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Two-dimensional wavelets for numerical solution of integral equations

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Abstract

Purpose: In this paper, we shall investigate the numerical solution of two-dimensional Fredholm integral equations (2D-FIEs).

Methods: In this work, we apply two-dimensional Haar wavelets, to solve linear two dimensional Fredholm integral equations (2D-FIEs). Using 2D Haar wavelets and their properties, 2D-FIEs of the second kind reduce to a system of algebraic equations.

Results: The numerical examples illustrate the efficiency and accuracy of the method.

Conclusions: In comparison with other bases (for example, polynomial bases), one of the advantages of this method is, although the involved matrices have a large dimension, they contain a large percentage of zero entries, which keeps computational effort within reasonable limits.

Keywords: Two-dimensional Fredholm integral equations; Two-dimensional Haar wavelets; Linear systems

Background

The integral equations provide an important tool for modeling a numerous phenomena and processes, and for solving boundary value problems for both ordinary and partial differential equations. Their historical development is closely related to the solution of boundary value problems in potential theory. In the last decades, there has been much interest in numerical solutions of integral equations. The Nystrom and collocation methods are probably the two most important approaches for the numerical solution of these integral equations [1,2]. While several numerical methods are known for one-dimensional integral equations, fewer methods are known for two-dimensional integral equations [3-6].

Recently, many different basic functions have been used to estimate the solution of integral equations, such as orthogonal functions and wavelets. Haar wavelets are the simplest orthogonal wavelet with compact support, and they have been used in different numerical approximation problems.

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In this work, we apply two-dimensional Haar wavelets, constructed on $D=[0,1)\times[0,1)$, to solve linear two-dimensional Fredholm integral equations (2D-FIEs) of the form:

$$u(x,y)=f(x,y)+\int_0^1\int_0^1 K(x,y,s,t)\times u(s,t)\,dsdt,\quad (x,y)\in D \tag{1}$$

where $u(x,y)$ is an unknown function to be found and the functions $f(x,y)$ and $K(x,y,s,t)$ are given continuous functions defined on D and D^2, respectively. The existence and uniqueness results for Equation 1 can be found in the classical theory of Fredholm integral equations.

Results and discussion

Two-dimensional Haar wavelets

We usually call the Haar wavelets containing one variable as one-dimensional, and those containing two variables as two-dimensional. One-dimensional Haar wavelets have been widely used for solving different problems [6-8]. Complete details for one-dimensional Haar wavelets is found in [9,10]. These discussions can also be extended to the two-dimensional one.

Definitions and properties

Definition 2.1. *The orthogonal basis* $\{h_n(t)\}$ *of one-dimensional Haar wavelets for the Hilbert space* $L^2[0,1)$ *consists of*

$$h_n(t) = 2^{\frac{j}{2}} H(2^j t - k)\big|_{[0,1]}, \quad n = 1, 2, \ldots, \quad n = 2^j + k, j \geq 0, 0 \leq k \leq 2^j - 1, \tag{2}$$

where

$$h_0(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & elsewhere, \end{cases} , \quad H(t) = \begin{cases} +1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1, \\ 0, & elsewhere. \end{cases} \tag{3}$$

The integer 2^j *indicates the level of the wavelet and* k *is the translation parameter.*

Simple calculations show that

$$\int_0^1 h_m(x) h_n(x) dx = \begin{cases} 1, & m = n; \\ 0, & m \neq n. \end{cases} \tag{4}$$

Also, it can be shown that any function $f(x) \in C[0,1]$ can be expressed as $\sum_n \langle f, h_n \rangle h_n$, where $\langle f, h_n \rangle = \int_0^1 f(x) h_n(x) dx$ [11].

Definition 2.2. *Let* $\{h_n(x)\}_{n=0}^{\infty}$ *be the one-dimensional Haar wavelets on* $[0,1)$*. We call* $\{h_{m,n}(x,y)\}_{m,n=0}^{\infty}$ *the two-dimensional Haar wavelets on* $[0,1)\times[0,1)$ *as:*

$$h_{m,n}(x,y) = 2^{\frac{i+j}{2}} H(2^i x - k_1)\big|_{[0,1]} H(2^j y - k_2)\big|_{[0,1]}, \tag{5}$$

where $m = 2^i + k_1, n = 2^j + k_2$*, with* $i, j \geq 0$ *and* $k_1 = 0, 1, \ldots, 2^i - 1,\ k_2 = 0, 1, \ldots, 2^j - 1$.

The family $\{h_{m,n}(x,y)\}_{m,n=0}^{\infty}$ is orthogonal on $[0,1)\times[0,1)$ and forms a basis for $L^2[0,1)^2$:

Theorem 2.3. *The basis* $\{h_{m,n}(x,y)\}_{m,n=0}^{\infty}$ *is orthonormal on* $[0,1)\times[0,1)$.

Proof. Let $m \neq l$ or $n \neq q$

$$\int_0^1 \int_0^1 h_{m,n}(x,y) h_{l,q}(x,y) dx dy = \int_0^1 h_m(x) h_l(x) dx \int_0^1 h_n(y) h_q(y) dy = 0.$$

□

Theorem 2.4.

$$\int_0^1 \int_0^1 [h_{m,n}(x,y)]^2 \, dx dy = 1.$$

Proof.

$$\int_0^1 \int_0^1 [h_{m,n}(x,y)]^2 \, dx dy = \int_0^1 h_m^2(x) dx \int_0^1 h_n^2(y) dy = 1.$$

□

The expansion of a function

A function $f(x,y)$ defined over $[0,1)\times[0,1)$ may be expanded by the two-dimensional Haar wavelets as

$$f(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{m,n} h_{m,n}(x,y), \tag{6}$$

where the wavelet coefficients, $f_{m,n}$, are obtained as

$$f_{m,n} = \langle h_m(x), \langle f(x,y), h_n(y) \rangle \rangle. \tag{7}$$

If the infinite series in Equation 6 is truncated up to their k terms, then it can be written as

$$f(x,y) \simeq \sum_{m=0}^{k-1} \sum_{n=0}^{k-1} f_{m,n} h_{m,n}(x,y) = \mathbf{F}^{\mathrm{T}} \mathbf{H}(x,y) = \mathbf{H}^{\mathrm{T}}(x,y) \mathbf{F}, \tag{8}$$

where $k = 2^{\alpha+1}$, and α is a nonnegative integer. Here, $\mathbf{F}$ and $\mathbf{H}(x,y)$ are the Haar wavelet coefficients and Haar wavelet functions vectors, respectively, and defined as:

$$\mathbf{F} = \left[f_{0,0}, f_{0,1}, \ldots, f_{0,(k-1)}, \ldots, f_{(k-1),0}, f_{(k-1),1} \ldots, f_{(k-1),(k-1)}\right]^{\mathrm{T}}, \tag{9}$$

$$\mathbf{H}(x,y) = \left[h_{0,0}, h_{0,1}, \ldots, h_{0,(k-1)}, \ldots, h_{(k-1),0}, h_{(k-1),1} \ldots, h_{(k-1),(k-1)}\right]^{\mathrm{T}}(x,y) \tag{10}$$

Similarly, a function of four variables, $k(x,y,s,t)$, on $([0,1)\times[0,1)\times[0,1)\times[0,1))$ may be approximated with respect to Haar wavelets such as:

$$k(x,y,s,t) \simeq \mathbf{H}^{\mathrm{T}}(x,y) \mathbf{K} \mathbf{H}(s,t) \tag{11}$$

where $\mathbf{H}(x,y)$ and $\mathbf{H}(s,t)$ are two-dimensional Haar wavelets vectors of dimension k^2, and $\mathbf{K}$ is the $(k^2) \times (k^2)$ two-dimensional Haar coefficient matrix.

Solution of 2D-FIEs of the second kind

Now, consider the second kind Fredholm integral equation of the form in Equation 1. Our goal is to reduce this equation to a linear system of algebraic equations by the method presented in this paper.

In order to approximate the solution of integral equation (Equation 1), we approximate functions $u(x,y), f(x,y)$ and $k(x,y,s,t)$ with respect to 2D-Haar wavelets by the way mentioned in 'Two-dimensional Haar wavelets' section as

$$\begin{aligned} u(x,y) &= \mathbf{H}^{\mathrm{T}}(x,y) \mathbf{U}, \\ f(x,y) &= \mathbf{H}^{\mathrm{T}}(x,y) \mathbf{F}, \\ K(x,y,s,t) &= \mathbf{H}^{\mathrm{T}}(x,y) \mathbf{K} \mathbf{H}(s,t), \end{aligned} \tag{12}$$

where $\mathbf{H}(x, y)$ is as defined in Equation 10, the vectors $\mathbf{U}$, $\mathbf{F}$ and matrix $\mathbf{K}$ are Haar wavelets coefficients of $u(x, y)$, $f(x, y)$ and $K(x, y, s, t)$, respectively.

By substituting the approximations (Equation 12) into Equation 1, we obtain

$$\mathbf{H}^{\mathrm{T}}(x, y)\mathbf{U} - \int_0^1 \int_0^1 \mathbf{H}^{\mathrm{T}}(x, y)\mathbf{K}\mathbf{H}(s, t)\mathbf{H}^{\mathrm{T}}(s, t)\mathbf{U}dsdt = \mathbf{H}^{\mathrm{T}}(x, y)\mathbf{F}, \tag{13}$$

which gives

$$\mathbf{H}^{\mathrm{T}}(x, y)\mathbf{U} - \mathbf{H}^{\mathrm{T}}(x, y)\mathbf{K}\left(\int_0^1 \int_0^1 \mathbf{H}(s, t)\mathbf{H}^{\mathrm{T}}(s, t)dsdt\right) \times \mathbf{U} = \mathbf{H}^{\mathrm{T}}(x, y)\mathbf{F}, \tag{14}$$

However, the orthonormality property of the sequence $\{h_{m,n}\}$ implies that

$$\int_0^1 \int_0^1 \mathbf{H}(s, t)\mathbf{H}^{\mathrm{T}}(s, t)dsdt = \mathbf{I}_{k^2 \times k^2}. \tag{15}$$

By substituting Equation 15 shown in Equation 14, we get the Equation below:

$$\mathbf{H}^{\mathrm{T}}(x, y)\mathbf{U} - \mathbf{H}^{\mathrm{T}}(x, y)\mathbf{K}\mathbf{U} = \mathbf{H}^{\mathrm{T}}(x, y)\mathbf{F} \tag{16}$$

By considering the inner product of the both sides of Equation 16 with $\mathbf{H}(x, y)$ and using the orthonormality property of the sequence $\{h_{m,n}\}$, we obtain

$$(\mathbf{I} - \mathbf{K})\mathbf{U} = \mathbf{F} \tag{17}$$

which is a linear system of algebraic equations that can be easily solved by direct or iterative methods.

Numerical examples

In this section, we applied the method presented in this paper for solving integral equation (Equation 1) and solved some examples. The computations associated with the examples were performed in a personal computer using Mathematica 7.

Example 1. Consider the following two-dimensional Fredholm integral equation of the second kind [12]

$$u(x, y) = f(x, y) + \int_0^1 \int_0^1 (s.\sin(t) + 1)u(s, t)\ dsdt, \quad 0 \le x, y < 1$$

where

$$f(x, y) = x.\cos(y) - \frac{1}{6}\sin(1)(3 + \sin(1))$$

and the exact solution is $u(x, y) = x.\cos(y)$. Table 1 shows the absolute values of error for $k = 4, 8, 16, 32$ using the present method in selected grid points. Better approximation is expected by choosing the optimal value $k = 32$.

Table 1 Absolute values of error for Example 1

$(x, y) = (\frac{1}{2^l}, \frac{1}{2^l})$	$k = 4$	$k = 8$	$k = 16$	$k = 32$
$l = 1$	5.1×10^{-2}	2.3×10^{-2}	1.1×10^{-2}	8.6×10^{-3}
$l = 2$	6.5×10^{-2}	3.2×10^{-2}	1.6×10^{-2}	1.2×10^{-2}
$l = 3$	2.4×10^{-3}	1.6×10^{-2}	8.0×10^{-3}	8.9×10^{-3}
$l = 4$	1.0×10^{-2}	9.5×10^{-3}	4.1×10^{-2}	2.0×10^{-2}
$l = 5$	4.7×10^{-2}	2.3×10^{-3}	2.1×10^{-4}	6.0×10^{-3}
$l = 6$	6.3×10^{-2}	3.2×10^{-2}	1.4×10^{-2}	4.3×10^{-5}

Table 2 Numerical results for Example 2

$(x, y) = (\frac{1}{2^l}, \frac{1}{2^l})$	$k = 4$	$k = 8$	$k = 16$	$k = 32$
$l = 1$	9.1×10^{-2}	6.3×10^{-2}	3.1×10^{-2}	3.6×10^{-4}
$l = 2$	6.5×10^{-2}	4.2×10^{-2}	2.6×10^{-2}	2.0×10^{-3}
$l = 3$	3.4×10^{-2}	1.6×10^{-2}	8.0×10^{-3}	3.2×10^{-3}
$l = 4$	1.0×10^{-2}	3.5×10^{-3}	5.1×10^{-2}	4.0×10^{-3}
$l = 5$	5.8×10^{-2}	1.5×10^{-2}	2.1×10^{-3}	5.0×10^{-3}
$l = 6$	3.6×10^{-2}	1.2×10^{-2}	6.4×10^{-3}	5.3×10^{-5}

Example 2. As the second example, consider the following linear two-dimensional integral equation

$$u(x, y) = f(x, y) + \int_0^1 \int_0^1 \frac{x}{1 + y} \times (1 + s + t)u(s, t)\ dsdt, \quad 0 \le x, y < 1$$

where

$$f(x, y) = \frac{1}{(1 + x + y)} - \frac{x}{1 + y}$$

and the exact solution $u(x, y) = \frac{1}{(1+x+y)}$. Numerical results are shown in Table 2. Better approximation is expected by choosing the optimal value $k = 32$.

Conclusion

Finding exact solutions for two-dimensional integral equations is often difficult, so approximating these solutions is very important. In this work, a computational method has been presented for numerical solution of 2D-FIEs based on Haar wavelet series. In comparison with other bases (for example, polynomial bases), one of the advantages of this method is, although the involved matrices have a large dimension, they contain a large percentage of zero entries, which keeps computational effort within reasonable limits. We can modify this method for the numerical solution of linear and nonlinear two-dimensional Volterra and Fredholm integral equations in the future.

Methods

We can modify this method for the numerical solution of linear and nonlinear two- Dimensional Volterra and Fredholm integral equations in the future.

Competing interests
The authors declare that they have no competing interests.

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Author's contributions
HD carried out the two-dimensional wavelets studies, participated in the sequence alignment and drafted the manuscript. SS carried out the necessary programing. SS participated in the sequence alignment. HH and SS participated in the design of the study and performed the error analysis. AA conceived of the study, and participated in its design and programing. All authors read and approved the final manuscript.

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