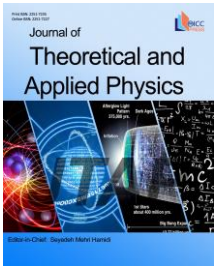

J. Theor. Appl. Phys., 2026; 20(3): 288-303

Journal of Theoretical and Applied Physics (JTAP)

https://doi.org/10.57647/jtap.2026.8614.0309

Research Article

Generalized Functional Analysis Approach for Bound State Solutions with Solvable and Quasi-Solvable Potentials

Ekwevugbe Omugbe[1*], Terkaa V Targema[2], Edwin Samson Eyube[3], Clement A Onate[4], Samuel O Ogundeji[5], Uchechukwu E Vincent[6]

[1]*Department of Physics and Engineering Physics, Faculty of Science, Obafemi Awolowo University Ile-Ife, Osun State, Nigeria*

[2]*Department of Physics, Taraba State University, Jalingo, Nigeria*

[3]*Department of Physics, Faculty of Physical Sciences, Modibbo Adama University, P.M.B. 2076, Yola, Adamawa State, Nigeria*

[4]*Physics Department, Bowen University, Iwo, Osun State, Nigeria*

[5]*Department of Physics and Astronomy, Iowa State University, Ames, United States of America*

[6]*Department of Physical Sciences, Redeemer's University Ede, Osun State, Nigeria*

***Corresponding author:** eomugbe@oauife.edu.ng

Article History:

Received:
07 December 2025

Revised:
05 January 2026

Accepted:
25 February 2026

Published Online:
05 March 2026

Published in Issue:
30 June 2026

Abstract

In this paper, a generalized functional analysis approach has been used to obtain the bound state solutions of the Schrödinger equation with solvable and quasi-solvable potentials. We extended the Nikiforov-Uvarov functional analysis method which has been limited to obtaining bound states solutions of exponential-type potentials. Here, we have generalized the approach to obtain the approximate and exact energy spectra and the corresponding wave functions of the Schrödinger equation under the Coulomb, Pseudo harmonic, hyperbolic, q-deformed hyperbolic and perturbed Coulomb potentials in closed forms. These potential functions with the Schrödinger equation were transformed into standard second-order linear differential equations such as the Whittaker, associated Laguerre, Gauss hypergeometric and the confluent Heun differential equations. The parameters of the wave functions and energy quantization conditions were derived from a direct comparison with the quantized systems of differential equations. Our results are in excellent agreement with the earlier works in which different approaches were utilized. Also, the quasi-exact energies for fixed potential parameters were found to be in good agreement with energy eigenvalues obtained via the matrix Numerov method.

Keywords: Associated Laguerre equation; Gauss hypergeometric equation; Functional analysis method; Nikiforov-Uvarov method; Confluent Heun equation

Cite this article: Omugbe, E., Targema, T.V., Samson Eyube, E., Onate, C.A., Ogundeji, S.O., Vincent, U.E., (2026), Generalized Functional Analysis Approach for Bound State Solutions with Solvable and Quasi-Solvable Potentials. *J. Theor. Appl. Phys.,* 20(3), 288-303. https://doi.org/10.57647/jtap.2026.8614.0307

1. Introduction

In quantum mechanics, the wave functions and their corresponding energy eigenvalues play vital roles in describing the behaviour of particles in various contexts and phenomena. From the non-relativistic to the relativistic regimes, the bound state solution has been a useful tool in quantum physics and chemistry.

Accordingly, the bound state problems under the influence of different potentials have attracted interest of researchers

over the years [1–8]. However, some potential models do not admit exact analytical solutions, leading to the formulation of mathematical tools to deal with the problem. Nikiforov and Uvarov [9] developed a method used in solving certain types of differential equations of hypergeometric-type with solutions as special orthogonal functions [10–18].

Subsequently, the Nikiforov and Uvarov (NU) method has been simplified [19] and extended to accommodate other special functions such the Heun function and its confluent variants [20].

Wentzel [21], Kramers [22], and Brillouin [23] (WKB) independently developed a semi-classical method for the special case where the potential varies slowly compared to the particle's wavelength.

This method allows for the calculation of approximate bound state problems, especially when finding exact solutions is challenging [24–28].

The supersymmetric WKB (SWKB) method has been adopted to improve upon the shortcomings of the WKB method [29–34], as it facilitates obtaining both ground and excited states via ladder operators in higher dimensions. Moreover, it admits the energy spectra of all shape-invariant potentials without necessarily modifying the centrifugal barrier as in the WKB approach. However, both the WKB and SWKB methods are insufficient to provide an exact quantization condition [2].

Several alternative methods for obtaining exact or approximate bound state solutions, include Super Symmetric Quantum Mechanics (SUSYQM) [35–38], which considers the symmetric behaviour of particles (bosons and fermions) confined in shape invariant potentials have been formulated.

The exact quantization rule, which has been developed based on fundamental quantum mechanics postulates, such as normalizability and boundary conditions, has been utilized to obtain the energy spectra of the wave equations [39–41].

Also, the Asymptotic Iteration Method (AIM) is a useful method in obtaining approximate solutions for bound state problems in various potential models, including those difficult to solve by other methods [42–51]. The path integral method developed by Feynman [52] for special cases, where the evolution of the particle in space over time can be studied through the path integral within the initial and final states of the particles, has been used to solve the bound state solutions [53, 54].

The factorization method [55], Ansatz methods [56, 57] have also been considered essential mathematical tools for bound state problems.

Recently, Horchani et al. [58] presented the analytical solutions for the energy equations of deformed Kratzer and hyperbolic potentials using a generalized functional analysis method.

They transformed the Schrödinger equation into a confluent Heun differential equation, which allows for quasi-exact energy and polynomial solutions of the wave function.

Their analytical solutions were consistent with earlier results in the literature, where methods such as the supersymmetric WKB(SWKB), functional Bethe ansatz and the extended Nikiforov-Uvarov methods were used to obtain bound state solutions.

Lévai and Soltész [59] studied polynomial solutions of the Heun-type equations and their application in generating exactly solvable Schrödinger potentials.

They focused on the Heun equation and its confluent variants such as the confluent, biconfluent, double confluent, and triconfluent, exploring conditions under which these equations reduce to polynomial forms. Kisoglu [60] used the AIM Method to derive quasi-exact analytical solutions for the one-dimensional Schrödinger equation under a general sextic oscillator and provided closed-form expressions for the energy eigenvalues. Additionally, a numerical scheme based on AIM was employed to verify the reliability of the analytical expressions across various potential parameters.

These methods have achieved considerable successes over the years but applying them requires significant mathematical knowledge and there have been several efforts to overcome this obstacle. Falaye et al. [61] formulated a parametric approach for obtaining the wave functions and energy spectra of relativistic and non-relativistic particles.

As a demonstration, the method was successfully used to obtain the wave functions and energy spectra of various potential models. Omugbe et al. [62] conjectured a method for obtaining energy spectra of solvable potentials and reproduced the energy spectra of the Schrodinger equation with the Hulthen, Pseudo harmonic and hyperbolic potentials.

Similarly, Ikot et al. [63] developed an approach based on the Nikiforov-Uvarov Functional Analysis (NUFA) for the bound state study of exponential-type potentials.

In this study, we are motivated by the absence of solutions for other potential types not treated in [63]. They applied their approach to exponential-type potentials and solved the second-order linear differential equation of a hypergeometric type.

In what follows, we generalized their approach that may not only apply to exponential-type potentials but also to all classes of solvable potentials such as the Coulomb, Pseudo harmonic, hyperbolic, q-deformed hyperbolic functions and also quasi-solvable potential like the perturbed

Coulomb potential function. The rest of the article is organized as follows:
in Sec. 2, we will study the solutions of some quantized standard second-order linear differential equations.
In Sec. 3, we will apply the solutions to obtain the energy quantization conditions for different classes of potentials.
The article is concluded in Sec. 4.

2. Quantized standard second-order linear differential equations

The Schrödinger equation may be transformed into a standard differential equation often encountered in applied Physics and mathematics.
The solutions of these equations are the usual special functions such as the Laguerre, Jacobi, confluent Heun and Gauss hypergeometric functions.
The standard equations of the associated Laguerre type are given as [64]:

$$x\frac{d^2\psi_n(x)}{dx^2}+(\alpha+1-x)\frac{d\psi_n(x)}{dx}+n\psi_n(x)=0 \tag{1}$$

$$\frac{d^2\psi_n(x)}{dx^2}+\left(\frac{n+\frac{\alpha}{2}+\frac{1}{2}}{x}+\frac{1-\alpha^2}{4x^2}-\frac{1}{4}\right)\psi_n(x)=0 \tag{2}$$

$$\frac{d^2\psi_n(x)}{dx^2}+\left(4n+2\alpha+2-x^2+\frac{\frac{1}{4}-\alpha^2}{x^2}\right)\psi_n(x)=0 \tag{3}$$

The Jacobi differential equation is given as [64, 65]

$$(1-x^2)\frac{d^2\psi_n(x)}{dx^2}+(\beta-\alpha-(\beta+\alpha+2)x)\frac{d\psi_n(x)}{dx}+n(n+\beta+\alpha+1)\psi_n(x)=0 \tag{4}$$

Eq. (2) and (4) may be transformed to the Whittaker and Gauss Hypergeometric differential equations respectively [65].

$$\frac{d^2W_{\kappa,\nu}(x)}{dx^2}+\left(\frac{\kappa}{x}+\frac{\frac{1}{4}-\nu^2}{x^2}-\frac{1}{4}\right)W_{\kappa,\nu}(x)=0 \tag{5}$$

$$y(1-y)\frac{d^2\psi(y)}{dy^2}+(\alpha+1-(\beta+\alpha+2)y)\frac{d\psi(y)}{dy}+n(n+\beta+\alpha+1)\psi(y)=0 \tag{6}$$

In Eq. (5), $\kappa=n+\nu+\frac{1}{2},\nu=\frac{\alpha}{2}$. The solution of (5) are the Whittaker functions [65]:

$$M_{\kappa,\nu}(x)=e^{-\frac{x}{2}}x^{\nu+\frac{1}{2}}M\left(\nu+\frac{1}{2}-\kappa,1+2\nu\,;x\right) \tag{7}$$

$$W_{\kappa,\nu}(x)=e^{-\frac{x}{2}}x^{\nu+\frac{1}{2}}U\left(\nu+\frac{1}{2}-\kappa,1+2\nu\,;x\right) \tag{8}$$

The functions $M(\cdot)$ and $U(\cdot)$ are confluent Hypergeometric functions. In Eq. (6), we have used the change of variable $(x=1-2y)$ and the solution is the Gauss Hypergeometric function given by

$$\psi_n(y)=\ {}_2F_1(-n,n+\alpha+\beta+1,\alpha+1;\ y) \tag{9}$$

The transformation of (2) and (4) into Eq. (5) and (6) allows the relationships between the associated Laguerre polynomial and the Confluent Hypergeometric function:

$$L_n^{(\alpha)}(x)=\binom{n+\alpha}{n}\ {}_1F_1(-n,\alpha+1,x) \tag{10}$$

On the other hand, the Jacobi function may be written as a function of the Gauss Hypergeometric function:

$$P_n^{(\alpha,\beta)}(x)=\binom{n+\alpha}{n}\ {}_2F_1\left(-n,\mathrm{n}+\alpha+\beta+1,\alpha+1,\frac{1-x}{2}\right) \tag{11}$$

These standard equations with their solutions play vital roles in determining the energy of quantum systems for a given potential function within the relativistic and non-relativistic wave equations.
In the context of the NU approach, the Schrödinger equation may be transformed into a second-order linear differential equation of a hypergeometric type [9]

$$\frac{d^2\psi_n(x)}{dx^2}+\frac{\tau(x)}{\sigma(x)}\frac{d\psi_n(x)}{dx}+\frac{\tilde{\sigma}(x)}{\sigma^2(x)}\psi_n(x)=0 \tag{12}$$

where $\tau(x)$ is a polynomial of at most degree one. The functions $\sigma(x)$ and $\tilde{\sigma}(x)$ are polynomial of at most second order. Eq. (12) has been reparameterized [19]

$$\frac{d^2\psi_n(x)}{dx^2}+\frac{(h_1-h_2x)}{x(1-h_3x)}\frac{d\psi_n(x)}{dx}+\frac{-\Lambda_1x^2+\Lambda_2x-\Lambda_3}{x^2(1-h_3x)^2}\psi_n(x)=0 \tag{13}$$

The solution of the energy and wave function in (12) requires a series of mathematical steps whereas in (13), the steps have been simplified by using closed form quantization conditions for the bound state solutions. The readers are referred to Refs. [9, 10, 19] for further details. It is worth stating that these methods have limitations in handling non-exactly solvable potentials whose solutions may be obtained by other approximation and numerical

methods. Also, quasi-solvable potentials with the Schrodinger equation may be transformed into the confluent Heun differential equation (CHDE) given by [20, 66]:

$$\frac{d^2\psi_n(x)}{dx^2}+\left(\alpha+\frac{\beta+1}{x}+\frac{\gamma+1}{x-1}\right)\frac{d\psi_n(x)}{dx} + \left(\frac{\kappa}{x}+\frac{\tau}{x-1}\right)\psi_n(x)=0 \tag{14}$$

Eq. (14) has singularities at $x = 0$ and $x = 1$ and the solution around $x = 0$ is the confluent Heun function given as

$$H_C(\alpha,\beta,\gamma,\delta,\eta,x)=\sum_{n=0}^{\infty}V_n(\alpha,\beta,\gamma,\delta,\eta)\,x^n \tag{15}$$

The parameters δ and η are given by

$$\delta=\kappa+\tau-\frac{\alpha}{2}(\beta+\gamma+2) \tag{16}$$

$$\eta=\frac{\alpha}{2}(\beta+1)-\kappa-\frac{1}{2}(\beta+\gamma+\beta\gamma) \tag{17}$$

The solution of (14) may be reduced to a polynomial of degree *n* provided the following constraints are obeyed [20, 66]:

$$\kappa+\tau+n\alpha=0 \tag{18}$$

$$\Delta_{n+1}(\kappa)=0 \tag{19}$$

where (19) is an $(n + 1) \times (n + 1)$ tridiagonal matrix determinant [20, 66]. In Sec 3, we will consider the CHDE and its solutions to obtain the energy equation of the perturbed Coulomb potential.

3. Solutions of the energy equations and wave functions of the schrödinger equation

3.1. The coulomb potential function

The Coulomb potential plays a pivotal role in accessing the binding energies of hydrogen-like atoms in atomic physics. The potential is asymptotic at large distances and also plays an important role together with confining potentials in the study of quark properties in high-energy physics [67]. The potential is given by

$$V=(r)=-\frac{A}{r} \tag{20}$$

where A is a constant parameter.

The radial Schrödinger equation with the Coulomb's potential can be written as

$$\frac{d^2\psi_{nl}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left(E_{nl}+\frac{A}{r}-\frac{l(l+1)\hbar^2}{2\mu r^2}\right)\psi_{nl}(r)=0 \tag{21}$$

where $\mu, \hbar, E_{nl}$ and $\psi_{nl}(r)$ are the respective reduced mass of a two-body particle, reduced Planck's constant, energy spectra and the wave function. The notations n and l are the radial and orbital quantum numbers respectively.

Eq. (21) may be simplified using new parameters:

$$\frac{d^2\psi_{nl}(r)}{dr^2}+\left(\varepsilon_{nl}+\frac{\chi_0}{r}-\frac{L}{r^2}\right)\psi_{nl}(r)=0 \tag{22}$$

where $\varepsilon_{nl}=-\frac{2\mu E_{nl}}{\hbar^2}, \chi_0=\frac{2\mu A}{\hbar^2}$ and $L=\delta^2-\frac{1}{4}$; $\delta=\left(l+\frac{1}{2}\right)$ Using the coordinate change $\rho=2\sqrt{\varepsilon_{nl}}r$, Eq. (22) transforms into a standard Whittaker equation

$$\frac{d^2\psi_{nl}(\rho)}{dx^2}+\left(\frac{\chi_1}{\rho}-\frac{\frac{1}{4}-\delta^2}{\rho^2}-\frac{1}{4}\right)\psi_{nl}(\rho)=0 \tag{23}$$

$$\chi_1=\frac{\chi_0}{2\sqrt{\varepsilon_{nl}}}$$

Comparing Eq. (23) with (5), the energy quantization condition may be written as

$$\frac{\chi_0}{2\sqrt{\varepsilon_{nl}}}=n+\delta+\frac{1}{2} \tag{24}$$

Inserting the terms for χ_0, ε_{nl} and δ into (24), the energy is obtained as

$$E_{nl}=-\frac{\mu A^2}{\hbar^2(n+l+1)^2} \tag{25}$$

It is worth stating that (23) may be solved using the parametric NU approach [19] where the wave function may be obtained in terms of the associated Laguerre polynomial.

Also, the Whittaker equation may be transformed into the associated Laguerre differential equation by utilizing the substitution

$$\psi_{nl}(\rho)=e^{-\frac{\rho}{2}}\rho^{\delta+\frac{1}{2}}F_n(\rho) \tag{26}$$

For large ρ, the wave function behaves as $\psi_{nl}(\rho)\sim e^{-\frac{\rho}{2}}$, while for small ρ, the wave function tends to $\rho^{\delta+\frac{1}{2}}$ such that the boundary conditions $\psi_{nl}(\rho)|_{\rho=0}=\psi_{nl}(\rho)|_{\rho=\infty}=0$ for a well behaved wave function is obeyed in the interval $(0\prec\rho\prec\infty)$. Inserting the function $\psi_{nl}(\rho)=e^{-\frac{\rho}{2}}\rho^{\delta+\frac{1}{2}}F_n(\rho)$ into (23) with algebraic simplification gives the standard equation

$$\rho\frac{d^2F_n(\rho)}{d\rho^2}+(2\delta+1-\rho)\frac{dF_n(\rho)}{d\rho} + \left(\chi_1-\left(\delta+\frac{1}{2}\right)\right)F_n(\rho)=0 \tag{27}$$

Comparing (27) with (1) gives the same energy quantization condition obtained in (22). The function $F_n(\rho)$ corresponds to the associated polynomial $L_n^{2\delta}(\rho)$.

3.2. Pseudo-harmonic potential

The bound state solutions of the Schrodinger equation with the pseudo harmonic potential have been obtained with several mathematical methods [11, 59, 68, 69]. This potential admits the exact energy equation regardless of the solution approach. The potential is given by

$$V(r) = D_0\left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2 \tag{28}$$

where D_0 and r_0 are potential parameters. Inserting the potential into the radial Schrödinger equation with algebraic simplification yields

$$\frac{d^2\psi_{nl}(r)}{dr^2} + \left(\varepsilon_{nl} - \eta_0 r^2 + \frac{\eta_1}{r^2}\right)\psi_{nl}(r) = 0 \tag{29}$$

where $\varepsilon_{nl} = \frac{2\mu(D_0+E_{nl})}{\hbar^2}, \eta_0 = \frac{2\mu D_0}{r_0^2\hbar^2}$ and $\eta_1 = \frac{1}{4} - \lambda^2$; $\lambda = \left(l+\frac{1}{2}\right)^2 - \frac{2\mu D_0 r_0^2}{\hbar^2}$. Using the change of variable $s^2 = r^2\sqrt{\eta_0}$, (29) reduces to the standard equation in (3)

$$\frac{d^2\psi_n(r)}{dr^2} + \left(\frac{\eta_1}{s^2} - s^2 + \frac{\varepsilon_{nl}}{\sqrt{\eta_0}}\right)\psi_n(r) = 0 \tag{30}$$

Comparing with (3), the energy quantization condition is obtained as

$$\frac{\varepsilon_{nl}}{\sqrt{\eta_0}} = 4n + 2\lambda + 2 \tag{31}$$

Inserting the terms for ε_{nl}, η_0 and λ in (29), the energy is obtained as

$$E_{nl} = -2D_0 + \hbar\sqrt{\frac{8D_0}{\mu r_0^2}}\left(n + \frac{1}{2} + \frac{1}{2}\sqrt{\frac{2\mu D_0 r_0^2}{\hbar^2} + \left(l+\frac{1}{2}\right)^2}\right) \tag{32}$$

In the context of the NUFA approach [63], using the coordinate transformation $y = s^2$ allows one to write Eq. (30) as

$$\frac{d^2\psi_{nl}(y)}{dy^2} + \frac{1}{2y}\frac{d\psi_{nl}(y)}{dy} + \frac{1}{4y}\left(\frac{\varepsilon_{nl}}{\sqrt{\eta_0}} - y + \frac{\frac{1}{4}-\lambda^2}{y}\right)\psi_{nl}(y) = 0 \tag{33}$$

For large y, the wave function behaves as $e^{-\frac{y}{2}}$ while as y approaches zero, the wave function becomes zero. This allows one to guess the function at the origin as y^a with a being a positive constant.

To transform (33) into a standard equation, we may use the trial wave function $\psi_{nl}(y) = y^a e^{-\frac{y}{2}}F_{nl}(y)$. Inserting this trial function into (33) yields

$$y\frac{d^2F_{nl}(y)}{dy^2} + \left(2a + \frac{1}{2} - y\right)\frac{dF_{nl}(y)}{dy} + \left(\frac{\varepsilon_{nl}}{4\sqrt{\eta_0}} - a - \frac{1}{4} + \frac{1}{y}\left(a^2 - \frac{a}{2} - \frac{\lambda^2}{4} + \frac{1}{16}\right)\right)F_{nl}(y) = 0 \tag{34}$$

To write (34) as a standard equation in (1), we must set the coefficients of $\frac{1}{y}$ to zero

$$a^2 - \frac{a}{2} - \frac{\lambda^2}{4} + \frac{1}{16} = 0 \tag{35}$$

Solving Eq. (35) gives $a = \frac{1}{4} + \frac{\lambda}{2}$. With the knowledge of a, Eq. (34) reduces to

$$y\frac{d^2F_{nl}(y)}{dy^2} + (\lambda + 1 - y)\frac{dF_{nl}(y)}{dy} + \left(\frac{\varepsilon_{nl}}{4\sqrt{\eta_0}} - \frac{\lambda}{2} - \frac{1}{2}\right)F_{nl}(y) = 0 \tag{36}$$

By comparing Eq. (36) with Eq. (1), we obtained the same energy quantization condition as in (31) with the function $F_{nl}(y) = L_n^{(\lambda)}(y)$.

3.3. Hyperbolic potential

The non-relativistic energy equation and wave function of the hyperbolic-type potential have been derived with another method in existing literature [70]. Solving the problem with this generalized approach allows us to demonstrate the extension of the approach in [63]. In this present work, we consider the hyperbolic potential given as [70]

$$V(r) = \frac{A}{cosh^2(\alpha r)} + \frac{B}{sinh^2(\alpha r)} + C\,tanh^2(\alpha r) + \frac{D}{tanh^2(\alpha r)} + E \tag{37}$$

Following Ref. [70] and using the appropriate coordinate transformation ($x = tanh^2(\alpha r)$) and centrifugal barrier approximation, the N-dimensional Schrödinger equation may be transformed into a second-order linear differential equation of a hypergeometric type given in (13)

$$\frac{d^2\psi_n(x)}{dx^2}+\frac{\left(\frac{1}{2}-\frac{3}{2}x\right)}{x(1-x)}\frac{d\psi_n(x)}{dx} + \frac{-\Lambda_1 x^2+\Lambda_2 x-\Lambda_3}{x^2(1-x)^2}\psi_n(x)=0 \tag{38}$$

Where

$$\Lambda_1=\frac{\mu}{2\alpha^2\hbar^2}(C-A-Ld_0)$$

$$\Lambda_2=\frac{\mu}{2\alpha^2\hbar^2}(E_{nl}-A+B-E+Ld_0)$$

$$\Lambda_3=\frac{\mu}{2\alpha^2\hbar^2}(B+D+L)$$

$$L=\frac{\alpha^2\hbar^2}{2\mu}\left(\left(l+\frac{N-2}{2}\right)^2-\frac{1}{4}\right)$$

All the constant parameters have the usual meanings given in Ref. [70]. Eq. (38) has singularities at $x=0$and $x=1$. Using the trial wave function $x^{a_0}(1-x)^b G_{nl}(x), a_0, b \succ 0$ in Eq. (38) with algebraic simplification we obtain:

$$x(1-x)\frac{d^2G_n(x)}{dx^2} + \left(2a_0+\frac{1}{2}-\left(2a_0+2b+\frac{3}{2}\right)x\right)\frac{dG_n(x)}{dx} - \left(a_0^2+b^2+2a_0b+\frac{1}{2}(a_0+b)-\Lambda_1\right)G_n(x) + \left(\frac{\Lambda_2-\Lambda_1-\Lambda_3+b^2}{1-x}+\frac{a_0^2-\frac{a_0}{2}-\Lambda_3}{x}\right)G_n(x)=0 \tag{39}$$

To obtain a standard equation of a Hypergeometric type given in (6), the terms $\Lambda_2-\Lambda_1-\Lambda_3+b^2$ and $a_0^2-\frac{a_0}{2}-\Lambda_3$ must independently vanish

$$\Lambda_2-\Lambda_1-\Lambda_3+b^2=0 \tag{40}$$

$$a_0^2-\frac{a_0}{2}-\Lambda_3=0 \tag{41}$$

Eq. (40) and (41) allows for the determination of the wave function parameters

$$a_0=\frac{1}{4}+\frac{1}{4}\sqrt{1+16\Lambda_3} \tag{42}$$

$$b=\sqrt{\Lambda_1-\Lambda_2+\Lambda_3} \tag{43}$$

In Eq. (42) and (43), we have chosen positive values to maintain boundary condition of the wave function. In a final step, we factorized the coefficient of $G_{nl}(x)$ to obtain a standard equation:

$$x(1-x)\frac{d^2G_n(x)}{dx^2} + \left(2a_0+\frac{1}{2}-\left(2a_0+2b+\frac{3}{2}\right)x\right)\frac{dG_n(x)}{dx} - \left(a_0+b+\frac{1}{4}-\sqrt{\Lambda_1+\frac{1}{16}}\right) \left(a_0+b+\frac{1}{4}+\sqrt{\Lambda_1+\frac{1}{16}}\right)G_n(x)=0 \tag{44}$$

Comparing (44) with (6), the conditions for energy quantization equation may be written as

$$\left(a_0+b+\frac{1}{4}-\sqrt{\Lambda_1+\frac{1}{16}}\right)=-n \tag{45}$$

$$\left(a_0+b+\frac{1}{4}+\sqrt{\Lambda_1+\frac{1}{16}}\right)=n+\alpha+\beta+1 \tag{46}$$

The parameters$\alpha$and β of the standard equation in (6) are obtained as

$$\alpha=\frac{1}{2}\sqrt{1+16\Lambda_3} \tag{47}$$

$$\beta=2\sqrt{\Lambda_1-\Lambda_2+\Lambda_3} \tag{48}$$

Using either (45) or (46), the energy and the wave function equation are obtained in closed form

$$E_{nl}=C+D+E-\frac{2\alpha^2\hbar^2}{\mu}\left(n+\frac{1}{2}+\frac{1}{2}\left(\sqrt{\frac{2\mu(B+D)}{\alpha^2\hbar^2}+\left(l+\frac{N-2}{2}\right)^2}-\sqrt{\frac{2\mu(C-A)}{\alpha^2\hbar^2}-\left(l+\frac{N-2}{2}\right)^2 d_0+\frac{1}{3}}\right)\right)^2 \tag{49}$$

$$\psi_n(x)= x^{a_0}(1-x)^b\ {}_2F_1(-n,n+\alpha+\beta+1,\alpha{+}1;x) \tag{50}$$

In the next section, we will apply this current approach to the q-deformed hyperbolic potential, which to our knowledge has not been reported.

10.57647/jtap.2026.8614.0307

3.4. q-deformed hyperbolic potential

The q-deformed hyperbolic function has been successfully used to model the vibrational energy spectra of diatomic molecules [71, 72]. The non-relativistic energy equation was obtained in [71] using the path integral approach, while Omugbe et al. [72] applied the NU and WKB approach to obtain the energy equation in closed form. The path integral approach is complex and rigorous, while the NU and WKB methods require careful mathematical manipulations to arrive at the energy solutions. We demonstrate the utility and the exactness of the generalized functional analysis approach in solving the problem. The q-deformed function is given by [71, 72]

$$V(x) = A + B\tanh_q^2(\alpha x) + \frac{C\tanh_q(\alpha x)}{\cosh_q(\alpha x)} \tag{51}$$

where

$$cosh_q(\alpha r) = \frac{e^{\alpha x} + qe^{-\alpha x}}{2} \tag{52}$$

$$sinh_q(\alpha r) = \frac{e^{\alpha x} - qe^{-\alpha x}}{2} \tag{53}$$

$$\tanh_q(\alpha x) = \frac{\sinh_q(\alpha x)}{\cosh_q(\alpha x)} \tag{54}$$

Using the change of variable $x = y + 1/\alpha In\sqrt{q}$, the one dimensional Schrodinger equation with Eq. (51) may be written as

$$\frac{d^2R_n(y)}{dy^2} + \frac{2\mu}{\hbar^2}\left(\epsilon_{nl} + \frac{B}{Cosh^2(\alpha y)} - \frac{\overline{C}Sinh(\alpha y)}{Cosh^2(\alpha y)}\right)R_n(y) = 0 \tag{55}$$

where

$$\epsilon_{nl} = E_{nl} - A - B$$

$$\overline{C} = \frac{C}{\sqrt{q}}$$

Using the change of variable $z = Sinh(\alpha y)$, Eq. (55) can be written in a standard form given in (12):

$$\frac{d^2R_n(z)}{dz^2} + \frac{z}{1+z^2}\frac{dR_n(z)}{dz} + \frac{-\Upsilon_1 z^2 + \Upsilon_2 z - \Upsilon_3}{(1+z^2)^2}R_n(z) = 0 \tag{56}$$

where

$$\Upsilon_1 = \frac{2\mu}{\alpha^2\hbar^2}(A + B - E_{nl})$$

$$\Upsilon_2 = -\frac{2\mu\overline{C}}{\alpha^2\hbar^2}$$

$$\Upsilon_3 = \frac{2\mu}{\alpha^2\hbar^2}(A - E_{nl})$$

Eq. (56) has singularities at $z = \pm i$ such that we may guess the wave function as $R_n(z) = (z + i)^{a_1}(z - i)^{b_1}H(z)$. With this wave function, Eq. (56) transforms as

$$(z^2+1)\frac{d^2H(z)}{dz^2} + h_1'(z)\frac{dH(z)}{dz} + h_2'H(z) + \frac{h_3'z + h_4'}{z^2+1}H(z) = 0 \tag{57}$$

where

$$h_1'(z) = (2b_1 + 2a_1 + 1)z + 2b_1 - 2a_1$$

$$h_2' = a_1^2 + b_1^2 + 2a_1b_1 - \Upsilon_1$$

$$h_3' = i(2b_1^2 - 2a_1^2 - b_1 + a_1) + \Upsilon_2$$

$$h_4' = -2a_1^2 - 2b_1^2 + a_1 + b_1 + \Upsilon_1 - \Upsilon_3$$

Following the same process adopted in Sec. 3.3 for obtaining the parameters of the wave function and a standard differential equation, the constant terms h_3' and h_4' must be equated to zero:

$$i(2b_1^2 - 2a_1^2 - b_1 + a_1) + \Upsilon_2 = 0 \tag{58}$$

$$-2a_1^2 - 2b_1^2 + a_1 + b_1 + \Upsilon_1 - \Upsilon_3 = 0 \tag{59}$$

Using (58) and (59), Eq. (57) reduces to

$$(z^2+1)\frac{d^2H(z)}{dz^2} + h_1'(z)\frac{dH(z)}{dz} + h_2'H(z) = 0 \tag{60}$$

Solving Eq. (58) and (59) completely, we obtained the values of a_1 and b_1 respectively

$$a_1 = \frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} + \Upsilon_1 - \Upsilon_3 - i\Upsilon_2} \tag{61}$$

$$b_1 = \frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} + \Upsilon_1 - \Upsilon_3 + i\Upsilon_2} \tag{62}$$

Using the substitution $z = i(2y - 1)$, Eq. (60) may be reduced to a standard Gauss hypergeometric equation:

$$y(1-y)\frac{d^2H(y)}{dy^2}+W_1(y)\frac{dH(y)}{dy} \\ -(a_1+b_1-\sqrt{\Upsilon_1})(a_1+b_1+\sqrt{\Upsilon_1})H(y)=0 \quad (63)$$

Where

$$W_1(y)=2a_1+\frac{1}{2}-(2a_1+2b_1+1)y$$

Comparing with the standard equation in (6), the necessary conditions for the energy spectra equation are obtained as

$$a_1+b_1-\sqrt{\Upsilon_1}=-n \quad (64)$$

$$a_1+b_1+\sqrt{\Upsilon_1}=n+\alpha+\beta+1 \quad (65)$$

The parameter $\alpha$and $\beta$are obtained as

$$\alpha=2a_1-\frac{1}{2} \quad (66)$$

$$\beta=2b_1-\frac{1}{2} \quad (67)$$

Using either Eq. (64) or (65), the energy is obtained in closed form

$$E_n=A+B-\frac{\alpha^2\hbar^2}{2\mu} \\ \left(n+\frac{1}{2}-\sqrt{\frac{\mu B}{\alpha^2\hbar^2}+\frac{1}{8}+\sqrt{\left(\frac{\mu B}{\alpha^2\hbar^2}+\frac{1}{8}\right)^2+\left(\frac{\mu}{\alpha^2\hbar^2}\left(\frac{C}{\sqrt{q}}\right)\right)^2}}\right)^2 \quad (68)$$

3.5. Solution of the Schrödinger equation with non-exactly solvable potential function

In this section, we will consider the perturbed Coulomb potential investigated with the WKB approximation [73], Lie algebra approach [74], and the Bethe ansatz method [75]:

$$V(r)=-\frac{A}{r}+\frac{B}{r+g} \quad (69)$$

Substituting (69) into the radial Schrodinger equation gives:

$$\frac{d^2\psi_{nl}(r)}{dr^2}+ \\ \frac{2\mu}{\hbar^2}\left(E_{nl}+\frac{A}{r}-\frac{B}{r+g}-\frac{l(l+1)\hbar^2}{2\mu r^2}\right)\psi_{nl}(r)=0 \quad (70)$$

In (70), it is difficult to obtain the analytical solutions of the energy spectra with most of the reviewed methods. Using the WKB approach in [73], the authors obtained a transcendental energy equation in terms of complete elliptic integrals, where the numerical results of the energy levels were presented for the s-wave states ($l=0$). In [74] and [75] rigorous approaches have been applied to the problem.

In this present work, we attempt to simplify the solution and also to obtain the energy equation in closed form for any l-states.

We introduce a variable change $x=-\frac{r}{g}$ to obtain:

$$\frac{d^2\psi_{nl}(x)}{dx^2}+ \\ \left(-\varepsilon_{nl}-\frac{k_0}{x}+\frac{k_1}{x-1}-\frac{k_2}{x^2}\right)\psi_{nl}(x)=0 \quad (71)$$

where

$$\varepsilon_{nl}=-\frac{2\mu g^2E_{nl}}{\hbar^2} \\ k_0=\frac{2\mu gA}{\hbar^2} \quad (72) \\ k_1=\frac{2\mu gB}{\hbar^2} \\ k_2=l(l+1)$$

Eq. (71) has singularities at $x=0$ and $x=1$, such that the wave function may have weighted functions $(-g)^{\lambda+\nu}x^{\lambda}(x-1)^{\nu}$.

Also, at large distances, the wave function possesses the behavior of an exponentially decaying function. The wave function may be written as:

$$\psi_n(x)=(-g)^{\lambda+\nu}x^{\lambda}(x-1)^{\nu}e^{cx}H_C(x) \quad (73)$$

Inserting the wave function into (71) with algebraic simplifications gives

$$x(x-1)\frac{d^2H_C(x)}{dx^2} \\ +(2cx^2+(2\nu+2\lambda-2c)x)\frac{dH_C(x)}{dx} \quad (74) \\ +\frac{Px^4+Qx^3+Rx^2+Sx+T}{x(x-1)}H_C(x)=0$$

where

$$P = c^2 - \varepsilon_{nl}$$

$$Q = k_1 - k_0 + 2\varepsilon_{nl} - 2c^2 + 2\lambda c + 2\nu c$$

$$R = 2k_0 - k_1 - k_2 - \varepsilon_{nl} + (\lambda + \nu)^2 - \lambda - \nu$$

$$-4\lambda c - 2\nu c + c^2$$

$$S = 2k_2 - k_0 + 2\lambda + 2\lambda c - 2\lambda\nu - 2\lambda^2$$

$$T = \lambda^2 - \lambda - k_2$$

To put (74) into a standard form, we decompose the functional coefficient of $H_C(x)$:

$$\frac{Px^4 + Qx^3 + Rx^2 + Sx + T}{x(x-1)} = Px^2 + (P+Q)x + P + Q + R + \frac{(P+Q+R+S)x + T}{x(x-1)} \tag{75}$$

The wave function parameters and a CHDE may be obtained from (74), provided that in (75), the coefficients of x^2, x in the last terms, and the constant T appearing on the right hand side must independently vanish:

$$P = 0$$

$$P + Q + R + S = 0 \tag{76}$$

$$T = 0$$

Solving (76) completely, we obtained

$$\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4k_2}$$

$$\nu = 1 \tag{77}$$

$$c = \pm\sqrt{\varepsilon_{nl}}$$

In the final step, using the results of (75) together with the conditions in (76), the CHDE is obtained as

$$\frac{d^2H_C(x)}{dx^2} + \left(2c + \frac{2\nu}{x-1} + \frac{2\lambda}{x}\right)\frac{dH_C(x)}{dx} + \left(\frac{P_2}{x-1} + \frac{P_1}{x}\right)H_C(x) = 0 \tag{78}$$

where

$$P_1 = k_2 - k_0 + \nu - \nu^2 - \lambda(2\nu - 2c - 1) - \lambda^2$$

$$P_2 = k_1 - k_2 - \nu + \nu^2 - \lambda(1 - 2\nu) + 2\nu c + \lambda^2$$

Comparing Eq. (78) with (14), and using (16) and (17), the parameters of the CHDE are obtained as

$$\alpha = 2c, \beta = 2\lambda - 1, \gamma = 2\nu - 1$$

$$\delta = k_1 - k_0, \eta = k_0 + \frac{1}{2}, \kappa = P_1, \tau = P_2$$

Utilizing the quantization condition in (18) , the energy of the perturbed Coulomb potential is obtained in closed form.

$$E_{nl} = -\frac{\mu(B-A)^2}{2\hbar^2(n+l+2)^2} \tag{79}$$

The radial wavefunction is obtained from Eq. (73) as

$$\psi_{nl}(r) = e^{-\sqrt{-\frac{2\mu E_{nl}}{\hbar^2}}r} r^{l+1}(r+g)H_C\left(\alpha, \beta, \gamma, \delta, \eta, -\frac{r}{g}\right) \tag{80}$$

Also, it is necessary to use the second constraint in Eq. (19) to obtain a relationship between the potential parameters which makes the solution quasi-exactly solvable. Using the tri-diagonal matrix determinant within Appendix A, for the ground state, the constraint equation is obtained as

$$\Delta_1(\kappa) = \kappa = k_2 - k_0 + \nu - \nu^2$$

$$-\lambda(2\nu - 2c - 1) - \lambda^2 = 0 \tag{81}$$

Using the notations; $k_i (i = 0, 2), \lambda, c, \mu = \hbar = 1$, and $\nu = 1$, Eq. (81) can be expressed in terms of the potential parameters

$$B = -\frac{l+2}{g} - \frac{A}{l+1} \tag{82}$$

For the first excited state ($n = 1$), the constraint equation is given by the matrix determinant

$$\Delta_2(\kappa) = \begin{vmatrix} \kappa - q_1 & 1 + \beta \\ n\alpha & \kappa - q_2 + \alpha \end{vmatrix} =$$

$$(k_2 - k_0 + \nu - \nu^2 - \lambda(2\nu - 2c - 1) - \lambda^2)$$

$$(k_2 - k_0 - \nu - \nu^2 - \lambda(2\nu - 2c) - \lambda^2 - \lambda + \alpha) \tag{83}$$

$$-4\alpha\lambda = 0$$

Using (82) and (83), we determined the potential parameter B for fixed values of A and g and l as shown in Tables 1-3 within the appendix B. These values were inserted into Eq. (79) to obtain quasi-exact energy spectra for $n = 0$ and $n = 1$ with varying orbital quantum number. The energy is bounded and increase with both l and g. Also, the results are in good agreement with the exact energy eigenvalues obtained using the matrix Numerov approach [76]. It is worth stating that the polynomial solution for $n = 1$, yielded two roots of B parameter (see Table 3).

The smallest of the roots causes the potential to be more attractive and reproduced energy spectra that matches the numerical energies obtained by the matrix Numerov approach at ground state ($n = 0$). This anomaly may be caused by the truncation effects and suggests a zero node due to the strong attractive nature of the potential.

Table1. Variations of energy spectra with orbital quantum number (l). $A = 5, g = 0.1, n = 0, \mu = 1, \hbar = 1$

state	Constraint parameter	Eq. (79)	Matrix Numerov approach [76]
l	B		
1	-32.50000000	-78.12500000	-78.12803062
2	-41.66666667	-68.05555555	-68.05556569
3	-51.25000000	-63.28125000	-63.28125216
4	-61.00000000	-60.50000000	-60.50000082
5	-70.83333333	-58.68055555	-58.68055595
6	-80.71428571	-57.39795920	-57.39795940
7	-90.62500000	-56.44531250	56.44531264
8	-100.5555556	-55.70987660	-55.70987668
9	-110.5000000	-55.12500000	-55.12500006
10	-120.4545455	-54.64876040	-54.64876041

Table 2. Variations of energy spectra with g. $A = 4, l = 1, n = 0, \mu = 1, \hbar = 1$

	Constraint parameter	Eq. (79)	Matrix Numerov approach [76]
g	B		
0.1	-32.00000000	- 72.00000000	- 72.00234215
0.2	-17.00000000	- 24.50000000	-24.50019380
0.3	-12.00000000	- 14.22222222	-14.22228130
0.4	- 9.500000000	- 10.12500000	-10.12502922
0.5	-8.000000000	-8.000000000	-8.000018368
0.6	-7.000000000	- 6.722222220	-6.722235467
0.7	- 6.285714286	- 5.877551030	-5.877561423
0.8	- 5.750000000	- 5.281250000	-5.281258661
0.9	- 5.333333333	- 4.839506172	-4.839513674
1.0	-5.000000000	-4.500000000	-4.500006686

Table3. Non-degenerate first excited state energy spectra. $A = 5, g = 0.1, n = 1, \mu = 1, \hbar = 1$

State	Constraint parameter	Eq. (79)	Matrix Numerov approach [76]	Constraint parameter	Eq. (79)	Matrix Numerov approach [76]
l	B			B		
1	-31.66666667	-42.01388890	-42.01546321	-95.00000000	-312.5000000	-312.5341972
2	-41.19224281	-42.67446592	-42.67447664	-113.3910905	-280.3290062	-280.3292955
3	-50.93869644	-43.46024666	-43.46025010	-132.5613036	-262.8210034	-262.8210823
4	-60.77852169	-44.15116241	-44.15116409	-152.0548116	-251.6960597	-251.6960939
5	-70.66713794	-44.73059190	-44.73059287	-171.7138144	-243.9669703	-243.9669883
6	-80.58472456	-45.21447579	-45.21447640	-191.4688469	-238.2716531	-238.2716637
7	-90.52104300	-45.62134828	-45.62134870	-211.2845126	-233.8949520	-233.8949587
8	-100.4702322	-45.96681768	-45.96681799	-231.1408789	-230.4236144	-230.4236190
9	-110.4286770	-46.26312316	-46.26312339	-251.0258684	-227.6015462	-227.6015494
10	-120.3940171	-46.51970272	-46.51970291	-270.9317404	-225.2613176	-225.2613200

4. Conclusion

In this paper, a generalized functional analysis approach has been used to obtain the bound state solutions of the wave equations with solvable potentials. We extended the previous Nikiforov-Uvarov functional analysis method which has been limited to obtaining exponential-type potentials [63]. We have generalized the approach to obtain the approximate and exact energy spectra and the corresponding wave functions of the Schrodinger equation under the Coulomb, Pseudo harmonic, hyperbolic, q-deformed hyperbolic and perturbed Coulomb potentials in

closed forms. These potential functions with the Schrodinger equation were transformed into a standard second-order differential equations such as the Whittaker, associated Laguerre, Gauss hypergeometric and confluent Heun equations. Parameters of the wave functions and also the energy quantization conditions were derived from a direct comparison with the quantized systems of differential equations. The obtained results are in excellent agreement with the works reported earlier for the Coulomb [62], Pseudo harmonic [62, 68, 69], hyperbolic-type [70–72], and the perturbed Coulomb [74, 75] potentials. Also, the quasi-exact energies for fixed potential parameters were found to be in good agreement with energy spectra obtained via the matrix Numerov method.

Funding statement
This research received no grant from any funding agency.

Authors Contribution
E. O. conceptualized the research and wrote the original draft, writing review and editing, data curation, T.V.T. conceptualized the research and wrote the original draft, E.S.E and C. A. O.:validation and methodology, S.O.O and U. E, V.: writing review and editing.

Availability of data and materials
All data were obtained numerically from the analytical solutions. Therefore no data was used in our paper.

Conflict of interests
There is no conflict of interest regarding the publication of the article.

References

[1] L. Eyges. “Bound-State Solutions of the Schrodinger Equation.” *Physical Review*, 154:1207-1217, 1967. DOI: https://doi.org/10.1103/PhysRev.154.1207

[2] Y. C. Ou, Z. Cao, and Q. Shen. “Formally exact quantization condition for nonrelativistic quantum systems.” *Journal of Chemical Physics*, 121:8175-9178, 2004. DOI: https://doi.org/10.1063/1.1799015

[3] W.C. Qiang and S.H. Dong. “Analytical approximations to the solutions of the Manning–Rosen potential with centrifugal term.” *Physics Letters A*, 368:13–17, 2007. DOI: https://doi.org/10.1016/j.physleta.2007.03.057

[4] K. J Oyewumi and K. D. Sen. “Exact solutions of the Schrodinger equation for the pseudoharmonic potential: an application to some diatomic molecules.” *Journal of Mathematical Chemistry*, 50:1039–1059, 2012. DOI: https://doi.org/10.1007/s10910-011-9967-4

[5] B. J. Falaye, K. J. Oyewumi, T. T. Ibrahim, M. A. Punyasena, and C. A. Onate. “Bound state solutions of the Manning–Rosen potential.” *Canadian. Journal. Physics*, 91:98–104, 2013. DOI: https://doi.org/10.1139/cjp-2012-0330

[6] J. Yu and S.H. Dong. “Exactly solvable potentials for the Schrodinger equation with spatially dependent mass.” *Physics Letters A*, 325:194–198, 2004. DOI: https://doi.org/10.1016/j.physleta.2004.03.056

[7] A. N. Ikot, U. S. Okorie, R. Sever and G. J. Rampho. “Eigensolution, expectation values and thermodynamic properties of the screened Kratzer potential.” *European Physical Journal Plus*, 134:386, 2019. DOI: https://doi.org/10.1140/epjp/i2019-12783-x

[8] A. Khare, F. Cooper, and J. F. Dawson. “Exact solutions of a generalized variant of the derivative nonlinear Schrodinger equation in a Scarff II external potential and their stability properties.” *Journal of Physics A: Mathematical and Theoretical*, 51:445203, 2018. DOI: https://doi.org/10.1088/1751-8121/aae1cf

[9] A. F. Nikiforov and V. B. Uvarov. *Special functions of mathematical physics*. Springer, 1988.

[10] F. Yasuk, C. Berkdemir, and A. Berkdemir. “Exact solutions of the Schrödinger equation with non-central potential by the Nikiforov-Uvarov method.” *Journal of Physics A Mathematical and General*, 38:6579, 2005. DOI: https://doi.org/10.1088/0305-4470/38/29/012

[11] S. M. Ikhdair and R. Sever. “Exact solutions of the D-dimensional Schrodinger equation for a ring-shaped pseu-doharmonic potential.” *Central. European Journal Physics*, 6:685–696, 2008. DOI: https://doi.org/10.2478/s11534-008-0024-2

[12] I. Okon, C. Onate, E. Omugbe, U. Okorie, A. Antia, M. Onyeaju, C. Wen-Li, and J. Araujo. “Approximate Solutions, Thermal Properties, and Superstatistics Solutions to Schrodinger Equation.” *Advances High Energy Physics*, 2022:1-18, 2022. DOI: https://doi.org/10.1155/2022/5178247

[13] C A Onate, L S Adebiyi, and D. T. Bankole. “Eigensolutions and theoretic quantities under the nonrelativistic wave equation.” *Journal of Theoretical and Computational Chemistry*, 19:2050007, 2020. DOI: https://doi.org/10.1142/S0219633620500078

[14] S. M Ikhdair and B. J. Falaye. “Two approximate analytic eigensolutions of the Hellmann potential with any arbitrary angular momentum.” *Zeitschrift für Naturforschung A*, 68:701–708, 2013. DOI: https://doi.org/10.5560/zna.2013-0054

[15] A. I. Ahmadov, S. M. Aslanova, M. Sh. Orujova, S. V. Badalov, and S. H. Dong. “Approximate bound state solutions of the Klein-Gordon equation with the linear

combination of Hulthen and Yukawa potentials." *Physics. Letters A*, 383:3010–3017, 2019. DOI: https://doi.org/10.1016/j.physleta.2019.06.043

[16] S. H. Dong, W. H. Huang, P. Sedaghatnia and H. Hassanabadi. "Exact solutions of an exponential type position dependent mass problem." *Results in Physics*, 34:105294, 2022. DOI: https://doi.org/10.1016/j.rinp.2022.105294

[17] C. O. Edet, P. O. Okoi, A. S. Yusuf, P. O. Ushie and P. O. Amadi. "Bound state solutions of the generalized shifted Hulthen potential." *Indian Journal of Physics*, 95:471–480, 2021. DOI: https://doi.org/10.1007/s12648-019-01650-0

[18] M. Demirci and R. Sever. "Arbitrary l-state solutions of the Klein–Gordon equation with the Eckart plus a class of Yukawa potential and its non-relativistic thermal properties." *European Physical Journal Plus*, 138:409, 2023. DOI: https://doi.org/10.1140/epjp/s13360-023-04030-0

[19] C. Tezcan and R. Sever. "A general approach for the exact solution of the Schrodinger equation." *Journal of Theoretical Physics,* 48:337–350, 2009. DOI: https://doi.org/10.1007/s10773-008-9806-y

[20] H. Karayer, D. Demirhan, and F. Büyükkılıç. "Extension of Nikiforov-Uvarov method for the solution of Heun equation." *Journal of Mathematical Physics* 56 063504 , 2015. DOI: https://doi.org/10.1063/1.4922601

[21] G. Wentzel. "Eine verallgemeinerung der quantenbedingungen fur die zwecke der wellenmechanik." *Zeitschrift für Physik* 38: 518–529, 1926. DOI: https://doi.org/10.1007/BF01397171

[22] H. A. Kramers. "Wellenmechanik und halbzahlige Quantisierung." *Zeitschrift für Physik*, 39:828–840, 1926. DOI: https://doi.org/10.1007/BF01451751

[23] L. Brillouin. "Remarques sur la mécanique ondulatoire." *Journal of physics radium*, 7:353–368, 1926. DOI: https://doi.org/10.1051/jphysrad:01926007012035300

[24] E. Omugbe, O.E. Osafile, I. B. Okon, E. S. Eyube, E. P. Inyang, U. S. Okorie, A. Jahanshir, and C. A. Onate. "Non-relativistic bound state solutions with α-deformed Kratzer-type potential using the super-symmetric WKB method: Application to theoretic information measures." *European Physical Journal D*, 76:72, 2022. DOI: https://doi.org/10.1140/epjd/s10053-022-00395-6

[25] S. S. Stepanov and R. S. Tutik. "Expansion with respect to $\hbar$ for bound states of the Schrodinger equation." *Theoretical and Mathematical Physics*, 90:139–145, 1992. DOI: https://doi.org/10.1007/BF01028437

[26] E. Omugbe, O. E. Osafile, E. P. Inyang, and A. Jahanshir. "Bound state solutions of the hyper-radial Klein-Gordon equation under the Deng-Fan potential by WKB and SWKB methods." *Physical Scipta*, 96:125408, 2021. Doi: https://doi.org/10.1088/1402-4896/ac38d4

[27] A. M. Bayoumi. "Use of WKB approximation for analytical boundary conditions in numerical solution of Schrodinger equation: application to semiconductor–high-k dielectric interfaces. *International Journal of Numerical Modelling: Electron. Networks Devices and Fields*, 29(3): 392–406, 2016. DOI: https://doi.org/10.1002/jnm.2084

[28] H. Yılmaz, D. Demirhan, and F. Büyükkılıç. "Solution of the Schr ¨ odinger equation for two q-deformed potentials by the SWKB method." *Journal of Mathematical Chemistry,* 47:539–547, 2010. DOI: https://doi.org/10.1007/s10910-009-9588-3

[29] A. Comtet, A.D. Bandrauk, D.K. Campbell. "Exactness of semiclassical bound state energies for supersymmetric quantum mechanics." *Physics Letters B*, 150:159–162, 1985. DOI: https://doi.org/10.1016/0370-2693(85)90160-1

[30] A. Khare and Y.P. Varshni. "Is shape invariance also necessary for lowest order supersymmetric WKB to be exact?" *Physics Letters A*, 142:1–4, 1989. DOI: https://doi.org/10.1016/0375-9601(89)90701-9

[31] R. Adhikari, R. Dutt, A. Khare, and U. Sukhatme. "Higher-order WKB approximations in supersymmetric quantum mechanics." *Physical Review A*, 38:1679-1686, 1988. DOI: https://link.aps.org/doi/10.1103/PhysRevA.38.1679

[32] H.K. Lau and P.T. Leung. "Application of supersymmetric WKB method to cyclic shape invariant potentials." *Journal of Physics A: Mathematical and Theoretical*, 41:075307, 2008. DOI: https://doi.org/10.1088/1751-8113/41/7/075307

[33] E. Kasap, B. Gonul and M. Sımsek. "Supersymmetric WKB solutions for pseudoharmonic and Mie-type potentials." *Chemical Physics Letters*, 172: 499-502, 1990. DOI: https://doi.org/10.1016/0009-2614(90)80146-5

[34] R. S. Bhalla, A. K. Kapoor, and P. K. Panigrahi. "Exactness of the supersymmetric WKB approximation scheme." *Physical. Review A*, 54:951-954, 1996. Doi: https://doi.org/10.1103/PhysRevA.54.951

[35] F. Cooper, A. Khare, and U. Sukhatme. "Supersymmetry and quantum mechanics." *Physcs Reports*, 251:267–385, 1995. DOI: https://doi.org/10.1016/0370-1573(94)00080-M

[36] A. Khare. "Supersymmetry in quantum mechanics." *Pramana Journal of Physics*, 49:41-64, 1997. DOI: https://doi.org/10.1007/BF02856337

[37] M. Eshghi, H. Mehraban and S. M. Ikhdair. "The relativistic bound states of a non-central potential." *Pramana Journal of Physics*, 88:73, 2017. DOI: https://doi.org/10.1007/s12043-017-1375-2

[38] A. Suparmi, J. Akbar and C. Cari. "Supersymmetric solutions of D-dimensional dirac equation for woods-saxon potential in minimal length formalism." *Journal of Physics: Conference Series*, 1825, 012104, 2021. DOI : https://doi.org/10.1088/1742-6596/1825/1/012104

[39] Z. Q. Ma and B. W. Xu. "Quantum correction in exact quantization rules." *Europhysics Letters*, 69:685, 2005. DOI: https://doi.org/10.1209/epl/i2004-10418-8.

[40] W. C. Qiang and S. H. Dong. "Arbitrary *l*-state solutions of the rotating Morse potential through the exact quantization rule method." *Physics Letters. A*, 363:169–176, 2007. DOI: https://doi.org/10.1016/j.physleta.2006.10.091

[41] S. M. Ikhdair and R. Sever. "Exact quantization rule to the Kratzer-type potentials: an application to the diatomic molecules." *Journal of Mathematical Chemistry*, 45:1137–1152, 2009. DOI: https://doi.org/10.1007/s10910-008-9438-8

[42] O. Bayrak and I. Boztosun. "Bound state solutions of the Hulthen potential by using the asymptotic iteration method." *Physica Scrpta*, 76:92, 2007. DOI: https://doi.org/10.1088/0031-8949/76/1/016

[43] T. Das. "Analytical approximate bound state solution of Schrodinger equation in D-dimensions with a new mixed class of potential for arbitrary *l*-state via asymptotic iteration method." *Chinese. Journal of Physics*, 54:850–858, 2016. DOI: https://doi.org/10.1016/j.cjph.2016.10.001

[44] A. N. Ikot, U. Okorie, A. T. Ngiangia, C. A. Onate, C. O. Edet, I. O. Akpan, and P. O. Amadi. "Bound state solutions of the Schrodinger equation with energy-dependent ¨ molecular Kratzer potential via asymptotic iteration method." *Eclética Química*, 45:65–77, 2020. DOI: https://doi.org/10.26850/1678-4618eqj.v45.1.2020.p65-77

[45] W. A Yahya, K. Issa, B. J. Falaye, and K. J. Oyewumi. "Nonrelativistic and relativistic bound state solutions of the molecular Tietz potential via the improved asymptotic iteration method." *Canadian Journal of Chemistry*, 92:215–220, 2014. DOI: https://doi.org/10.1139/cjc-2013-0479

[46] K. S. Nayek. "Bound state solutions of diatomic molecules with screened modified Kratzer potential by asymptotic iteration method." *European Physical Journal Plus*, 137:1205, 2022. DOI: https://doi.org/10.1140/epjp/s13360-022-03425-9

[47] B. J. Falaye. "Arbitrary *l*-state solutions of the hyperbolical potential by the asymptotic iteration method." *Few-Body Systems*, 53: 557–562, 2012. DOI: https://doi.org/10.1007/s00601-012-0440-0

[48] O. Bayrak and I. Boztosun. Arbitrary *l*-state solutions of the rotating Morse potential by the asymptotic iteration method. *Journal of Physics A: Mathematical and General*, 39:6955, 2006. DOI: https://doi.org/10.1088/0305-4470/39/22/010

[49] M. A.ygun, O. Bayrak, and I. Boztosun. "Solution of the radial Schrodinger equation for the potential family using the asymptotic iteration method." *Journal of Physics B: Atomic, Molecular and Optical Physics*, 40:537, 2007. DOI: https://doi.org/10.1088/0953-4075/40/3/009

[50] O. Ozer and G. Levai. "Asymptotic iteration method applied to bound-state problems with unbroken and broken supersymmetry." *Romanian Journal of Physics*, 57:582–593, 2012.

[51] V. H. Badalov. "The bound state solutions of the *D*-dimensional Schrödinger equation for the Woods-Saxon potential." *International Journal of Modern Physics E*, 25:1650002, 2016. DOI: https://doi.org/10.1142/S0218301316500026

[52] R. P. Feynman. "Forces in molecules." *Physical Review*, 56:340-343, 1939. DOI: https://doi.org/10.1103/PhysRev.56.340

[53] A. Sarsa, K.E. Schmidt, and W.R. Magro. "A path integral ground state method." , *Journal of Chemical Physics*, 113:1366–1371, 2000. DOI: https://doi.org/10.1063/1.481926

[54] A. Khodja, A. Kadja, F. Benamira, and L. Guechi. "Complete non-relativistic bound state solutions of the Tietz-Wei potential via the path integral approach." *European Physical Journal Plus*, 134:1–12, 2019. DOI: https://doi.org/10.1140/epjp/i2019-12430-8

[55] U. S. Okorie, A. N. Ikot, M. C Onyeaju, and E. O. Chukwuocha. "Bound state solutions of Schrodinger equation with modified Mobius square potential (MMSP) and its thermodynamic properties." *Journal of Molecular Modeling*, 24:1–12, 2018. DOI: https://doi.org/10.1007/s00894-018-3811-8

[56] N. Tazimi. Determination of the Energy Eigenvalues of the Varshni-Hellmann Potential. *Advances in High Energy Physics*, 2024:1, 2024. DOI: https://doi.org/10.1155/2024/5584682

[57] J. M. Zhang, D. Braak, and M. Kollar. "Bound states in the one-dimensional two-particle Hubbard model with an impurity." *Physical Review A*, 87:023613, 2013. DOI: https://doi.org/10.1103/PhysRevA.87.023613

[58] R. Horchani, E. Omugbe, T.V. Targema and E.M. Khokha. "Nonrelativistic bound state solutions and the fundamental properties of semi-exactly solvable potentials*." Nonlinear Science*, 6: 100105, 2026. DOI: https://doi.org/10.1016/j.nls.2025.100105

[59] G. Lévai and T. Soltész. T. "Schrödinger Potentials with Polynomial Solutions of Heun-Type Equations." *Mathematics*, 13: 1963, 2025. DOI: https://doi.org/10.3390/math13121963

[60] H.F. Kisoglu. "A Closed-form Quasi-exact Eigenvalues of Schrödinger Equation for a General Sextic Oscillator via Asymptotic Iteration Method." *International Journal of Theoretical Physics*, 64:327, 2025. DOI: https://doi.org/10.1007/s10773-025-06195-z

[61] B. J. Falaye, S. M. Ikhdair, and M. Hamzavi. "Formula method for bound state problems." *Few-Body Systems*, 56: 63-78, 2015. DOI: https://doi.org/10.1007/s00601-014-0937-9

[62] E. Omugbe, J. N. Aniezi, S. O. Ogundeji, C. Mbamara, R.M. Obodo, I. J. Njoku, C.A. Onate, E. S. Eyube, M. C. Onyeaju, I. B. Okon, et al. "A new conjecture for obtaining the energy spectra of the wave equations under solvable potentials." *Quantum Studies: Mathematics and Foundations*, pages 11:379–389, 2024. DOI: https://doi.org/10.1007/s40509-024-00327-6

[63] A.N. Ikot, U.S. Okorie, P.O. Amadi, C.O. Edet, G.J. Rampho, and R. Sever. The Nikiforov–Uvarov-Functional Analysis (NUFA) Method: A new approach for solving exponential-type potentials. *Few-Body Syst.*, 62:1–16, 2021. DOI: https://doi.org/10.1007/s00601-021-01593-5

[64] G. Szego. "Orthogonal Polynomials." American Mathematical Society: In *Colloquium Publications.*, 23:59, 1939.

[65] I. Gradshteyn and I. Ryzhik. "Table of integrals, series, and products.: 7th edition, academic. press, 2007.

[66] P.P. Fiziev. "Novel relation and new properties of confluent Heun functions aand their derivatives of arbitrary order." *Journal of Physics A: Mathematical and Theoretical*, 43:035203, 2010. DOI: https://doi.org/10.1088/1751-8113/43/3/035203

[67] E. Omugbe, O.E. Osafile, I.B. Okon, E.P. Inyang, E.S. William and A. Jahanshir. Any *l*-State Energy of the Spinless Salpeter Equation Under the Cornell Potential by the WKB Approximation Method: An Application to Mass Spectra of Mesons. *Few-Body Systems*, 63:1-7, 2022. DOI: https://doi.org/10.1007/s00601-021-01705-1.

[68] R. Rani, S. B. Bhardwaj, and F. Chand. "Bound state solutions to the Schrodinger equation for some diatomic molecules." *Pramana Journal of Physics*, 91:1–8, 2018. DOI: https://doi.org/10.1007/s12043-018-1622-1

[69] A. R. Amani and H. Ghorbanpour. "Supersymmetry approach and shape invariance for pseudo-harmonic potential." *Acta Physica Polonica B*, 43:1795, 2012. DOI: https://dx.doi.org/10.5506/APhysPolB.43.1795

[70] E. Omugbe, O.E. Osafile, and I.B. Okon. "Approximate eigensolutions, thermodynamic properties and expectation values of a mixed hyperbolic Poschl-Teller potential (MHPTP)." *European Physical Journal Plus*, 136:1–18, 2021. DOI: https://doi.org/10.1140/epjp/s13360-021-01712-5

[71] M. M. Ezzine, M. Hachama, and A. Diaf. "Feynman kernel analytical solutions for the deformed hyperbolic barrier potential with application to some diatomic molecules." *Phys. Scr.*, 96:125260, 2021. DOI: https://doi.org/10.1088/1402-4896/ac3c57

[72] E. Omugbe, E.S. Eyube, C.A. Onate, I.J. Njoku, A. Jahanshir, E.P. Inyang, and K.O. Emeje. "Non-relativistic energy equations for diatomic molecules constrained in a deformed hyperbolic potential function." *Journal of Molecular Modeling*, 30:74, 2024. DOI: https://doi.org/10.1007/s00894-024-05855-x

[73] E. Omugbe, O.E. Osafile, E.A. Enaibe, M.C. Onyeaju and E. Akpata. "Approximate non-relativistic s-wave energy spectra with non-polynomial potentials within the framework of the WKB approximation." *Quantum Studies: Mathematics and Foundations*, 8:261–270, 2021. DOI: https://doi.org/10.1007/s40509-021-00244-y

[74] H. Panahi and M. Baradaran. Unified treatment of a class of spherically symmetric potentials; Quasi-exact solution." *Advances in High Energy Physics*, 2016:1,

2016.
DOI: https://doi.org/10.1155/2016/8710604

[75] D. Agboola and Y. Z. Zhang. "Unified derivation of exact solutions for a class of quasi-exactly solvable models." *Journal of Mathematical Physics*, 53:043518, 2012.
DOI: https://doi.org/10.1063/1.3701833

[76] M. Pillai, J. Goglio, and T. G. Walker. "Matrix Numerov method for solving Schrödinger's equation." *American Journal of Physics* 80:1017-1019, 2012.
DOI: https://doi.org/10.1119/1.4748813

Appendix A: Second termination condition

The tridiagonal matrix determinant in eq. (19) is given by the equation [66] by

$$\begin{vmatrix} \kappa - q_1 & (1+\beta) & 0 & \cdots & 0 & 0 & 0 \\ n\alpha & \kappa - q_2 + \alpha & 2(2+\beta) & \cdots & 0 & 0 & 0 \\ 0 & (n-1)\alpha & \kappa - q_3 + 2\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \kappa - q_{n-1} + (n-2)\alpha & (n-1)\alpha(n-1+\beta)\alpha & 0 \\ 0 & 0 & 0 & \cdots & 2\alpha & \kappa - q_n + (n-1)\alpha & n(n+\beta) \\ 0 & 0 & 0 & \cdots & 0 & \alpha & \kappa - q_{n+1} + n\alpha \end{vmatrix} = 0 \quad \text{(A1)}$$

Where

$$q_n = (n-1)(n+\beta+\gamma)$$

Appendix B

Energy spectra of the potential

$$V(r) = -\frac{A}{r} + \frac{B}{g+r}$$

10.57647/jtap.2026.8614.0307