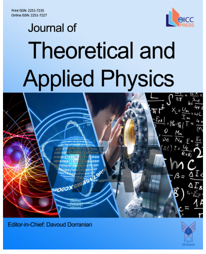

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A numerical solution for a nonlinear inverse stochastic parabolic problem based on Legendre wavelets bases

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Original Research

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Abstract:

In this paper, we consider a 1D nonlinear stochastic partial differential equation of parabolic type. In this problem, values of the function on a part of physical boundary of the domain are unknown. A numerical approach has been developed to approximate the exact solution for this issue by employing Legendre wavelets and their operational matrix. This method is combined with the Levenberg-Marquardt regularization technique. In continuation, the error analysis is given. Finally, a numerical sample confirms the efficiency and accuracy of this method.

Keywords: Stochastic partial differential equation (SPDE); Legendre wavelets; Levenberg-Marquardt regularization

1. Introduction

Inverse deterministic problems are commonly used in mathematics and engineering. These problems belong to the class of ill-posed problems. Therefore, Many researchers have made great efforts to find a stable approximate of the solution for them. J. V. Beck et al. have considered the one-dimensional linear inverse heat conduction problem and have suggested a numerical method that compares with the classic Beck's function specification method [1].
Furthermore, W. Rundell and H. M. Yin have examined a parabolic inverse problem in which an unknown function was involved in the boundary condition, and they attempted to recover this function by measuring the temperature at a fixed point on the boundary [2].
In recent years, the use of stochastic equations in scientific texts has increased across various fields. Due to their high efficiency in modeling physical phenomena, stochastic equations have recently attracted the attention of researchers. Therefore, inverse stochastic equations are of great importance. Badri Narayana and Zabaras in [3] are restated a conditionally well-posed L_2 optimization problem for the ill-posed stochastic inverse problem. Crisan, Otobe and Peszat analyzed stochastic linear transport equations that depend on an unknown potential and have either additive noise or multiplicative noise [4].
Orthogonal wavelet basis are commonly utilized in the numerical solution of partial differential equations. The use of these basis was developed by Meyer and his group [5]. The application of these foundations can be seen in many papers. For example, we refer [6] and [7].
Also, wavelet methods are used to solve stochastic partial differential equations (PDEs). In recent years, several methods have been proposed to address these problems, Such as the Chebyshev wavelets method [8] and the Legendre wavelets method [9].
Legendre wavelets have been extensively applied in engineering and physics research as a valuable mathematical tool. An efficient computational method, based on Legendre and Chebyshev wavelets, has been developed to find an approximate solution for one-dimensional partial differential equations with specified initial conditions in [10]. Furthermore, a new computational method has been proposed based on a new class of orthogonal Chebyshev cardinal wavelets to solve a class of nonlinear stochastic differential equations driven by fractional Brownian motion in [8].
In this paper, we present a computational method for solving inverse nonlinear SPDEs. The outline of this paper is as

follows.
Section 2 is devoted to introducing preliminaries. In section 3, a brief review of Legendre wavelets and their properties is provided. Then we present the main result and solve the inverse problem using Legendre wavelets bases. In continuation, the error analysis is given in section 4. Finally, in section 5, a numerical example is given to demonstrate the applicability and accuracy of this method. In the final section, the conclusion is stated.

2. Preliminaries

First, we need to introduce some necessary preliminaries about spaces and norm notations. Let $(\Omega,\ F,\ P)$ be a probability space. In this space, Ω is the set of outcomes, F and P are the $\sigma-$ algebra of events and the probability measure, respectively. For a Hilbert space H, let u be a random variable taking values in $L_2(\Omega,\ H)$, with the corresponding norm defined as follows

$$\|u\|_{L_2(\Omega,\ H)} = \left(E\|u\|_H^2\right)^{\frac{1}{2}},$$

where E is expectation operator. Let $\Lambda \subset \mathbb{R}^n$, $(n \geq 1)$ is a bounded open set, then for $u \in L_2(\Omega,\ L_2(\Lambda))$, we set

$$\|u\|^2_{L_2(\Omega,\ L_2(\Lambda))} := E\left(\int_\Lambda u^2 d\Lambda\right).$$

3. Solving inverse stochastic problem

The main aim of this paper is to use the Legendre wavelets bases for solving a nonlinear inverse stochastic partial differential equation (NISPDE) of the form

$$du = -F(u)dt + B(x)dw(t),\ \ t \in (0,\ T),\ \ x \in \Lambda = (0,\ 1), \tag{1}$$

subject to the boundary and initial conditions

$$\begin{aligned} u(x,\ 0) &= \phi(x),\ \ x \in= (0,\ 1) \\ u(0,\ t) &= \lambda(t),\ \ t \in= (0,\ 1) \\ u(1,\ t) &= q(t),\ \ t \in= [0,\ T] \end{aligned} \tag{2}$$

and the overspecified condition

$$u(a,\ t) = g(t),\ \ t \in= [0,\ T] \tag{3}$$

where $0 < a < 1$. Here F is a nonlinear differential operator in $(0,\ 1)$ that describes an elliptic boundary-value problem (for more details see [11, 12]). In this problem, $u \in L_2(\Omega,\ L_2(Q_T))$, where $Q_T = (0,\ 1) \times (0,\ T)$ and we assume that the operator B belongs to $L_2(\Omega,\ L_2(\Lambda))$. The noise term $w(t)$ is a Wiener process defined on a probability space $(\Omega,\ F,\ P)$, where Ω is the set of outcomes and F and P are the $\sigma-$ algebra of events and the probability measure, respectively.
Moreover, ϕ is a continuous function, g and q are differentiable functions and T represents the final time. While the function λ is unknown, the problem is an inverse.

3.1 Legendre wavelets bases

According to [13] the family of continuous wavelets is obtained as

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right),\ \ a,b \in \mathbb{R},\ \ a \neq 0,$$

where a, b are dilation and translation parameters, respectively.
Legendre wavelets functions are defined as $\psi_{n,m}(x) = \psi(k,\ \hat{n},\ m,\ x)$, where $k \in \mathbb{N}$,

$$\hat{n} = 2n-1,\ \ n = 1,2,\ldots,2^{k-1}$$

and $m = 0,1,2,\ldots,M-1$ are their arguments. Note that m is the degree of the Legendre polynomial. They are defined on the interval $[0,\ 1]$ as

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}} d_m(2^k x - 2n + 1) & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ 0 & \text{otherwise.} \end{cases}$$

In the above definition

$$d_m(x),\ (m = 0,1,2,\ldots,M-1)$$

are the well-known Legendre polynomials of degree m. One of the important properties of these polynomials is orthogonality with respect to the weight function $w(x) = 1$ on the interval $[-1,\ 1]$ and satisfy the following recursive formula

$$\begin{aligned} d_0(x) &= 1, \\ d_1(x) &= x, \\ d_{m+1}(x) &= \frac{2m+1}{m+1} x d_m(x) - \frac{m}{m+1} d_{m-1}(x) \end{aligned}$$

Any function $f \in L_2([0,\ 1])$ can be expressed by the Legendre wavelets as follows

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x), \tag{4}$$

where

$$C_{n,m} = < f(x), \psi_{n,m}(x) > = \int_0^1 f(x)\psi_{n,m}(x)dx. \tag{5}$$

Equation (4), may be truncated and we use an approximation of this, in the following series form

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\Psi(x),$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C^T = [c_{10}, c11, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, c_{2^{k-1}1}, \ldots, c_{2^{k-1}M-1}] \tag{6}$$

$$\Psi(x) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \ldots, \psi_{2M-1}, \ldots, \psi_{2^{k-1}0}, \psi_{2^{k-1}1}, \ldots, \psi_{2^{k-1}M-1}]. \tag{7}$$

For the integration of the vector $\Psi(x)$ as $\int_0^x \psi(s)ds \cong \mathbf{P}\Psi(x)$, the operational matrix $\mathbf{P}$ of order $2^{k-1}M \times 2^{k-1}M$ is defined as

$$\mathbf{P} = \frac{1}{2^k}\begin{bmatrix} \mathbf{L} & \mathbf{G} & \mathbf{G} & \cdots & \mathbf{G} \\ 0 & \mathbf{L} & \mathbf{G} & \cdots & \mathbf{G} \\ 0 & 0 & \mathbf{L} & \cdots & \mathbf{G} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{L} \end{bmatrix} \tag{8}$$

where $\mathbf{G}$ and $\mathbf{L}$ are $M\times M$ matrices given by $\mathbf{G} = diag(2,0,\cdots,0)$ and

$$\mathbf{L} = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{15}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{15}} & 0 & \frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \zeta_1 & 0 & \zeta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 & 0 \end{bmatrix}$$

where

$$\zeta_1 = \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-5}}, \quad \zeta_2 = \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}},$$

and

$$\zeta_3 = \frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}}$$

3.2 The computational algorithm

In this subsection, we solve an inverse nonlinear stochastic partial differential equation with idea of the established method. A computational algorithm is given to find the function λ of the NISPDE (1)-(3).

Algorithm: Identification of the unknown function λ.

Step 1. Suppose that $u_{xxt}(x,t)$ expanded in terms of Legendre wavelet as

$$u_{xxt}(x,t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\Psi(x). \tag{9}$$

Let $\Delta t = T/N$, where $t_i = i\Delta t$, $(i = 0,1,...,N)$. So, the vector C^T is constant in each subinterval $[t_i, t_{i+1}]$.

Step 2. Similar to the method given in [9] to solve the NISPDE, by integrating from Equation (9) twice with respect to x from a to x and respect to t, from t_n to t, we obtain

$$u_{xx}(x,t) = (t-t_n)C^T\Psi(x) + u_{xx}(x,t_n), \tag{10}$$

$$u_t(x,t) = C^T\mathbf{P}(\mathbf{P}\Psi(x) - \mathbf{P}\Psi(a) - (x-a)\Psi(a)) + u_t(a,t) + (x-a)u_{xt}(a,t) \tag{11}$$

and

$$u_t(x,t) = (t-t_n)C^T\mathbf{P}(\mathbf{P}\Psi(x) - \mathbf{P}\Psi(a) - (x-a)\Psi(a)) + u(x,t_n) + u(a,t) - u(a,t_n) + (x-a)(u_x(a,t) - u_x(a,t_n)). \tag{12}$$

Putting $x = 1$ in (11) and (12). Using boundary conditions (11) and overspecified condition (3), we obtain

$$u_{xt}(a,t) = C^T\mathbf{P}\left(\frac{1}{a-1}\mathbf{P}\Psi(1) - \frac{1}{a-1}\mathbf{P}\Psi(a) + \Psi(a)\right) + \frac{1}{1-a}\acute{q}(t) - \frac{1}{1-a}\acute{g}(t) \tag{13}$$

and

$$u_x(a,t) = u_x(a,t_n) - C^T(t-t_n)\mathbf{P}\left(\frac{1}{1-a}\mathbf{P}\Psi(1) - \frac{1}{1-a}\mathbf{P}\Psi(a) - \Psi(a)\right) + \frac{1}{1-a}(q(t)-q(t_n)) - \frac{1}{1-a}(g(t)-g(t_n)). \tag{14}$$

Assume that $x = x_l$ and $t = t_{n+1}$, where

$$x_l = \frac{l-0.5}{2^{k-1}M}$$

are collocation points. Substituting Equations (13) and (14) into (11) and (12), respectively, we have

$$u_{xx}(x_l,t_{n+1}) = (t_{n+1}-t_n)C^T\Psi(xl) + u_{xx}(x_l,t_n), \tag{15}$$

$$u_t(x_l,t_{n+1}) = \frac{x_l-a}{1-a}\acute{q}(t_{n+1}) + \frac{1-x_l}{1-a}\acute{g}(t_{n+1}) + C^T\mathbf{P}^2\left(\Psi(x_l) + \frac{x_l-1}{1-a}\Psi(a) - \frac{x_l-a}{1-a}\Psi(1)\right), \tag{16}$$

and

$$u(x_l,t_{n+1}) = (t_{n+1}-t_n)C^T\mathbf{P}^2\left(\Psi(x_l) + \frac{x_l-1}{1-a}\Psi(a) - \frac{x_l-a}{1-a}\Psi(1)\right) + u(x_l,t_n) + \frac{x_l-a}{1-a}(q(t_{n+1})-q(t_n)) + \frac{1-x_l}{1-a}(g(t_{n+1})-g(t_n)), \tag{17}$$

where $\mathbf{P}$ is obtained from (8).

Step 3. From Equation (1), we have

$$du(x_l,t_{n+1}) = -F(u(x_l,t_n))dt + B(x_l)dw(t_n),$$

so

$$u_t(x_l,t_{n+1}) = -F(u(x_l,t_n)) + B(x_l)\dot{w}(t_n). \tag{18}$$

Substituting Equation (16) into Equation (18) gives

$$C^T\mathbf{P}^2\left(\Psi(x_l) + \frac{x_l-1}{1-a}\Psi(a) - \frac{x_l-a}{1-a}\Psi(1)\right) = -F(u(x_l,t_n)) - \frac{x_l-a}{1-a}\acute{q}(t_{n+1}) - \frac{1-x_l}{1-a}\acute{g}(t_{n+1}) + B(x_l)\dot{w}(t_n). \tag{19}$$

Now, using the collocation points $x_l = (l-0.5)/(2^{k-1}M)$, $l = 1,2,...,2^{k-1}M$. In order to identify the unknown coefficients C^T within the interval $[t_n, t_{n+1}]$, it is imperative to

solve the system of equations. Since the obtained system of algebraic equations is ill-conditioned [14], the computational algorithm for the Levenberg-Marquardt regularization is provided in the next step and the coefficient C^T can be calculated.
After the computation of the unknown coefficients C^T, we inserted $x=0$ into the Equation (17). This way, the value of $u(0,\,t)$ is found. Following this procedure, the value of function λ can be obtained at any time.
Step 4. Levenberg-Marquardt regularization
At first, consider an initial guess for the vector of unknown coefficients C and denote it with C^0.
1. Let μ_0 be an arbitrary regularization parameter (e.g., $\mu_0 = 0.001$) and set $k=0$.
2. Compute Φ^k, where

$$\Phi = C^T\left(\mathbf{P}^2\Psi(x_l)+\frac{x_l-1}{1-a}\mathbf{P}^2\Psi(a)-\frac{x_l-a}{1-a}\mathbf{P}^2\Psi(1)+ F(u(x_l,\,t_n))\right)+\frac{x_l-a}{1-a}\acute{q}(t_{n+1})+\frac{1-x_l}{1-a}\acute{g}(t_{n+1})-B(x_l)\dot{w}(t_n).$$

3. Compute the sensitivity matrix $\mathbf{J}^k$ [15] and $\Omega^k = diag((\mathbf{J}^k)^T\mathbf{J}^k)$. 4. Solve the system of algebraic equations

$$((\mathbf{J}^k)^T(\mathbf{J}^k)+\mu^k\Omega^k)\Delta(C)^k = (\mathbf{J}^k)^T(\Phi),$$

and compute $(C)^{k+1} = \Delta(C)^k + (C)^k$.
5. If $\Phi^{k+1}\geq\Phi^k$ replace μ^k with $10\mu^k$ and go to item 4, otherwise, accept $(C)^{k+1}$ and replace μ^k with $0.1\mu^k$.
6. Assume that "tol" (tolerance) is given. If $\|(C)^{k+1}-(C)^k\|_2\leq$ tol, then an acceptable approximation is obtained. Otherwise, replace k with $k+1$ and go to item 3.

4. Error analysis

Similar to what was mentioned in [9], to assess the convergence in the Legendre wavelet method, we proceed as follows.
Suppose that $u_{xxt}(x,\,t) = \Gamma(x)$. The Legendre wavelets method is convergence if the function $\Gamma(x)$ be N times continuously differentiable. We assume that $\bar{\Gamma}(x)$ is an approximation of $\Gamma(x)$ with Legendre wavelet basis

$$\bar{\Gamma}(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{nm}\psi_{nm}(x) = C^T\Psi(x). \tag{20}$$

We have

$$\begin{aligned}\|\Gamma(x)-\bar{\Gamma}(x)\|^2_{L_2(\Omega,\,L_2(\Lambda))} &= E\left(\|\Gamma(x)-\bar{\Gamma}(x)\|^2_{L_2(\Lambda)}\right)\\ &= E\left(\int_0^1|\Gamma(x)-C^T\Psi(x)|^2dx\right)\\ &\leq \sum_{n=1}^{2^{k-1}}E\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|\Gamma(x)-\hat{\Gamma}(x)|^2dx\right),\end{aligned}$$

where $\hat{\Gamma}(x)$ denotes the interpolating polynomial of $\Gamma(x)$ see [15]. If β is an upper bound for Γ hence

$$\begin{aligned}|\Gamma(x)-\hat{\Gamma}(x)| &\leq \frac{(x-x_0)(x-x_1)\cdots(x-x_{N-1})}{N!}\Gamma^N(x)\\ &\leq \frac{\beta}{(N!)2^{k-1}}\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Gamma(x)-\bar{\Gamma}(x)\|^2_{L_2(\Omega,\,L_2(\Lambda))} &\leq E\left(\int_0^1\frac{\beta^2}{(N!)^2 2^{2k-2}}dx\right)\\ &= \frac{\beta^2}{(N!)^2 2^{2k-2}}\end{aligned}$$

In continuation, the Legendre wavelets method will be convergent when $N,\,k\to\infty$.

5. Numerical results

In this section, we have applied the proposed method to solve NISPDE. To illustrate the applicability and accuracy of this method, we compare the results obtained from the exact solution with those obtained from the approximate solution using the method presented with Legendre wavelets as the basis. Due to the non-existence of the exact solution to this problem, we consider a numerical approximation obtained using Legendre wavelets with $k=4$ and $M=5$ as the exact solution.
A special case of the problem (1) is

$$\begin{aligned}u_t(x,\,t) = u_{xx}(x,\,t)+u(x,\,t)(1-u(x,\,t))+\theta\dot{w}(t),\\ x\in(0,\,1),\,t\in(0,\,T).\end{aligned} \tag{21}$$

Obviously, if $\theta=0$, Equation (21) is the famous Fisher's equation in the deterministic case. We solve problem (21) with following known functions for the initial and boundary conditions (2) and overspecified condition (3) as below

$$\begin{aligned}\phi(x) &= \left(1+e^{\frac{x}{\sqrt{6}}}\right)^{-2}, & x\in[0,\,1]\\ q(t) &= \left(1+e^{\sqrt{\frac{x}{6}}-\frac{5}{6}t}\right)^{-2}, & t\in[0,\,T]\\ g(t) &= \left(1+e^{\sqrt{\frac{x}{6}}a-\frac{5}{6}t}\right)^{-2}, & t\in[0,\,T],\end{aligned} \tag{22}$$

where $a=0.5$. Here $\dot{w}(t) = dw/dt$ denotes white noise process on the probability space $(\Omega;\,F;\,P)$ and we consider various fixed values for $\theta\in[0,\,1]$.
Applying the our numerical algorithm that presented in the subsection 3.2, let

$$u_t(x_l,\,t_{n+1}) = u_{xx}(x_l,\,t_n)+u(x_l,\,t_n)(1-u(x_l,\,t_n))+\theta\dot{w}(t_n), \tag{23}$$

where

$$x_l = \frac{l-0.5}{2^{k-1}M},\;\; l=1,\,2,\cdots,\,2^{k-1}M$$

are collocation points. It is obtained similar to Equation (19)

$$\begin{aligned}C^T\mathbf{P}^2\left(\Psi(x_l)+\frac{x_l-1}{1-a}\Psi(a)-\frac{x_l-a}{1-a}\Psi(1)\right) &=\\ u_{xx}(x_l,\,t_n)+u(x_l,\,t_n)(1-u(x_l,\,t_n))&-\\ \frac{x_l-a}{1-a}\acute{q}(t_{n+1})-\frac{1-x_l}{1-a}\acute{g}(t_{n+1})+\theta\dot{w}(t_n).\end{aligned} \tag{24}$$

Using system of Equations (24) and Levenberg-Marquardt regularization, the coefficient C^T can be calculated. Thus,

we obtain the unknown function $\lambda(t)$ with approximate the value of $\lambda(t)$ for all $t \in [0, T]$. The results obtained for $u(0, t) = \lambda(t)$ are presented in the following Figures 1- 4 and Tables 1-4 with $T = 1$, $\Delta t = 0.01$.
Figures 1 and 2 display both the exact solution and the approximate solution $u(0, t)$ for the different values of θ and the plots of the error function S which is mentioned in the subsection 5-1, are shown in Figures 3 and 4. The numerical results are listed in Tables 1-4 provide a comparison between the exact solution and the approximate solution for $k = 1$ and $M = 5$, and we have also listed the error function S in these tables. As we observe, the error value increases at the endpoints of the interval $[0, 1]$.
The results demonstrate that the proposed method is effective, and we see that the desired solution is also provided with an approximate solution in both the deterministic and stochastic cases.

5.1 Remark

To compare the exact and approximate solutions for NIS-PDE, we consider the resulting variance of measurement errors. Therefore, we compare exact and approximate solutions by considering the total error S defined as

$$S = \left(\frac{1}{N-1} \sum_{i=1}^{N} (\hat{\lambda}_i - \lambda_i)^2 \right)^{\frac{1}{2}},$$

where n, λ, and $\hat{\lambda}$ are the number of estimated values, the estimated values and the exact values, respectively.

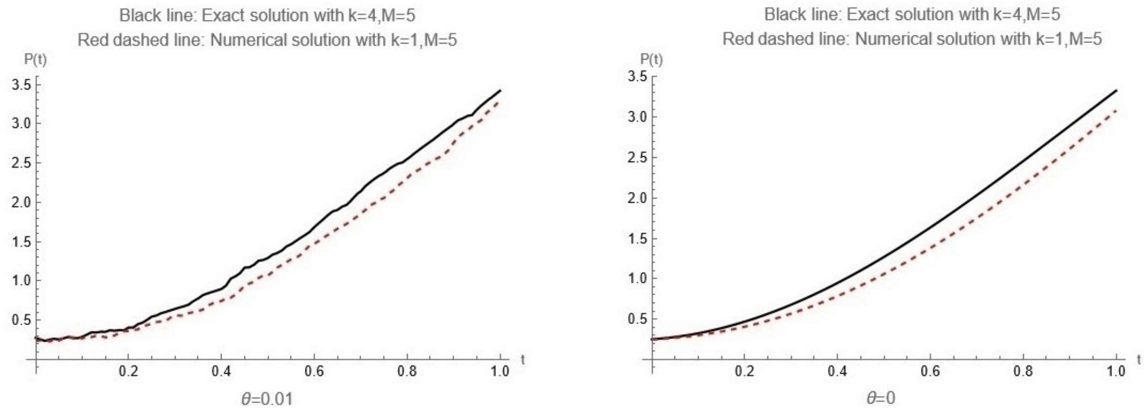

Figure 1. The plots of approximation $\lambda(t)$ with Legendre Wavelets for $\theta = 0$ and 0.01.

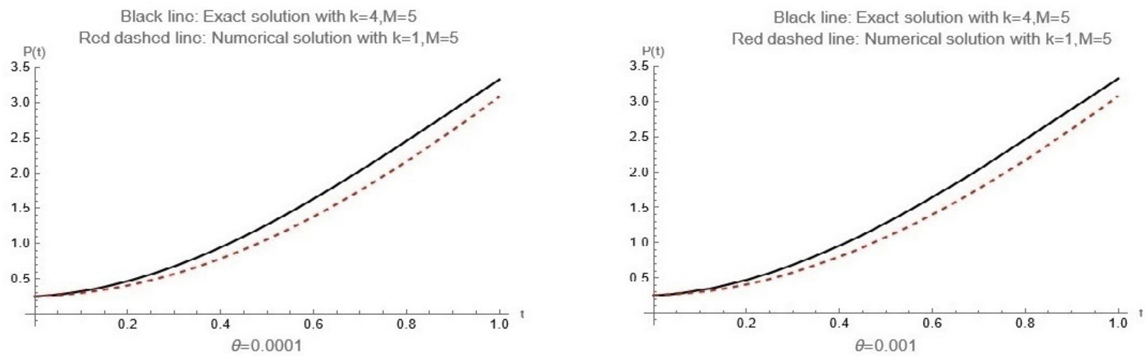

Figure 2. The plots of approximation $\lambda(t)$ with Legendre Wavelets for $\theta = 0.001$ and 0.0001.

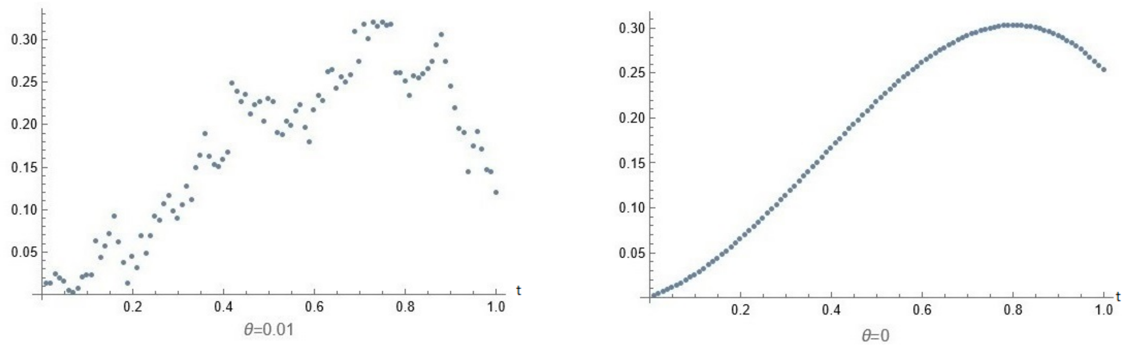

Figure 3. The error function S for $\theta = 0$ and 0.01.

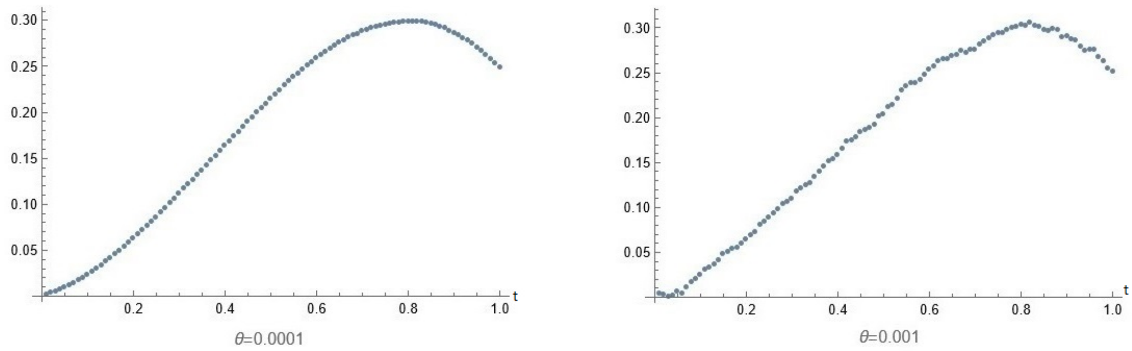

Figure 4. The error function S for $\theta = 0.001$ and 0.0001.

Table 1. The exact solution, approximate solution and error function S for $\theta = 0$.

t	Exact solution	Approximate solution	Error function S
0.01	0.254173	0.252484	1.7×10^{-3}
0.08	0.303525	0.285239	1.8×10^{-2}
0.1	0.324024	0.299687	2.5×10^{-2}
0.19	0.450485	0.391951	6.005×10^{-2}
0.38	0.888529	0.898668	1.5×10^{-1}
0.5	1.26872	1.0569	2.1×10^{-1}
0.62	1.71182	1.45052	2.6×10^{-1}
0.8	2.45653	2.16144	3.02×10^{-1}
0.94	3.06626	2.79396	2.7×10^{-1}
1	3.32725	3.08076	2.5×10^{-1}

Table 2. The exact solution, approximate solution and error function S for $\theta = 0.01$.

t	Exact solution	Approximate solution	Error function S
0.01	0.24638	0.240226	1.9×10^{-2}
0.08	0.313103	0.311820	6.8×10^{-3}
0.1	0.287934	0.308411	2.1×10^{-2}
0.19	0.390715	0.355019	1.3×10^{-2}
0.38	0.888529	0.700753	1.5×10^{-1}
0.5	1.30141	1.10034	2.2×10^{-1}
0.62	1.76571	1.56044	2.2×10^{-1}
0.8	2.53457	2.28597	2.5×10^{-1}
0.94	3.13037	2.97688	1.4×10^{-1}
1	3.43947	3.26978	1.1×10^{-1}

Table 3. The exact solution, approximate solution and error function S for $\theta = 0.001$.

t	Exact solution	Approximate solution	Error function S
0.01	0.253522	0.251574	2.7×10^{-3}
0.08	0.304385	0.290410	1.7×10^{-2}
0.1	0.337868	0.310422	2.5×10^{-2}
0.19	0.450380	0.393358	5.9×10^{-2}
0.38	0.900921	0.753228	1.5×10^{-1}
0.5	1.28117	1.08062	2.04×10^{-1}
0.62	1.72031	1.46880	2.6×10^{-1}
0.8	2.46157	2.16929	3.03×10^{-1}
0.94	3.06494	2.79688	2.7×10^{-1}
1	3.32153	3.08292	2.5×10^{-1}

Table 4. The exact solution, approximate solution and error function S for $\theta = 0.0001$.

t	Exact solution	Approximate solution	Error function S
0.01	0.253988	0.252408	1.6×10^{-3}
0.08	0.302515	0.286081	1.7×10^{-2}
0.1	0.322744	0.300115	2.3×10^{-2}
0.19	0.449093	0.392261	5.8×10^{-2}
0.38	0.886323	0.737384	1.5×10^{-1}
0.5	1.26622	1.05718	2.1×10^{-1}
0.62	1.70991	1.45017	2.6×10^{-1}
0.8	2.45293	2.16227	2.9×10^{-1}
0.94	3.06252	2.79516	2.7×10^{-1}
1	3.32326	3.08197	2.4×10^{-1}

6. Conclusion

In this paper we study the inverse nonlinear stochastic problem. A numerical method has been developed to approximate values of the function on a part of physical boundary of the domain. This method approximates the exact solution for this issue by employing Legendre wavelets and their operational matrix combining with the Levenberg-Marquardt regularization technique. It can be concluded from the obtained results and error analysis that the above method has the appropriate efficiency. The numerical results provide a comparison between the exact solution and the approximate solution and we observe, the error value increases at the endpoints of the interval $[0,\ 1]$.

Authors contributions
Authors have contributed equally in preparing and writing the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work

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