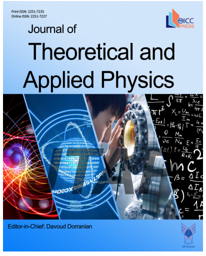

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On the conformable fractional derivative and its applications in physics

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Original Research

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Abstract:

This research reviews the basics of the conformable fractional derivative (CFD) and explores various applications of it in physics. The conformable fractional versions of path integral approach, divergence and Green's theorem are thoroughly discussed. Additionally, the basics of Newtonian mechanics in the context of CFD are covered, including velocity, acceleration, Newton's law, Yank, the classical Doppler effect, work, energies and the theory of conservation of momentum. Alongside some fundamentals of special relativity and its postulates are formulated within the frame of CFD, including the Lorentz transformation and fundamental four-vectors. Conformable fractional non-relativistic and relativistic quantum mechanics are also extensively explored, covering the ordinary and angular Schrödinger equations, as well as the Pauli equation. Moreover, a particle-in-a-box model scenario within CFD is investigated, along with the Klein-Gordon, Dirac, and Fisk-Tait equations. Additionally, the continuity equation and a classical limit using Ehrenfest's theorem are derived from the conformable fractional Dirac equation. Some graphics are also included to enhance understanding the behaviors of certain models within CFD, such as the divergence theorem and spherical harmonic.

Keywords: Conformable fractional derivative; Path integral; Divergence theorem; Classical doppler effect; Dirac equation; Conformable fractional quantum mechanics; Conformable fractional spherical harmonic; Conformable fractional special relativity

1. Introduction

In recent times, there has been great interest in the concept of fractional derivatives [1–3], largely attributed to their wide-ranging application in various fields of science, engineering, finance and optimal problem. Consequently, numerous explanations of fractional derivatives have been proposed, owing to the benefits offered by this concept for modeling real-world problems. Note that, the fractional derivative, dating back to as early as calculus itself, traces its origins to 1695 when L'Hospital posed inquiries to Leibniz about $d^n f/dx^n$ when n equals $1/2$. However, Leibniz responded that this would be "an apparent paradox, from which one day useful consequences will be drawn" [4]. Since then, researchers have worked to elucidate the concept of fractional derivatives, predominantly employing integral formulations. Over time, various definitions have emerged, including those by Riemann–Liouville, Caputo, Riesz, Weyl, Riesz–Caputo, Chen, Grünwald, and Hadamard, [5–7]. Among these, the Riemann–Liouville and Caputo formulations stand out as the most prevalent. So, the Riemann–Liouville definition, for $\alpha \in [n-1, n)$, the α derivative of f is:

$$D_x^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx. \tag{1}$$

But, the Caputo definition, for $\alpha \in [n-1, n)$, the α derivative of f is:

$$D_x^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}} dx. \tag{2}$$

Mathematicians prefer Riemann–Liouville fractional derivative because it is amenable to many mathematical manipulations. However, this type of fractional derivative of a

constant is not zero, and it requires fractional initial conditions, which are not generally specified. Conversely, Caputo derivative of a constant is zero, and a fractional differential equation expressed in terms of Caputo fractional derivative requires standard boundary condition. For this reason, physicists and engineers often prefer Caputo fractional derivative. For further insight into diverse mathematical aspects of fractional calculus, refer to the seminal works in [8, 9]. The fractional derivative has played an essential role across various domains including physics, chemistry, biology, and engineering. See, for example, [10–12].
However, in the past few years, Khalil et al. [13] introduced a new concept of derivative termed the conformable fractional derivative (CFD) depending just on the basic limit definition of the derivative. They claimed that this new definition is the simplest, most natural, and efficient approach to fractional derivative of order $\alpha \in (0, 1]$, with properties that closely align with the classical Newtonian derivative, making it easier to solve fractional differential equations. Subsequently, significant research efforts have been devoted to advancing CFD calculus and investigating its properties. Specifically, by Katugampola and T. Abdeljawad. For instance, conformable fractional forms of fundamental mathematical tools such as exponential functions, the chain rule, Taylor power series expansions, integration by parts, Gronwall's inequality, Laplace or Fourier transforms, linear differential systems [14, 15], PT symmetry [16] and Nikiforov–Uvarov method [17] have been proposed and studied. Furthermore, applications of CFD in various physical contexts have been explored [18–31].
Although fractional derivatives have been criticized by contemporary mathematicians for exhibiting unusual properties that can complicate their application in physics and mechanics, some physicists continue to use them. Physicists consider fractional derivatives, including CFD to be very valuable, powerful, and effective tools for modelling nonlinear system and investigating the behavior of systems characterized by power-law non-locality, long-term memory, or fractal properties, among other phenomena. For the most recent references on this topic, see [32–40].
In the following, we outline some critiques of fractional derivatives and CFD. In [41–43], the violation of the Leibniz rule (Equation (6)) is identified as a defining characteristic of fractional derivatives, specifically those of non-integer orders. Therefore, any derivative that satisfies the Leibniz rule must be of integer order, implying that fractional derivatives of non-integer orders, including the CFD, cannot satisfy the Leibniz rule, meaning the CFD must correspond to an integer-order derivative. In addition, in [44], the concept of fractional derivatives and the properties that such operators should satisfy were extensively discussed. It was argued that the CFD fails in certain aspects. From a mathematical perspective, it is demonstrated that the conformable derivative is not truly fractional. Additionally, in [45], it was confirmed that the CFD is not a fractional derivative but rather a controlled or conformable derivative. Furthermore, in [46], a major flaw in the so-called conformable calculus was pointed out. The authors claimed to demonstrate why it fails to define a true fractional-order derivative and explained where exactly these tempting conformability properties come from.
In this work, we review the postulates, basic formulas and proprieties of the CFD and consider its application in physics, encompassing mathematical physics, Newtonian mechanics, special relativity as well as the ordinary and relativistic quantum mechanics. In section 2, we review the basic formulas of the CFD. Following that, section 3 delves into the conformable fractional version of the path integral approach, the divergence and Green's theorems. Section 4 focuses on deriving conformable fractional expressions for velocity and acceleration, which are then utilized to present Newton's second law, Yank, works, and energies. Additionally, we investigate the conformable fractional form of the classical Doppler effect and explore the conservation of conformable fractional momentum. In section 5, we present the postulates of conformable fractional special relativity, derive the conformable fractional version of Lorentz transformation, and apply it in space-time. Additionally, we define the conformable fractional versions of the fundamental four-vectors. In section 6, we investigate the conformable fractional versions of the ordinary, angular Schrödinger and Pauli equations. Additionally, we explore the infinite potential well problem within the context of CFD. In section 7, our focus lies in the exploration of conformable fractional versions of the Klein-Gordon, Dirac equations, and the Fisk-Tait equation. Furthermore, we derive the conformable fractional continuity equation from the Dirac equation and delve into the classical limit of the latter by employing Ehrenfest's theorem. We present the conclusion and remarks in section 8.

2. Overview on the conformable fractional derivative

Let briefly examine the postulates and basic formulas of the CFD utilized in this study. Therefore, for a smooth function in x, the CFD can be defined as follows [13]:

$$D_x^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon |x|^{1-\alpha}) - f(x)}{\varepsilon} = |x|^{1-\alpha} \partial_x f(x), \tag{3}$$

where $0 < \alpha \leq 1$ is assumed. Note that D^α is the CFD operator. However, at $x = 0$, the CFD is $D_x^\alpha f(0) = \lim_{x \to 0} D_x^\alpha f(x)$. Then, the conformable fractional partial derivative of f in x_i is expressed as [47]:

$$\frac{\partial^\alpha}{\partial x_i^\alpha} f(x_1, \cdots, x_m) = \lim_{\varepsilon \to 0} \frac{f(x_1, \cdots, x_{i-1}, \cdots, x_i + \varepsilon |x_i|^{1-\alpha}, \cdots, x_m) - f(x_1, \cdots, x_m)}{\varepsilon}. \tag{4}$$

The CFD satisfies the following properties: For f, g be α-differentiable functions, and real constants a, b, we have:
The linearity:

$$D_x^\alpha \{a f(x) + b g(x)\} = a D_x^\alpha f(x) + b D_x^\alpha g(x), \tag{5}$$

the product rule, also called the Leibniz rule:

$$D_x^\alpha \{f(x) g(x)\} = [D_x^\alpha f(x)] g(x) + f(x) D_x^\alpha g(x), \tag{6}$$

the chain rule:

$$D_x^\alpha f(g(x)) = \frac{df}{du}(D_x^\alpha u). \tag{7}$$

Also

$$D_x^\alpha(x^\alpha) = \alpha, \tag{8}$$

$$D_x^\alpha(x^p) = px^{p-\alpha}, \text{ for } p \in R, \tag{9}$$

$$D_x^\alpha\left(\frac{f(x)}{g(x)}\right) = \frac{(g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x))}{g(x)^2}, \tag{10}$$

$$D_x^\alpha(\lambda) = 0 \text{ for all constant functions } f(x) = \lambda. \tag{11}$$

All of the classical derivative rules, such as sum, product, division, etc. are same as the conformable fractional derivative.

The conformable fractional integral is specified by:

$$\mathfrak{I}_{a|x}^\alpha f(x) = \int_a^x |W|^{\alpha-1} f(W)dW, \tag{12}$$

where $f(x)$ is any continuous function. Moreover, the CFD and conformable fractional integral obey the following relations:

$$D_x^\alpha \mathfrak{I}_{a|x}^\alpha f(x) = f(x), \tag{13}$$

$$\mathfrak{I}_{a|x}^\alpha D_x^\alpha f(x) = f(x) - f(a). \tag{14}$$

With all the aforementioned definitions, one can present the following properties for the various operations on scalar and vector fields [47]:

$$\mathbf{D}_x^\alpha \cdot (a\mathbf{F} + b\mathbf{G}) = a\mathbf{D}_x^\alpha \cdot \mathbf{F} + b\mathbf{D}_x^\alpha \cdot \mathbf{G}, \tag{15}$$

$$\mathbf{D}_x^\alpha \times \{a\mathbf{F} + b\mathbf{G}\} = a\mathbf{D}_x^\alpha \times \mathbf{F} + b\mathbf{D}_x^\alpha \times \mathbf{G}, \tag{16}$$

$$\mathbf{D}_x^\alpha \cdot (\mathbf{D}_x^\alpha \times \mathbf{F}) = 0, \tag{17}$$

$$\mathbf{D}_x^\alpha \times (\mathbf{D}_x^\alpha \cdot \mathbf{F}) = 0, \tag{18}$$

where $\mathbf{F}(x)$, $\mathbf{G}(x)$ are two arbitrary vectors.

It should be emphasized that in [48], a new version of the CFD was introduced, called the general conformable fractional derivative (GCFD), designed to better describe the physical world. The GCFD is a generalization of the original CFD, achieved through a new framework called the Extended Gâteaux derivative and the Linear Extended Gâteaux derivative, both of which are natural extensions of the traditional Gâteaux derivative. The authors argue that the $t^{1-\alpha}$ term in the CFD definition is not essential and is merely a type of conformable fractional function. Additionally, they provided physical and geometrical interpretations of the GCFD, suggesting its potential applications in physics and engineering. They also claimed that it is straightforward to show that CFD is a special case of GCFD.

This study will highlight the significance of exploring the CFD and its properties across diverse domains.

3. Mathematics

In this section, we will employ CFD to explore some significant mathematical methods, tools, and approaches.

3.1 Conformable fractional divergence theorem

The divergence theorem, also known as Gauss's theorem, is a fundamental theorem in vector calculus. It relates the flux of a vector field through a closed surface to the divergence of the field within the volume enclosed.

It is an important tool to use for physics, engineering, and particularly in electrostatics and fluid dynamics. It is usually applied in three dimensions. However, it generalizes to any number of dimensions.

Let V represent a volume in three-dimensional space with $\partial V = S$ where S is the surfae. If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of V, then [49]:

$$\int\int\int(\nabla \cdot \mathbf{F})dV = \int\int(\mathbf{F} \cdot \mathbf{n})dS, \tag{19}$$

$\mathbf{n}$ is the outward unit normal of the surface. Now, we obtain the conformable fractional form of the divergence theorem, so in CFD, and using Equation (19), we have:

$$\int\int\int(\mathbf{D}_x^\alpha \cdot \mathbf{F})d^\alpha V = \int\int |x|^{1-\alpha}(\mathbf{F} \cdot \mathbf{n})dS, \tag{20}$$

where $\mathbf{D}_x^\alpha \cdot \mathbf{F} = |x|^{1-\alpha}\boldsymbol{\partial}_x \cdot \mathbf{F}(x)$. With $d^\alpha S = x^{\alpha-1}y^{\alpha-1}dxdy$, and $d^\alpha V = x^{\alpha-1}y^{\alpha-1}z^{\alpha-1}dxdydz$.

Similarly, the theorems of Green and Stokes from vector calculus may be suitably adapted to easily accommodate the concept of the CFD. As an example, the conformable fractional Green's theorem can be formulated as follows:

$$\int \partial_y^{\alpha-1} f d^\alpha x + \partial_x^{\alpha-1} g d^\alpha y = \int\int(\partial_x^\alpha g - \partial_y^\alpha f)d^\alpha S. \tag{21}$$

In physics, the theorem of Green boasts numerous applications. One such application involves solving $2D$ flow integrals. It asserts that the total fluid outflow from a volume is equivalent to the total outflow summed around a bounding area. Note that the conformable divergence and Green's theorems are considered in [47].

In Figure 1, we plot the divergence theorem of the $2D$ vector field $[x^2, y^2]$.

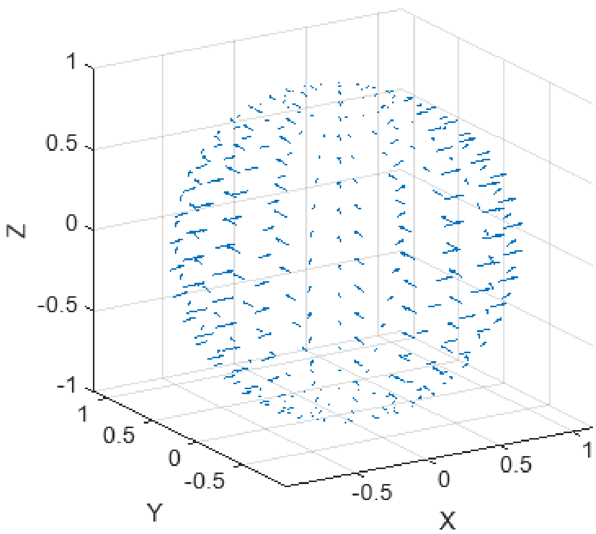

Figure 1. Vector field on the surface of the region, $R = 1$.

In Figures 2, 3 we plot the conformable fractional divergence theorem of the 2D vector field $[x^2, y^2]$ for the cases of $\alpha = 0.1$, and $\alpha = 0.5$.

3.2 Conformable fractional path integral

In quantum mechanics, if a particle at an initial time t_a, starts from the point x_a and goes to a final point x_b at time t_b, we have a quantum-mechanical amplitude called a kernel. To get from point a to point b, we may write $\mathscr{K}_a(x_b, t_b|x_a, t_a)$. This is the sum over all paths that go between those end-points and their contributions [50]. So, for the following Hamiltonian:

$$\mathscr{H}_\alpha(x,p) = \frac{1}{2m^\alpha}\hat{p}_\alpha^2 + V_\alpha(\hat{x}_\alpha), \tag{22}$$

then, the definition of the kernel $\mathscr{K}_a(x_b, t_b|x_a, t_a)$ in terms of path integral in the phase-space representation is as follows

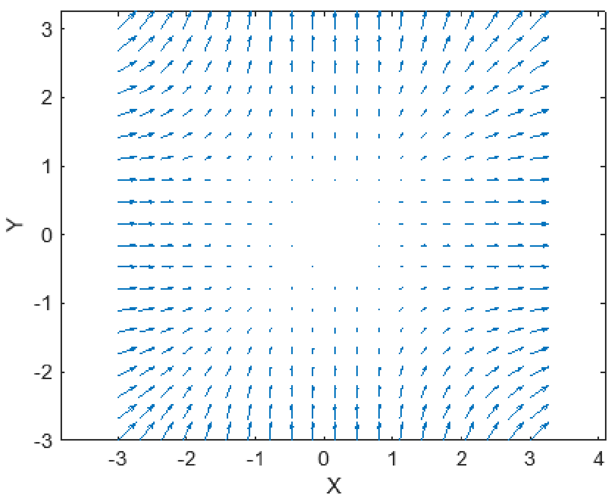

Figure 2. Vector field on the surface of the region, $\alpha = 0.1$.

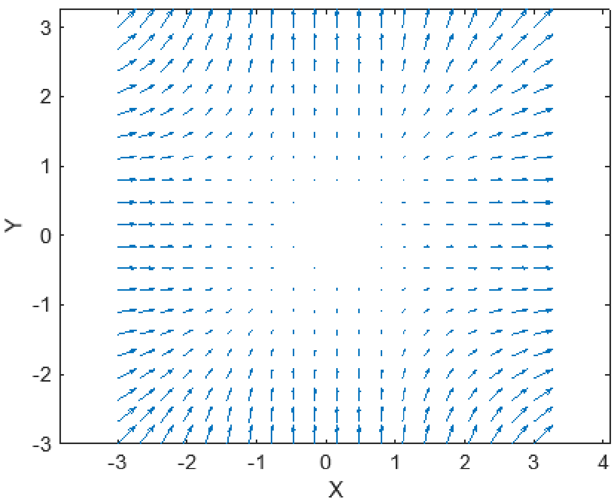

Figure 3. Vector field on the surface of the region, $\alpha = 0.5$.

[51]:

$$\begin{aligned} &\mathscr{K}_a(x_b, t_b|x_a, t_a) = \\ &\lim_{N\to\infty}\int_{-\infty}^{\infty} dx_1 \cdots dx_{N-1}\frac{1}{(2\pi\hbar)^N}\int_{-\infty}^{\infty} dp_1\cdots dp_N \times \\ &\exp\{\frac{i}{\hbar}\Sigma_{j=1}^N p_j(x_j - x_{j-1})\}\times \\ &\exp\{\frac{i}{\hbar}\frac{1}{2m^\alpha}\varepsilon\Sigma_{j=1}^N|p_j|^\alpha - \frac{i}{\hbar}\varepsilon\Sigma_{j=1}^N V(x_j, j\varepsilon)\}. \end{aligned} \tag{23}$$

where $\varepsilon = (t_b - t_a)/N$, and $x_j = x(j\varepsilon)$, $p_j = p(j\varepsilon)$, also $x_0 = x_a$, $x_N = x_b$. Now, in the limits of $N \to \infty$, $\varepsilon \to 0$ (continuum limit), one can have

$$\begin{aligned} \mathscr{K}_a(x_b, t_b|x_a, t_a) = &\int_{x(t_a)=x_a}^{x(t_b)=x_b} Dx(\tau)\int Dp(\tau) \\ &\exp\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau|p(\tau)\dot{x} - \mathscr{H}_\alpha(p(\tau),x(\tau),\tau)|\}. \end{aligned} \tag{24}$$

Note that in Equation (24), the conformable fractional Hamiltonian (22), we made the replacement $p \to p(\tau)$, $x \to x(\tau)$, and this replacement stands for the particle trajectory in phase-space. Therefore, the phase-space path integral in Equation (23) is:

$$\begin{aligned} &\int_{x(t_a)=x_a}^{x(t_b)=x_b} Dx(\tau)\int Dp(\tau)\exp\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau|p(\tau)\dot{x} - \\ &\mathscr{H}_\alpha(p(\tau),x(\tau),\tau)|\} = \lim_{N\to\infty}\int_{-\infty}^{\infty} dx_1\cdots dx_{N-1}\frac{1}{(2\pi\hbar)^N} \\ &\int_{-\infty}^{\infty} dp_1\cdots dp_N \times \exp\{\frac{i}{\hbar}(p_1(x_1 - x_a) - \frac{1}{2m^\alpha}\varepsilon|p_1|^\alpha)\}\times \\ &\cdots\exp\{\frac{i}{\hbar}(p_N(x_b - x_{N-1})\frac{1}{2m^\alpha}\varepsilon|p_N|^\alpha)\}\cdots \end{aligned} \tag{25}$$

Note that, for the path $p(t)$, $x(t)$ in phase-space, the conformable fractional canonical classical mechanical action is given as:

$$S_\alpha = \int_{t_a}^{t_b} d\tau[p(\tau)\dot{x} - \mathscr{H}_\alpha(p(\tau),x(\tau),\tau)]. \tag{26}$$

The wave function as well can be given by:

$$\psi(x_b,t_b) = \int_{-\infty}^{\infty} dx_a\mathscr{K}_a(x_b, t_b|x_a, t_a)\psi(x_a,t_a), \tag{27}$$

where $\psi(x_b,t_b)$, $\psi(x_a,t_a)$ are the wave function of the final and initial states respectively.

It is important to mention that the associated kernel can be acquired by following the same procedures as previously outlined, but working in the momentum representation. Note that the fractional path integral approach, where $1 < \alpha \leq 2$, is discussed in [52].

4. Classical mechanics

In this section, we define the Newtonian mechanics in terms of the CFD, subsequently employ it to study some classical cases.

4.1 Conformable fractional velocity & acceleration

In the context of CFD, the average velocity of a particle that lies on $x(t)$ at time t and on $x(\acute{t})$ at time $\acute{t}$ (with $\acute{t} > t$) is

$$v_{ave} = \frac{(x(\acute{t}))^\alpha - (x(t))^\alpha}{\acute{t} - t}. \quad (28)$$

Note that the numerator in Equation (28) cannot be regarded as the deformed subtraction because its corresponding deformed addition is not associative. Thus, in [53], the conformable fractional addition and subtraction for the case of $x > y > 0$ are defined as:

$$\begin{cases} x \oplus_\alpha y = (x^\alpha + y^\alpha)^{\frac{1}{\alpha}}, \\ \qquad\qquad\qquad\qquad \text{with } 0 < \alpha \le 1, \\ x \ominus_\alpha y = (x^\alpha - y^\alpha)^{\frac{1}{\alpha}}, \end{cases} \quad (29)$$

where the conformable fractional addition is commutative and associative and preserves the dimension. Then, the conformable fractional average velocity becomes

$$v_{ave} = \frac{(x(\acute{t}) \ominus_\alpha x(t))^\alpha}{\acute{t} - t} = \frac{\Delta_\alpha x}{\Delta t}, \quad (30)$$

with $\Delta_\alpha x$ is the α-displacement. As reported in [53], there is an issue regarding the possibility of the position $x(t)$ taking on negative values. There, the conformable fractional addition or subtraction becomes a complex number. This was fixed by modifying the definitions of conformable fractional addition and subtraction so that they can be hold for any $x,\ y \in R$ [53]:

$$\begin{cases} x \oplus_\alpha y = \| x|^{\alpha-1}x + |y|^{\alpha-1}y|^{\frac{1}{\alpha-1}} (|x|^{\alpha-1}x + |y|^{\alpha-1}y), \\ x \ominus_\alpha y = \| x|^{\alpha-1}x - |y|^{\alpha-1}y|^{\frac{1}{\alpha-1}} (|x|^{\alpha-1}x - |y|^{\alpha-1}y). \end{cases} \quad (31)$$

Note that the relation between the conformable fractional addition and subtraction is as follows:

$$x \ominus_\alpha y = x \oplus_\alpha (-y), \quad (32)$$

Furthermore, one can distinguish the properties of conformable fractional addition and subtraction as follows [53]:
1. Distributivity:

$$\begin{cases} \lambda(x \oplus_\alpha y) = (\lambda x) \oplus_\alpha (\lambda y), \\ \qquad\qquad\qquad\qquad\qquad \lambda \in \mathbb{R} \\ \lambda(x \ominus_\alpha y) = (\lambda x) \ominus_\alpha (\lambda y), \end{cases} \quad (33)$$

2. Expansion:

$$(A \oplus_\alpha B)(C \oplus_\alpha D) = AC \oplus_\alpha BC \oplus_\alpha AD \oplus_\alpha BD. \quad (34)$$

Now, the conformable fractional velocity is given as:

$$v(t) = D_t^\alpha x(t) = \lim_{\acute{t} \to t} \frac{|x(\acute{t}) \ominus_\alpha x(t)|^{\alpha-1}(x(\acute{t}) \ominus_\alpha, x(t))}{\alpha(\acute{t} - t)} \quad (35)$$

or [54]:

$$v(t) = \alpha |x|^{\alpha-1} \dot{x}. \quad (36)$$

This velocity has a dimension of $(\text{Length})/(\text{Time})^\alpha$. Note that the conformable fractional velocity is not invariant under ordinary translation $x \to x + \varepsilon$, but it is invariant under the α-translation $x \to x \oplus_\alpha \varepsilon$. Additionally, we can write the conformable fractional momentum as follows:

$$p = mv = m\alpha |x|^{\alpha-1} \dot{x}. \quad (37)$$

Now, with the same manner, we define the conformable fractional average acceleration as follows:

$$a_{ave} = \frac{v(\acute{t}) - v(t)}{\acute{t} - t}, \quad (38)$$

subsequently, the conformable fractional instantaneous acceleration is given by [54]:

$$a = D_t^\alpha v(t) = \alpha |x|^{\alpha-1} \ddot{x} - \alpha(1-\alpha)|x|^{\alpha-2} x \dot{x}^2, \quad (39)$$

which has dimension of $(\text{Length})^\alpha/(\text{Time})^2$.
Last and not least, having Equations (36), (39) enables the exploration of diverse phenomena such as the Keplerian planetary motion, pendulum dynamics, or rigid body motion in the context of CFD.

4.2 Conformable fractional Newton's Second law

By using Equation (39), the conformable fractional version of the Newton's Second law reads (m is time independent):

$$F = D_t^\alpha p = m(D_t^\alpha)^2 x(t) = m\{\alpha |x|^{\alpha-1} \ddot{x} - \alpha(1-\alpha)|x|^{\alpha-2} x \dot{x}^2\}. \quad (40)$$

In [54], the invariance of Equation (40) under both the ordinary Galilei ($x \to x - ut$) and the conformable fractional Galilei transformations ($|x|^{\alpha-1}x \to |x|^{\alpha-1}x - u_\alpha t$, u_α is the constant conformable fractional velocity) was examined. It was discovered that while conformable fractional Newton's law is not invariant under the ordinary Galilei transformation, it remains invariant under the conformable fractional one. There, for $x,\ \acute{x},\ u > 0$, we obtained:

$$\acute{x} = (x^\alpha + \alpha ut)^{\frac{1}{\alpha}} = x \oplus_\alpha (\alpha ut)^{\frac{1}{\alpha}}. \quad (41)$$

Having Equation (40) also enables the derivation of the conformable fractional Yank, which represents the rate of change of the conformable fractional force. It can be expressed as follows:

$$Y = D_t^\alpha F = m(D_t^\alpha)^3 x(t) = mD_t^\alpha \{\alpha |x|^{\alpha-1} \ddot{x} - \alpha(1-\alpha)|x|^{\alpha-2} x \dot{x}^2\}. \quad (42)$$

4.3 Conservation of conformable fractional momentum (1dimensional)

In a closed system consisting of two interacting particles, the forces between them are equal in magnitude but opposite in direction, in accordance with the third law. So, with $F_1 = D_t^\alpha p_1$ and $F_2 = D_t^\alpha p_2$ then

$$D_t^\alpha p_1 = -D_t^\alpha p_2. \quad (43)$$

However, if the velocities of the bodies are v_{A1} and v_{B1} before the collision, and afterwards they are v_{A2} and v_{B2}, so the conservation of momentum before and after the collision is expressed as:

$$m_A v_{A1} + m_B v_{B1} = m_A v_{A2} + m_B v_{B2}, \tag{44}$$

then, employing Equation (44), one can have

$$m_A |x_{A1}|^{\alpha-1} \dot{x}_{A1} + m_B |x_{B1}|^{\alpha-1} \dot{x}_{B1} = m_A |x_{A2}|^{\alpha-1} \dot{x}_{A2} + m_B |x_{B2}|^{\alpha-1} \dot{x}_{B2}, \tag{45}$$

■ Elastic collision case: In any collision, momentum remains conserved; however, in an elastic collision, the conservation of kinetic energy is also conserved. By solving Equation (45), we can simply find:

$$\dot{x}_{A2} = \frac{m_A - m_B}{m_A + m_B} \left| \frac{x_{A1}}{x_{A2}} \right|^{\alpha-1} \dot{x}_{A1} + \frac{2m_B}{m_A + m_B} \left| \frac{x_{B1}}{x_{A2}} \right|^{\alpha-1} \dot{x}_{B1}, \tag{46}$$

$$\dot{x}_{B2} = \frac{m_B - m_A}{m_A + m_B} \left| \frac{x_{B1}}{x_{B2}} \right|^{\alpha-1} \dot{x}_{B1} + \frac{2m_A}{m_A + m_B} \left| \frac{x_{A1}}{x_{B2}} \right|^{\alpha-1} \dot{x}_{A1}, \tag{47}$$

■ Inelastic collision case: In a perfectly inelastic collision, both bodies have the same motion afterwards (with velocity v_2). So that, Equation (44) expressing conservation of momentum becomes:

$$m_A |x_{A1}|^{\alpha-1} \dot{x}_{A1} + m_B |x_{B1}|^{\alpha-1} \dot{x}_{B1} = (m_A + m_B) |x_2|^{\alpha-1} \dot{x}_2. \tag{48}$$

Once one body is motionless to begin, one can have

$$\dot{x}_2 = \frac{m_A}{m_A + m_B} \left| \frac{x_{A1}}{x_2} \right|^{\alpha-1} \dot{x}_{A1}. \tag{49}$$

4.4 Conformable fractional from of classical Doppler effect

The Doppler effect is the change in frequency or wavelength of a wave observed when the source or observer is in motion relative to each other. This effect is commonly experienced with sound waves, i.e., acoustic Doppler effect and light waves, i.e., optical Doppler effect. The Doppler effect finds numerous applications across various branches of physics, including sonar, Doppler effect for electromagnetic waves (particularly in astronomy), radar, medical imaging (utilizing Doppler ultrasound), and satellite navigation (NAVSAT). So, the essential formulas for classical Doppler effect [55] in the context of CFD are as follows:

■ For moving source:

$$\frac{\acute{f}^{\alpha}}{f^{\alpha}} = \frac{v^{\alpha} \pm v_0}{v^{\alpha} \pm v_s} = \frac{v^{\alpha} \pm \alpha |x_s|^{\alpha-1} \dot{x}_s}{v^{\alpha} \pm \alpha |x_0|^{\alpha-1} \dot{x}_0}. \tag{50}$$

■ For moving observer:

$$\frac{\acute{f}^{\alpha}}{f^{\alpha}} = \frac{v^{\alpha}}{v^{\alpha} \pm v_s} = \frac{v^{\alpha}}{v^{\alpha} \pm \alpha |x_0|^{\alpha-1} \dot{x}_0}. \tag{51}$$

And

$$v^{\alpha} = f^{\alpha} \lambda^{\alpha}, \tag{52}$$

where $\acute{f}$, f are observed and source actual frequencies. v is speed of sound in the medium, v_s is velocity of the source (positive if moving towards the observer, negative if moving away), v_0 velocity of the observer (positive if moving towards the source, negative if moving away). λ is the wavelength.

4.5 Conformable fractional work and energies

In [53], the conformable fractional work is defined as the product of the force and conformable fractional displacement as follows:

$$W = F \Delta_{\alpha} x = F (\acute{x} \ominus_{\alpha} x)^{\alpha}, \tag{53}$$

But, if the force varies during a motion from x_i to x_f, the work will be:

$$W = \int_{x_i}^{x_i} d_{\alpha} x F(x) = \int_{x_i}^{x_i} dx \alpha |x|^{\alpha-1} F(x). \tag{54}$$

The conformable fractional kinetic energy K from Equation (54), so, employing Equation (40), one can have

$$W = \alpha \int_{x_i}^{x_i} |x|^{\alpha-1} m \frac{dv}{dt} \frac{dx}{dt} dt = \alpha \int_{x_i}^{x_i} |x|^{\alpha-1} m \frac{dv}{dt} v dt = K(x_f) - K(x_i). \tag{55}$$

Then, we define the conformable fractional potential energy $U(x)$ through:

$$F(x) = -D_x^{\alpha} U = |x|^{\alpha-1} \partial_x U. \tag{56}$$

The force derived from such a potential function is said to be conservative. Now, with Equations (54) and (56), we have:

$$W = \int_{x_i}^{x_i} d_{\alpha} x F(x) = -\int_{x_i}^{x_i} d_{\alpha} x |x|^{\alpha-1} \partial_x U = -U(x_f) + U(x_i). \tag{57}$$

Knowing that the conformable fractional kinetic energy reads:

$$K = \frac{1}{2} m (\alpha |x|^{\alpha-1} \dot{x})^2. \tag{58}$$

Last but not least, with Equations (36), (54) in hand, we can delve into the study of various classical phenomena. For example, we can explore Keplerian Planetary Motion, Pendulum Dynamics, or rigid body dynamics within the context of CFD as in [56].

5. Special relativity

In conformable fractional special relativity, the following postulates are established:

■ Postulate 1: Constancy of the speed of light - the speed of light remains constant and the same across all conformable fractional inertial frames of references. This is verified in Ref. [57].

■ Postulate 2: Invariance principle - the laws of physics remain unchanged and invariant under conformable fractional Lorentz transformations.

5.1 Lorentz transformation

Here, we discuss the Lorentz transformations utilizing CFD, as reported in Refs. [57, 58]. So, the conformable fractional Lorentz transformations (α-Lorentz transformations) between two inertial frames S and $\acute{S}$ are defined as follows:

$$\acute{x}^{\alpha} = \Gamma_{\alpha} (x^{\alpha} - \mathrm{v}_{\alpha} t^{\alpha}), \tag{59}$$

$$\acute{t}^{\alpha} = \Gamma_{\alpha}(t^{\alpha} - \frac{\mathrm{v}_{\alpha}}{c^{2\alpha}}x^{\alpha}), \tag{60}$$

$$\acute{y}^{\alpha} = y^{\alpha}, \tag{61}$$

$$\acute{z}^{\alpha} = z^{\alpha}. \tag{62}$$

where v_{α} is the conformable fractional relative velocity between S and $\acute{S}$ frames.

$$\Gamma_{\alpha} = \frac{1}{\sqrt{1-(\frac{\mathrm{v}_{\alpha}}{c^{\alpha}})^2}}, \text{ with } \beta^{\alpha} = \frac{\mathrm{v}_{\alpha}}{c^{\alpha}} \tag{63}$$

is the conformable fractional Lorentz factor (Fig. 4).

5.2 Conformable fractional Lorentz transformation in space-time

Minkowski space-time is a combination of three-dimensional Euclidean space and time, resulting in a unified four-dimensional manifold. The α-Lorentz transformation in Minkowski space-time is given by:

$$\acute{x}^{\mu\alpha} = (\Lambda^{\mu}_{\nu})^{\alpha} x^{\nu\alpha}, \tag{64}$$

$$\acute{x}^{\alpha}_{\mu} = (\Lambda^{\nu}_{\mu})^{\alpha} x^{\alpha}_{\nu}, \tag{65}$$

where $(\Lambda^{\mu}_{\nu})^{\alpha}$, $(\Lambda^{\nu}_{\mu})^{\alpha}$ are α-tensors, defined as:

$$(\Lambda^{\mu}_{\nu})^{\alpha} = \begin{bmatrix} \Gamma_{\alpha} & -\Gamma_{\alpha}\beta^{\alpha} & 0 & 0 \\ -\Gamma_{\alpha}\beta^{\alpha} & \Gamma_{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{66}$$

$$(\Lambda^{\nu}_{\mu})^{\alpha} = \begin{bmatrix} \Gamma_{\alpha} & \Gamma_{\alpha}\beta^{\alpha} & 0 & 0 \\ \Gamma_{\alpha}\beta^{\alpha} & \Gamma_{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{67}$$

with $((\Lambda^{\mu}_{\nu})^{\alpha})^{-1} = (\Lambda^{\nu}_{\mu})^{\alpha}$.

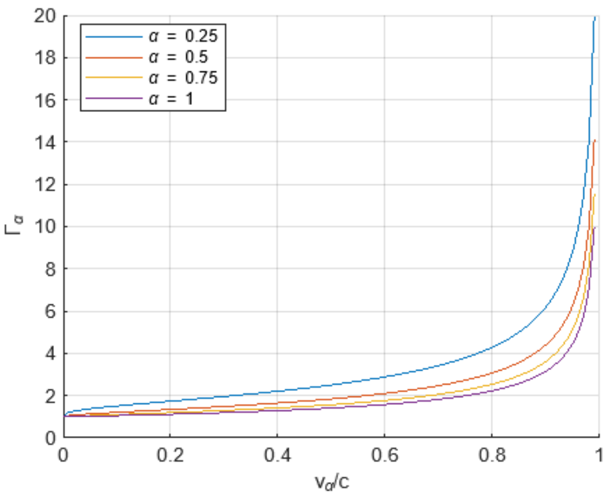

Figure 4. Plot of $\mathbf{\Gamma}_{\alpha}$ vs $\mathbf{v}_{\alpha}$; $c = 1$ (normalized to 1).

5.3 Conformable fractional four-vector

Here, we define the conformable fractional version of the fundamental four-vectors as follows:
1. The covariant position four-vector: It is defined as:

$$x^{\alpha}_{\mu} = (x^{\alpha}_{0}, x^{\alpha}_{1}, x^{\alpha}_{2}, x^{\alpha}_{3}) = (c^{\alpha}t^{\alpha}, -x^{\alpha}, -y^{\alpha}, -z^{\alpha}), \tag{68}$$

and its contravariant notation is:

$$(x^{\mu})^{\alpha} = (c^{\alpha}t^{\alpha}, x^{\alpha}, y^{\alpha}, z^{\alpha}), \tag{69}$$

where $x^{\alpha}_{\mu} = g_{\mu\nu}(x^{\mu})^{\alpha}$, with $g_{\mu\nu}$ is the metric tensor. Furthermore, the conformable fractional norm using Equations (68)−(69) is given as:

$$\|d^{\alpha}r\|^2 = d^{\alpha}x_{\mu}d^{\alpha}x^{\mu} = c^{2\alpha}d^{2\alpha}t - (d^{2\alpha}x + d^{2\alpha}y + d^{2\alpha}z), \tag{70}$$

2. The four-gradient operator in a conformable fractional form: Its conformable fractional covariant form is defined as:

$$D^{\alpha}_{\mu} = \partial^{\alpha}_{\mu} = \frac{\partial^{\alpha}}{\partial(x^{\mu})^{\alpha}} = \left(\frac{\partial^{\alpha}}{c^{\alpha}\partial t^{\alpha}}, \nabla^{\alpha}\right), \tag{71}$$

and the its conformable fractional contravariant form is defined as:

$$(D^{\mu})^{\alpha} = (\partial^{\mu})^{\alpha} = \left(\frac{\partial^{\alpha}}{c^{\alpha}\partial t^{\alpha}}, -\nabla^{\alpha}\right). \tag{72}$$

The above equations guide us to derive the conformable fractional D'Alembert operator as follows:

$$\Box^{\alpha} = D^{\alpha}_{\mu}(D^{\mu})^{\alpha} = -\nabla^{\alpha}\nabla^{\alpha} + \frac{\partial^{2\alpha}}{c2^{\alpha}\partial t2^{\alpha}}. \tag{73}$$

3. The energy-momentum four vector in conformable fractional form can be expressed as follows:

$$P^{\alpha}_{\mu} = \left(\frac{E^{\alpha}}{c^{\alpha}}, -i\hbar_{\alpha}{}^{\alpha}\nabla^{\alpha}\right) \text{ and } (P^{\mu})^{\alpha} = \left(\frac{E^{\alpha}}{c^{\alpha}}, i\hbar_{\alpha}{}^{\alpha}\nabla^{\alpha}\right). \tag{74}$$

Knowing that using Equation (74), we have

$$\|P^{\alpha}_{\mu}\|^2 = \frac{E^{2\alpha}}{c^{2\alpha}} + \hbar_{\alpha}{}^{2\alpha}\nabla^{\alpha}\nabla^{\alpha}, \tag{75}$$

which is the conformable fractional version of energy–momentum relation.

6. Quantum mechanics

The integration of CFD into quantum mechanics, as described in several sources [18–20, 26], results in conformable fractional quantum mechanics. However, comprehensive understanding of its postulates, fundamentals, and essential properties has been meticulously developed [13, 20, 22].
The inner product in Hilbert space associated with CFQM is expressed as follows:

$$\langle f|g\rangle = \int_{-\infty}^{\infty} g^*(x)f(x)|x|^{\alpha-1}dx. \tag{76}$$

The definition of the expectation value of a physical operator $\mathscr{O}$ in relation to the state ψ is defined as follows:

$$\langle\psi|\mathscr{O}\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\mathscr{O}\psi(x,t)|x|^{\alpha-1}dx, \tag{77}$$

however, for $\mathscr{O}$ to be a Hermitian operator, one may obey

$$\langle\psi|\mathscr{O}\psi\rangle = \langle\mathscr{O}\psi|\psi\rangle. \quad (78)$$

Note that the realization of α-position $\hat{x}^{\alpha}$, and α-momentum $\hat{p}^{\alpha}$ are:

$$\hat{x}_{\alpha} = x \qquad \hat{p}_{\alpha} = -i\hbar_{\alpha}^{\alpha} D_{x}^{\alpha}, \quad (79)$$

Here, one can simply verify that both position and momentum operators are Hermitian. Therefore, utilizing the de Broglie relation $p = h/\lambda$ and Planck relation $E = h/T$, as outlined in [20], one can have $\hat{x}_{\alpha}\psi = x\psi$, $\hat{p}_{\alpha}\psi = p^{\alpha}\psi$, which leads to the following x-representation:

$$\hat{x}_{\alpha} = x, \;\; \hat{p}_{\alpha} = -i\hbar_{\alpha}^{\alpha} D_{x}^{\alpha}, \text{ and } \hat{\mathscr{H}}_{\alpha} = i\hbar_{\alpha} D_{t}^{\alpha}, \quad (80)$$

with

$$D_{x}^{\alpha} = |x|^{1-\alpha}\frac{\partial}{\partial x} \text{ and } D_{t}^{\alpha} = |t|^{1-\alpha}\frac{\partial}{\partial t}, \quad (81)$$

where $\hbar_{\alpha} = h/(2\pi)^{1/\alpha}$, and $\hat{\mathscr{H}}_{\alpha}$ is a α-Hamiltonian operator. Note that the α-position and α-momentum operators have dimensions of length and momentum$^{\alpha}$ respectively while α-Hamiltonian operator has dimension of energy$^{\alpha}$.
In the FCQM, the commutator between the α-position operator and α-momentum operator is

$$[\hat{x}_{\alpha},\, \hat{p}_{\alpha}] = i\hbar_{\alpha}^{\alpha}|\hat{x}|^{1-\alpha}. \quad (82)$$

6.1 Conformable fractional Schrödinger equation

Let us begin by considering the conformable fractional wave function. In Ref. [20], it is expressed in relation to α-wavelength and α-period as follows:

$$\psi(x,t) = \psi_{0} e^{i\left(\frac{k^{\alpha}}{\alpha}|x|^{\alpha-1}x - \frac{w^{\alpha}}{\alpha}|t|^{\alpha-1}t\right)} = \psi_{0} e^{i\left(\frac{x}{\lambda_{\alpha}} - \frac{t}{T_{\alpha}}\right)}, \quad (83)$$

where $\psi_{0} = Ae^{i\phi}$, which represents the initial or amplitude term of the wave function. A is a real constant that gives the amplitude of the wave function, and $e^{i\phi}$ represents the phase factor, where ϕ is the phase angle of ψ_{0}. Here the factor $e^{i\phi}$ helps capture any initial phase information that might be relevant to how ψ interacts with other waves or fields in our system.
Then, in the conformable fractional quantum mechanics, the time-dependent Schrödinger equation is as follows:

$$i\hbar_{\alpha}^{\alpha} D_{t}^{\alpha}\psi(\mathbf{x},t) = \hat{\mathscr{H}}_{\alpha}\psi(\mathbf{x},t), \quad (84)$$

where the α-Hamiltonian operator is

$$\hat{\mathscr{H}}_{\alpha} = \frac{\hat{p}_{\alpha}^{2}}{2m^{\alpha}} + V_{\alpha}(\hat{x}_{\alpha}). \quad (85)$$

Knowing that in 3-dimensions, $\hat{p}_{\alpha}^{2} = -\hbar_{\alpha}^{2\alpha}(D_{x}^{\alpha}D_{x}^{\alpha} + D_{y}^{\alpha}D_{y}^{\alpha} + D_{z}^{\alpha}D_{z}^{\alpha})$.
Then, if, setting

$$\psi(\mathbf{x},t) = \psi(\mathbf{x})e^{-\frac{i}{\hbar^{\alpha}}E^{\alpha}\frac{t^{\alpha}}{\alpha}}, \quad (86)$$

we have the time-independent Schrödinger equation:

$$\{-\frac{\hbar_{\alpha}^{2\alpha}}{2m^{\alpha}} D_{x}^{\alpha}D_{x}^{\alpha} + V_{\alpha}(\hat{x}_{\alpha})\}\psi(\mathbf{x}) = E^{\alpha}\psi(\mathbf{x}). \quad (87)$$

However, the conformable fractional Schrödinger Equation (87) does not exhibit invariance under the α-Lorentz transformations, as confirmed in Ref. [57], from which we obtain:

$$-\frac{\hbar_{\alpha}^{2\alpha}}{2m^{\alpha}}\{\Gamma_{\alpha}^{2}D_{\acute{x}}^{\alpha}D_{\acute{x}}^{\alpha} - 2\Gamma_{\alpha}^{2}\frac{v_{\alpha}}{c^{2\alpha}}D_{\acute{x}}^{\alpha}D_{\acute{t}}^{\alpha} + \Gamma_{\alpha}^{2}\frac{v_{\alpha}^{2}}{c^{4\alpha}}D_{\acute{t}}^{\alpha}D_{\acute{t}}^{\alpha} + D_{\acute{y}}^{\alpha}D_{\acute{y}}^{\alpha} + D_{\acute{z}}^{\alpha}D_{\acute{z}}^{\alpha} + V_{\alpha}(\hat{x}_{\alpha})\}\psi = \{-v_{\alpha}\Gamma_{\alpha}D_{\acute{x}}^{\alpha} - \Gamma_{\alpha}D_{\acute{t}}^{\alpha}\}\psi. \quad (88)$$

Additionally, through Equation (84), one can define the conformable fractional probability density ρ_{α} and conformable fractional probability flux $\mathfrak{J}_{\alpha}$ as follows:

$$D_{t}^{\alpha}\rho_{\alpha}(x,t) + D_{x}^{\alpha}\mathfrak{J}_{\alpha}(x,t) = 0, \quad (89)$$

where

$$\begin{cases} \rho_{\alpha}(x,t) = \psi^{\dagger}\psi, \\ \mathfrak{J}_{\alpha}(x,t) = -i\frac{\hbar_{\alpha}^{\alpha}}{2m^{\alpha}}(\psi^{\dagger}D_{x}^{\alpha}\psi - \psi D_{x}^{\alpha}\psi^{\dagger}). \end{cases} \quad (90)$$

Our obtained conformable fractional Schrödinger equation can be extended to involve polynomial law nonlinearity, as in [59], where the authors considered two models of the nonlinear Schrödinger equation with polynomial law nonlinearity using the variational principle and amplitude ansatz methods. The solutions were extracted based on the Jost function. Additional applications of Equation (89) may involve computing the conservation laws for the Chen–Lee–Liu equation of nonlinear optics [60].

6.2 Conformable fractional spherical harmonics

In this part of our work, we present the angular equation of the conformable fractional Schrödinger equation. In terms the CFD, the Schrodinger equation in spherical coordinates can be stated as:

$$\{\Delta^{\alpha} + \frac{2m^{\alpha}}{\hbar_{\alpha}^{2\alpha}}(-V_{\alpha}(r_{\alpha}) + E^{\alpha})\}\psi(r_{\alpha},\theta_{\alpha},\varphi_{\alpha}) = 0, \quad (91)$$

where $\Delta^{\alpha} = D_{r}^{\alpha}D_{r}^{\alpha}$ in spherical coordinates is given by:

$$\Delta^{\alpha} = \frac{1}{r^{2\alpha}}D_{r}^{\alpha}[r^{2\alpha}D_{r}^{\alpha}] + \frac{1}{r^{2\alpha}\sin(\theta_{\alpha})}D_{\theta}^{\alpha}[\sin(\theta_{\alpha})D_{\theta}^{\alpha}] + \frac{1}{r^{2\alpha}\sin^{2}(\theta_{\alpha})}D_{\varphi}^{\alpha}D_{\varphi}^{\alpha}, \quad (92)$$

We consider that the solutions of the form:

$$\psi(r_{\alpha},\theta_{\alpha},\varphi_{\alpha}) = R(r_{\alpha})Y(\theta_{\alpha},\varphi_{\alpha}). \quad (93)$$

By separation of variables and upon substituting Equation (92) into Equation (91), we obtain two conformable fractional equations: one pertaining to radial aspects, in which its solutions depend on V_{α} and the other to angular components:

$$\frac{1}{R(r_{\alpha})}D_{r}^{\alpha}[r^{2\alpha}D_{r}^{\alpha}R(r_{\alpha})] + \frac{2m^{\alpha}r^{2\alpha}}{\hbar_{\alpha}^{2\alpha}}(E^{\alpha} - V_{\alpha}(r_{\alpha})) = \alpha^{2}l(l+1), \quad (94)$$

$$\frac{1}{Y(\theta_\alpha,\varphi_\alpha)\sin(\theta_\alpha)}D_\theta^\alpha[\sin(\theta_\alpha)D_\theta^\alpha Y(\theta_\alpha,\varphi_\alpha)]+ \frac{1}{Y(\theta_\alpha,\varphi_\alpha)\sin^2(\theta_\alpha)}D_\varphi^\alpha D_\varphi^\alpha Y(\theta_\alpha,\varphi_\alpha) = -\alpha^2 l(l+1), \tag{95}$$

Now, again by using separation of the variable $Y(\theta_\alpha,\varphi_\alpha) = \Theta(\theta_\alpha)\Phi(\varphi_\alpha)$ in order to solve Equation (95), one can get:

$$\frac{\sin(\theta_\alpha)}{\Theta(\theta_\alpha)}D_\theta^\alpha[\sin(\theta_\alpha)D_\theta^\alpha\Theta(\theta_\alpha)] + \frac{1}{\Phi(\varphi_\alpha)}D_\varphi^\alpha D_\varphi^\alpha\Phi(\varphi_\alpha) = -\alpha^2 l(l+1)\sin^2(\theta_\alpha), \tag{96}$$

with $\Phi(\varphi_\alpha) = e^{im\varphi_\alpha}$, and

$$\frac{1}{\Phi(\varphi_\alpha)}D_\varphi^\alpha D_\varphi^\alpha\Phi(\varphi_\alpha) = -\alpha^2 m^2. \tag{97}$$

Then, we have

$$\frac{\sin(\theta_\alpha)}{\Theta(\theta_\alpha)}D_\theta^\alpha[\sin(\theta_\alpha)D_\theta^\alpha\Theta(\theta_\alpha)] + \alpha^2 l(l+1)\sin^2(\theta_\alpha) = \alpha^2 m^2, \tag{98}$$

Now, making use of associated conformable fractional Legendre differential equation [61]:

$$(1-a^{2x})A^xA^xP_{x,n}^{x,m}(a) - 2xa^xA^xP_{x,n}^{x,m}(a) + x^2\{n(n+1) - \frac{m^2}{1-a^{2x}}P_{x,n}^{x,m}(a) = 0, \tag{99}$$

with $0 < x \leq 1$, Equation (98) becomes

$$(1-x^{2\alpha})D_x^\alpha D_x^\alpha P_{\alpha,l}^{\alpha,m}(x) - 2\alpha x^\alpha D_x^\alpha P_{\alpha,l}^{\alpha,m}(x) + \alpha^2\{l(l+1) - \frac{m^2}{1-x^{2\alpha}}\}P_{\alpha,l}^{\alpha,m}(x) = 0, \tag{100}$$

and its solution is given as [61]:

$$P_{\alpha,l}^{\alpha,m} = \frac{1}{l!}\frac{(-1)^m(1-x^{2\alpha})^{\frac{m}{2}}}{\alpha^l 2^l}D^{(l+m)\alpha}(x^{2\alpha}-1)^l. \tag{101}$$

Now, using Equation (101), the solution of Equation (95) is given as follows [62]:

$$Y_{\alpha,l}^{\alpha,m}(\theta_\alpha,\varphi_\alpha) = N_{\alpha,l}^{\alpha,m}e^{-im\varphi_\alpha}P_{\alpha,l}^{\alpha,m}(\cos(\theta_\alpha)). \tag{102}$$

where $N_{\alpha,l}^{\alpha,m}$ is the normalization constant, can be simply calculated using normalization condition $\int|Y_\alpha(x,m,l)|^2\sin(\theta_\alpha)d^\alpha\theta d^\alpha\varphi = 1$, as follows:

$$N_{\alpha,l}^{\alpha,m} = \sqrt{\frac{(2l+1)(l-m)!}{\alpha^{2m-2}2(l+m)!(2\pi)^\alpha}}. \tag{103}$$

Finally, the orthonormal spherical harmonic (102) is:

$$Y_{\alpha,l}^{\alpha,m}(\theta_\alpha,\varphi_\alpha) = \sqrt{\frac{(2l+1)(l-m)!}{\alpha^{2m-2}2(l+m)!(2\pi)^\alpha}}e^{-im\varphi_\alpha}P_{\alpha,l}^{\alpha,m}(\cos(\theta_\alpha)). \tag{104}$$

with

$$(Y_{\alpha,l}^{\alpha,m})^* = (-1)^m Y_{\alpha,l}^{\alpha,-m}, \tag{105}$$

where the superscript '$*$' denotes complex conjugation. Below are provided some conformable fractional spherical harmonic functions obtained using Equation (104):

l	m	$Y_\alpha(x,l,m)$
0	0	$\alpha\sqrt{\frac{1}{(2\pi)^\alpha}}$
1	-1	$\frac{\alpha}{2}\sqrt{\frac{3}{(2\pi)^\alpha}}e^{-i\varphi_\alpha}\sin(\theta_\alpha)$
	0	$\alpha\sqrt{\frac{3}{2(2\pi)^\alpha}}\cos(\theta_\alpha)$
	1	$-\frac{\alpha}{2}\sqrt{\frac{3}{(2\pi)^\alpha}}e^{i\varphi_\alpha}\sin(\theta_\alpha)$

Figure 5, presents some plots of the conformable fractional spherical harmonic.
Note that the conformable spherical harmonic is extensively studied in [62].

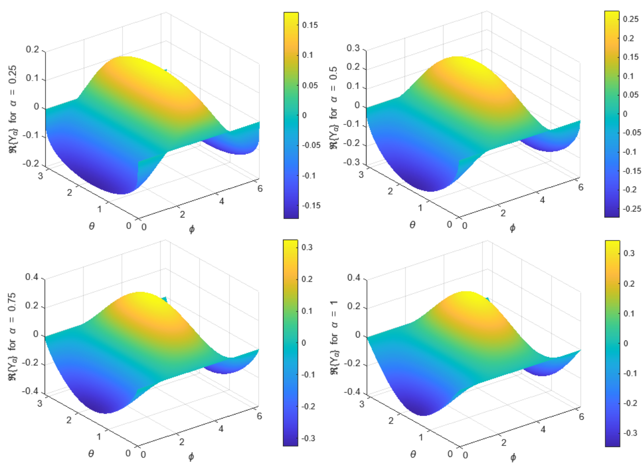

Figure 5. Plot of conformable fractional spherical harmonic $Y_\alpha(\theta_\alpha,\varphi_\alpha)$ for the cases of $\alpha = 0.25, 0.5, 0.75, 1$.

6.3 Infinite potential well (particle in a box)

The particle in a box model, known as 'infinite potential well' characterizes a scenario wherein a particle is allowed to move within a limited space enclosed by impenetrable barriers. This model is one of the very few problems in quantum mechanics which can be solved analytically, without approximations. The potential energy within this model is defined as:

$$V_\alpha(x) = \begin{cases} 0, & \text{for } 0 < x < L \\ \infty, & \text{otherwise.} \end{cases} \tag{106}$$

where L is the length of the box. Now, we by solving the following conformable fractional time-independent Schrodinger (for $0 < x < L$):

$$\{(D_x^\alpha)^2 + k_\alpha^2\}\psi(\mathbf{x}) = 0, \text{ with } k_\alpha = \sqrt{\frac{2m^\alpha E^\alpha}{\hbar_\alpha^{2\alpha}}}, \tag{107}$$

we find

$$\psi(\mathbf{x}) = A\cos_\alpha(kx) + B\sin_\alpha(kx), \tag{108}$$

where A and B are arbitrary complex numbers. With $\psi(0) = 0$, $A = 0$. Then equation (q) becomes

$$\psi(\mathbf{x}) = B\sin_\alpha(kx). \tag{109}$$

Subsequently, from $\psi(L) = 0$, one can have

$$\sqrt{\frac{2m^\alpha E^\alpha}{\hbar_\alpha^{2\alpha}}} L^\alpha = \alpha(n\pi), \quad n = 0, 1, 2, \cdots \tag{110}$$

Finally, the energy levels are given as follows:

$$E_n^\alpha = \frac{\hbar_\alpha^{2\alpha}(\alpha n\pi)^2}{2m^\alpha L^{\alpha 2}}, \quad n = 0, 1, 2, \cdots \tag{111}$$

Variations of the energy levels versus n for different αs is presented in Fig. 6. Note that the particle in a box model within CFD is considered in [58].

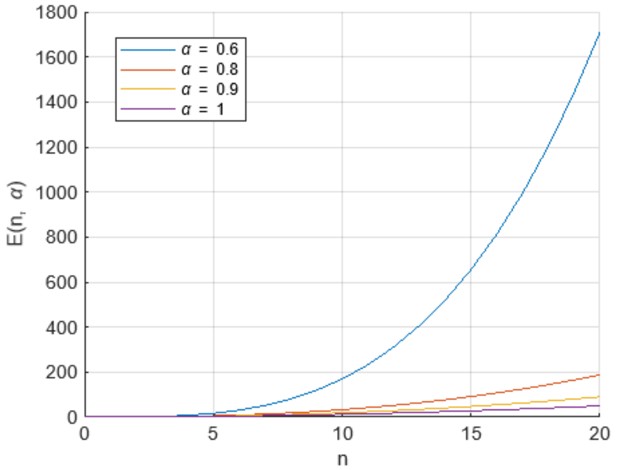

Figure 6. Plot of E_n vs n for the cases of $\alpha = 0.6$, $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 1$. With $a = m = L = 1$.

6.4 Conformable fractional Pauli equation

The Pauli equation is a non-relativistic quantum mechanical equation. It is an extension of the Schrödinger equation that incorporates spin. Therefore, we present the conformable fractional time-dependent Pauli equation as follows:

$$\frac{1}{2m^\alpha}(\boldsymbol{\sigma} \cdot \widehat{\mathbf{\Pi}}_\alpha)^2 \psi(\mathbf{x},t) + e\phi_\alpha \psi(\mathbf{x},t) = i\hbar_\alpha^\alpha D_t^\alpha \psi(\mathbf{x},t). \tag{112}$$

with $\widehat{\mathbf{\Pi}}_\alpha = \hat{\mathbf{p}}_\alpha - e\mathbf{A}_\alpha$. Knowing that the conformable fractional electromagnetic field is described by the conformable fractional magnetic vector potential $\mathbf{A}_\alpha$ and the conformable fractional electric scalar potential ϕ_α. Then, by using the following Pauli vector identity:

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}), \tag{113}$$

Equation (112) becomes:

$$\frac{1}{2m^\alpha}(\hat{\mathbf{p}}_\alpha - e\mathbf{A}_\alpha(x))^2 \psi(\mathbf{x},t) + e\phi_\alpha(x)\psi(\mathbf{x},t) - \mu_B^\alpha \boldsymbol{\sigma} \cdot \mathbf{B}_\alpha \psi(\mathbf{x},t) = i\hbar_\alpha^\alpha D_t^\alpha \psi(\mathbf{x},t), \tag{114}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Also, $\mu_B^\alpha = (|e|\hbar_\alpha^\alpha)/(2m^\alpha)$ is the conformable fractional Bohr magneton.

Equation (114) can be stated also as:

$$\frac{1}{2m^\alpha}(\hat{\mathbf{p}}_\alpha - e\mathbf{A}_\alpha(x))^2 \psi(\mathbf{x},t) + e\phi_\alpha(x)\psi(\mathbf{x},t) - \boldsymbol{\mu}_s^\alpha \cdot \mathbf{B}_\alpha \psi(\mathbf{x},t) = i\hbar_\alpha\, {}^\alpha D_t^\alpha \psi(\mathbf{x},t), \tag{115}$$

where $\boldsymbol{\mu}_s^\alpha = \mu_B^\alpha \boldsymbol{\sigma}$ is the conformable fractional electron magnetic moment vector.

If $\alpha = 1$, we have the ordinary Pauli equation [63, 64]. However, it is crucial for us to subsequently derive the non-relativistic limit of the Dirac equation within the framework of the CFD (a work in progress). Thereafter, we aim to compare the actual conformable fractional Pauli equation with the one obtained using the non-relativistic limit of Dirac equation.

7. Relativistic quantum mechanics

Relativistic quantum mechanics is a framework that combines quantum mechanics with special relativity. It introduces relativistic wave equations, like the Klein-Gordon, DKP, Dirac equations...

7.1 Conformable fractional Klein-Gordon equation

The Klein-Gordon equation describes the behavior of spinless particles such as mesons, incorporating special relativity. The conformable fractional Klein-Gordon equation will be obtained by setting conformable fractional operators $E^\alpha = i\hbar_\alpha\, {}^\alpha D_t^\alpha$ and $\hat{\mathbf{p}}_\alpha = -i\hbar_\alpha\, {}^\alpha D_x^\alpha$ in the following conformable fractional relativistic energy:

$$(E^\alpha)^2 = (\hat{\mathbf{p}}_\alpha c^\alpha)^2 (m^\alpha c^{2\alpha})^2. \tag{116}$$

So, we have

$$\frac{1}{c^{2\alpha}} D_t^\alpha D_t^\alpha \psi - D_x^\alpha D_x^\alpha \psi + \frac{m^{2\alpha}c^{2\alpha}}{\hbar_\alpha^{2\alpha}} D_x^\alpha \psi = 0. \tag{117}$$

Note that the invariance of the conformable fractional Klein-Gordon equation under the α-Lorentz transformations is examined in Ref. [57]. It remains invariant under such transformations.

7.2 Conformable fractional Dirac equation

The conformable fractional form of the Dirac equation is presented as follows [20, 26]:

$$\{i\hbar^{\alpha}\gamma^{\mu}\partial_{\mu}^{\alpha} - m^{\alpha}c^{\alpha}\}\psi(x^{\mu}) = 0, \tag{118}$$

where $0 < \alpha \leq 1$, is what we call the fractional, and $\gamma^{\mu} = (\gamma^0, \gamma^k)$ are the Dirac matrices. So, by multiplying Equation (118) from the left by γ^0 and separating the time and the spatial parts, one can obtain:

$$\{i\hbar^{\alpha}(\gamma^0)^2\partial_0^{\alpha} - i\hbar^{\alpha}\gamma^0\gamma^k\partial_k^{\alpha} - \gamma^0 m^{\alpha}c^{\alpha}\}\psi(\mathbf{x},t) = 0, \tag{119}$$

knowing that $(\gamma^0)^2 = 1$. Then, by employing Equation (86) and with $i\hbar^{\alpha}(1/c)\partial_0^{\alpha} = i\hbar^{\alpha}(1/c)D_t^{\alpha}$, the time-independent Dirac equation in interaction with an electromagnetic four-potential $A_{\alpha}^{\mu}(\Phi_{\alpha}, \mathbf{A}_{\alpha})$ (in SI units) reads

$$\{c^{\alpha}\boldsymbol{\alpha}\cdot(\hat{\mathbf{p}}_{\alpha} - e\mathbf{A}_{\alpha}(\mathbf{x})) + e\Phi_{\alpha}(\mathbf{x}) + \beta m^{\alpha}c^{2\alpha}\}\psi(\mathbf{x}) = E^{\alpha}\psi(\mathbf{x}), \tag{120}$$

where $\psi(\mathbf{r},t) = (\phi(\mathbf{r},t)\chi(\mathbf{r},t))^T$ is the bispinor in the Dirac representation. The Dirac matrices $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$ and $\beta = \gamma^0$ satisfy the following anticommutation relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \; \{\alpha_i, \beta\} = 0, \; \alpha_i^2 = \beta^2. \tag{121}$$

Then in more elegant simple form, we have

$$\{c^{\alpha}\boldsymbol{\alpha}\cdot\widehat{\boldsymbol{\Pi}}_{\alpha} + e\Phi_{\alpha} + \beta m^{\alpha}c^{2\alpha}\}\psi(\mathbf{x}) = E^{\alpha}\psi(\mathbf{x}), \tag{122}$$

where the minimal substitution is

$$\hat{\mathbf{p}}_{\alpha} - e\mathbf{A}_{\alpha}(\mathbf{x}) = \widehat{\boldsymbol{\Pi}}_{\alpha}. \tag{123}$$

Next, we move to employ the obtained conformable fractional Dirac Equation (122) to explore the classical limit through Ehrenfest's theorem.

7.2.1 Conformable fractional continuity equation

We define $\bar{\psi} \equiv \psi^{\dagger}\gamma^0$, where $\psi^{\dagger}$ is the complex conjugate of the row vector corresponding to the column vector ψ. Then, by taking the adjoint of Equation (119):

$$\bar{\psi}\{-i\hbar^{\alpha}\gamma^0\partial_0^{\alpha} + i\hbar^{\alpha}(\gamma^k)^{\dagger}\overleftarrow{\partial}^{\alpha} - m^{\alpha}c^{\alpha}\} = 0, \tag{124}$$

with $\overleftarrow{\partial}^{\alpha} = \partial_k^{\alpha}$ acts from the right on $\bar{\psi}$, also, from the Hermicity of γ^{μ}, we have

$$(\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0, \;\text{ and }\; (a^k)^{\dagger} = a^k \tag{125}$$

Besides, the Dirac equation and its adjoint can be obtained using the variation of the action through the conformable fractional Lagrangian density, which is given as follows:

$$\mathfrak{L}_{\alpha} = -i\hbar^{\alpha}c^{\alpha}\bar{\psi}\gamma^{\mu}\partial_{\mu}{}^{\alpha}\psi - m^{\alpha}c^{2\alpha}\bar{\psi}\psi \tag{126}$$

Then, if one performs variation with respect to ψ, the result is the adjoint Dirac equation. Conversely, varying it with respect to $\bar{\psi}$ yields the Dirac equation.

Now, from Equations (119), (124), the α-continuity equation is given as follows [65]:

$$\bar{\psi}\gamma^k\overleftarrow{\partial}^{\alpha}\psi - \bar{\psi}\gamma^0\partial_0{}^{\alpha}\psi + \bar{\psi}\gamma^0\partial_0{}^{\alpha}\psi - \bar{\psi}\gamma^k\partial_k{}^{\alpha}\psi = 0 \tag{127}$$

In a more and concise manner, elegantly, Equation (127) transforms to:

$$D_{\mu}^{\alpha}\mathfrak{J}_{\alpha}^{\mu} = D_t^{\alpha}\rho_{\alpha} + D_x^{\alpha}\mathfrak{J}_{\alpha}^k = 0, \tag{128}$$

where the four-vector α-current density (the α-probability density and the α-probability flux) is given as follows:

$$\mathfrak{J}_{\alpha}^{\mu} : \begin{cases} \rho_{\alpha} = \bar{\psi}\gamma^0\psi = \psi^{\dagger}\psi = |\psi|^2, \\ \mathfrak{J}_{\alpha}^k = \bar{\psi}\gamma^k\psi = \psi^{\dagger}\alpha^k\psi. \end{cases} \tag{129}$$

From Equation (129), it is evident that CFQM does not affect the four-vector current. However, the continuity equation undergoes alteration.

7.2.2 A classical limit of the Dirac equation

In our work [65], in the context of CFD, a classical limit of the Dirac equation by using Ehrenfest's theorem [66] is investigated. However, the time derivatives of position and kinetic momentum operators for Dirac particles interacting with an electromagnetic field were computed and deformed classical equations were obtained. These classical conformable fractional equations are as follows:

■ For the position operator $\hat{\mathbf{x}}$, we have:

$$\frac{d\hat{\mathbf{x}}}{dt} = \frac{1}{\hbar}\hbar_{\alpha}{}^{\alpha}c^{\alpha}|x|^{1-\alpha}\hat{\boldsymbol{\alpha}}, \tag{130}$$

where the eigenvalues of $\hat{\boldsymbol{\alpha}}$ are ± 1. This result confirms that there is no classical analogy due to the Dirac particle is still moving at the speed of light c^{α}.

■ The equation of motion for the kinetic momentum operator $\widehat{\boldsymbol{\Pi}} = \hat{\mathbf{p}} - e\mathbf{A}$ is:

$$\frac{d\widehat{\boldsymbol{\Pi}}}{dt} = e\mathbf{E}_{\alpha} + e\frac{1}{\hbar}\hbar_{\alpha}{}^{\alpha}c^{\alpha}|x|^{1-\alpha}\Sigma_{i,j}(\hat{\boldsymbol{\alpha}}_i)(\nabla_j A_i - \nabla_i A_j)e_j + \frac{1}{\hbar}c^{\alpha}\hbar_{\alpha}^{\alpha}[|x|^{1-\alpha}, \boldsymbol{\nabla}]\boldsymbol{\alpha}.\hat{\mathbf{p}}, \tag{131}$$

where $\mathbf{E}_{\alpha} = -(\partial\mathbf{A}/\partial t) - (\boldsymbol{\nabla}\Phi_{\alpha})$, and $A_{\alpha} = \hbar_{\alpha}^{\alpha-1}|x|^{1-\alpha}\mathbf{A}$. Then after minor simplifications, we have:

$$\frac{d\widehat{\boldsymbol{\Pi}}}{dt} = e\{\mathbf{E}_{\alpha} + \mathbf{v}\times\mathbf{B}\} + \frac{1}{\hbar}\hbar_{\alpha}^{\alpha}c^{\alpha}[|x|^{1-\alpha}, \boldsymbol{\nabla}]\boldsymbol{\alpha}.\hat{\mathbf{p}}, \tag{132}$$

Equation (132) is a conformable fractional Lorentz force in the classical case.

As in the case of velocity in Equation (130), the effect of CFQM on the Lorentz force appears widely. In the limit of $\alpha \to 1$, we have

$$\frac{d\widehat{\boldsymbol{\Pi}}}{dt} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \tag{133}$$

which is similar to the form of Lorentz force in the classical case [67].

It is found that $\hat{\mathbf{x}}$ does not comply with classical equations of motion. But, a classical equation of motion can be established for the operator $\widehat{\boldsymbol{\Pi}}$.

7.3 Conformable fractional Fisk-Tait equation

The covariant form of Fisk-Tait equation, which is a spin-3/2 relativistic equation, in an external magnetic field is given as follows [68]:

$$(\Gamma^{\lambda})^{\mu\nu}_{\rho\sigma}\left(i\hbar\frac{\partial}{\partial x^{\lambda}}+eA_{\lambda}\right)\Psi^{\rho\sigma}+mB^{\mu\nu}_{\rho\sigma}\Psi^{\rho\sigma}=0, \tag{134}$$

where the matrices $(\Gamma^{\lambda})^{\mu\nu}_{\rho\sigma}$ and $B^{\mu\nu}$ are given as follows:

$$B^{\mu\nu}_{\rho\sigma}=g^{\mu}_{\rho}g^{\nu}_{\sigma}, \tag{135}$$

$$(\Gamma^{\lambda})^{\mu\nu}_{\rho\sigma}=-\frac{4}{3}\gamma^{\lambda}g^{\mu}_{\rho}g^{\nu}_{\sigma}-\frac{1}{3}\gamma^{\lambda}(\gamma^{\mu}\gamma_{\sigma}g^{\nu}_{\rho}-\gamma^{\nu}\gamma_{\sigma}g^{\mu}_{\rho})+ \frac{1}{3}(\gamma^{\mu}g^{\lambda}_{\rho}g^{\nu}_{\sigma}-\gamma^{\nu}g^{\lambda}_{\rho}g^{\mu}_{\sigma}-g^{\lambda\mu}\gamma_{\rho}g^{\nu}_{\sigma}+g^{\lambda\nu}\gamma_{\rho}g^{\mu}_{\sigma}). \tag{136}$$

Here

$$\gamma^{0}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \boldsymbol{\gamma}=\begin{pmatrix}0&\boldsymbol{\sigma}\\-\boldsymbol{\sigma}&0\end{pmatrix}, \tag{137}$$

Note that the wave function employed is a 24-component anti-symmetric tensor-spinor, i.e., $\Psi^{\mu\nu}_{\lambda}=-\Psi^{\nu\mu}_{\lambda}$. Now, in the context of CFD, Equation (134) becomes:

$$(\Gamma^{\lambda})^{\mu\nu}_{\rho\sigma}(i\hbar^{\alpha}_{\alpha}D^{\alpha}_{\lambda}+eA_{\alpha\lambda})\Psi^{\rho\sigma}=-m^{\alpha}B^{\mu\nu}_{\rho\sigma}\Psi^{\rho\sigma}. \tag{138}$$

8. Conclusion and remarks

In this work, we have presented the basic formulas and properties of CFD and extensively considered its application in physics, including mathematical physics, Newtonian mechanics, special relativity, as well as ordinary and relativistic quantum mechanics.
Based on the idea of conformable fractional calculus, we have thoroughly discussed the conformable fractional version of the path integral approach, divergence and Green's theorems, and included some graphics to enhance understanding of the behavior of the conformable fractional divergence theorem. Additionally, conformable fractional forms for velocity and acceleration are obtained through conformable fractional addition and subtraction, and then utilized to present Newton's II law, work, kinetic and potential energies. Subsequently, the conformable fractional form of the classical Doppler effect is formulated, and the conservation of momentum is studied within the framework of CFD, encompassing elastic and inelastic collisions.
Furthermore, this study presents the postulates of conformable fractional special relativity, obtains the conformable fractional version of Lorentz transformation, and applies it in Minkowski space-time, along with obtaining conformable fractional versions of the fundamental four-vectors. The study also investigates the conformable fractional versions of the ordinary and angular Schrödinger equations, as well as the Pauli equations. The angular solutions of the conformable fractional Schrödinger equation are plotted in terms of the conformable fractional spherical harmonics. As an example, the infinite potential well problem is explored within the context of CFD, along with the exploration of conformable fractional variations of the Klein-Gordon, Dirac equations, and the Fisk-Tait equation. Additionally, the conformable fractional continuity equation is derived from the Dirac equation, and a classical limit of the latter is presented by employing Ehrenfest's theorem.
It is known that, in the limit of $\alpha\to 1$, the α-deformed studied models and equations reduce to those of ordinary physics, confirming consistently the reducibility to those found and discussed in the literature.
This work serves as a valuable resource for researchers working with CFD, and our results may prove useful for exploring further investigations. These could include studies on traveling wave structures of the perturbed Fokas–Lenells model [69] or the Zakharov–Kuznetsov equation [70], an isotropic nonlinear evolution equation describing weakly nonlinear ion-acoustic waves in a strongly magnetized, lossless plasma in $2D$.

Authors contributions
Not applicable.

Availability of data and materials
There is no data associated with this study.

Conflict of interests
The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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