^2}}, \tag{16}$$

$$\hat{k}_{opt} = \frac{\hat{q}_{opt} \sum_{i=1}^p \frac{\hat{\sigma}^2}{\lambda_i} + (\hat{q}_{opt} - 1) \sum_{i=1}^p \hat{\alpha}_i^2 \lambda_i^2}{\sum_{i=1}^p \hat{\alpha}_i^2 \lambda_i}. \tag{17}$$

iii. [26] modified two-parameter ridge estimators to minimize the affect of multicollinearity in data. These estimators are denoted as, YAM, YGM, and YHM. These estimators adjust the ridge regression approach by proposing alternative methods for calculating the regularization parameter k . The adjusted k -values for each estimator are specified as follows:

$$\hat{k}_{YAM}^* = \frac{\sum_{i=1}^p k_i^*}{p}, \tag{18}$$

$$\hat{k}_{YGM}^* = \left(\prod_{i=1}^p k_i^* \right)^{\frac{1}{p}}, \tag{19}$$

$$\hat{k}_{YHM}^* = \frac{p}{\sum_{i=1}^p \frac{1}{k_i^*}}. \tag{20}$$

In each case, the modified k-value for the i^{th} predictor is calculated as:

$$k_i^* = w_i \hat{k}_{opt},$$

where, w_i represents weight based on the ratio of the eigenvalue λ_i to the absolute value of the estimated coefficient α_i : $w_i = \frac{\lambda_i}{|\alpha_i|}$.

iv. [27] modified two-parameter ridge estimators to mitigate the multicollinearity in data, denoted by MLCR2. The k-value for MLCR2 is given as:

$$\hat{k}_{MLCR2} = \min(\lambda_i |\hat{\alpha}_i|) \tag{21}$$

To obtain the value of the second parameter (q) Eq. (5) is proposed to be used.

2.2 Proposed Ridge Estimators

The newly two-parameter estimators are introduced by combining different ridge parameters \hat{k}_i , (where $i = 1, 2, 3$) as defined in Eqs. (22) - (24). Using these \hat{k}_i values, the corresponding \hat{q} values are determined using Eq. (6). Based on these calculated \hat{k} and \hat{q} values, three new ITPR estimators-HITPR1, HITPR2, and HITPR3 are introduced for which the estimated k values are respectively given as below.

$$\hat{k}_1 = \left(\sqrt{\frac{1}{p} \sum_{i=1}^p (\lambda_i |\hat{\alpha}_i|)^3 + \frac{1}{p} \sum_{i=1}^p (\lambda_i |\hat{\alpha}_i|)^7} \right) \left(\frac{\hat{\sigma}^2}{\hat{\alpha}_{\max}^2} \right), \tag{22}$$

$$\hat{k}_2 = \left(\sum_{i=1}^p w_i \sum_{i=1}^p (\lambda_i |\hat{\alpha}_i|) \right) \left(\frac{\hat{\sigma}^2}{\hat{\alpha}_{\max}^2} \right), \tag{23}$$

$$\hat{k}_3 = \sqrt{\frac{\sum_{i=1}^p w_i \sum_{i=1}^p (\lambda_i (|\hat{\alpha}_i| + \frac{1}{|\hat{\alpha}_i|}))}{\sum_{i=1}^p (w_i \cdot |\hat{\alpha}_i| + \frac{1}{|\hat{\alpha}_i|})}}, \tag{24}$$

where, $w_i = \frac{1}{|\hat{\alpha}_i|+1}$.

3. Simulation Study

A comprehensive simulation study is conducted to compare the relative performance of the proposed ITPR estimators with existing ridge estimators and the OLS estimator.

3.1 Monte Carlo Simulation Technique

Simulation Methodology: The study employed a Monte Carlo simulation technique to generate predictor variables exhibiting different levels of collinearity, following the approach used by [28].

Expression for Predictor Variables: The expression for x_{ji} is defined in Eq. (25) as follows:

$$x_{ji} = \sqrt{1 - \rho^2} z_{ji} + \rho z_{jp+1}, i = 1, 2, \dots, p \text{ and } j = 1, 2, \dots, n. \tag{25}$$

In this context, ρ denotes the correlation among the predictors, p indicates the total number of predictors, and n signifies the sample size. The values for ρ are

established at 0.80, 0.90, and 0.99, with sample sizes 20, 50, and 100, with p 4 and 10. The random variables z_{ji} are produced from a standard normal distribution.

Response Variable: The response variable y_j is formulated in Eq. (19) as:

$$y_j = \alpha_0 + \alpha_1 x_{j1} + \alpha_2 x_{j2} + \dots + \alpha_p x_{jp} + \epsilon_j, \quad \text{where, } j = 1, 2, \dots, n. \quad (26)$$

Here, ϵ_j follows a normal distribution, specifically $\epsilon_j \sim N(0, \sigma^2 I)$, the error variance, σ^2 , was set to 0.40, 5, and 10, and the regression coefficients α_j were aligned with the optimal direction described in [29]

MSE Calculation: The estimated MSE is calculated over 1000 simulations using the following equation:

$$MSE(\hat{\alpha}) = \frac{1}{1000} \sum_{j=1}^{1000} (\hat{\alpha}_j - \alpha)' (\hat{\alpha}_j - \alpha), \quad (27)$$

where, the calculations are performed using R software.

3.2 Performance Metrics

The relative performances of the newly proposed are evaluated using the MSE criterion in different conditions, including varying levels of predictor collinearity and error variances following methodologies from previous studies [7, 30, 29]. The estimated MSE for an estimator $\hat{\alpha}$ of the parameter α is as follow.

$$MSE(\hat{\alpha}) = E \left[(\hat{\alpha}_j - \alpha)' (\hat{\alpha}_j - \alpha) \right]. \quad (28)$$

All simulation results are presented in Table 6-Table 11 in Appendix A, providing a detailed comparison between the newly proposed estimators and the existing ones. The bold values in the tables represent the lowest MSEs of the estimators.

3.3 Discussion of Simulation Results

The simulation study evaluated the performance of the newly proposed ridge estimators (HITPR1, HITPR2, and HITPR3) compared to OLS and existing ridge estimators under varying conditions of sample size, multicollinearity, error variance, and number of predictors. The results, summarized in Table 6 to Table 11, highlight several key findings.

Performance at Different Variance Levels ($\sigma^2 = 0.40, 5, \text{ and } 10$): In small variance setting ($\sigma^2 = 0.40$), the proposed estimators, HITPR1, HITPR2, and HITPR3, consistently perform well across the different correlation values ($\rho = 0.80, 0.90, \text{ and } 0.99$). Notably, HITPR1 demonstrated superior accuracy, particularly when compared to OLS and some of the existing estimators like KGM and YAM. For $\rho = 0.80$, HITPR1 has very small MSE than OLS, HK, and KGM, while performing similarly to HKB and KAM.

When the variance is increased ($\sigma^2 = 5$), the newly proposed estimators continue to exhibit higher performance, especially in moderate to high correlation scenarios (ρ

$= 0.90$ and $\rho = 0.99$). HITPR1 and HITPR3 are generally more robust than OLS, yielding smaller MSE. As expected, the MSE of all the estimators increases with increasing variance, but HITPR1 still offers a notable reduction in MSE in comparison to OLS, HK, and several other estimators across all the considered correlation levels. HITPR2 also showed promising results, second to HITPR1, particularly at higher correlation level.

For higher variance level ($\sigma^2 = 10$), efficiency of all the estimators decrease, which is consistent with the expectation that estimation accuracy declines as noise increases. However, HITPR1, HITPR2, and HITPR3 still outperform the existing estimators. For instance, at $\rho = 0.90$ and $\rho = 0.99$, HITPR1 exhibited much lower MSE values than OLS, HK, and KGM. Despite the increase in error, HITPR1's efficiency over most of the considered estimators is still evident, highlighting its robustness under high noise conditions.

Performance at Different Correlation Levels ($\rho = 0.80, 0.90, \text{ and } 0.99$): The correlation between variables (ρ) played a significant role in shaping the performance of the estimators. At $\rho = 0.80$, which indicates a moderate correlation, the newly proposed estimators consistently outperform OLS, especially, for small $\sigma^2(0.40)$. However, as the correlation increases to $\rho = 0.99$, the relative performance of HITPR1, HITPR2, and HITPR3 show even more significant improvement in comparison to other estimators. In particular, HITPR1 provides a significant reduction in MSE compared to most of the existing methods, particularly in high-variance conditions ($\sigma^2 = 5$ and $\sigma^2 = 10$).

The larger correlation value ($\rho = 0.99$) allowed for better identification of underlying patterns, where, the proposed estimators leveraged their higher performance in stronger multicollinearity. While OLS struggled in higher correlation, HITPR1, HITPR2, and HITPR3 continued to show higher efficiency, particularly at higher variance levels, where traditional estimators were prone to larger errors.

Comparisons with Existing Estimators: Among the existing estimators, KGM, KMed, and YAM demonstrated relatively higher performance but none outperformed the newly proposed HITPR estimators, especially when considering higher variance values. The MSE values for HITPR1, in particular, were consistently lower across various settings.

The HK and HKB showed relatively stable performance, but they were outperformed by the HITPR estimators in several cases, particularly when the correlation was higher. The KMS and LC possess similar results One of the notable improvements is observed in situations with higher correlation ($\rho = 0.99$) and higher variance ($\sigma^2 = 5$ and $\sigma^2 = 10$).

Different estimators performed better under various scenarios, influenced by factors such as sample sizes, error variances, and multicollinearity levels. To portray this fact, we have created a combined summary Table 1.

From the summary Table 1, it is clear that the newly proposed estimator HITPR1 performed well in most of

Table 1. Summary of Ridge Estimator Performance across Different Scenarios

n	$\sigma^2 \downarrow \rho$	$p = 4$			$p = 10$			
		\rightarrow	0.80	0.90	0.99	0.80	0.90	0.99
20	0.4		YHM	HITPR1	HITPR1	YHM	HITPR1	HITPR1
	5		HITPR1	HITPR1	HITPR1	HITPR1	HITPR1	HITPR1
	10		HITPR1	HITPR1	HITPR1	HITPR1	HITPR1	HITPR1
50	0.4		HITPR1	HITPR1	HITPR1	YHM	YHM	HITPR1
	5		HITPR1	HITPR1	HITPR1	HITPR1	HITPR1	HITPR1
	10		HITPR1	HITPR1	HITPR1	HITPR1	HITPR1	HITPR1
100	0.4		HITPR1	HITPR1	HITPR1	HITPR1	HITPR1	HITPR1
	5		HITPR1	HITPR1	HITPR1	HITPR1	HITPR1	HITPR1
	10		HITPR1	HITPR1	HITPR1	HITPR1	HITPR1	HITPR1

the scenarios.

4. Real life Applications

This section illustrates the use of the proposed estimators and methodology with two real-world datasets. The first data is sourced from [31]). The second dataset is the Hospital manpower dataset from [17]. These datasets share structural similarities with those analyzed in the simulation study presented in Section 3.1, providing a basis for practical validation of the methodology.

4.1 Income Dataset

The dataset provided contains 35 observations with 9 variables. The dependent variable is Hours (y), which represents the target outcome, while the independent variables include Rate (X_1), ERSP (X_2), ERNO (X_3), NEIN (X_4), Assets (X_5), Age (X_6), DEP (X_7), and School (X_8). The multiple linear regression with eight independent variables takes the following form:

$$y = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5 + \alpha_6 X_6 + \alpha_7 X_7 + \alpha_8 X_8 + \epsilon$$

To assess multicollinearity, we use three indicators i.e. Variance Inflation Factor (VIF), Condition Number (CN), and correlation heatmap.

The VIF values show how much the variance of a regression coefficient is inflated due to multicollinearity with other independent variables. A high VIF (> 10) indicates severe multicollinearity.

Table 2 shows that variables X_4 and X_5 have very high VIFs (180.507 and 192.565), and X_1 and X_8 have VIFs greater than 10 (17.082 and 25.402, respectively). This indicates severe multicollinearity, which can negatively affect the reliability of the regression model.

The CN of the variables are used to assess multicollinearity. A large CN (above 30) often indicates high multicollinearity. Here, $CN = \frac{\lambda_{\max}}{\lambda_{\min}} \cong 80.19$ which is greater than 30 indicating the presence of severe multicollinearity among the variables which could cause instability and unreliable regression results. A correlation heatmap is a visual representation of the correlation matrix between the independent variables. High correla-

tions between variables (near +1 or -1) indicate higher multicollinearity.

In Figure 1, the heatmap visual shows that correlation between X_4 and X_5 is 0.99, and X_1 and X_8 is 0.88, indicating severe multicollinearity in the real dataset. All three indicators VIFs, CN, and the heatmap display show severe multicollinearity in the dataset. Therefore, to address these issues, we apply newly proposed and existing estimators to assess the relative performance.

The results in Table 3 cement the findings of the simulation findings.

4.2 Patients Dataset

The dataset contains 17 observations on 6 variables. The variables are: Y(Hours) (dependent variable), X_1 (Xray), X_2 (BedDays), X_3 (Length), X_4 (Load), and X_5 (Pop). These variables represent healthcare-related metrics for each observation. Linear regression is used to model the relationship between the dependent and independent variables.

$$y = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5 + \epsilon$$

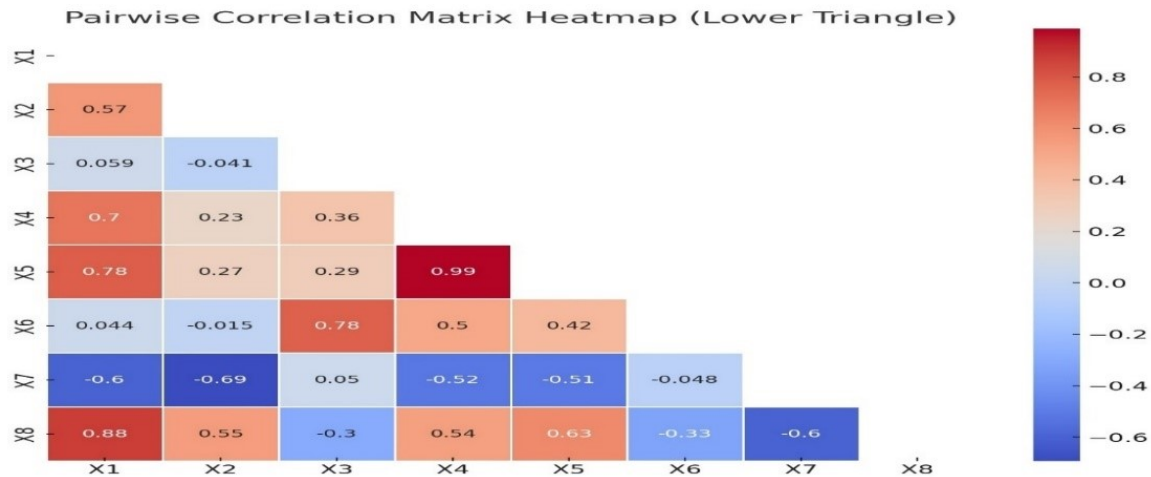
The CN value for this data is 58.31, which is greater than 10, indicating higher multicollinearity in this dataset. Figure 2 shows a heatmap illustrating the correlations among variables in the hospital manpower data, highlighting strong relationships, particularly between X-ray, AreaPop, and Stay, further confirming the presence of higher multicollinearity.

To address these challenges, we applied our newly proposed and existing ridge estimators to evaluate their effectiveness in mitigating multicollinearity, compared their performance based on the estimated MSE. Table 4 presents the estimated MSE for each estimator applied to the data. The results clearly indicate that the proposed HITPR1 estimator achieved lower MSE compared to OLS and other estimators.

The results showcased in Table 4 further strengthen the applicability and efficiency of the proposed estimators.

Table 2. VIF of Income Dataset

Variable	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
VIF	17.082	3.498	3.1379	180.507	192.565	9.72114	4.5250	25.402

**Figure 1.** Heatmap Display for Income Dataset**Table 3.** MSE and Coefficient of the Estimators

Estimators	MSE	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\alpha}_6$	$\hat{\alpha}_7$	$\hat{\alpha}_8$
OLS	2.043205	-0.3089	-0.5405	-0.2829	-0.3748	-0.3546	-0.3978	-0.022	-0.0041
HK	1.165115	-0.1112	-0.2867	-0.4955	0.0521	-0.0714	-0.1387	-0.021	0.0002
HKB	1.347428	-0.5594	-0.3033	-0.1978	-0.3539	-0.1441	-0.0965	-0.0043	-0.3342
KAM	1.991284	-0.3084	-0.565	0.0171	-0.1165	-0.037	-0.1036	0.0003	-0.1169
KGM	1.216633	-0.347	-0.2613	-0.3047	-0.4548	-0.0219	-0.0226	-0.3093	-0.5375
KMed	1.177232	-0.7486	0.0262	-0.1085	-0.1753	-0.021	0.0014	-0.1113	-0.2542
KMS	1.556477	-0.5948	-0.3088	-0.5262	-0.1311	-0.0042	-0.3379	-0.5598	-0.2309
LC	0.709316	0.3323	-0.1111	-0.2714	-0.1497	0.0003	-0.1173	-0.3083	-0.3298
TK	0.708807	-0.3045	-0.5591	-0.2755	-0.0342	-0.3598	-0.5263	-0.3462	-0.0944
YAM	0.444678	-0.1083	-0.3081	-0.4721	0.0022	-0.0812	-0.2403	-0.743	0.0066
YGM	0.452991	-0.5248	-0.3462	-0.1802	-0.354	-0.1805	-0.2098	-0.575	-0.3546
YHM	0.618662	-0.27	-0.7448	0.015	-0.1165	-0.048	-0.2841	0.2498	-0.1164
MLCR2	0.477919	-0.8019	-0.2917	-0.2301	-0.3291	-0.1607	-0.1091	-0.203	-0.2840
HITPR1	0.444001	0.0144	-0.1088	-0.2981	-0.1494	0.0003	-0.0715	-0.0356	-0.1456
HITPR2	0.981646	-0.3065	-0.5302	-0.3254	-0.0342	-0.3614	-0.1443	-0.0211	-0.0332
HITPR3	0.701924	-0.1096	-0.2756	-0.6512	0.0022	-0.1134	-0.037	-0.0202	0.0021

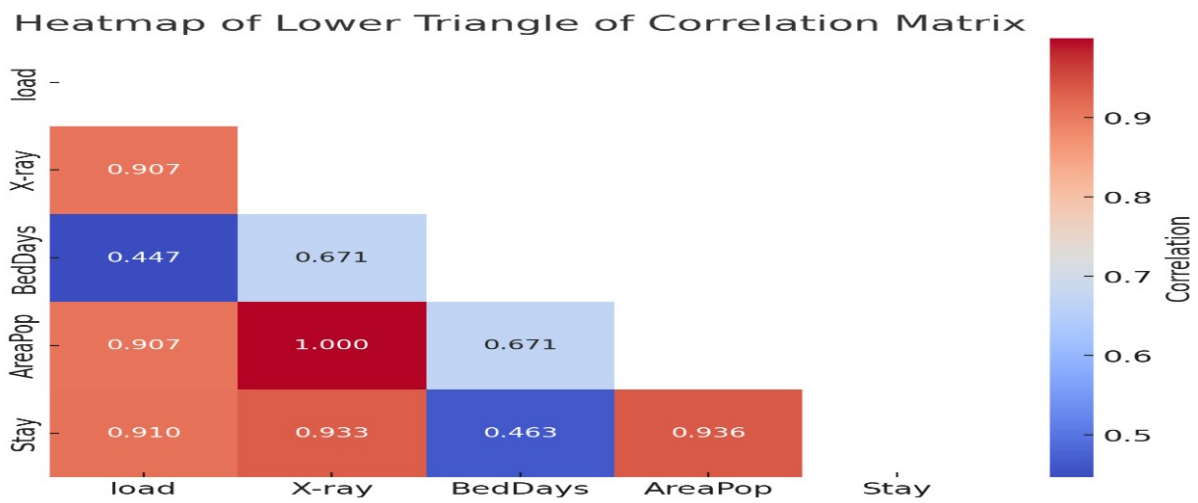


Figure 2. Heatmap of Patients Dataset

Table 4. MSE and Regression coefficients

Methods	MSE	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$
OLS	18.7264	-0.37755	-0.37755	-0.37755	-0.37776	-0.37846	-0.37798
HK	0.166682	-0.0007	-0.0007	-0.0007	-9.2E-05	-0.00069	-0.00017
HKB	0.471371	-0.44371	-0.44356	-0.44333	-0.00816	-0.42951	-0.01735
KAM	7.698355	-0.33035	-0.33015	-0.32984	-0.00331	-0.31071	-0.00711
KGM	0.328807	-0.76722	-0.76307	-0.75682	-0.00086	-0.48239	-0.00186
KMed	0.319938	-10.4674	-6.58821	-4.20717	-0.00011	-0.16042	-0.00024
KMS	3.41267	-0.00045	-0.35627	-0.37776	-0.37776	-0.37884	-0.37814
LC	0.316382	-0.00008	-0.00048	-9.2E-05	-9.2E-05	-0.00038	-0.0007
TK	0.316382	-0.00098	-0.10445	-0.00816	-0.00816	-0.05743	-0.43767
YAM	0.316382	-0.00007	-0.04747	-0.00331	-0.00331	-0.02456	-0.32171
YGM	0.316382	-0.00004	-0.01411	-0.00086	-0.00086	-0.00676	-0.61132
YHM	0.316411	-0.00010	-0.00181	-0.00011	-0.00011	-0.00086	-0.36337
MLCR2	0.63986	-0.43848	-0.19056	-0.00816	-0.01012	-0.44208	-0.0914
HTPR1	0.316053	-0.32328	-0.0961	-0.00331	-0.00411	-0.32778	-0.0406
HTPR2	0.450009	-0.6416	-0.03361	-0.00086	-0.00107	-0.71018	-0.01171
HTPR3	0.318837	-0.47157	-0.00443	-0.00011	-0.00013	-1.06444	-0.0015

4.3 Comparisons of the Estimator Based on Confidence Interval

To assess the performance of estimators on a real dataset, we analyze their 99% confidence intervals (CIs). Narrower CIs indicate higher precision, while wider CIs suggest more uncertainty. This helps identify the most reliable estimator.

The 99% CI for each estimator is calculated as:

$$CI = \text{Estimated Coefficient} \pm t_{\frac{\alpha}{2}(v)} \text{Standard Error}$$

$$v = n - 1$$

Where the standard error (SE) is calculated as:

$$S.E = \sqrt{\frac{MSE}{n}}$$

Here, $n=17$ is the sample size. Smaller MSE values lead to smaller standard errors, resulting in narrower, more precise confidence intervals.

Table 5 shows that the proposed estimators (HTPR1-3) provide narrower and more precise confidence intervals compared to OLS and other ridge estimators, indicating superior performance in terms of estimation accuracy.

5. Conclusion

This study is devoted to propose methodologies to encounter multicollinearity in data sets. Extensive simulation study under different conditions has been carried out. The numerical results in this study demonstrate that the newly proposed HITPR estimators, particularly HITPR1, consistently outperform OLS and existing ridge estimators. Through both simulation and real-world applications, the newly proposed estimators exhibit greater robustness and achieve lower MSEs across a range of scenarios in presence of multicollinearity. Moreover, the newly proposed estimators provide more reliable regression coefficient estimates even in the presence of highly correlated predictors, ultimately enhancing decision-making in fields such as economics and healthcare. The superior performance of HITPR1, in particular, underscores its potential as an effective solution for regression problems challenged by high multicollinearity.

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Authors contributions

All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials

The authors have no conflicts of interest to declare.

Conflict of interests

The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Table 5. Confidence Intervals of the Estimator for the patients' Dataset

Methods	$C.I(\hat{\alpha}_0)$	$C.I(\hat{\alpha}_1)$	$C.I(\hat{\alpha}_2)$	$C.I(\hat{\alpha}_3)$	$C.I(\hat{\alpha}_4)$	$C.I(\hat{\alpha}_5)$
OLS	(-3.4431, 2.688)	(-3.431, 2.682)	(-3.44, 2.6890)	(-3.4433, 2.6877)	(-3.4440, 2.6870)	(-3.4435, 2.6875)
HK	(-0.2899, 0.288)	(-0.289, 0.285)	(-0.289, 0.2885)	(-0.2893, 0.2891)	(-0.2899, 0.2885)	(-0.2894, 0.2890)
HKB	(-0.9301, 0.042)	(-0.929, 0.0428)	(-0.997, 0.0430)	(-0.4945, 0.4782)	(-0.9159, 0.0568)	(-0.5037, 0.4690)
KAM	(-2.2959, 1.635)	(-2.297, 1.654)	(-2.29, 1.6357)	(-1.9688, 1.9622)	(-2.2762, 1.6548)	(-1.9726, 1.9584)
KGM	(-1.1734, -0.361)	(-1.1693, -.369)	(-1.160, -0.356)	(-0.4071, 0.4053)	(-0.8886, -0.0762)	(-0.4081, 0.4043)
KMed	(-10.861, -10.16)	(-6.98, -6.85)	(-4.679, -3.806)	(-0.4008, 0.4006)	(-0.5611, 0.2403)	(-0.4009, 0.4004)
KMS	(-1.3087, 1.3086)	(-1.6649, 0.954)	(-1.6864, 0.9309)	(-1.6864, 0.9309)	(-1.6875, 0.9298)	(-1.6868, 0.9305)
LC	(-0.3985, 0.3984)	(-0.3989, 0.398)	(-0.3985, 0.3984)	(-0.3985, 0.3984)	(-0.3988, 0.3981)	(-0.3992, 0.3978)
TK	(-0.3986, 0.3984)	(-0.5029, 0.294)	(-0.4066, 0.3903)	(-0.4066, 0.3903)	(-0.4559, 0.3410)	(-0.8361, -0.0392)
YAM	(-0.3985, 0.3984)	(-0.449, 0.351)	(-0.4018, 0.3951)	(-0.4018, 0.3951)	(-0.4230, 0.3739)	(-0.7202, 0.0767)
YGM	(-0.3985, 0.3984)	(-0.426, 0.383)	(-0.3993, 0.3976)	(-0.3993, 0.3976)	(-0.4052, 0.3917)	(-1.0098, -0.2129)
YHM	(-0.3986, 0.3984)	(-0.4013, 0.397)	(-0.3986, 0.3984)	(-0.3986, 0.3984)	(-0.3993, 0.3976)	(-0.7618, 0.0351)
MLCR2	(-1.0051, 0.1282)	(-0.752, 0.376)	(-0.5748, 0.5585)	(-0.5768, 0.5565)	(-1.0087, 0.1246)	(-0.6581, 0.4753)
HTPR1	(-0.7215, 0.0750)	(-0.493, 0.301)	(-0.4016, 0.3949)	(-0.4024, 0.3941)	(-0.7260, 0.0705)	(-0.4388, 0.3576)
HTPR2	(-1.1168, -0.1664)	(-0.508, 0.441)	(-0.4763, 0.4741)	(-0.4763, 0.4741)	(-1.1854, -0.2350)	(-0.4869, 0.4635)
HTPR3	(-0.8716, -0.0716)	(-0.404, 0.395)	(-0.4001, 0.3999)	(-0.4001, 0.3999)	(-1.4644, -0.6644)	(-0.4015, 0.3985)

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Table 6. MSE of Ridge Estimators for $n = 20$ and $p = 4$

σ^2	ρ	OLS	HK	HKB	KAM	KGM	KMed	KMS	LC	TK	YAM	YGM	YHM	MLCR2	HITPR1	HITPR2	HITPR3
0.40	0.80	0.10049	0.09272	0.07113	0.09874	0.06671	0.11149	0.08667	0.00954	1.15004	0.00329	0.00423	0.00308	2.76868	0.00351	0.07272	0.02560
	0.90	0.21976	0.18409	0.12117	0.21233	0.07242	0.08129	0.16077	0.01495	0.56554	0.00263	0.00333	0.01142	0.269099	0.00280	0.11061	0.02390
	0.99	2.41963	0.98791	0.66283	2.14835	0.42150	0.49213	1.2989	0.11074	0.20059	0.00408	0.01915	1.07484	0.00698	0.00201	0.44933	0.00634
5	0.80	15.5605	5.29746	4.38437	13.4584	2.07673	2.04489	11.2588	1.82057	6.60616	1.47125	2.29627	4.41419	11.0200	0.89701	1.00160	5.79230
	0.90	33.42761	10.7134	8.71386	28.5900	3.122084	3.33497	25.1304	1.610488	7.877929	1.32020	3.352395	8.72111	64.4492	0.907965	1.37007	6.59921
	0.99	405.697	129.751	95.4656	343.859	16.08012	25.4656	356.617	0.927373	8.587046	0.91098	31.8997	195.8962	7.24483	0.76893	2.747113	2.88459
10	0.80	65.5912	21.6809	16.5257	56.8053	7.130941	7.84954	54.2610	11.29015	27.74272	10.4893	16.1272	25.70489	48.1950	8.024162	8.208648	32.9016
	0.90	135.164	45.3811	36.5670	115.489	10.76597	11.3372	113.978	8.450041	32.25605	8.79180	22.61472	52.60088	60.3456	6.34471	6.943788	36.7366
	0.99	1588.45	514.526	377.967	1348.25	52.36574	108.047	1475.27	3.293638	24.52788	3.43057	182.6788	906.0819	243.093	2.973881	11.35973	34.3484

Table 7. MSE of Ridge Estimators for $n = 50$ and $p = 4$

σ^2	ρ	OLS	HK	HKB	KAM	KGM	KMed	KMS	LC	TK	YAM	YGM	YHM	MLCR2	HITPR1	HITPR2	HITPR3
0.40	0.80	0.05759	0.14742	0.04479	0.05690	0.03685	0.07913	0.05172	0.00525	0.04082	0.00727	0.0028	0.01473	0.19997	0.00112	0.03550	0.01601
	0.90	0.12071	0.10873	0.07554	0.11818	0.04312	0.05883	0.09664	0.00632	0.2089	0.00094	0.0012	0.00097	0.43345	0.00120	0.04533	0.01648
	0.99	1.20262	0.62163	0.36094	1.09224	0.23293	0.26250	0.61210	0.06654	0.09563	0.00098	0.0039	0.17529	0.01678	0.00094	0.07311	0.00480
5	0.80	9.60650	3.18910	2.5988	8.40690	1.3864	1.33010	6.52010	0.6570	3.6102	0.25962	0.40734	0.98633	6.5828	0.18083	0.18428	2.70242
	0.90	18.4659	5.68180	4.30888	15.8515	1.93089	1.73192	12.9563	0.54875	3.79258	0.36569	0.59673	2.69282	31.7643	0.15270	0.15492	2.22370
	0.99	185.520	58.3201	41.2022	156.193	9.84124	12.7280	155.010	0.24484	4.22993	0.13518	6.12902	56.5754	4.06687	0.11602	0.11853	0.59354
10	0.80	36.3839	13.0159	9.58830	31.3013	3.5652	3.59110	28.0799	2.63226	12.5939	1.99844	3.40133	6.95682	17.1794	1.55871	1.61150	14.8629
	0.90	72.6071	24.4012	16.3873	61.6741	5.34459	6.27030	57.4833	1.60554	12.8435	1.55134	3.48313	13.0702	98.6252	0.97701	1.01681	16.3222
	0.99	719.390	246.336	157.262	604.651	22.3382	42.7533	645.637	0.82049	11.8824	0.73692	24.4842	262.4326	50.7518	0.69884	1.91240	12.5213

Table 8. MSE of Ridge Estimators for $n = 100$ and $p = 4$

σ^2	ρ	OLS	HK	HKB	KAM	KGM	KMed	KMS	LC	TK	YAM	YGM	YHM	MLCR2	HITPR1	HITPR2	HITPR3
0.40	0.80	0.02642	0.02594	0.02420	0.02637	0.02618	0.06099	0.02518	0.00233	0.11471	0.00064	0.00069	0.00060	0.18319	0.00062	0.01724	0.01063
	0.90	0.0534	0.05120	0.04410	0.05290	0.0248	0.03820	0.04810	0.0036	1.3406	0.00043	0.00049	0.00047	0.08768	0.00040	0.02333	0.01269
	0.99	0.49018	0.33557	0.16673	0.46022	0.11490	0.12409	0.27219	0.02340	0.00461	0.00050	0.00212	0.0049	0.02747	0.00046	0.01732	0.00485
5	0.80	3.78112	1.52650	1.01664	3.3529	0.71516	0.79420	2.28642	0.34286	2.59446	0.12675	0.15262	0.21321	10.3350	0.09346	0.09385	1.40318
	0.90	8.06616	2.66176	2.13650	6.99605	1.07963	1.02822	5.05665	0.37983	2.15592	0.07494	0.14021	0.42226	11.1366	0.06384	0.06418	1.38279
	0.99	84.79698	26.0345	19.3826	72.3269	5.68593	6.33599	67.5721	0.1981	1.43505	0.06602	1.03104	9.34856	4.85521	0.06561	0.06603	0.49043
10	0.80	15.48501	5.40319	3.62722	13.3988	1.9663	1.97146	10.9882	1.25965	6.30600	0.66219	1.0069	1.78626	27.0018	0.59642	0.59925	7.56544
	0.90	33.00224	10.6450	8.12184	28.4320	3.41819	3.41796	24.6301	1.1945	6.08562	0.65056	0.97705	2.34637	40.7855	0.6328	0.63769	9.20181
	0.99	325.357	109.348	72.2989	274.068	13.4623	22.6345	280.368	0.9488	11.0575	0.87683	11.1217	71.6128	35.0896	0.84871	1.35073	6.14443

Table 9. MSE of Ridge Estimators for $n = 20$ and $p = 10$

σ^2	ρ	OLS	HK	HK	HKB	KAM	KGM	KMed	KMS	LC	TK	YAM	YGM	YHM	MLCR2	HITPR1	HITPR2	HITPR3
0.40	0.80	0.40909	0.3397	0.1868	0.40381	0.06351	0.06553	0.2838	0.00426	0.79495	0.00217	0.00249	0.00262	0.07641	0.00225	0.1656	0.0363	
	0.90	0.88903	0.60711	0.25971	0.8681	0.07216	0.08012	0.4635	0.00344	0.19697	0.00133	0.00156	0.03924	0.04147	0.00138	0.20347	0.0383	
	0.99	9.0141	3.88281	1.77362	8.59661	0.5274	0.66842	4.91752	0.02182	0.21771	0.00104	0.00903	2.62582	0.00628	0.00093	1.22716	0.01395	
	5	63.832	27.93877	12.83883	61.29795	3.79541	4.58931	52.1895	1.36263	17.6879	2.10027	5.4273	15.69016	18.4516	0.78973	0.99594	9.55706	
10	0.80	129.670	47.8006	24.3334	123.695	5.83233	6.51739	105.417	0.84556	15.5676	0.80105	6.53227	34.0943	11.8418	0.4190	0.74830	8.12561	
	0.90	1391.17	589.057	237.859	1319.38	48.3714	65.2504	1251.47	0.46450	14.9928	0.49033	108.939	839.064	1.30326	0.13795	1.82261	1.76591	
	0.80	265.989	120.450	46.0518	254.944	12.1539	15.5448	236.012	11.1489	76.7165	15.4894	38.5643	98.8187	88.5484	8.99379	9.39323	53.9791	
	0.90	528.156	211.152	103.264	503.859	21.0123	28.4623	469.698	7.46399	82.9815	13.8949	65.3506	199.910	77.9358	5.87883	7.03675	61.5943	
0.99	6135.23	2411.21	1101.22	5821.69	150.459	257.337	5799.45	5.41541	106.499	12.9822	802.670	3810.02	58.7446	5.15269	21.8849	41.9887		

Table 10. MSE of Ridge Estimators for $n = 50$ and $p = 10$

σ^2	ρ	OLS	HK	HK	HKB	KAM	KGM	KMed	KMS	LC	TK	YAM	YGM	YHM	MLCR2	HITPR1	HITPR2	HITPR3
0.40	0.80	0.18544	0.16984	0.10021	0.18409	0.03014	0.0345	0.14196	0.00129	0.2398	0.00068	0.00068	0.0007	0.00063	0.08164	0.00068	0.06065	0.02674
	0.90	0.40986	0.33579	0.16232	0.40415	0.03905	0.04251	0.24673	0.00134	0.03121	0.00042	0.00044	0.00041	0.04897	0.00043	0.06658	0.03058	
	0.99	4.12103	1.93992	0.78401	3.96883	0.28216	0.38438	2.03589	0.00843	0.13449	0.00124	0.00828	0.22487	0.01003	0.00035	0.14984	0.01026	
	5	29.1694	11.976	5.43104	28.0099	1.70419	2.26323	21.0721	0.19821	3.43472	0.15594	0.39087	1.13585	14.0493	0.0933	0.09609	4.16455	
10	0.80	61.9006	25.5525	11.6001	59.2484	3.46854	4.16329	46.3442	0.25358	3.7056	0.16197	0.95290	3.47138	10.3090	0.07165	0.07459	3.72757	
	0.90	591.852	224.055	105.921	563.139	24.9635	38.0943	509.753	0.22311	2.68065	0.05847	7.16754	71.9677	1.77498	0.0482	0.05437	0.99733	
	0.80	120.532	55.1927	22.9919	115.759	6.4397	8.17232	99.4101	1.38221	32.10622	2.51021	7.93556	22.554	61.3405	0.91268	0.93675	26.3589	
	0.90	251.605	105.565	43.8153	240.698	10.1762	13.8138	212.33	0.94302	34.37411	1.78662	13.16942	50.1457	57.5800	0.63102	0.65881	25.6772	
0.99	2397.10	944.385	417.062	2281.15	83.3718	140.579	2208.53	0.32931	12.1677	0.32519	153.632	892.509	20.8980	0.21041	0.21685	6.25129		

Table 11. MSE of Ridge Estimators for $n = 100$ and $p = 10$

σ^2	ρ	OLS	HK	HK	HKB	KAM	KGM	KMed	KMS	LC	TK	YAM	YGM	YHM	MLCR2	HITPR1	HITPR2	HITPR3
0.40	0.80	0.05716	0.05596	0.04555	0.05704	0.02029	0.02402	0.05238	0.00058	2.5226	0.00032	0.00032	0.00033	0.00031	0.04636	0.00032	0.02779	0.01761
	0.90	0.11343	0.10829	0.07185	0.11297	0.01839	0.01925	0.09413	0.00053	0.00904	0.00022	0.00023	0.00022	0.00022	0.04069	0.00022	0.03088	0.02260
	0.99	1.20024	0.82610	0.33194	1.17545	0.12071	0.14863	0.60681	0.00492	0.0013	0.00016	0.00018	0.00084	0.01667	0.00016	0.01981	0.01254	
	5	10.0017	5.03413	2.24400	9.73587	0.96134	1.33468	6.76576	0.11439	1.66964	0.05461	0.07328	0.09053	8.12751	0.05293	0.05341	2.94630	
10	0.80	19.1735	8.48966	3.80884	18.5512	1.45538	1.99112	12.93804	0.1354	0.84472	0.03664	0.04817	0.09522	7.43769	0.03645	0.0368	2.84342	
	0.90	187.569	81.1612	36.8138	180.803	10.73433	15.24958	153.9703	0.20077	0.19221	0.02497	0.09637	1.44282	2.84209	0.02500	0.02548	0.90147	
	0.80	37.9636	17.2787	7.81329	36.8421	2.99282	3.77823	29.4788	0.56972	12.5730	0.39118	0.89643	1.79069	34.0860	0.22778	0.22856	13.9463	
	0.90	74.3219	32.1298	13.7258	71.8019	4.5151	6.01729	58.20807	0.41654	10.98332	0.27324	1.18373	3.87753	37.2692	0.19686	0.19942	15.1666	
0.99	766.592	336.667	140.098	738.459	31.5273	51.6629	687.298	0.22051	13.2347	0.10472	17.6295	75.9573	15.9171	0.09897	0.09933	4.81604		