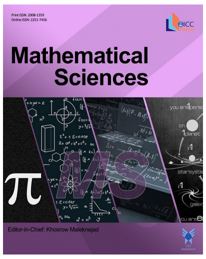

Math. Sci. 2026; 20(2): 108-117

Mathematical Sciences (MS)

https://doi.org/10.57647/mathsci.2026.92501

Research Article

New Bounded Distribution: Unit-Weighted Lomax Distribution with Application to COVID-19 Recovery Rates Data

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Article History

Received:
9 September 2025
Revised:
26 December 2025
Accepted:
30 January 2026

Abstract:

This paper proposes a new bounded probability model, the Unit-Weighted Lomax (UWLx) distribution, constructed via transformation of the weighted Lomax distribution. The UWLx distribution is designed for data restricted to the unit interval and provides greater flexibility in representing diverse behaviors of density and hazard functions, including monotonic and bathtub-shaped forms. Key statistical properties such as survival and hazard rate functions, moments, and order statistics are derived. Parameter estimation is addressed through the maximum likelihood method, and a simulation study confirms the consistency and efficiency of the estimators. The practical utility of the model is demonstrated through an application to COVID-19 recovery rate data, where the UWLx distribution is compared against well-known unit distributions, including the Beta, Kumaraswamy, and Unit Weibull. Model selection criteria and goodness-of-fit tests show that the UWLx distribution yields a superior fit, underscoring its potential as a versatile tool for analyzing bounded data in medical, reliability, and related fields.

Keywords: Weighted Lomax distribution; Unit distribution; Maximum likelihood estimation; Goodness of fit; COVID-19

Cite this article: N. M. Kilany, A. M. A. Aldawsari, R. Alotaibi, L. H. El-Refai New Bounded Distribution: Unit-Weighted Lomax Distribution with Application to COVID-19 Recovery Rates Data. Math. Sci 2026; *20*(2): 108-117 https://doi.org/10.57647/mathsci.2026.92501

1. Introduction

Modeling bounded data has long been an important area of study in applied statistics, particularly for variables constrained to the unit interval such as proportions, probabilities, and recovery rates. Traditional distributions like the Beta, Kumaraswamy, and Unit Logistic have been widely used for such cases, yet their flexibility is sometimes limited when data exhibit more complex behaviors such as heavy tails or bathtub-shaped hazard functions. To address these challenges, researchers have increasingly developed new families of unit distributions by transforming or extending existing models. Over time, numerous unit distributions have been proposed to enhance flexibility and model diverse data behaviors within this domain. For example, the unit inverse power lomax distribution was derived by Abdelall et al. [1]. Fayomi et al. [2] investigated Bayesian and data analysis of the unit power Burr X distribution. The unit logistic distribution was introduced by Menezes et al. [3], while the unit Lindley distribution was derived by Mazucheli et al. [4]. Hassan et al. [5] focused on Bayesian and non Bayesian inference for unit Exponentiated half- logistic distribution. Saeed et al. [6] discussed bounded sine hyperbolic distribution, highlighting its potential in modeling data with restricted support. Similarly, Saeed et al. [7], Novel parameter based trigonometric probability distribution was examined with bounded support. Mazucheli et al. [8] studied the unit Gompertz distribution with various applications, and Modi and Gill [9] presented the unit Burr-III model. The unit exponentiated Lomax distribution was implemented by Fayomi et al. [10] ,and Ul Haq et al. [11] later proposed a modified version, the unit modified Burr-III distribution. Hassan and Alharbi [12] provided different estimation methods for the unit inverse exponentiated Weibull distribution. Additionally, Karakaya et al. [13] conducted a regression analysis on the power new power function distribution for data on (0,1), and Ahmed et al. [14] introduced the unit exponential Pareto distribution, demonstrating its practical utility in modeling bounded datasets (see also [15],[16]).

The complexity of bounded data, especially when exhibiting patterns such as bathtub-shaped hazard rates, presents a challenge for statistical traditional distributions. In this research, a new model, termed the *Unit Weighted Lomax (UWLx) distribution*, is proposed to address this issue. This new model is derived from the Weighted Lomax (WLx) distribution through a suitable transformation. The UWLx distribution provides a good fit for modeling data that is known to be bounded, such as survival times or waiting times. Moreover, it demonstrates greater flexibility in modeling complex behaviors, including the capability of capturing bathtub-shaped hazard rate patterns.

The importance of WLx distribution [17], which is an extension of Lomax distribution [18],lies in its ability to provide accurate modeling for complex lifetime and survival data. The WLx model is particularly valuable in biomedical research and engineering applications. In addition, it offers statisticians a flexible tool for simulating and analyzing heavy-tailed data, making it both a practical and theoretically appealing distribution (see also, Aljohani [19]).

The probability density function (pdf) and cumulative distribution function (cdf), assuming Y is a random variable with a WLx distribution, are as follows:

$$g(y) = \frac{\Gamma(\alpha+1)\lambda^{1+\alpha-\beta}}{\Gamma(1+\alpha-\beta)\Gamma(\beta)}\left(\frac{y^{\beta-1}}{(y+\lambda)^{\alpha+1}}\right), \tag{1}$$
$$y \geq 0,\ \ \lambda > 0,\ \ \ \alpha > 0,\ \ 0 < \beta < \alpha + 1.$$

$$G(y) = \frac{\Gamma(\alpha+1)\lambda^{-\beta}y^{\beta}\ {}_2F_1(\alpha+1,\beta,\beta+1;-\frac{y}{\lambda})}{\beta\Gamma(1+\alpha-\beta)\Gamma(\beta)}, \tag{2}$$
$$y \geq 0,\ \ \lambda > 0,\ \ \ \alpha > 0,\ \ 0 < \beta < \alpha + 1.$$

where, $\Gamma(.)$ is a complete gamma function and ${}_2F_1(a, b, c; z)$ is the hypergeometric function.

This research is outlined in many sections: In Section 2, the statistical characteristics and the reliability measures are evaluated. In Section 3, the maximum likelihood method is utilized in determining the maximum likelihood estimators. In Section 4, the distribution of order statistics and extreme values of UWLx distribution is discussed. In Section 5, Simulation study is introduced. Finally, in Section 6 we test the advantage of using UWLx for modeling bounded data through using COVID-19 data and discussing the comparison of its results with other distributions.

2. Unit Weighted Lomax Distribution

Let Y be a random variable following the Weighted Lomax (WLx) distribution with parameters α, β, and λ, and pdf

$$f_Y(y) = \frac{\lambda^{\alpha-\beta+1}\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)\Gamma(\beta)}\, y^{\beta-1}\left(1+\frac{y}{\lambda}\right)^{-(\alpha+1)} \quad y > 0.$$

Consider the transformation

$$X = \frac{Y}{1+Y}.$$

Using the change-of-variables formula,

$$f_X(x) = f_Y\left(\frac{x}{1-x}\right)\left|\frac{d}{dx}\left(\frac{x}{1-x}\right)\right| = f_Y\left(\frac{x}{1-x}\right)\frac{1}{(1-x)^2}.$$

Substituting $y = \frac{x}{1-x}$ into $f_Y(y)$ gives

$$\begin{aligned} f_X(x) &= \frac{\lambda^{\alpha-\beta+1}\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)\Gamma(\beta)}\left(\frac{x}{1-x}\right)^{\beta-1} \\ &\quad \times\left(1+\frac{x}{\lambda(1-x)}\right)^{-(\alpha+1)}\frac{1}{(1-x)^2} \\ &= \frac{\lambda^{\alpha-\beta+1}\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)\Gamma(\beta)}\,\frac{x^{\beta-1}(1-x)^{\alpha-\beta}}{(\lambda(1-x)+x)^{\alpha+1}}, \quad 0 < x < 1. \end{aligned}$$

Hence, the pdf of the unit-weighted Lomax (UWLx) distribution is

$$f(x) = \frac{\lambda^{\alpha-\beta+1}x^{\beta-1}(1-x)^{\alpha-\beta}\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)\Gamma(\beta)\,(\lambda(1-x)+x)^{\alpha+1}},$$
$$0 < x < 1,\ \alpha > 0,\ \beta > 0,\ \lambda > 0. \tag{3}$$

By integrating the pdf $f(x)$ of the UWLx distribution, the cumulative distribution function (CDF) can be expressed in closed form in terms of the Gauss hypergeometric function ${}_2F_1(\cdot)$. After applying the appropriate substitution and simplifying the resulting integral, the CDF is obtained as

$$F(x) = \frac{(\frac{x}{1-x})^{\beta}\lambda^{\beta}\Gamma(1+\alpha){}_2F_1\left(1+\alpha,\beta,1+\beta;\frac{-x}{(1-x)\lambda}\right)}{\beta\Gamma(1+\alpha-\beta)\Gamma(\beta)}. \tag{4}$$

where, the Gauss hypergeometric Function ${}_2F_1(a, b; c; z)$ is defined from equation

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad |z| < 1,$$

All special-function computations are performed using Mathematica program. We notice that there are many shapes of the Pdf for UWLx such as negative skewness and positive skewness as well as decreasing curve. The UWLx distribution's many pdf forms are displayed in Figure 1.

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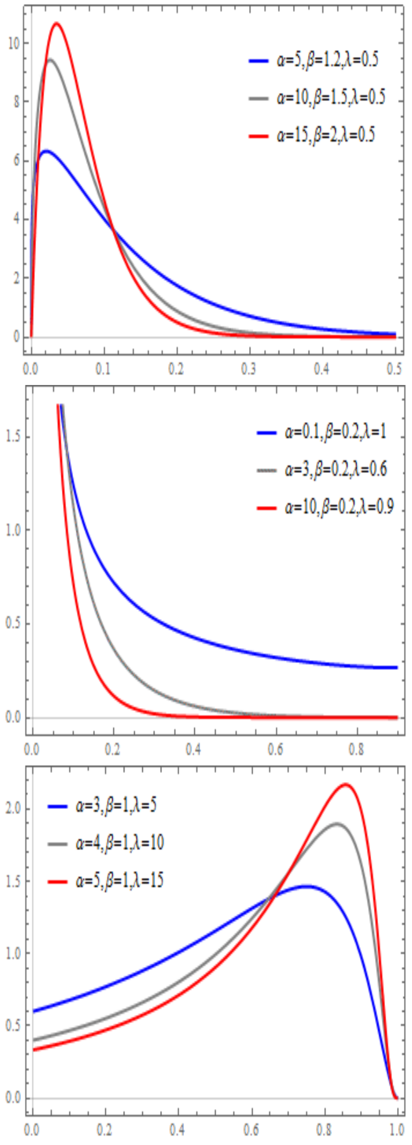

Figure 1. Density function of UWLx distribution.

2.1 Survival and Hazard Rate Functions

Survival function provides the probability that the random variable X exceeds the detected time x which is observed as follows

$$S(x) = P(X > x) = 1 - F(x) = 1 - \frac{\left(\frac{x}{1-x}\right)^{\beta} \lambda^{\beta}\Gamma(1+\alpha){}_2F_1\left(1+\alpha, \beta, 1+\beta; \frac{-x}{\lambda(1-x)}\right)}{\beta\Gamma(1+\alpha-\beta)\Gamma(\beta)}. \tag{5}$$

In addition, The hazard rate is a conditional probability, meaning it depends on the condition that the event hasn't occurred yet at time t.It's widely used in fields like engineering, medicine, finance, and insurance to model and analyze the time until an event occurs. Reliability and survival analysis frequently use it to evaluate the risk of failure over time. The hazard rate function $h(x)$ of UWLx model is created as follows,

$$h(x) = \frac{f(x)}{S(x)} \tag{6}$$

$$= \frac{\lambda^{\alpha-\beta+1}x^{\beta-1}(1-x)^{\alpha-\beta}\,\Gamma(1+\alpha)(\lambda(1-x)+x)^{\alpha+1}}{\Gamma(1+\alpha+\beta)\Gamma(\beta)\left(1 - \frac{\left(\frac{x}{1-x}\right)^{\beta}\lambda^{\beta}\Gamma(1+\alpha){}_2F_1\left(1+\alpha,\beta,1+\beta;\frac{-x}{\lambda(1-x)}\right)}{\beta\Gamma(1+\alpha-\beta)\Gamma(\beta)}\right)}.$$

Figure 2 clarifies the shapes of survival function of UWLx distribution. Also, Figure 3 displays the hazard rate function shapes for UWLx distribution which are bathtub and increasing shaped.

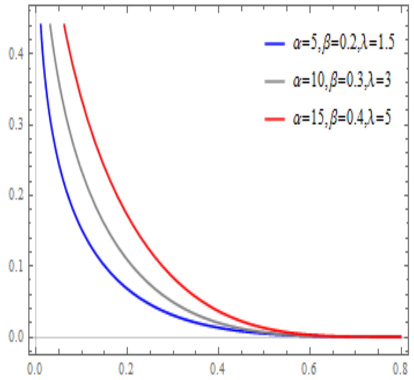

Figure 2. Survival function of UWLx model.

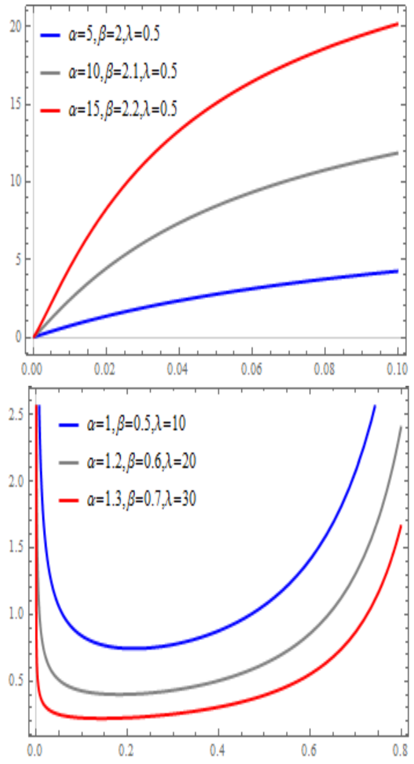

Figure 3. Hazard ratefunction of UWLx model.

2.2 Mean Residual Life Function

A helpful tool for survival and reliability analysis is the mean residual life function. It can be used to assess the risk of failure over time, and it can also be used to make decisions about maintenance and replacement. Given that the age of an object is X, the remaining life after age X is called the residual life. In a different way , if the life of the object is represented by X, then the mean residual life is calculated from the equation,

$$MRL = E(X - x/X > x) = \frac{1}{S(x)} \int_x^1 y f(y) dy - x,$$

$$MRL = -x + \frac{\beta \Gamma(1+\alpha-\beta)\Gamma(\beta)(W+V)}{\lambda\beta\Gamma(1+\alpha-\beta)\Gamma(\beta) - \left(\frac{x}{1-x}\right)^{\beta} \lambda^{-\beta+1} \Gamma(1+\alpha)\, {}_2F_1\left(1+\alpha, \beta, 1+\beta; \frac{x}{(x-1)\lambda}\right)}. \tag{7}$$

where, $W = -\frac{x^{\beta+1}\left(1+x\left(\frac{1}{\lambda}-1\right)\right)^{\alpha} F_1\left(\beta+1; -\alpha+\beta, \alpha+1; \beta+2; x, \frac{x(\lambda-1)}{\lambda}\right)}{\beta+1}$
and
$V = \left(\frac{1}{\lambda}\right)^{\alpha} \Gamma(1+\alpha-\beta)\Gamma(1+\alpha)\, {}_2\tilde{F}_1\left(\alpha+1, \beta+1, \alpha+2; \frac{\lambda}{1-\lambda}\right).$
Note that, ${}_2\tilde{F}_1(a, b, c; z)$ is the regularized hypergeometric function and $F_1(a; b_1, b_2; c; z_1, z_2)$ is the appell hypergeometric function of two random variables. The forms of the mean residual life function for the UWLx model at different parameter values are shown in Figure 4. Figure 4 shows that MRL curves are increasing for many parameter settings.

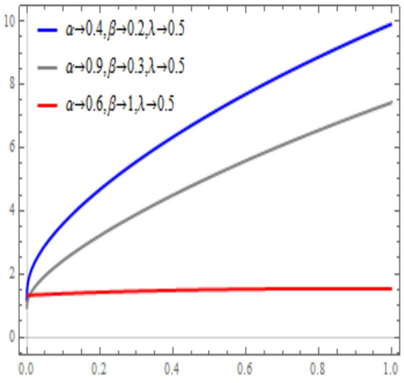

Figure 4. Mean Residual Life function of UWLx distribution.

2.3 Moments

Moments are a set of measures which are used to characterize different statistical features of a frequency distribution. The rth moment about the origin is calculated from

$$\mu_r' = E(X^r) = \int_0^1 x^r f(x) dx, r = 1, 2, ..., n,$$

The r^{th} moment about the origin of UWLx distribution is provided as

$$\mu_r' = \frac{\lambda^{\alpha-\beta+1}\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)\Gamma(\beta)} \times \int_0^1 x^{r+\beta-1}(1-x)^{\alpha-\beta}(\lambda(1-x)+x)^{-(\alpha+1)} dx, \quad r = 1, 2, ...n,$$

let

$$I = \int_0^1 x^{r+\beta-1}(1-x)^{\alpha-\beta}(\lambda(1-x)+x)^{-(\alpha+1)} dx \tag{8}$$

$$= \lambda^{-(\alpha+1)} \int_0^1 x^{r+\beta-1}(1-x)^{\alpha-\beta}\left(1 - \frac{\lambda-1}{\lambda}x\right)^{-(\alpha+1)} dx$$

$$= B(r+\beta, \alpha-\beta+1)\, {}_2F_1\left(\alpha+1, r+\beta; r+\alpha+1; \frac{\lambda-1}{\lambda}\right).$$

Since

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},$$

we obtain

$$I = \frac{\Gamma(r+\beta)\Gamma(\alpha-\beta+1)}{\Gamma(r+\alpha+1)} \times {}_2F_1\left(\alpha+1, r+\beta; r+\alpha+1; \frac{\lambda-1}{\lambda}\right).$$

then,

$$\mu_r' = \frac{\lambda^{-\beta}\Gamma(1+\alpha)\Gamma(r+\beta)}{\Gamma(\beta)} \times {}_2F_1\left(\alpha+1,\ r+\beta;\ r+\alpha+1;\ \frac{\lambda-1}{\lambda}\right), \quad r = 1, 2, ...n,$$

where, ${}_2F_1(a, b; c; z)$ is the hypergeometric function.

The first raw moment and the second raw moment supply some information about the distribution's location, variability, and appearance .The third and the forth raw moments introduce some information on the shape of distribution. The r^{th} moment of UWLx model is defined by

$$\mu_1' = \frac{\lambda^{-\beta}\Gamma(\alpha+1)\Gamma(\beta+1)\, {}_2F_1(\alpha+1, \beta+1, \alpha+2; \frac{\lambda-1}{\lambda})}{\Gamma(\beta)}.$$

$$\mu_2' = \frac{\lambda^{-\beta}\Gamma(\alpha+1)\Gamma(\beta+2)\, {}_2F_1(\alpha+1, \beta+2, \alpha+3; \frac{\lambda-1}{\lambda})}{\Gamma(\beta)}.$$

$$\mu_3' = \frac{\lambda^{-\beta}\Gamma(\alpha+1)\Gamma(\beta+3)\, {}_2F_1(\alpha+1, \beta+3, \alpha+4; \frac{\lambda-1}{\lambda})}{\Gamma(\beta)}.$$

$$\vdots$$

$$\mu_n' = \frac{\lambda^{-\beta}\Gamma(\alpha+1)\Gamma(\beta+n)\, {}_2F_1(\alpha+1, \beta+n, \alpha+(n+1); \frac{\lambda-1}{\lambda})}{\Gamma(\beta)}.$$

The mean, variance, skewness, and kurtosis attitudes in relation to α are shown in Figure 5. Figure 5 show that that the mean decreases, indicating shorter expected lifetime when the scale parameter becomes larger. The variance widens , reflecting higher variability in the distribution. The plot illustrate that the distribution tends to be right-skewed while higher α reduces skewness and makes the shape more symmetric. Similarly, kurtosis values are high, indicating a heavy-tailed distribution.

3. Maximum Likelihood Estimation

Using a random sample from the UWLx distribution of size n, denoted as $x_1, x_2, ..., x_n$, the likelihood function is deduced from the formula

$$L = \prod_{i=1}^{n} f(x_i, \alpha, \beta, \lambda),$$

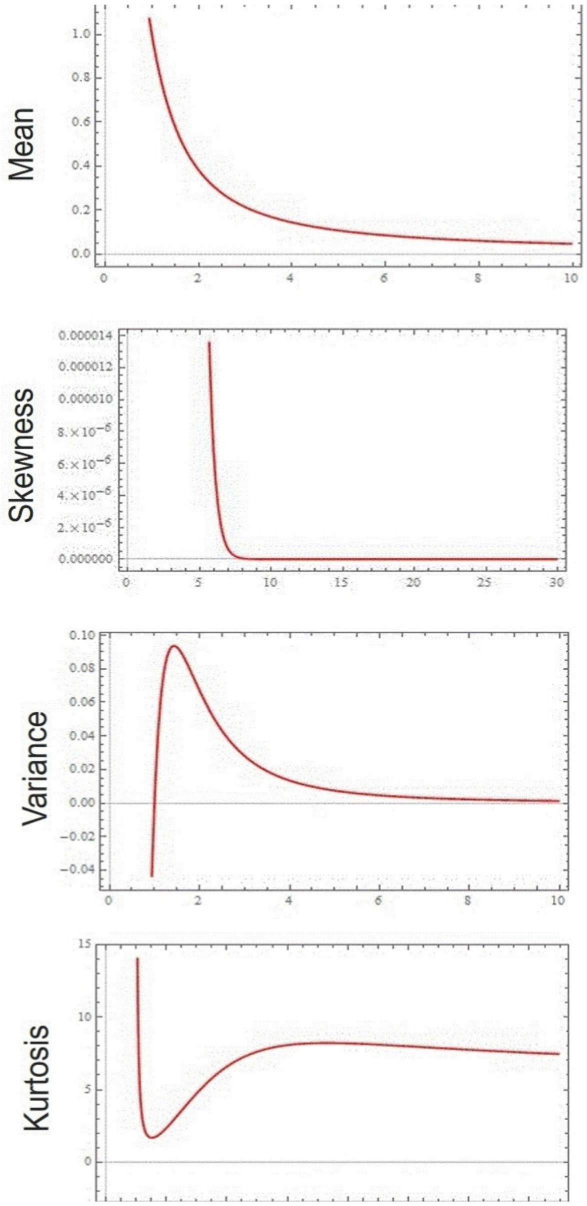

Figure 5. The mean, variance, skewness, and kurtosis attitudes as a function ofα.

Using the Pdf Eq. (3), the likelihood function is shown as,

$$L = \prod_{i=1}^{n} \frac{\lambda^{\alpha-\beta+1}\Gamma(\alpha+1)(1-x_i)^{\alpha-\beta}x_i^{\beta-1}(x_i+(1+x_i)\lambda)^{-1-\alpha}}{\Gamma(\alpha-\beta+1)\Gamma(\beta)}. \tag{9}$$

By using the natural logarithm for Eq. (9), the log-likelihood function can be formed as

$$\begin{aligned} l(\alpha,\beta,x) &= n(\alpha-\beta+1)\ln(\lambda) + n\ln(\Gamma(\alpha+1)) \\ &-n\ln(\Gamma(\alpha-\beta+1)) - n\ln(\Gamma(\beta)) \\ &+(\beta-1)\sum_{i=1}^{n}\ln(x_i) + (\alpha-\beta)\sum_{i=1}^{n}\ln(1-x_i) \\ &-(\alpha+1)\sum_{i=1}^{n}\ln(\lambda(1-x_i)+x_i). \end{aligned} \tag{10}$$

By having a solution for the following equations. we get the maximum likelihood estimators $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$.

$$\begin{aligned} &n\ln(\lambda) + n\Psi(1+\alpha) - n\Psi(1+\alpha-\beta) \\ &+\sum_{i=1}^{n}\ln(1-x_i) - \sum_{i=1}^{n}\ln(\lambda(1-x_i)+x_i) = 0. \end{aligned} \tag{11}$$

$$\begin{aligned} &-n\ln(\lambda) + n\Psi(1+\alpha-\beta) \\ &+n\Psi(\beta)\sum_{i=1}^{n}\ln(1-x_i) + \sum_{i=1}^{n}\ln(x_i) = 0. \end{aligned} \tag{12}$$

where, $\Psi(z)$ is the digamma function.

$$\frac{n(1+\alpha-\beta)}{\lambda} - (1+\alpha)\sum_{i=1}^{n}\frac{1-x_i}{\lambda(1-x_i)+x_i} = 0. \tag{13}$$

4. Order Statistics and Extreme Values

In statistical applications, Extreme value distributions are crucial tools in statistics for studying and modeling very large or very small values within a dataset. This section explains the cumulative distribution function's and the probability density function's order values. The purpose of extreme value distributions is to construct the minimum and maximum limiting distribution from the UWLx model.

4.1 Probability and Cumulative Function of Order Statistics

If $X_1, X_2, ..., X_n$ is a random sample drawn from the UWLx distribution, then $X_{1:n} < X_{2:n} < ...... < X_{n:n}$ is the order statistics of the UWLx sample.The jth order statistics' pdf and cdf are displayed as follows:

$$\begin{aligned} f_Y(y) &= \frac{n!}{(j-1)!(n-j)!}F^{j-1}(y)\{1-F(y)\}^{n-j}f(y), \\ &= \frac{\lambda^{\alpha+1}\Gamma(n+1)\left(\frac{1}{1-y}\right)^{-\beta}(1-y)^{\alpha-\beta}(\lambda-\lambda y+y)^{-\alpha-1}}{y\Gamma(\beta)\Gamma(j)\Gamma(-j+n+1)\,{}_2\tilde{F}_1\left(\alpha+1,\beta;\beta+1;\frac{y}{(y-1)\lambda}\right)} \\ &\times\left(\frac{\Gamma(\alpha+1)\lambda^{-\beta}\left(\frac{y}{1-y}\right)^{\beta}{}_2\tilde{F}_1\left(\alpha+1,\beta;\beta+1;\frac{y}{(y-1)\lambda}\right)}{\Gamma(\alpha-\beta+1)}\right)^{j} \\ &\times\left(1-\frac{\Gamma(\alpha+1)\lambda^{-\beta}\left(\frac{y}{1-y}\right)^{\beta}{}_2\tilde{F}_1\left(\alpha+1,\beta;\beta+1;\frac{y}{(y-1)\lambda}\right)}{\Gamma(\alpha-\beta+1)}\right)^{n-j}. \end{aligned} \tag{14}$$

and

$$\begin{aligned} F_Y(y) &= \sum_{m=j}^{n}\left(\begin{array}{c} n \\ m \end{array}\right)F^m(y)\times[1-F(y)]^{n-m}, \\ &= \frac{\Gamma(n+1)\left(\frac{\Gamma(\alpha+1)\lambda^{-\beta}\left(\frac{y}{1-y}\right)^{\beta}{}_2\tilde{F}_1\left(\alpha+1,\beta;\beta+1;\frac{y}{(y-1)\lambda}\right)}{\Gamma(\alpha-\beta+1)}\right)^{j}}{\Gamma(-j+n+1)} \\ &\times{}_2\tilde{F}_1\left(1,j-n;j+1;\frac{1}{1-\frac{\left(\frac{y}{1-y}\right)^{-\beta}\lambda^{\beta}\Gamma(\alpha-\beta+1)}{\Gamma(\alpha+1)\,{}_2\tilde{F}_1\left(\alpha+1,\beta;\beta+1;\frac{y}{(y-1)\lambda}\right)}}\right) \end{aligned}$$

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$$\times\left(1-\frac{\Gamma(\alpha+1)\lambda^{-\beta}\left(\frac{y}{1-y}\right)^{\beta}\ {}_2\tilde{F}_1\left(\alpha+1,\beta;\beta+1;\frac{y}{(y-1)\lambda}\right)}{\Gamma(\alpha-\beta+1)}\right)^{n-j}. \tag{15}$$

4.2 Limiting Distribution of Extreme Values

Suppose that M_n and m_n are the maximum and minimum of the sample generated from UWLx distribution. Limiting distributions for extreme values essentially describe the behavior of the maximum or minimum values of a dataset as the sample size grows infinitely large. Theorem 4.1 will show this.

Theorem 4.1 *A random sample drawn from the UWLx model, the limiting distribution of the minimum and maximum, m_n and M_n, are given by, respectively*

(i) $\lim\limits_{n\to\infty}\left(\frac{m_n-a_n}{b_n}\le x\right)=1-\exp(-x^{\beta})$.

(ii) $\lim\limits_{n\to\infty}\left(\frac{M_n-c_n}{d_n}\le x\right)=\exp(-e^{-x})$.

where,

$$a_n=0,\ b_n=\frac{1}{F^{-1}(n)},\ c_n=F^{-1}(1-\tfrac{1}{n}),\ d_n=\frac{1}{nf(c_n)}.$$

Proof: For the UWLx distribution we have

(i) $\lim\limits_{\varepsilon\to0^+}\frac{F(F^{-1}(0)+\varepsilon x)}{F(F^{-1}(0)+\varepsilon)}=\lim\limits_{\varepsilon\to0^+}\frac{F(\varepsilon x)}{F(\varepsilon)}$

Applying L′Hospital rule, we have

$$\lim_{\varepsilon\to0^+}\frac{xf(\varepsilon x)}{f(\varepsilon)}=\lim_{\varepsilon\to0^+}\frac{x^{\beta}(1-\varepsilon x)(\lambda(1-\varepsilon)+\varepsilon)^{\alpha+1}}{(\lambda(1-\varepsilon x)+\varepsilon x)^{\alpha+1}(1-\varepsilon)^{\alpha-\beta}}=x^{\beta}.$$

Thus, based on Theorem 8.3.6 of Arnold et al. [20], The Weibull distribution, which provides part (i), is the smallest domain of attraction of the UWLx distribution.

(ii) $\lim\limits_{x\to\infty}\frac{d}{dx}\{\frac{1}{h(x)}\}=$

$$\lim_{x\to\infty}\frac{\beta\lambda^{\alpha+1}\Gamma(\alpha+1)x^{\beta-1}(1-x)^{\alpha-\beta}(\lambda-\lambda x+x)^{-\alpha-1}}{\beta\lambda^{\beta}\Gamma(\beta)\Gamma(\alpha-\beta+1)-\Gamma(\alpha+1)\left(\frac{x}{1-x}\right)^{\beta}\ {}_2F_1\left(\alpha+1,\beta;\beta+1;\frac{x}{(x-1)\lambda}\right)}=0.$$

According to Arnold et al. [20], Theorem 8.3.6, the standard Gumbel distribution providing part (ii) is the maximal domain of attraction of the UWLx distribution.

5. Simulation Study

The simulation aims to study the behavior of the UWLx distribution under varying sample sizes. The following steps provide a summary of the simulation study:

- **Step 1**: The UWLx distribution is defined by parameters α, β, and λ, which are fixed at 0.06, 0.2, and 0.01 respectively for this study.

- **Step 2**: The simulation considers multiple sample sizes (n) ranging from 50 to 190.

- **Step 3**: For each sample size, 1000 samples (N) are generated.

- **Step 4**: The inverse transform method is used to generate random variates from the UWLx distribution. In order to achieve this, we must solve the equation $F(x)-u=0$ for x, in where u is a random uniform integer and $F(x)$ is the UWLx CDF.

The listed measures below are concluded:

- The average bias of the parameters α, β, and λ is, respectively, $\overset{\wedge}{\alpha}$, $\overset{\wedge}{\beta}$, and $\overset{\wedge}{\lambda}$.

$$\frac{1}{N}\sum_{i=1}^{N}(\overset{\wedge}{\alpha}-\alpha),\quad \frac{1}{N}\sum_{i=1}^{N}(\overset{\wedge}{\beta}-\beta)\quad\text{and}\quad \frac{1}{N}\sum_{i=1}^{N}(\overset{\wedge}{\lambda}-\lambda)$$

- The following are the mean square errors (MSE) of $\overset{\wedge}{\alpha}$, $\overset{\wedge}{\beta}$ and $\overset{\wedge}{\lambda}$ for the parameters α, β and λ:

$$\frac{1}{N}\sum_{i=1}^{N}(\overset{\wedge}{\alpha}-\alpha)^2,\quad \frac{1}{N}\sum_{i=1}^{N}(\overset{\wedge}{\beta}-\beta)^2\quad\text{and}\quad \frac{1}{N}\sum_{i=1}^{N}(\overset{\wedge}{\lambda}-\lambda)^2$$

Table 1 presents the estimated bias and mean squared error (MSE) of the maximum likelihood estimators for the parameters (α,β,λ) of the UWLx distribution across different sample sizes. The results indicate that all three estimators perform well, with biases close to zero and MSE values that steadily decrease as the sample size increases. This monotonic decline in MSE demonstrates the desirable large-sample property of consistency for the MLEs. Furthermore, the reduction in both bias and variability with larger n confirms that the estimation procedure is stable and numerically well-behaved even for small parameter values. Overall, the simulation provides strong empirical support for the reliability and accuracy of the proposed estimation method for the UWLx model.

6. COVID-19 Application

The rapid spread of Corona Virus (COVID-19) all over the world, makes its extremely important to limit its Suffusion . Tracking the true recovery rates for Corona Virus patients is a very tricky point through the world. Table 2, shows the data from 3th March to 7th May, 2020 (see [21]) on Spain for COVID-19 patients recovery rates.

This section, aims to assess the suitability of the newly proposed UWLx distribution for modeling COVID-19 data, which is inherently bounded (likely between 0 and 1) . Some well known existing distribution such as Beta distribution, Kumaraswamy distribution, unit Burr III distribution, unit Weibull distribution, unit Lindley distribution, unit improved second-degree Lindley distribution. distribution are compared with UWLx distribution to interpret the flexibility of UWLx model in modeling bounded data. The UWLx distribution was fitted to Covid-19 data using MLE, Kolmogorov-Smirnov (K-s), Anderson-Darling (A_n^2) , Cramér-von Mises (W_n^2) and Watson (U_n^2) tests statistics. For comparison purpose, the following fundamental life distributions are used this comparison

- Beta distribution [22]

$$f(x)=\frac{\Gamma(\delta+\gamma)}{\Gamma(\delta).\Gamma(\gamma)}x^{\delta-1}(1-x)^{\gamma-1},\qquad 0<x<1,\delta>0,\gamma>0. \tag{16}$$

- Kumaraswamy distribution [23]

Table 1. Bias and MSE for estimation of parameters α, β and λ.

α	β	λ	n	$Bias(\alpha)$	$MSE(\alpha)$	Bias(β)	MSE(β)	Bias(λ)	MSE(λ)
0.06	0.2	0.01	50	−0.005	0.0010	0.010	0.0030	0.018	0.0040
			70	−0.004	0.0008	0.009	0.0020	0.016	0.0030
			90	−0.003	0.0006	0.007	0.0015	0.014	0.0025
			150	−0.002	0.0004	0.006	0.0012	0.012	0.0020
			170	−0.002	0.0003	0.005	0.0010	0.011	0.0018
			190	−0.001	0.0002	0.005	0.0010	0.010	0.0016

Table 2. COVID-19 Patients recovery rates

0.6670	0.5000	0.5000	0.4286	0.7500	0.6531
0.5161	0.7895	0.7689	0.6873	0.5200	0.7251
0.6375	0.6078	0.6289	0.5712	0.5923	0.6061
0.5924	0.5921	0.5592	0.5954	0.6164	0.6455
0.6725	0.6838	0.6850	0.6947	0.7210	0.7315
0.7412	0.7508	0.7519	0.7547	0.7645	0.7715
0.7759	0.7807	0.7838	0.7847	0.7871	0.7902
0.7934	0.7913	0.7962	0.7971	0.7977	0.8007
0.8038	0.8289	0.8322	0.8354	0.8371	0.8387
0.8456	0.8490	0.8535	0.8547	0.8564	0.8580
0.8604	0.8628	0.6586	0.7070	0.7963	0.8516

$$f(x) = abx^{a-1}\left(1-x^{a}\right)^{b-1}, \qquad (17)$$
$$0 < x < 1,\ a > 0,\ b > 0.$$

- unit Burr III (UBIII) distribution

$$f(x) = \alpha\beta\frac{1}{x^2}\left(\frac{1}{x}-1\right)^{\beta-1}\left[1+\left(\frac{1}{x}-1\right)^{\beta}\right]^{-\alpha-1}, (18)$$
$$0 < x < 1, \alpha, \beta > 0.$$

- unit Weibull (UW) distribution [24]

$$f(x) = \alpha\beta\frac{1}{x}(-\log\ x)^{\beta-1}\exp\left[-\alpha(-\log\ x)^{\beta}\right], (19)$$
$$0 < x < 1, \alpha, \beta > 0.$$

- unit Lindley (UL) distribution

$$f(x) = \frac{\alpha^2}{1+\alpha}(1-x)^{-3}\exp\left[-\frac{\alpha x}{1-x}\right], \qquad (20)$$
$$0 < x < 1,\ \alpha > 0.$$

- unit improved second-degree Lindley (UISDL) distribution [25]

$$f(x) = \frac{\alpha^3(1-x)^{-2}}{\alpha^2+2\alpha+2}\left(1+\frac{x}{1-x}\right)^2\exp\left[\frac{-\alpha x}{1-x}\right], (21)$$
$$0 < x < 1, \alpha > 0.$$

Utilizing the Akaike information criteria (AIC), Bayesian information criteria (BIC), Corrected Akaike information criteria (AICC), Hannan-Quinn information criteria (HQIC), and Consistent Akaike information criteria (CAIC), one can compare the UWLx distribution with alternative distributions; Chen [26] offered additional details on this. The information criteria are constructed as follows,

$$AIC = -2l(\hat{\theta}) + 2q.$$
$$BIC = -2l(\hat{\theta}) + q\log(n).$$
$$AICC = AIC + \tfrac{2q(q+1)}{n-q-1}.$$
$$HQIC = -2l(\hat{\theta}) + 2q\log(\log(n)).$$
$$CAIC = -2l(\hat{\theta}) + \tfrac{2qn}{n-q-1}.$$

where q denotes the number of parameters and n is the number of data.

A statistical term called "goodness of fit" is used to determine how well a particular probability distribution fits a given set of data. As shown in Table 3, smaller values of the reported criteria indicate a better alignment between the model and the observed data. A lower value implies less information loss and a potentially better model. The UWLx distribution is an excellent fit for modeling COVID-19 data since it performs better than alternative distributions according to information criteria (AIC, BIC, AICC, CAIC, and HQIC) and goodness-of-fit tests.

Table 4 illustrates the parameters estimation of UWLx distribution and other distributions. The UWLx model has fewest statistics, as Table 5 illustrates. As a result, it is clear from the result of the Kolmogrov-Simrnov (K-S), Anderson-Darling (A_n^2), Cramér-von Mises (W_n^2) and Watson (U_n^2) tests that the UWLx distribution fits COVID-19 recovery rates the best when compared to other models.

Conclusion

In this study, a new bounded distribution called the Unit Weighted Lomax (UWLx) distribution was proposed as an extension of the Weighted Lomax model. The UWLx distribution offers greater flexibility for modeling bounded and complex lifetime data, especially those exhibiting bathtub-shaped hazard rates. Several statistical properties were derived, and parameters were efficiently estimated using the maximum likelihood method. Simulation results and real data applications demonstrated that the UWLx model provides a superior fit compared to classical unit distributions such as the Beta and Kumaraswamy distributions. Therefore, the UWLx distribution represents a valuable addition to the family of unit distributions and has potential applications across various scientific fields. For future work, several poten-

10.57647/mathsci.2026.92501

Table 3. The Model Criteria for COVID-19 Data

Distribution	-Logl	AIC	BIC	AICC	HQIC	CAIC
UWLx	−60.2041	−114.408	−107.839	−114.021	−111.812	−114.021
Beta	−57.5673	−111.135	−106.755	−110.944	−109.404	−110.944
Kumaraswamy	−58.8282	−113.656	−109.277	−113.466	−111.926	−113.466
UBIII	−53.7889	−103.578	−99.198	−103.387	−101.847	−103.387
UW	−53.9588	−101.918	−95.349	−101.531	−99.322	−101.531
UL	−46.1192	−90.238	−88.049	−90.176	−89.373	−90.176
UISDL	−52.0396	−102.079	−99.889	−102.017	−101.214	−102.017

Table 4. Estimates of Parameters for COVID-19 Data

Distribution	Estimates		
	α	β	λ
UWLx	57.815	4.159	41.113
Beta	12.791	4.897	–
Kumaraswamy	8.078	7.734	–
UBIII	5.439	2.061	–
UW	8.642	2.232	–
UL	0.519	–	–
UISDL	0.740	–	–

tial extensions of the Unit Weighted Lomax (UWLx) distribution may be considered. We may investigate the properties of the UWLx distribution under right, left, or interval censoring, making it applicable to reliability and survival related proportion data. Bayesian estimation of the UWLx parameters using informative and non-informative priors can be developed. Markov Chain Monte Carlo (MCMC) methods may be used to compare Bayesian estimators with classical methods in terms of bias and mean squared error. Also, A multivariate version of the UWLx distribution may be constructed using copula functions, enabling the modeling of dependent unit-valued data arising in finance, environmental studies, and medicine.

Acknowledgment

This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2026/R/1447).

Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Abdelall YY, Ismail GM, and Nagy HF. A New Bounded Distribution: COVID-19 Application. The Egyptian Statistical Journal 2025; 69:1–29. DOI: 10.21608/esju.2025.330520.1047. Available from: https://doi.org/10.21608/esju.2025.330520.1047
2. Fayomi A, Hassan AS, Baaqeel H, and Almetwally EM. Bayesian Inference and Data Analysis of the Unit–Power Burr X Distribution. Axioms 2023; 12:297. DOI: 10.3390/axioms12030297. Available from: https://doi.org/10.3390/axioms12030297
3. Menezes AFB, Mazucheli J, and Dey S. The Unit-Logistic Distribution: Different Methods of Estimation. Pesquisa Operacional 2018; 38:555–78. DOI: 10.1590/0101-7438.2018.038.03.0555. Available from: https://doi.org/10.1590/0101-7438.2018.038.03.0555
4. Mazucheli J, Bapat SR, and Menezes AFB. A New One-Parameter Unit-Lindley Distribution. Chilean Journal of Statistics 2020; 11:53–67. Available from: https://www.chjs.mat.utfsm.cl/volumes/11/ChJS_v11_n01.pdf
5. Hassan AS, Fayomi A, Algarni A, and Almetwally EM. Bayesian and Non-Bayesian Inference for Unit-Exponentiated Half-Logistic Distribution with Data Analysis. Applied Sciences 2022; 12:11253. DOI: 10.3390/app122111253. Available from: https://doi.org/10.3390/app122111253
6. Saeed A, Saboor A, Jamal F, Alsadat N, Balogun OS, Faal A, and Elgarhy M. Bounded Sine Hyperbolic Distribution. Kuwait Journal of Science 2025; 52:100467. DOI: 10.1016/j.kjs.2025.100467.

Table 5. Goodness of Fit Tests and P-value for COVID-19 Data

Distribution	P-value	K-S	A_n^2	W_n^2	U_n^2
UWLx	0.7848	0.0805	0.6939	0.1074	0.1051
Beta	0.3507	0.1147	1.0538	0.1786	0.1592
Kumaraswamy	0.5296	0.9958	0.9338	0.1523	0.1443
UBIII	0.2183	0.1295	1.3736	0.2210	0.2021
UW	0.2107	0.1305	1.3841	0.2239	0.2014
UL	0.0168	0.1903	4.2462	0.6731	0.4538
UISDL	4.0849×10^{-7}	0.3416	17.5336	3.5088	0.6656

Available from: https://doi.org/10.1016/j.kjs.2025.100467

7. Saeed A, Elbatal I, Saboor A, Jamal F, Khan S, and Ben Ghorbal A. Novel Parameters Based Trigonometric Probability Distribution Having Bathtub Shape with Bounded Support. Journal of Radiation Research and Applied Sciences 2025; 18:101443. DOI: 10.1016/j.jrras.2025.101443. Available from: https://doi.org/10.1016/j.jrras.2025.101443
8. Mazucheli J, Menezes AFB, and Dey S. Unit Gompertz Distribution with Applications. Statistica 2019; 79:25–43. DOI: 10.6092/issn.1973-2201/8497. Available from: https://doi.org/10.6092/issn.1973-2201/8497
9. Modi K and Gill V. Unit Burr-III Distribution with Application. Journal of Statistics and Management Systems 2020; 23:579–92. DOI: 10.1080/09720510.2019.1646503. Available from: https://doi.org/10.1080/09720510.2019.1646503
10. Fayomi A, Hassan AS, and Almetwally EM. Inference and Quantile Regression for the Unit-Exponentiated Lomax Distribution. PLOS ONE 2023; 18:e0288635. DOI: 10.1371/journal.pone.0288635. Available from: https://doi.org/10.1371/journal.pone.0288635
11. Ul Haq MA, Hashmi S, Aidi K, Ramos PL, and Louzada F. Unit Modified Burr-III Distribution: Estimation, Characterizations and Validation Test. Annals of Data Science 2023; 10:415–40. DOI: 10.1007/s40745-020-00298-6. Available from: https://doi.org/10.1007/s40745-020-00298-6
12. Hassan AS and Alharbi RS. Different Estimation Methods for the Unit Inverse Exponentiated Weibull Distribution. Communications for Statistical Applications and Methods 2023; 30:191–213. DOI: 10.29220/CSAM.2023.30.2.191. Available from: https://doi.org/10.29220/CSAM.2023.30.2.191
13. Karakaya K, Rajitha CS, Sağlam Ş, Tashkandy YA, Bakr ME, Muse AH, Kumar A, Hussam E, and Gemeay AM. A New Unit Distribution: Properties, Estimation and Regression Analysis. Scientific Reports 2024; 14:7214. DOI: 10.1038/s41598-024-57390-7. Available from: https://doi.org/10.1038/s41598-024-57390-7
14. Haj Ahmad HH, Almetwally EM, Elgarhy M, and Ramadan DA. On Unit Exponential Pareto Distribution for Modeling the Recovery Rate of COVID-19. Processes 2023; 11:232. DOI: 10.3390/pr11010232. Available from: https://doi.org/10.3390/pr11010232
15. Hassan AS, Khalil AM, and Nagy HF. Versatile Extension of the Unit Gompertz: Efficient Estimation and Application. Gazi University Journal of Science 2025; 38:1540–64. DOI: 10.35378/gujs.1541941. Available from: https://doi.org/10.35378/gujs.1541941
16. Hassan AS, Abdalla GSS, Faal A, and Saudi OA. Novel Unit Distribution for Enhanced Modeling Capabilities: Healthcare and Geological Applications. Engineering Reports 2025; 7:e70277. DOI: 10.1002/eng2.70277. Available from: https://doi.org/10.1002/eng2.70277
17. Kilany NM. Weighted Lomax Distribution. SpringerPlus 2016; 5:1862. DOI: 10.1186/s40064-016-3489-2. Available from: https://doi.org/10.1186/s40064-016-3489-2
18. Lomax KS. Business Failures: Another Example of the Analysis of Failure Data. Journal of the American Statistical Association 1954; 49:847–52. DOI: 10.1080/01621459.1954.10501239. Available from: https://doi.org/10.1080/01621459.1954.10501239
19. Aljohani HM. The New Explanation of Lomax Distribution: Its Properties, Inference, and Applications to Real-Life Data. Contemporary Mathematics 2025; 6:2541–69. DOI: 10.37256/cm.6220255778. Available from: https://doi.org/10.37256/cm.6220255778
20. Arnold BC, Balakrishnan N, and Nagaraja HN. A First Course in Order Statistics. Reprinted by SIAM (Classics in Applied Mathematics), 2008. New York: John Wiley & Sons, 1992. DOI: 10.1137/1.9780898719062. Available from: https://doi.org/10.1137/1.9780898719062
21. Afify AZ, Nassar M, Kumar D, and Cordeiro GM. A New Unit Distribution: Properties, Inference, and Applications. Electronic Journal of Applied Statistical Analysis 2022; 15:460–84

22. Şenol Ç and Korkmaz M. Beta Distribution and Inferences about the Beta Functions. Asian Journal of Science and Technology 2016; 7:2960–70

23. Kumaraswamy P. A Generalized Probability Density Function for Double Bounded Random Processes. Journal of Hydrology 1980; 46:79–88. DOI: 10.1016/0022-1694(80)90036-0. Available from: https://doi.org/10.1016/0022-1694(80)90036-0

24. Mazucheli J, Menezes AF, and Ghitany ME. The Unit Weibull Distribution and Associated Inference. Journal of Applied Probability and Statistics 2018; 13:1–22

25. Altun E and Cordeiro GM. The Unit-Improved Second-Degree Lindley Distribution: Inference and Regression Modeling. Computational Statistics 2020; 35:259–79. DOI: 10.1007/s00180-019-00921-y. Available from: https://doi.org/10.1007/s00180-019-00921-y

26. Chen G and Balakrishnan N. A General Purpose Approximate Goodness-of-Fit Test. Journal of Quality Technology 1995; 27:154–61. DOI: 10.1080/00224065.1995.11979578. Available from: https://doi.org/10.1080/00224065.1995.11979578