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Solving nonlinear two-dimensional Volterra integro-differential equations by block-pulse functions

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Abstract

In this paper, an effective numerical method is introduced for the treatment of nonlinear two-dimensional Volterra integro-differential equations. Here, we use the so-called two-dimensional block-pulse functions.
First, the two-dimensional block-pulse operational matrix of integration and differentiation has been presented. Then, by using this matrices, the nonlinear two-dimensional Volterra integro-differential equation has been reduced to an algebraic system. Some numerical examples are presented to illustrate the effectiveness and accuracy of the method.

Keywords: Nonlinear equations, Two-dimensional Volterra integro-differential equations, Two-dimensional block-pulse functions, Operational matrix

Introduction

An area of increasing scientific interest over the past decades is the study of Volterra integro-differential equation. This equation is encountered in various applications such as physics, mechanics, and applied science [1-4].

A general form of the Volterra integral equation can be written as

$$u_{xx} + u_{tx} + u_{tt} + u(t,x) = g(t,x) + \int_0^t \int_0^x k(t,s,x,y) \times F(u(s,y))dyds, \quad (t,x) \in [0,A_1] \times [0,A_2], \tag{1}$$

with given supplementary conditions, where $u(t,x)$ is an unknown function which should be determined; $g(t,x)$ and $k(t,s,x,y)$ are analytical functions, respectively[5]. In this paper, we consider the nonlinear function $F(u(s,y))$ in the following form

$$F(u(s,y)) = u^p(s,y),$$

where p is a positive integer. With regard to the fact that every finite interval can be transformed to $[0,1]$ by linear map, without loss of generality, we can consider $A_1 = A_2 = 1$.

As we know, the block-pulse functions (BPFs) presented by Harmuth [6] are a powerful mathematical tool for solving various kinds of integral equations. These functions are a set of orthogonal functions with piecewise constant values which are defined in the time interval $[0, T_1]$ as

$$\phi_i(t) = \begin{cases} 1, & (i-1)\frac{T_1}{m} \le t \le i\frac{T_1}{m}, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $i = 0, \dots, m-1$ with m as a positive integer.

The solution of Fredholm and Volterra integral equations of the second kind have been approximated using BPFs in [7]. Maleknejad and Mahmoudi in [8] have applied a combination of Taylor and block-pulse functions to solve linear Fredholm integral equation. The BPFs and Lagrange interpolating polynomials have been used to approximate the solution of Volterra's population model by Marzban et al. [9]. Recently, Maleknejad and Mahdiani have applied two dimensional (2D-BPFs) for solving nonlinear mixed Volterra-Fredholm integral equations [10]. In this paper, we use 2D-BPFs to approximate the solution of Equation 1.

This paper is organized as follows. In section 'Properties of the 2D-BPFs', the definition and some properties

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of the 2D-BPFs are presented. The 2D-BPFs are applied to solve Equation 1 in 'Applying the method' section. The error analysis of the proposed method has been investigated in section 'The error analysis'. Some numerical results have been presented in section 'Numerical results' to show accuracy and efficiency of the proposed method. Finally, some concluding remarks are given in 'Conclusion' section.

Properties of the 2D-BPFs

We usually call the block-pulse functions containing two variables as two-dimensional block-pulse functions. An (m_1m_2) set of 2D-BPFs are defined in region $t \in [0, T_1)$ and $x \in [0, T_2)$ as

$$\phi_{i_1,i_2}(t,x) = \begin{cases} 1, & (i_1-1)h_1 \leqslant x \leqslant i_1h_1 \text{ and} \\ & (i_2-1)h_2 \leqslant y \leqslant i_2h_2, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where $i_1 = 1, 2, \ldots, m_1$ and $i_2 = 1, 2, \ldots, m_2$ with positive integer values for m_1, m_2, and $h_1 = \frac{T_1}{m_1}, h_2 = \frac{T_2}{m_2}$. There are some properties for 2D-BPFs, e.g. disjointness, orthogonality, and completeness.

1. Disjointness

   The two-dimensional block-pulse functions are disjoined with each other, i.e.

$$\phi_{i_1,i_2}(t,x)\phi_{j_1,j_2}(t,x) = \begin{cases} \phi_{i_1,i_2}(t,x), & i_1 = j_1 \text{ and } i_2 = j_2, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

2. Orthogonality

   The two-dimensional block-pulse functions are orthogonal with each other, i.e.

$$\int_0^{T_1}\int_0^{T_2} \phi_{i_1,i_2}(t,x)\phi_{j_1,j_2}(t,x)dxdt = \begin{cases} h_1h_2, & i_1 = j_1 \text{ and} \\ & i_2 = j_2, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

   in the region of $t \in [0, T_1)$ and $x \in [0, T_2)$ where $i_1, j_1 = 1, 2, \ldots, m_1$ and $i_2, j_2 = 1, 2, \ldots, m_2$.

3. Completeness

   For every $f \in L^2([0, T_1) \times [0, T_2))$ when m_1 and m_2 go to infinity, Parseval identity holds:

$$\int_0^{T_1}\int_0^{T_2} f^2(t,x)dxdt = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty} f_{i_1,i_2}^2 \|\phi_{i_1,i_2}(t,x)\|^2, \tag{6}$$

where

$$f_{i_1,i_2} = \frac{1}{h_1h_2}\int_0^{T_1}\int_0^{T_2} f(t,x)\phi_{i_1,i_2}(t,x)dxdt. \tag{7}$$

The set of 2D-BPFs may be written as a (m_1m_2) vector $\Phi(t,x)$:

$$\Phi(t,x) = [\phi_{1,1}(t,x), \ldots, \phi_{1,m_2}(t,x), \ldots, \phi_{m_1,1}(t,x), \ldots, \times \phi_{m_1,m_2}(t,x)]^T, \tag{8}$$

where $(t,x) \in [0, T_1) \times [0, T_2)$. From the above representation and disjointness property, it follows that

$$\Phi(t,x)\Phi^T(t,x) = \begin{pmatrix} \phi_{1,1}(t,x) & 0 & \cdots & 0 \\ 0 & \phi_{1,2}(t,x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \phi_{m_1,m_2}(t,x) \end{pmatrix}, \tag{9}$$

$$\Phi^T(t,x)\Phi(t,x) = 1, \tag{10}$$

$$\Phi(t,x)\Phi^T(t,x)V = \widetilde{V}\Phi(t,x), \tag{11}$$

where V is an m_1m_2 vector and $\widetilde{V} = \text{diag}(V)$. Moreover, it can be clearly concluded that for every $(m_1m_2) \times (m_1m_2)$ matrix A

$$\Phi^T(t,x)A\Phi(t,x) = \widehat{A}^T\Phi(t,x), \tag{12}$$

where $\widehat{A}$ is an m_1m_2 vector with elements equal to the diagonal entries of matrix A.

2D-BPFs expansion

A function $f(t,x) \in L^2([0, T_1) \times [0, T_2))$ may be expanded by the 2D-BPFs as

$$f(t,x) \simeq \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2} f_{i_1,i_2}\phi_{i_1,i_2}(t,x) = F^T\Phi(t,x) = \Phi^T(t,x)F, \tag{13}$$

where F is a $(m_1m_2) \times 1$ vector given by

$$F = [f_{1,1}, \ldots, f_{1,m_2}, \ldots, f_{m_1,1}, \ldots, f_{m_1,m_2}]^T, \tag{14}$$

and $\Phi(t,x)$ is defined in (8).

The block-pulse coefficients f_{i_1,i_2} are obtained as

$$f_{i_1,i_2} = \frac{1}{h_1h_2}\int_{(i_1-1)h_1}^{i_1h_1}\int_{(i_2-1)h_2}^{i_2h_2} f(t,x)dxdt, \tag{15}$$

such that the error between $f(t,x)$ and its block-pulse expansion (13) in the region of $t\in[0,T_1)$, $y\in[0,T_2)$, i.e,

$$\varepsilon=\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\left(f-\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}f_{i_1,i_2}\phi_{i_1,i_2}(t,x)\right)^2dxdt \tag{16}$$

is minimal.

A function of four variables $k(t,s,x,y)$ on $[0,T_1)\times[0,T_2)\times[0,T_3)\times[0,T_4)$ may be approximated with respect to BPFs such as

$$k(t,s,x,y)=\Phi^T(t,x)K\Phi(s,y), \tag{17}$$

where $\Phi(t,x)$ and $\Phi(s,y)$ are 2D-BPF vectors of dimension m_1m_2 and m_3m_4, respectively, and K is a $(m_1m_2)\times(m_3m_4)$ two dimensional block-pulse coefficient matrix. Also, the positive integer powers of a function $u(s,y)$ may be approximated by 2D-BPFs as

$$[u(s,y)]^p=[\Phi^T(s,y)U]^p=\Phi^T(s,y)\Lambda, \tag{18}$$

where Λ is a column vector, the elements of which are pth power of the elements of the vector U.

Operational matrix of integration

The integration of the vector $\Phi(t,x)$ defined in (3) may be obtained as

$$\int_0^t\int_0^x\Phi(\tau_1,\tau_2)d\tau_1d\tau_2\simeq\Upsilon\Phi(t,x) \tag{19}$$

$$=[E_{(m_1\times m_1)}\otimes E_{(m_2\times m_2)}]\,\Phi(t,x), \tag{20}$$

where Υ is a $(m_1m_2)\times(m_1m_2)$ operational matrix of integration for 2D-BPFs, and E is the operational matrix of 1D-BPFs defined over $[0,1)$ with $h=\frac{1}{m}$ as follows

$$E=\frac{h}{2}\begin{pmatrix}1&2&2&\dots&2\\0&1&2&\dots&2\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&1\end{pmatrix}. \tag{21}$$

In (20), $\otimes$ denotes the Kronecker product defined as

$$A\otimes B=(a_{ij}B). \tag{22}$$

So, the 2D integral of every function $f(t,x)$ can be approximated as follows:

$$\begin{aligned}\int_0^t\int_0^x f(\tau_1,\tau_2)d\tau_1d\tau_2&\simeq\int_0^t\int_0^x F^T\Phi(\tau_1,\tau_2)d\tau_1d\tau_2\\&\simeq F^T\Upsilon\Phi(t,x).\end{aligned} \tag{23}$$

Operational matrix of differentiation

We now need to compute the operational matrix of differentiation. For this, let

$$\begin{aligned}u(t,x)&=U^T\Phi(t,x),\\u(t,0)&=U_{t0}^T\Phi(t,x),\\u(0,x)&=U_{0x}^T\Phi(t,x),\\u_x(t,x)&=U_x^T\Phi(t,x),\\u_t(t,x)&=U_t^T\Phi(t,x),\\u_x(t,0)&=U_{xt0}^T\Phi(t,x),\\u_{xx}(t,x)&=U_{xx}^T\Phi(t,x),\\u_t(0,x)&=U_{t0x}^T\Phi(t,x),\\u_{tt}(t,x)&=U_{tt}^T\Phi(t,x),\\u_{tx}(t,x)&=U_{tx}^T\Phi(t,x).\end{aligned} \tag{24}$$

Now, we can write

$$u(t,x)-u(t,0)=\int_0^x u_x(t,\tau)d\tau. \tag{25}$$

Then, from (24) and (25), we obtain

$$\begin{aligned}U^T\Phi(t,x)-U_{t0}^T\Phi(t,x)&=\int_0^x U_x^T\Phi(t,\tau)d\tau\\&=U_x^T\int_0^x\Phi(t,\tau)d\tau\\&=U_x^TE\Phi(t,x).\end{aligned}$$

So we get

$$U^T-U_{t0}^T=U_x^TE.$$

Hence,

$$U_x^T=(U^T-U_{t0}^T)E^{-1}. \tag{26}$$

Similarly, for the partial derivative of $u(t,x)$ with respect to t, it can be shown that

$$U_t^T=(U^T-U_{0x}^T)E^{-1}. \tag{27}$$

Moreover, for the second-order partial derivatives of $u(t,x)$, the following equations can be written as

$$u_x(t,x)-u_x(t,0)=\int_0^x u_{xx}(t,\tau)d\tau. \tag{28}$$

Using (24) and (28), we have

$$\begin{aligned}U_x^T\Phi(t,x)-U_{xt0}^T\Phi(t,x)&=\int_0^x U_{xx}^T\Phi(t,\tau)d\tau,\\&=U_{xx}^T\int_0^x\Phi(t,\tau)d\tau,\\&=U_{xx}^TE\Phi(t,x),\end{aligned}$$

so we get

$$U_x^T-U_{xt0}^T=U_{xx}^TE.$$

Then,

$$U_{xx}^T = (U_x^T - U_{xt0}^T)E^{-1}. \qquad (29)$$

In similar way, to approximate the second-order partial derivatives of $u(t,x)$ with respect to t, the following equation has been obtained:

$$U_{tt}^T = (U_t^T - U_{t0x}^T)E^{-1}. \qquad (30)$$

Finally, the following procedure can be applied to approximate $u_{tx}(t,x)$:

$$u_t(t,x) - u_t(t,0) = \int_0^x u_{tx}(t,\tau)d\tau. \qquad (31)$$

Hence,

$$\begin{aligned} U_t^T\Phi(t,x) - U_{t0x}^T\Phi(t,x) &= \int_0^x U_{tx}^T\Phi(t,\tau)d\tau, \\ &= U_{tx}^T\int_0^x \Phi(t,\tau)d\tau, \\ &= U_{tx}^T E\Phi(t,x), \end{aligned}$$

so we can obtain

$$U_t^T - U_{t0x}^T = U_{tx}^T E.$$

Then, we have

$$U_{tx}^T = (U_t^T - U_{t0x}^T)E^{-1}. \qquad (32)$$

Applying the method

In this section, we solve the nonlinear two-dimensional Volterra integro-differential equations using 2D-BPFs. As we have shown before, we can write

$$\begin{aligned} u(t,x) &= U^T\Phi(t,x), \\ g(t,x) &= G^T\Phi(t,x), \\ [u(s,y)]^p &= \Phi^T(s,y)\Lambda, \\ u_x(t,x) &= U_x^T\Phi(t,x), \\ u_t(t,x) &= U_t^T\Phi(t,x), \\ u_{xx}(t,x) &= U_{xx}^T\Phi(t,x), \\ u_{tt}(t,x) &= U_{tt}^T\Phi(t,x), \\ u_{tx}(t,x) &= U_{tx}^T\Phi(t,x), \\ k(t,s,x,y) &= \Phi^T(t,x)K\Phi(s,y), \end{aligned}$$

where the m_1m_2 vectors $U, G, \Lambda, U_x, U_t, U_{xx}, U_{tt}, U_{tx}$ and matrix K are the BPF coefficients of $u(t,x), g(t,x)$, $[u(s,y)]^p$, $u_x(t,x)$, $u_t(t,x)$, $u_{xx}(t,x)$, $u_{tt}(t,x), u_{tx}(t,x)$ and $k(x,y,s,t)$, respectively; Λ is a column vector, the elements of which are pth power of the elements of the vector U. Now, consider the following equation:

$$\begin{aligned} u_{xx} + u_{tx} + u_{tt} + u(t,x) = g(t,x) + \int_0^t\int_0^x k(t,s,x,y) \\ \times u^p(s,y)dyds, \\ (t,x) \in [0,A_1]\times[0,A_2]. \end{aligned}$$

Using the proposed equations in section 'Properties of the 2D-BPFs' to approximate the partial derivatives, we have

$$\begin{aligned} &\Phi^T(t,x)(U_{xx} + U_{tx} + U_{tt} + U) \\ &= \Phi^T(t,x)G + \int_0^t\int_0^x k(t,s,x,y)u^p(s,y)dyds, \\ &= \Phi^T(t,x)G + \int_0^t\int_0^x \Phi^T(t,x)K\Phi(s,y)\Phi^T(s,y)\Lambda dyds \\ &= \Phi^T(t,x)G + \Phi^T(t,x)K\int_0^t\int_0^x \Phi(s,y)\Phi^T(s,y)\Lambda dyds \\ &= \Phi^T(t,x)G + \Phi^T(t,x)K\int_0^t\int_0^x \tilde{\Lambda}\Phi(s,y)dyds, \\ &= \Phi^T(t,x)G + \Phi^T(t,x)K\tilde{\Lambda}\int_0^t\int_0^x \Phi(s,y)dyds, \\ &= \Phi^T(t,x)G + \Phi^T(t,x)K\tilde{\Lambda}\Upsilon\Phi(t,x). \end{aligned}$$

If we put $B = K\tilde{\Lambda}\Upsilon$, then it can be written from Equation 12,

$$\begin{aligned} \Phi^T(t,x)(U_{xx} + U_{tx} + U_{tt} + U) &= \Phi^T(t,x)G + \widehat{B}^T\Phi(t,x) \\ &= \Phi^T(t,x)(G + \widehat{B}). \end{aligned}$$

Hence, we have

$$U_{xx} + U_{tx} + U_{tt} + U = G + \widehat{B}. \qquad (33)$$

Now, using Equations 26, 27, 29, 30, 32 and 33, we can obtain a nonlinear system, where the solution can be obtained from Newton-Raphson method.

The error analysis

Here, we investigate the representation error of a differentiable function $f(t,x)$ when it is represented in a series of 2D-BPFs over the region $D = [0,1)\times[0,1)$. For this, we briefly review and use some results from [10,11]. For details, see the mentioned references. We put $m_1 = m_2 = m$, so $h_1 = h_2 = \frac{1}{m}$.

We define the representation error between $f(t,x)$ and its 2D-BPF expansion over every subregion D_{i_1,i_2} as follows:

$$e_{i_1,i_2}(t,x) = f_{i_1,i_2}\phi_{i_1,i_2}(t,x) - f(t,x) = f_{i_1,i_2} - f(t,x), \quad t,x \in D_{i_1,i_2}, \tag{34}$$

where

$$D_{i_1,i_2} = \left\{(t,x) : \frac{i_1-1}{m} \le t < \frac{i_1}{m},\ \frac{i_2-1}{m} \le t < \frac{i_2}{m}\right\}. \tag{35}$$

Using mean value theorem, it can be shown that

$$\|e_{i_1,i_2}\|^2 \le \frac{2}{m^4}M^2, \tag{36}$$

where $\|f'(t,x)\| \le$ M [10,11]. Error between $f(t,x)$ and its 2D-BPF expansion, $f_m(t,x)$, over the region D can be obtained as follows:

$$e(t,x) = f_m(t,x) - f(t,x). \tag{37}$$

Using Equations 36 and 37, it can be shown that (see [10,11])

$$\|e(t,x)\|^2 \le \frac{2}{m^2}M^2. \tag{38}$$

Hence, $\|e(t,x)\| = O(\frac{1}{m})$. Similar to the proposed method in [10,11], suppose that $f(t,x)$ is approximated by

$$f_m(t,x) = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} f_{i_1,i_2}\phi_{i_1,i_2}(t,x),$$

we get $\bar{f}_{i_1,i_2}$, the approximation of f_{i_1,i_2}, and

$$\bar{f}_m(t,x) = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} \bar{f}_{i_1,i_2}\phi_{i_1,i_2}(t,x).$$

Then, from Equation 38 for $(t,x) \in D_{i_1,i_2}$, we have

$$\|\bar{f}_{i_1,i_2}\phi_{i_1,i_2} - f(t,x)\| \le \frac{\sqrt{2}M}{m} + \frac{\|\bar{f}_m - f\|_\infty}{m}. \tag{39}$$

Therefore, from Equation 39, it can be shown that

$$\lim_{x\to\infty} f_m(t,x) = f(t,x).$$

For an error estimation, reconsider the following nonlinear two-dimensional Volterra integro-differential equation

$$u_{xx}(t,x) + u_{tx}(t,x) + u_{tt}(t,x) + u(t,x) = g(t,x) + \int_0^t \int_0^x k(t,s,x,y) \times u^p(s,y)dyds, \quad (t,x) \in [0,1]\times[0,1]. \tag{40}$$

Let $e_m^p(t,x) = u^p(t,x) - u_m^p(t,x)$ be the error function of the approximate solution $u_m(t,x)$ to $u(t,x)$, where $u(t,x)$ is the exact solution of Equation 40. Then, we consider

$$R_m(t,x) + \big(u_{xx}(t,x) + u_{tx}(t,x) + u_{tt}(t,x) + u(t,x)\big)_m = g(t,x) + \int_0^t\int_0^x k(t,s,x,y)u_m^p(s,y)dyds, \tag{41}$$

where $R_m(t,x)$ is the perturbation function that depends on $u_m(t,x)$, $(u_{xx}(t,x))_m$, $(u_{tx}(t,x))_m$ and $(u_{tt}(t,x))_m$. It can be obtained by substituting $u_m(t,x)$, $(u_{xx}(t,x))_m$, $(u_{tx}(t,x))_m$ and $(u_{tt}(t,x))_m$ into Equation 40 as

$$R_m(t,x) = g(t,x) + \int_0^t\int_0^x k(t,s,x,y)u_m^p(s,y)dyds - \big(u_{xx}(t,x) + u_{tx}(t,x) + u_{tt}(t,x) + u(t,x)\big)_m.$$

Subtracting (41) from (40) gives

$$\int_0^t\int_0^x k(t,s,x,y)e_m^p(s,y)dyds = -R_m(t,x) - \big(e_{xx}(t,x) + e_{tx}(t,x) + e_{tt}(t,x) + e(t,x)\big)_m. \tag{42}$$

Finally, the proposed method in this paper can be applied to approximate $e_m(t,x)$ in Equation 42.

Numerical results

In this section, three examples are given to show the the accuracy of the proposed method. For the all examples, we consider the supplementary conditions from the exact solution. The absolute error is computed for $m = m_1 = m_2$ terms of 2D-BPF series in all examples. All computations are implemented in MATLAB software on a personal computer.

Example 1. For the first example, consider the following equation [1]:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^2 u(x,t)}{\partial t^2} + \int_0^x\int_0^t x^2 t u(s,y)dsdy = g(x,t), \quad x,t \in [0,1],$$

Table 1 Absolute errors for example 1

(x,t)	$\lvert e_{5,5}(x,y)\rvert$ [1]	$\lvert e_{6,6}(x,y)\rvert$ [1]	$m=32$	$m=64$
(0.01, 0.01)	1.666×10^{-7}	1.666×10^{-7}	2.238×10^{-8}	2.124×10^{-8}
(0.02, 0.02)	1.333×10^{-6}	1.333×10^{-6}	7.980×10^{-8}	7.742×10^{-8}
(0.1, 0.1)	1.670×10^{-4}	1.666×10^{-4}	6.809×10^{-6}	4.714×10^{-6}
(0.2, 0.2)	1.347×10^{-3}	1.332×10^{-3}	2.334×10^{-4}	2.230×10^{-4}

Table 2 Absolute errors for example 2

(x,t)	$m=16$	$m=32$	$m=64$
(0.01, 0.01)	5.136×10^{-7}	8.903×10^{-8}	7.378×10^{-8}
(0.02, 0.02)	1.307×10^{-6}	2.918×10^{-7}	8.703×10^{-8}
(0.1, 0.1)	5.563×10^{-5}	1.809×10^{-6}	1.714×10^{-6}
(0.2, 0.2)	2.973×10^{-3}	1.334×10^{-4}	1.230×10^{-4}

where

$$g(x,t)=x\,exp(t)-\frac{1}{2}x^4t+\frac{1}{2}x^4t\,exp(t),$$

with subject to the initial conditions

$$u(0,t)=0,\quad \frac{\partial u(x,1)}{\partial x}=exp(1).$$

The exact solution of this problem is $u(x,t)=x\,exp(t)$. The numerical results of this problem is shown in Table 1.

Example 2. In this example, we consider a two-dimensional nonlinear Volterra integro-differential equation as follows:

$$\frac{\partial^2 u(x,t)}{\partial t^2}+u(x,t)-\int_0^t\int_0^x (y+\cos z)u^2(y,z)dydz=g(x,t),\qquad x,t\in[0,1],$$

where

$$g(x,t)=\frac{1}{8}x^4\sin t\cos t-\frac{1}{8}x^4t-\frac{1}{9}x^3\sin^3 t.$$

With supplementary conditions,

$$u(x,0)=0,\quad \frac{\partial u}{\partial t}(x,0)=x.$$

The exact solution of this problem is $u(x,t)=x\sin t$. In Table 2, the numerical results are presented.

Conclusion

In this paper, we have successfully approximated the solution of the form (1) of nonlinear Volterra integro-differential equations. To this end, we have used some orthogonal functions called block-pulse functions. Moreover, the error of the proposed method is analyzed. For more investigation, some examples have been presented. As the numerical results showed, the proposed method is an effective method to solve the Volterra integro-differential equations.

Competing interest

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed suitably and significantly in writing this article. Both authors read and approved the final manuscript.

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