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Convergence in simultaneous approximation for Srivastava-Gupta operators

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Abstract

Purpose: The purpose of the present paper is to introduce the generalized form of Srivastava-Gupta operators and study their approximation properties.

Methods: We use analytical method to obtain our results.

Results: We have established the rate of convergence, in simultaneous approximation, for functions having derivatives of bounded variation.

Conclusions: The results proposed here are new and have a better rate of convergence.

Keywords: Bounded variation, Srivastava-Gupta operators, Simultaneous approximation, Rate of convergence

AMS Subject Classification: 2010; 26A45, 41A28

Introduction

In the year 2003, Srivastava and Gupta [1] introduced a general family of summation-integral type operators which includes some well-known operators as special cases. They estimated the rate of convergence for functions of bounded variation. For the details of special cases in [1], we refer the readers to [2-7]. Ispir and Yuksel [8] considered the Bezier variant of the operators studied in [1] and estimated the rate of convergence for functions of bounded variation. Very recently, Deo [9] studied Srivastava-Gupta operators and obtained the faster rate convergence as well as Voronovskaja type results for these operators by using the King approach. In the last section, he considered Stancu variant of these operators and established some approximation properties.

The operators $G_{n,c}$ is defined as follows:

$$G_{n,c}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^{\infty} p_{n+c,k-1}(t,c)f(t)\,dt + p_{n,0}(x,c)f(0), \tag{1.1}$$

where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x) \tag{1.2}$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0, \\ (1+cx)^{-n/c}, & c = 1,2,3,.... \end{cases}$$

Here $\{\phi_{n,c}(x)\}_{n=1}^{\infty}$ is a sequence of functions defined on the closed interval $[0,b]$, $b > 0$, satisfying the following properties. For each $n \in \mathbb{N}$ and $k \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$:

(i) $\phi_{n,c} \in C^{\infty}[a,b]$, $b > a \geq 0$,
(ii) $\phi_{n,c}(0) = 1$,
(iii) $\phi_{n,c}$ is completely monotone so that $(-1)^k\phi_{n,c}^{(k)}(x) \geq 0$, $x \in [0,b]$, and
(iv) there exists an integer c such that

$$\phi_{n,c}^{(k+1)}(x) = -n\phi_{n+c,c}^{(k)}(x), \quad n > \max\{0,-c\};\ x \in [0,b]$$

(see [1]).

Nowadays, the rate of convergence for the functions having the derivatives of bounded variation (BV) is an interesting area of research. Bai et al.[10] worked in this direction and estimated the rate of convergence for the several operators. Gupta [4] estimated the rate of convergence for functions of BV on certain Baskakov-Durrmeyer type operators. Ispir et al. [11] estimated the rate of convergence for the Kantorovich type operators

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for functions having derivatives of BV. Recently, Acar et al. [12] introduced the general integral modification of the Szász-Mirakyan operators having the weight functions of Baskakov basis functions. The rate of convergence for functions having the derivatives of bounded variation is obtained. This motivated us to study the rate of convergence for the generalized Srivastava-Gupta operators as follows: For a function $f \in BV_\alpha[0,\infty)$, the class of bounded variation functions satisfying the growth condition $|f(t)| \leq M(1+t)^\alpha, M>0,\ \alpha \geq 0$, the operators $G_{n,r,c}$ are defined by

$$G_{n,r,c}(f,x) = \frac{n\,\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r+1)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})}\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \int_0^\infty p_{n-(r-1)c,k+r-1}(t,c)f(t)\,dt, \qquad (1.3)$$

where $p_{n,k}(x,c)$ is given by Equation 1.2 and $n > (r-1)c$.

Remark 1. *For the special case of $c=1$, the operators in Equation 1.3 are reduced to the following operators:*

$$G_{n,r,1}(f,x) = \frac{(n+r-1)!\,(n-r)!}{((n-1)!)^2}\sum_{k=0}^{\infty} p_{n+r,k}(x,1) \int_0^\infty p_{n-(r-1),k+r-1}(t,1)f(t)\,dt,$$

where $p_{n,k}(x,1) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$.

We denote that the class of absolutely continuous functions f on $(0,\infty)$ by $DB_q(0,\infty)$, (where q is some positive integer) are satisfied:

(i) $|f(t)| \leq C_1 t^q, \quad C_1 > 0$ and
(ii) the function f has the first derivative on interval $(0,\infty)$ which coincide, a.e., with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that for all $f \in DB_q(0,\infty)$, we can have the representation

$$f(x) = f(c) + \int_c^x \psi(t)dt, \quad x \geq c > 0.$$

In the present paper, we study the rate of convergence for the operators $G_{n,r,c}$ for functions having the derivatives of bounded variation. We also mention a corollary which provides the result in simultaneous approximation.

Methods

The principal methods used in the present work involve the application of the theory of functions having the derivatives of bounded variation to analyze and study the rate of convergence, in simultaneous approximation, for the Srivastava-Gupta operators.

Results and discussion

In the sequel we shall need the following lemmas:

Lemma 1. *If we define the moments as*

$$T_{n,r,m}(x,c) = (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \int_0^\infty p_{n-(r-1)c,k+r-1}(t,c)(t-x)^m\,dt,$$

and then, $T_{r,n,0}(x,c) = 1$, $T_{r,n,1}(x,c) = \frac{(1+2r)cx+r}{n-(r+1)c}$ *and for* $n > (m+r+1)c$, *we have the following recurrence relation:*

$$[\,n-(m+r+1)c]\,T_{n,r,m+1}(x,c) = x(1+cx)[\,T'_{n,r,m}(x,c) + 2mT_{n,r,m-1}(x,c)] + [\,(m+r)(1+2cx)+cx]\,T_{n,r,m}(x,c).$$

Consequently,

$$T_{n,r,2}(x,c) = \frac{x(1+cx)(2n-c) + [\,(1+r)(1+2cx)+cx]\cdot[\,(1+2r)cx+r]}{(n-(r+1)c)(n-(r+2)c)}.$$

Furthermore, $T_{n,r,m}(x,c)$ is polynomial of degree m in x and

$$T_{n,r,m}(x,c) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right).$$

Proof. Taking the derivative of $T_{n,r,m}(x,c)$ with respect to x and using the identity $x(1+cx)p'_{n+rc,k}(x,c) = [\,k-(n+rc)x]\,p_{n+rc,k}(x,c)$, we have

$$\begin{aligned} x(1+cx)[\,T'_{n,r,m}(x,c) + mT_{n,r,m-1}(x,c)] &= (n-rc)\sum_{k=0}^{\infty}[\,k-(n+rc)x]\,p_{n+rc,k}(x,c) \int_0^\infty p_{n-(r-1)c,k+r-1}(t,c)\,(t-x)^m\,dt \\ &= (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \int_0^\infty [\,(k+r-1)-(n-(r-1)c)t]\, p_{n-(r-1)c,k+r-1}(t,c)\,(t-x)^m\,dt \\ &\quad + [\,n-(r-1)c]\,(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \int_0^\infty p_{n-(r-1)c,k+r-1}(t,c)t\,(t-x)^m\,dt \\ &\quad - [\,(n+rc)x+(r-1)]\,T_{n,r,m}(x,c) \\ &= I_1 + I_2 - [\,(n+rc)x+(r-1)]\,T_{n,r,m}(x,c). \end{aligned}$$

To compute I_2 we have

$$\begin{aligned} I_2 =& [\,n-(r-1)c]\,(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)\\ &\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t,c)\\ &\times\left[\,(t-x)^{m+1}+x\,(t-x)^{m}\right]dt\\ =&[\,n-(r-1)c]\left[\,T_{n,r,m+1}(x,c)+xT_{n,r,m}(x,c)\right]. \end{aligned}$$

Using $t(1+ct)p'_{n-(r-1)c,k}(t,c) = [\,k-(n-(r-1)c)t]\, p_{n-(r-1)c,k}(t,c)$, we can write I_1 as

$$\begin{aligned} I_1 =& (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)\\ &\int_0^{\infty}p'_{n-(r-1)c,k+r-1}(t,c)t\,(t-x)^{m}\,dt\\ &+(n-rc)c\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)\\ &\int_0^{\infty}p'_{n-(r-1)c,k+r-1}(t,c)t^{2}\,(t-x)^{m}\,dt\\ =& J_1+J_2. \end{aligned}$$

Again using $t(t-x)^m = (t-x)^{m+1}+x(t-x)^m$ and integrating by parts, we get

$$\begin{aligned} J_1 =& (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)\\ &\int_0^{\infty}p'_{n-(r-1)c,k+r-1}(t,c)\left[\,(t-x)^{m+1}+x\,(t-x)^{m}\right]dt\\ =& (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)\\ &\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t,c)\\ &\times\left[\,-(m+1)\,(t-x)^{m}-mx\,(t-x)^{m-1}\right]dt\\ =& -(m+1)T_{n,r,m}(x,c)-mxT_{n,r,m-1}(x,c). \end{aligned}$$

Proceeding in a similar manner, we obtain J_2 as

$$\begin{aligned} J_2 = c\,\big[&-(m+2)T_{n,r,m+1}(x,c)-2(m+1)xT_{n,r,m}(x,c)\\ &-mx^2T_{n,r,m-1}(x,c)\big]. \end{aligned}$$

Combining I_1, I_2, J_1, and J_2, we have

$$\begin{aligned} &x(1+cx)[\,T'_{n,r,m}(x,c)+mT_{n,r,m-1}(x,c)]\\ &\quad= -(m+1)T_{n,r,m}(x,c)-mxT_{n,r,m-1}(x,c)\\ &\qquad+c\left[-(m+2)T_{n,r,m+1}(x,c)-2(m+1)xT_{n,r,m}(x,c)\right.\\ &\qquad\quad\left.-mx^2T_{n,r,m-1}(x,c)\right]\\ &\qquad+[\,n-(r-1)c]\left[T_{n,r,m+1}(x,c)+xT_{n,r,m}(x,c)\right]\\ &\qquad-[\,(n+rc)x+(r-1)]\,T_{n,r,m}(x,c), \end{aligned}$$

$$\begin{aligned} &x(1+cx)[\,T'_{n,r,m}(x,c)+mT_{n,r,m-1}(x,c)]\\ &\qquad=[\{n-(r-1)c\}-(m+2)c]\,T_{n,r,m+1}(x,c)\\ &\qquad\quad+[-(m+1)-2(m+1)cx+\{n-(r-1)c\}x\\ &\qquad\qquad-\{(n+rc)x+(r-1)\}]\,T_{n,r,m}(x,c)\\ &\qquad\quad+\left[-mx-mcx^2\right]T_{n,r,m-1}(x,c), \text{ and} \end{aligned}$$

$$\begin{aligned} &[\,n-(m+r+1)c]\,T_{n,r,m+1}(x,c)\\ &\quad= x(1+cx)[\,T'_{n,r,m}(x,c)+2mT_{n,r,m-1}(x,c)]\\ &\qquad+[\,(m+r)(1+2cx)+cx]\,T_{n,r,m}(x,c). \end{aligned}$$

□

Remark 2. *Let* $x\in(0,\infty)$ *and* $\lambda>2$*; then for* n *sufficiently large, Lemma 1 yields that*

$$T_{n,r,2}(x,c)\le\frac{\lambda x(1+cx)}{n}\qquad(c\in\mathbb{N}_0).$$

Lemma 2. *Let* $x\in(0,\infty)$ *and* $\lambda>2$*; then for* n *sufficiently large, we have*

$$\begin{aligned} \mu_{n,r}(x,y) =& (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)\\ &\int_0^{y}p_{n-(r-1)c,k+r-1}(t,c)dt\\ \le& \frac{\lambda x(1+cx)}{n(x-y)^2},\quad 0\le y<x,\\ 1-\mu_{n,r}(x,z) =& (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)\\ &\int_z^{\infty}p_{n-(r-1)c,k+r-1}(t,c)dt\\ \le& \frac{\lambda x(1+cx)}{n(z-x)^2},\quad x<z<\infty. \end{aligned}$$

Proof. The proof of the lemma follows easily by Remark 2. For instance, for the first inequality for n sufficiently large and $0 \leq y < x$, we have

$$\begin{aligned}\mu_{n,r}(x,y) &= (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \int_0^y p_{n-(r-1)c,k+r-1}(t,c)dt \\ &\leq (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \int_0^y p_{n-(r-1)c,k+r-1}(t,c)\frac{(t-x)^2}{(y-x)^2}dt \\ &\leq \frac{T_{n,r,2}(x)}{(y-x)^2} \leq \frac{\lambda x(1+cx)}{n(y-x)^2}.\end{aligned}$$

The proof of the second inequality follows along the similar lines. □

Lemma 3. *Let f be s times differentiable on $[0,\infty)$ such that $f^{(s-1)}(t) = O(t^q)$ as $t \to \infty$ where q is a positive integer. Then for any $r, s \in \mathbb{N}_0$ and $n > max\{q, r+s+1\}$, we have*

$$D^s G_{n,r,c}(f,x) = G_{n,r+s,c}(D^s f,x), \quad D = \frac{d}{dx}.$$

Proof. We prove this result by applying the principle of mathematical induction and using the following identity:

$$\begin{aligned}p'_{n,k}(x,c) &= n[p_{n+c,k-1}(x,c) - p_{n+c,k}(x,c)] \quad \text{and} \\ p'_{n,0}(x,c) &= -np_{n+c,0}(x,c).\end{aligned} \tag{2.1}$$

The above identities are true even for the case of $k = 0$, as we observe that $p_{n+c,k} = 0$ for $k < 0$. Using Equation 2.1 and integrating by parts, we have

$$\begin{aligned}D[G_{n,r,c}](f,x) &= \frac{n\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r+1)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})}\sum_{k=0}^{\infty} Dp_{n+rc,k}(x,c) \int_0^{\infty} p_{n-(r-1)c,k+r-1}(t,c)f(t)\,dt \\ &= \frac{n\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r+1)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})} \cdot \sum_{k=0}^{\infty}(n+rc)[p_{n+(r+1)c,k-1}(x,c) - p_{n+(r+1)c,k}(x,c)] \int_0^{\infty} p_{n-(r-1)c,k+r-1}(t,c)f(t)\,dt \\ &= \frac{n(n+rc)\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r+1)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})} \cdot \sum_{k=0}^{\infty} p_{n+(r+1)c,k}(x,c)\int_0^{\infty}[p_{n-(r-1)c,k+r}(t,c) - p_{n-(r-1)c,k+r-1}(t,c)]f(t)\,dt \\ &= -\frac{n(n+rc)\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r+1)}{(n-rc)\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})} \cdot \sum_{k=0}^{\infty} p_{n+(r+1)c,k}(x,c) \int_0^{\infty} Dp_{n-rc,k+r}(t,c)f(t)\,dt \\ &= \frac{n\Gamma(\frac{n}{c}+r+1)\Gamma(\frac{n}{c}-r)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})} \cdot \sum_{k=0}^{\infty} p_{n+(r+1)c,k}(x,c) \int_0^{\infty} p_{n-rc,k+r}(t,c)Df(t),\, dt \\ &= [G_{n,r+1,c}](Df,x),\end{aligned}$$

which shows that the result holds for $s = 1$. Let us suppose that the result holds for $s = m$ i.e.,

$$\begin{aligned}D^m[G_{n,r,c}](f,x) &= [G_{n,r+m,c}](D^m f,x) \\ &= \frac{n\Gamma(\frac{n}{c}+r+m)\Gamma(\frac{n}{c}-r-m+1)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})} \sum_{k=0}^{\infty} p_{n+(r+m)c,k}(x,c) \int_0^{\infty} p_{n-(r+m-1)c,k+r+m-1}(t,c)D^m f(t)\,dt.\end{aligned}$$

Now by Equation 2.1,

$$\begin{aligned}&D^{m+1}[G_{n,r,c}](f,x) \\ &= \frac{n\Gamma(\frac{n}{c}+r+m)\Gamma(\frac{n}{c}-r-m+1)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})}\sum_{k=0}^{\infty} Dp_{n+(r+m)c,k}(x,c) \int_0^{\infty} p_{n-(r+m-1)c,k+r+m-1}(t,c)D^m f(t)\,dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{n\,\Gamma(\frac{n}{c}+r+m)\Gamma(\frac{n}{c}-r-m+1)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})}\sum_{k=0}^{\infty}[\,n+(r+m)c]\\
&\quad\times\left(p_{n+(r+m+1)c,k-1}(x,c)-p_{n+(r+m+1)c,k}(x,c)\right)\\
&\quad\int_0^\infty p_{n-(r+m-1)c,k+r+m-1}(t,c)D^m f(t)\,dt\\
&= \frac{nc\,\Gamma(\frac{n}{c}+r+m+1)\Gamma(\frac{n}{c}-r-m+1)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})}\\
&\quad\sum_{k=0}^{\infty}p_{n+(r+m+1)c,k}\int_0^\infty\Bigg(p_{n-(r+m-1)c,k+r+m}(t,c)\\
&\qquad\qquad -p_{n-(r+m-1)c,k+r+m-1}(t,c)\Bigg)D^m f(t)\,dt\\
&= -\frac{nc\,\Gamma(\frac{n}{c}+r+m+1)\Gamma(\frac{n}{c}-r-m+1)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})}\\
&\quad\sum_{k=0}^{\infty}p_{n+(r+m+1)c,k}\\
&\quad\int_0^\infty\frac{Dp_{n-(r+m)c,k+r+m}(t,c)}{n-(r+m-1)c}D^m f(t)\,dt.
\end{aligned}
$$

Integrating by parts the last integral, we have

$$
\begin{aligned}
D^{m+1}[\,G_{n,r,c}]\,(f,x) &= \frac{n\,\Gamma(\frac{n}{c}+r+m+1)\Gamma(\frac{n}{c}-r-m)}{\Gamma(\frac{n}{c}+1)\Gamma(\frac{n}{c})}\\
&\quad\sum_{k=0}^{\infty}p_{n+(r+m+1)c,k}\\
&\quad\int_0^\infty p_{n-(r+m)c,k+r+m}(t,c)D^{m+1}f(t)\,dt.
\end{aligned}
$$

Therefore,

$$D^{m+1}G_{n,r,c}(f,x)=G_{n,r+m+1,c}(D^{m+1}f,x).$$

Thus, the result is true for $s=m+1$; hence, by mathematical induction the proof of the lemma is completed. ☐

Main results

In this subsection we prove our main results.

Theorem 1. *Let $f\in DB_q(0,\infty)$, $q>0$ and $x\in(0,\infty)$. The for $\lambda>2$ and n sufficiently large, we have*

$$
\begin{aligned}
&\left|\frac{(\Gamma(\frac{n}{c}))^2}{\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r)}G_{n,r,c}(f;x)-f(x)\right|\\
&\le\frac{\lambda(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)\\
&\quad+\frac{\lambda(1+cx)}{n}(|f(2x)-f(x)-xf'(x^+)|+|f(x)|)\\
&\quad+O(n^{-q})+\frac{\lambda(1+cx)}{n}|f'(x^+)|\\
&\quad+\frac{|f'(x^+)-f'(x^-)|}{2}\sqrt{\frac{\lambda x(1+cx)}{n}}\\
&\quad+\frac{|f'(x^+)+f'(x^-)|}{2}\frac{r(1+2cx)+cx}{n-(r+1)c},
\end{aligned}
$$

where $\bigvee_a^b f_x$ denotes the total variation of f_x on $[\,a,b]$, and the auxiliary function f_x is defined by

$$
f_x(t)=\begin{cases} f(t)-f(x^-), & 0\le t<x,\\ 0, & t=x,\\ f(t)-f(x^+), & x<t<\infty.\end{cases}
$$

Proof. Using the mean value theorem, we have

$$
\begin{aligned}
&\frac{(\Gamma(\frac{n}{c}))^2}{\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r)}G_{n,r,c}(f;x)-f(x)\\
&\quad=(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)\int_0^\infty p_{n-(r-1)c,k+r-1}(t,c)\\
&\quad\times[f(t)-f(x)]\,dt\\
&\quad=\int_0^\infty\Bigg(\int_x^t(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}\\
&\qquad\times(t,c)f'(u)du\Bigg)\,dt.
\end{aligned}
$$

Also it is a valid identity that

$$
\begin{aligned}
f'(u)&=\frac{f'(x^+)+f'(x^-)}{2}+(f')_x(u)+\frac{f'(x^+)-f'(x^-)}{2}sgn(u-x)\\
&\quad+\left[f'(x)-\frac{f'(x^+)+f'(x^-)}{2}\right]\chi_x(u),
\end{aligned}
$$

where

$$
\chi_x(u)=\begin{cases}1, & u=x,\\ 0, & u\ne x.\end{cases}
$$

Obviously, we have

$$
\begin{aligned}
&(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x,c)\int_0^\infty\Bigg(\int_x^t\left[f'(x)-\frac{f'(x^+)+f'(x^-)}{2}\right]\\
&\qquad\chi_x(u)du\Bigg)p_{n-(r-1)c,k+r-1}(t,c)dt=0.
\end{aligned}
$$

Thus, using the above identities, we can write

$$\left|\frac{(\Gamma(\frac{n}{c}))^2}{\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r)}G_{n,r,c}(f;x)-f(x)\right|$$

$$\le\left|\int_0^\infty\left(\int_x^t(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}(t,c)\right.\right.$$
$$\left.\left.\times\left(\frac{f'(x^+)+f'(x^-)}{2}+(f')_x(u)\right)du\right)\right|\,dt$$
$$+\left|\int_0^\infty\left(\int_x^t(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}(t,c)\right.\right.$$
$$\left.\left.\frac{[f'(x^+)-f'(x^-)]}{2}sgn(u-x)du\right)dt\right|. \qquad (3.1)$$

Also it can be verified that

$$\left|\int_0^\infty\left(\int_x^t\frac{[f(x^+)-f(x^-)]}{2}sgn(u-x)du\right)(n-rc)\right.$$
$$\left.\sum_{k=0}^\infty p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}(t,c)dt\right|$$
$$\le\frac{|f'(x^+)-f'(x^-)|}{2}[\,T_{n,r,2}(x,c)]^{1/2} \qquad (3.2)$$

and

$$\left|\int_0^\infty\left(\int_x^t\frac{[f'(x^+)+f'(x^-)]}{2}du\right)(n-rc)\right.$$
$$\left.\sum_{k=0}^\infty p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}(t,c)dt\right|$$
$$\le\frac{|f'(x^+)+f'(x^-)|}{2}T_{n,r,1}(x,c). \qquad (3.3)$$

Combining Equations 3.1–3.3, we get

$$\left|\frac{(\Gamma(\frac{n}{c}))^2}{\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r)}G_{n,r,c}(f;x)-f(x)\right|$$
$$\le\left|\int_x^\infty\left(\int_x^t(f')_x(u)du\right)(n-rc)\right.$$
$$\sum_{k=0}^\infty p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}(t,c)dt$$
$$+\int_0^x\left(\int_x^t(f')_x(u)du\right)(n-rc)$$
$$\left.\sum_{k=0}^\infty p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}(t,c)dt\right|$$
$$+\frac{|f'(x^+)-f'(x^-)|}{2}[\,T_{n,r,2}(x,c)]^{1/2}$$
$$+\frac{|f'(x^+)+f'(x^-)|}{2}T_{n,r,1}(x,c)$$
$$=|A_{n,r}(f,x)+B_{n,r}(f,x)|$$
$$+\frac{|f'(x^+)-f'(x^-)|}{2}[\,T_{n,r,2}(x,c)]^{1/2}$$
$$+\frac{|f'(x^+)+f'(x^-)|}{2}T_{n,r,1}(x,c). \qquad (3.4)$$

Applying Remark 2 and Lemma 1 in Equation 3.4, we have

$$\left|\frac{(\Gamma(\frac{n}{c}))^2}{\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r)}G_{n,r,c}(f;x)-f(x)\right|$$

$$\le|A_{n,r}(f,x)|+|B_{n,r}(f,x)|+\frac{|f'(x^+)-f'(x^-)|}{2}\sqrt{\frac{\lambda x(1+cx)}{n}}$$
$$+\frac{|f'(x^+)+f'(x^-)|}{2}\frac{r(1+2cx)+cx}{n-(r+1)c}. \qquad (3.5)$$

In order to complete the proof of the theorem, it suffices to estimate the terms $A_{n,r}(f,x)$ and $B_{n,r}(f,x)$ as follows:

$$|A_{n,r}(f,x)|=\left|\int_x^\infty\left(\int_x^t(f')_x(u)du\right)(n-rc)\right.$$
$$\left.\sum_{k=0}^\infty p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}(t,c)dt\right|$$
$$=\left|\int_{2x}^\infty\left(\int_x^t(f')_x(u)du\right)(n-rc)\right.$$
$$\sum_{k=0}^\infty p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}(t,c)dt$$
$$+\int_x^{2x}\left(\int_x^t(f')_x(u)du\right)(n-rc)$$
$$\left.\sum_{k=0}^\infty p_{n+rc,k}(x,c)p_{n-(r-1)c,k+r-1}(t,c)dt\right|$$
$$\le\left|(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x,c)\right.$$
$$\left.\int_{2x}^\infty\big(f(t)-f(x)\big)\,p_{n-(r-1)c,k+r-1}(t,c)dt\right|$$
$$+|f'(x^+)|\left|(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x,c)\right.$$
$$\left.\int_x^{2x}p_{n-(r-1)c,k+r-1}(t,c)(t-x)dt\right|$$
$$+\left|\int_x^{2x}(f')_x(u)du\right|\,\big|1-\mu_{n,r}(x,2x)\big|$$
$$+\left|\int_x^{2x}|(f')_x(t)|\cdot\big|1-\mu_{n,r}(x,t)\big|\,dt\right|$$

$$\begin{aligned}
&\le (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \\
&\quad \int_{2x}^{\infty} p_{n-(r-1)c,k+r-1}(t,c)C_1 t^q dt \\
&\quad + \frac{|f(x)|}{x^2}(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \\
&\quad \int_{0}^{\infty} p_{n-(r-1)c,k+r-1}(t,c)(t-x)^2 dt \\
&\quad + |f'(x^+)|(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \\
&\quad \int_{2x}^{\infty} p_{n-(r-1)c,k+r-1}(t,c)|t-x| dt \\
&\quad + \frac{\lambda(1+cx)}{nx}(|f(2x) - f(x) - xf'(x^+)| \\
&\quad + \frac{\lambda(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} ((f')_x). \qquad (3.6)
\end{aligned}$$

For estimating the integral $(n - rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \int_{2x}^{\infty} p_{n-(r-1)c,k+r-1}(t,c)C_1 t^q dt$ above, we proceed as follows: since $t \ge 2x$ implies that $t \le 2(t-x)$ Schwarz inequality which follows from Lemma 1,

$$\begin{aligned}
&(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \int_{2x}^{\infty} p_{n-(r-1)c,k+r-1}(t,c)C_1 t^q dt \\
&\le C_1 2^q (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \\
&\quad \int_{0}^{\infty} p_{n-(r-1)c,k+r-1}(t,c)C_1 (t-x)^q dt \\
&= C_1 2^q T_{n,r,q}(x,c) = O(n^{-q/2}), \quad \text{as } n \to \infty \text{ and} \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
&\frac{|f(x)|}{x^2}(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \int_{0}^{\infty} p_{n-(r-1)c,k+r-1}(t,c) \\
&\quad \times (t-x)^2 dt = |f(x)| \frac{\lambda(1+cx)}{nx}. \qquad (3.8)
\end{aligned}$$

By using the Hölder's inequality and Remark 2, we get the estimate as follows:

$$\begin{aligned}
&|f'(x^+)|(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \\
&\quad \int_{2x}^{\infty} p_{n-(r-1)c,k+r-1}(t,c)|t-x| dt \\
&\le |f'(x^+)| \left((n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x,c) \right. \\
&\quad \left. \int_{0}^{\infty} p_{n-(r-1)c,k+r-1}(t,c)(t-x)^2 dt \right)^{1/2} \\
&= |f'(x^+)| \sqrt{\frac{\lambda x(1+cx)}{n}}. \qquad (3.9)
\end{aligned}$$

Collecting the estimates from Equations 3.6–3.9, we obtain

$$\begin{aligned}
|A_{n,r}(f,x)| &= O(n^{-q}) + |f'(x^+)| \sqrt{\frac{\lambda x(1+cx)}{n}} \\
&\quad + \frac{\lambda(1+cx)}{nx}(|f(2x) - f(x) - xf'(x^+)| + |f(x)|) \\
&\quad + \frac{\lambda(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} ((f')_x). \qquad (3.10)
\end{aligned}$$

On the other hand, to estimate $B_{n,r}(f,x)$ by applying the Lemma 2 with $y = x - \frac{x}{\sqrt{n}}$ and integration by parts, we have

$$\begin{aligned}
|B_{n,r}(f,x)| &= \left| \int_0^x \int_x^t (f')_x(u) d_t(\mu_{n,r}(x,t)) \right| \\
&= \left| \left(\int_0^x (f')_x(u) du \right) \mu_{n,r}(x,t) \right| \\
&\le \left(\int_0^y + \int_y^x \right) |(f')_x(t)||\mu_{n,r}(x,t)| dt \\
&\le \frac{\lambda x(1+cx)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \int_y^x \bigvee_t^x ((f')_x) dt \\
&\le \frac{\lambda x(1+cx)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} ((f')_x) \\
&= \frac{\lambda x(1+cx)}{n} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{u}}}^{x} ((f')_x) du + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} ((f')_x) \\
&\le \frac{\lambda(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} ((f')_x), \qquad (3.11)
\end{aligned}$$

where $u = \frac{x}{x-t}$.

Through combining the Equations 3.4, 3.10, and 3.11, we get the desired results. ☐

As a consequence of Lemma 3, we can easily prove the following corollary for the derivatives of the operators $G_{n,r,c}$.

Corollary 1. *Let* $f^s \in DB_q(0,\infty)$, $q > 0$ *and* $x \in (0,\infty)$. *The for* $\lambda > 2$ *and* n *sufficiently large, we have*

$$\begin{aligned}
&\left| \frac{(\Gamma(\frac{n}{c}))^2}{\Gamma(\frac{n}{c}+r)\Gamma(\frac{n}{c}-r)} D^s G_{n,r,c}(f;x) - f^s(x) \right| \\
&\quad \le \frac{\lambda(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+x/k} ((D^{s+1}f)_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} ((D^{s+1}f)_x) \\
&\qquad + \frac{\lambda(1+cx)}{n} (|D^s f(2x) - D^s f(x) \\
&\qquad - xD^{s+1} f(x^+)| + |D^s f(x)|) + O(n^{-q}) \\
&\qquad + \frac{\lambda(1+cx)}{n} |D^{s+1} f(x^+)| \\
&\qquad + \frac{|D^{s+1} f(x^+) - D^{s+1} f(x^-)|}{2} \sqrt{\frac{\lambda x(1+cx)}{n}} \\
&\qquad + \frac{|D^{s+1} f(x^+) + D^{s+1} f(x^-)|}{2} \frac{r(1+2cx)+cx}{n-(r+1)c},
\end{aligned}$$

where $\bigvee_a^b f_x$ *denotes the total variation of* f_x *on* $[a,b]$, *and the auxiliary function* $D^{s+1} f_x$ *is defined by*

$$D^{s+1} f_x(t) = \begin{cases} D^{s+1} f(t) - D^{s+1} f(x^-), & 0 \le t < x, \\ 0, & t = x, \\ D^{s+1} f(t) - D^{s+1} f(x^+), & x < t < \infty. \end{cases}$$

Conclusions

We have obtained the rate of convergence for the generalized Srivastava-Gupta operators for the functions having the derivatives of bounded variation which gives a better rate of convergence than the classical Srivastava-Gupta operators.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

DKV and PNA contributed equally to this work. Both authors read and approved the final manuscript.

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