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Research Article

Effects of q-Deformation on the Thermal Deconfinement Phase Transition in High- Energy Physics

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Abstract

The influence of the q-deformation parameter on the thermal deconfinement phase transition (DPT) from a hadronic gas phase to a Quark-Gluon Plasma (QGP) phase is investigated in this paper using analytical derivations and numerical simulations within the phase coexistence model. By studying key thermodynamic parameters such as specific heat, entropy, energy density, and the order parameter, we demonstrate how the phase transition dynamics are modified by the deformation parameter q. Our results indicate that increasing q lowers the critical transition temperature $T_C(q)$, while enhancing thermodynamic quantities like entropy and energy densities. Furthermore, the phase transition exhibits first-order characteristics, as evidenced by latent heat and discontinuities in thermodynamic functions near the critical point. Additionally, we study finite-size effects, which result in smoothed transitions in smaller system volumes. These findings underscore the relevance of q-deformation in high-energy physics, particularly in understanding the thermodynamics of QCD phase transitions and non-extensive statistical contributions.

Keywords q-deformation; thermodynamic quantities; quark-gluon plasma; phase transition; non-extensive statistics; hadronic matter.

1 Introduction

One of the main problems in quantum chromodynamics (QCD) is still comprehending the microscopic dynamics that control the transition from confined hadronic matter to the deconfined quark–gluon plasma (QGP) [1-3]. Strong correlations, memory effects, and collective behavior are present in the system formed in high-energy nuclear collisions near the deconfinement temperature, and neither perturbative QCD nor traditional Boltzmann-Gibbs statistics can adequately describe these features. The search for generalized statistical frameworks that can characterize non-ideal and non-extensive features of strongly interacting matter is motivated by these deviations [4-5].

One promising method is based on q-deformation, an algebraic generalization of quantum mechanics where a deformation parameter q modifies the canonical commutation relations. Standard quantum statistics and the standard Heisenberg algebra are recovered when $q \to 1$; when $q \neq 1$, the algebra introduces an efficient description of correlations, non-local interactions, and finite-size effects and encodes deviations from extensivity. Physically, the parameter q measures the degree of correlation or non-Markovianity in the system rather than a new fundamental constant. Therefore, a consistent mathematical tool for incorporating long-range color interactions and collective behavior that might arise close to the QCD phase boundary is provided by q-deformed quantum and statistical algebras. [6-15].

In high-energy nuclear collisions, the extreme initial conditions described by the Color Glass Condensate [16] lead to a complex evolution toward a partially thermalized quark-gluon plasma, where turbulent transport and memory effects are significant [17]. To effectively capture these non-perturbative, non-extensive features within a thermodynamic framework, q-deformed statistics [6] has been successfully applied, providing a powerful phenomenological tool for describing particle spectra and phase structure [9, 18]. Previous works on q-deformed quantum mechanics, statistical mechanics, and field theory have demonstrated that this formalism naturally reproduces several non-extensive features of many-body systems. At the algebraic level, the q-deformed oscillator leads to deformed spectra [19-20]. In statistical ensembles, it yields modified occupation numbers and alters collective excitation spectra, with applications extending to deformed field theories and hadronic systems [9]. Therefore, the motivation for this study is not to solve a specific conceptual problem in quantum mechanics, but to investigate how the deformation parameter q influences the thermodynamic characteristics of the deconfinement phase transition and how such deformations could mimic medium-induced correlations near criticality.

In theoretical physics, the concept of q -deformation has therefore attracted considerable attention in recent decades because of its capacity to generalize algebraic structures in quantum theory [21–27]. By introducing the deformation parameter q, one modifies the standard algebraic relations and structures of the Heisenberg algebra [28–32]. Numerous applications have been achieved using this framework, including the q-deformed Schrödinger equation [33–35], the q-deformed hydrogen atom and harmonic oscillator [36–38], and the extension of statistical mechanics and thermodynamics using q-calculus [24, 39–41].

Furthermore, q-deformation has been applied in diverse areas of physics, for example to the q-deformed Klein–Gordon and Dirac equations, to nonlinear systems, and to the study of deformed oscillators exhibiting intermediate statistics [24, 42]. Thermodynamics formulated on the basis of q-calculus remains consistent provided that ordinary derivatives are replaced with the Jackson derivative and q-integrals [24]. The resulting q-thermostatistics yields thermodynamic functions that reveal purely quantum-statistical effects stronger than those of undeformed bosonic or fermionic systems [24, 43].

In high-energy nuclear collisions, extreme conditions of temperature and density necessitate the consideration of non-standard statistical effects, including memory effects and long-range interactions. These phenomena suggest that q-deformed statistics may provide a more accurate description of the system's behavior near the phase transition. This paper explores the impact of q-deformation on the thermal deconfinement phase transition DPT, focusing on thermodynamic quantities such as pressure, energy density, and susceptibility. We use the phase coexistence model, assuming that the mixed HG-QGP phase system has a finite volume: $V = V_{HG} + V_{QGP}$, and the parameter h representing the fraction of volume occupied by the HG phase is then defined: $V_{HG} = hV$. Thus, the value $h = 1$ corresponds to a total HG phase while $h = 0$ corresponds to a total QGP phase. We derive analytical expressions for these quantities and investigate their dependence on q and temperature. Our work bridges gaps in understanding how non-extensive statistics influence QCD phase transitions, offering insights into the thermodynamics of strongly interacting matter. In the present work, we examined the effects of deformed parameter q for the thermal deconfinement phase transition (DPT) from a hadronic gas phase consisting of massless pions to a Quark-Gluon Plasma phase containing gluons, massless up and down quarks within the phase coexistence model [44, 45]. For this, we examine analytically the behavior of the most important thermodynamical quantities describing the system (energy density ε, order parameter h, … etc) with temperature T and the deformation parameter q for a vanishing chemical potential. Because of the extreme density and temperature conditions in high-energy nuclear collisions, non-standard statistical effects are important in nuclear and high-energy physics. These conditions suggest that non-Markovian processes are involved in the kinetic equation, influencing the thermalization process leading to equilibrium and the typical equilibrium distribution [46, 47]. Microscopic computations in relation to the parton plasma created during the high energy collisions should serve as the foundation for a thorough identification of conditions that result in a non-extensive behavior, caused by memory effects and/or long-range interactions. At this moment, we are only able to examine the issue from a qualitative perspective. Nonetheless, it is significant to observe that non-perturbative QCD calculations become crucial in proximity of the hadronic-QGP phase transition. There are very few partons in the Debye sphere: Memory effects are not insignificant, and the plasma's normal mean field approximation is no longer accurate. Additionally, we establish that the color magnetic field remains unscreened (at leading order) at high densities and that long-range color magnetic interaction should be present at all temperatures [48].

The present analysis also connects to a broader class of recent studies in which generalized or modified thermodynamic frameworks have been applied to gravitational and high-energy systems. Several works have explored how non-standard statistical features, quantum corrections, or extended thermodynamic structures affect phase transitions and stability criteria in black-hole and related systems [49–56]. These studies demonstrate that modified statistical or thermodynamic assumptions can significantly influence critical behavior, thermodynamic stability, and response functions—paralleling the type of sensitivity analysis performed here with q-deformed quantum statistics. While the physical settings differ, the underlying theme is similar: deviations from ideal thermodynamics can provide insight into how microscopic correlations or non-extensive effects modify macroscopic phase structure.

Recent lattice QCD studies and theoretical analyses have established that the QCD phase transition at zero baryon chemical potential, in the chiral limit, is of second order or a crossover rather than a first-order transition [57]. This understanding provides a critical benchmark for models studying deconfinement and chiral symmetry restoration. Our present work employs a q-deformation framework within a phase coexistence model to investigate how deviations from standard quantum statistics influence thermodynamic behavior and signatures of the phase transition. While the model exhibits sharp first-order-like features arising from the deformation and finite-size effects, these should not be interpreted as contradicting the established crossover nature but as effective manifestations of non-extensive statistical modifications. This motivates a careful interpretation of the results in the context of current theoretical consensus.

Before proceeding, it is important to clarify the origin of the deformation parameter q used in this work. Our analysis is based on the algebraic q-deformation of the bosonic/fermionic oscillator algebra, where the commutation relations are generalized according to a quantum-group (q-algebra) structure. In this framework, q modifies the underlying algebra of creation and annihilation operators and consequently the associated quantum-statistical distribution functions. This interpretation is distinct from Tsallis non-extensive statistics, in which q arises from generalized entropy, and from approaches that modify the phase-space volume. The deformation considered here is purely algebraic and reflects changes in the microscopic quantum structure rather than in the thermodynamic entropy functional or the geometry of phase space.

The present paper is organized as follows. In section 2, we give a brief overview on q-deformed oscillator algebra. In section 3, we will look for the partition function which serves to determine, explicitly the different thermodynamic functions such as the pressure, order parameter, energy and entropy densities, speed of sound, susceptibility and specific heat density. Section 4, will be devoted to the numerical results and discussions as well as comparison with literature. We conclude our results in section 5.

2 q-deformed oscillator algebra

In this work, the parameter q stems from the q-deformed Heisenberg algebra of creation and annihilation operators introduced by Biedenharn [19] and McFarlane [20]. It represents an algebraic deformation of the oscillator structure, rather than a non-extensive entropy index or a geometric deformation of phase-space volume. The thermodynamic effects studied here arise solely from this q-deformed oscillator algebra and its associated Jackson derivative

The q-oscillators algebra is defined through q-deformed Heisenberg algebra by introducing a deformation parameter q, into the oscillator algebra. The commutation relations for the creation c^{+}, annihilation c operators and the number N operator are given via [58, 59]:

$$[c,c]_k = [c^{+},c^{+}]_k = 0,\ cc^{+} - kq^{k}c^{+}c = q^{-N},$$

$$[N, c^{+}] = c^{+},\ [N, c] = -c, \tag{1}$$

where $k = 1$ for q-bosons with commutators and $k = -1$ for q-fermions with anticommutators.

The operators obey the relations:

$$c^{+}c = [N],\ \ cc^{+} = [1 + kN], \tag{2}$$

where the q-basic number $[x]$ is given by:

$$[x] = \frac{q^{x} - q^{-x}}{q - q^{-1}}. \tag{3}$$

The q-Fock space is constructed using these relations. The number operator becomes a nonlinear function $N = f(c^{+}c)$, expressible in closed form:

$$N = \frac{1}{\ln q} \ln\left(1 + (q - 1)c^{+}c\right). \tag{4}$$

Operators can be mapped into configuration space using the Bargmann representation [60, 61]:

$$c^{+} = x,\ \ c = \partial_x^{(q)}, \tag{5}$$

where $\mathcal{D}_x^{(q)}$ is the Jackson derivative (JD) [62]:

$$\mathcal{D}_x^{(q)} f(x) = \frac{f(qx) - f(x)}{x(q - 1)}. \tag{6}$$

We apply the Jackson derivative to thermodynamic quantities using the standard assumption in q-deformed thermostatistics that the q-dependence of all thermodynamic variables is derived from the grand partition function's q-exponential structure. As a result, the Jackson derivative is only used in place of the ordinary derivatives in the definitions of particle number, energy, entropy, and related quantities with regard to variables that explicitly enter through the q deformed exponential or q-logarithm. This guarantees that the differentiation maintains the thermodynamics Legendre-transform structure and is consistent with the underlying q-oscillator algebra. The Jackson derivative smoothly reduces to the ordinary derivative in the limit $q \longrightarrow 1$, restoring the conventional Bose–Einstein and Fermi–Dirac framework.

The q-deformation of the Bose and Fermi distributions arises directly from the modified oscillator algebra and the use of the Jackson derivative in evaluating thermodynamic quantities. For a q-deformed bosonic (fermionic) mode with energy ϵ, the mean occupation number takes the form:

$$n_q(\epsilon) = \frac{1}{exp_q(\beta\epsilon) \mp 1}, \quad (7)$$

where the function $exp_q(x)$ is generated by the q-calculus structure associated with the algebra and reduces to the ordinary exponential when $q \longrightarrow 1$. These q-modified distributions result from the altered creation-annihilation operator algebra and have been discussed in Refs. [24, 39–41], to which we refer for explicit derivations. In the present work, these expressions serve as the basis for constructing the q-deformed partition functions and the associated thermodynamic quantities.

Before turning to thermodynamic quantities, it is useful to comment on the physically relevant range of the deformation parameter q. In the algebraic q-deformation discussed here, values of q greater than 1 enhance quantum-statistical effects, increasing bosonic coherence and strengthening fermionic exclusion, while values of q less than 1 weaken these effects. The modifications dependent on q alter the effective occupation numbers in both phases, leading to a somewhat stiffer equation of state for $q > 1$ and a softer equation for $q < 1$. As a result, the relative pressure difference between the hadronic and QGP phases is modified: $q > 1$ encourages an earlier onset of deconfinement and a more distinct transition, while $q < 1$moves the transition to higher temperatures and leads to a more gradual phase change. The thermodynamic results presented in Section 4 illustrate these qualitative behaviors.

3 q-deformed thermodynamic quantities

To evaluate the thermodynamic properties of each phase, we apply the *q*-deformed formalism to the hadronic gas (HG) and quark–gluon plasma (QGP) sectors separately. In the present work, the hadronic gas is modeled as an ideal, non-interacting gas of massless pions. This approximation is widely used in coexistence models because it provides an analytically tractable baseline while allowing us to isolate the specific influence of the deformation parameter *q*. We note that interactions in the hadronic sector may become important close to the deconfinement temperature; however, including such effects would require additional assumptions about hadronic potentials and resonance contributions, which would obscure the role of *q*-deformation in the present analysis. The ideal-pion-gas approximation therefore serves as a controlled and standard reference point.

3-1 Grand partition function

The grand canonical partition function for a single quantum state (mode) with energy ε_i is defined as:

$$\mathcal{Z}_i = \sum_{n_i=0}^{\infty} \langle n_i | e^{-\beta(\varepsilon_i-\mu)n_i} | n_i \rangle \tag{8}$$

However, in q-deformed statistics, the energy of a state with $n_i$particles is not simply $n_i\varepsilon_i$. The deformation changes the spectrum. A more robust approach is to use the density matrix.

The Hamiltonian for the non-interacting gas is $H = \sum_i (\varepsilon_i - \mu) n_i$ The mean occupation number for a mode is given by the general formula:

$$\langle n_i \rangle = \frac{1}{\mathcal{Z}_i} Tr\left(e^{-\beta H} n_i\right). \tag{9}$$

Calculating this trace within the q-deformed Fock space is complex. A standard result derived in the literature (e.g., from the algebra using the Jackson derivative) is that the q-deformed mean occupation number for bosons ($k = 1$) and fermions ($k = -1$) is:

$$\langle n_i \rangle = \frac{1}{lnq}\left[ln\left(\frac{1-kq^{-k}ze^{-\beta\varepsilon_i}}{1-kq^{k}ze^{-\beta\varepsilon_i}}\right)\right], \tag{10}$$

where $z = e^{\beta\mu}$ is the fugacity.

Let's define $y_i = e^{-\beta\varepsilon_i}$. For bosons ($k = 1$), this becomes:

$$\langle n_i \rangle^{(b)} = \frac{1}{ln\, q}\left[ln\left(\frac{1-q^{-1}zy_i}{1-qzy_i}\right)\right], \tag{11}$$

For fermions ($k = -1$), we get:

$$\langle n_i \rangle^{(f)} = \frac{1}{ln\, q}\left[ln\left(\frac{1+qzy_i}{1+q^{-1}zy_i}\right)\right]. \tag{12}$$

In standard thermodynamics, for a grand canonical ensemble, we have the relation:

$$\langle n_i \rangle = -\frac{1}{\beta}\frac{\partial\, ln\, \mathcal{Z}}{\partial \varepsilon_i}. \tag{13}$$

In q-deformed thermodynamics, this ordinary derivative is replaced by the Jackson Derivative (JD) with respect to the variable $y_i = e^{-\beta\varepsilon_i}$. Using the JD definition Eq. (6), thus, the q-analogue of the relation is:

$$\langle n_i \rangle = -\frac{1}{\beta}\mathcal{D}_{y_i}^{(q)}\, ln\, \mathcal{Z}. \tag{14}$$

We can now integrate this (using the q-integral, which is the inverse of the JD) to find $ln\, \mathcal{Z}$.

Let's do this for a single bosonic mode. We have:

$$\mathcal{D}_{y_i}^{(q)}\, ln\, \mathcal{Z}_i^{(b)} = -\frac{1}{ln\, q} ln\left(\frac{1-q^{-1}zy_i}{1-qzy_i}\right). \tag{15}$$

To find $ln\, \mathcal{Z}_i^{(b)}$ we need a function whose JD gives the right-hand side. It can be shown (by verification or using q-calculus tables) that:

$$ln\, \mathcal{Z}_i^{(b)} = -\frac{1}{ln\, q} Li_2(q^{-1}zy_i) + \frac{1}{ln\, q} Li_2(qzy_i), \tag{16}$$

where $\mathrm{Li}_2(x)$ is the polylogarithm function, and we have used the property that $\mathcal{D}_{y_i}^{(q)} \mathrm{Li}_2(ax) = \frac{1}{x ln\, q} ln(1-ax)$.

A more common and useful form is the series expansion. Using the Taylor expansion for $ln(1-x)$, one can find the expansion for $\langle n_i \rangle$ and then integrate it term-by-term. The result for the single-mode partition function is:

$$ln\, \mathcal{Z}_i^{(b)} = -\sum_{l=1}^{\infty} \frac{(zy_i)^l}{l^2} \frac{q^l - q^{-l}}{ln\, q} = \sum_{l=1}^{\infty} \frac{(zy_i)^l}{l^2} [l], \tag{17}$$

where $[l]$ is the q-number defined in Eq. (3). This is the q-deformed version of the standard result $ln\, \mathcal{Z}_i = -ln(1-zy_i)$

To get the total partition function $ln\, \mathcal{Z}$ for a gas, we sum over all momentum states i.

$$ln\, \mathcal{Z} = \sum_i ln\, \mathcal{Z}_i^{(b)} = \sum_i \sum_{l=1}^{\infty} \frac{z^l}{l^2} [l] \left(e^{-\beta\varepsilon_i}\right)^l. \tag{18}$$

In the thermodynamic limit, we replace the sum over states i with an integral over phase space. For a relativistic particle in a volume V:

$$\sum_i \longrightarrow \frac{gV}{(2\pi)^3} \int d^3p, \tag{19}$$

where g is the degeneracy factor (e.g., $g = 1$for a scalar pion).

For a massless particle, $\varepsilon_p = |\vec{p}|$. Switching to spherical coordinates in momentum space:

$$\sum_i e^{-l\beta\varepsilon_i} \longrightarrow \frac{gV}{2\pi^2} \int_0^{\infty} p^2 dp\, e^{-l\beta p}. \tag{20}$$

This integral is: $\frac{gV}{2\pi^2} \frac{2}{(l\beta)^3} = \frac{gV}{\pi^2} \frac{1}{(l\beta)^3}$.

Substituting this back in Eq. (18):

$$ln\,\mathcal{Z}^{(b)} = \sum_{l=1}^{\infty} \frac{z^l}{l^2}[l]\left(\frac{gV}{\pi^2}\frac{1}{(l\beta)^3}\right) = \frac{gV}{\pi^2\beta^3}\sum_{l=1}^{\infty}\frac{z^l}{l^5}[l]. \tag{21}$$

We can now recognize with the q-deformed function $h_n^{(k)}(z,q)$, for bosons we have:

$$h_n^{(b)}(z,q) = \frac{1}{\ln q}\left(\sum_{l=1}^{\infty}\frac{(qz)^l}{l^{n+1}} - \sum_{l=1}^{\infty}\frac{(q^{-1}z)^l}{l^{n+1}}\right) = \sum_{l=1}^{\infty}\frac{z^l}{l^{n+1}}[l]. \tag{22}$$

For $n = 4$, we have:

$$h_4^{(b)}(z,q) = \sum_{l=1}^{\infty}\frac{z^l}{l^5}[l]. \tag{23}$$

Therefore, our derived partition function becomes:

$$\ln \mathcal{Z}^{(b)} = \frac{gV}{\pi^2}T^3 h_4^{(b)}(z,q). \tag{24}$$

For the HG consisting of massless pions. Pions come in three charge states (π^+, π^-, π^0), so the degeneracy is $g = 3$. The chemical potential for pions is zero $\mu = 0, z = 1$. Applying the formula above:

$$\ln Z_{HG} = \frac{3V_{HG}}{\pi^2}T^3 h_4^{(b)}(q), \tag{25}$$

where $h_4^{(b)}(q) \equiv h_4^{(b)}(1,q)$.

3.2 Quark-Gluon Plasma (QGP) Partition function:

The QGP is modeled as a non-interacting gas of quarks and gluons confined by the bag constant *B*.

Gluons: These are massless bosons with zero chemical potential. Their degeneracy is $g_{gluon} = 2(polarizations) \times 8(color) = 16$ Using the bosonic formula:

$$\ln Z_{gluon} = \frac{16\ \ _{QGP}}{\pi^2}T^3 h_4^{(b)}(q) \tag{26}$$

Quarks (u and d): These are massless fermions. We must account for particle and antiparticle contributions (z and z^{-1}). For a single quark flavor with degeneracy, and chemical potential μ, the fermionic partition function is derived similarly to the bosonic one, yielding:

$$\ln Z_{quark} = \frac{g_{quark}V_{QGP}}{\pi^2}T^3 h_4^{(f)}(z,q). \tag{27}$$

For the antiquarks, we replace z with z^{-1}. For $\mu = 0$, the total quark contribution is $12V_{QGP}T^3/\pi^2.\left(h_4^{(f)}(z,q) + h_4^{(f)}(z^{-1},q)\right)$, where the factor $g_{quark} = 12$.

Adding all contributions, we get the total QGP partition function:

$$\ln Z_{QGP} = \frac{4V_{QGP}T^3}{\pi^2}\left(3(h_4^{(f)}(z,q) + h_4^{(f)}(z^{-1},q)) + 4h_4^{(b)}(q)\right) - \frac{BV_{QGP}}{T}, \tag{28}$$

where $h_n^{(k)}(z,q)$ is the q-deformed function defined as:

$$h_n^{(k)}(z,q) = \frac{1}{\Gamma(n)}\int_0^\infty dx x^{n-1}\frac{1}{\ln q}\ln\left(\frac{z^{-1}e^x + q^{-1}}{z^{-1}e^x + q}\right)$$

$$h_n^{(k)}(z,q) = \frac{1}{\ln q}\left(\sum_{i=1}^{\infty}\frac{(kq^k z)^i}{i^{n+1}} - \sum_{i=1}^{\infty}\frac{(kq^{-k}z)^i}{i^{n+1}}\right). \tag{29}$$

The second line of Eq. (29) is simply the same function $h_n^{(k)}(z,q)$ in its series (summation) form, derived from expanding the integral using q-calculus definitions.

In the limit $q \to 1$, the deformed $h_n^{(k)}(z,q)$ functions reduce to the standard $g_n(z)$ functions for bosons and to the $f_n(z)$ functions for fermions.

We note that Eq. (28) and the subsequent expressions for f_{HG} and f_{QGP} are dimensionally consistent in natural units $\hbar = c = \mathrm{k_B} = 1$, the pressure and free-energy carry dimension of energy density T^4.

Now, we examine the behavior of some thermodynamic quantities with temperature for varying deformed parameter q, at a vanishing chemical potential $\mu = 0 \ or\ z = e^{\beta\mu} = 1$, at fixed volume, using the common value $B^{1/4} = 145\ \mathrm{MeV}$ for the bag constant.

For the QGP phase, we employ the standard MIT bag model, in which the deconfined medium consists of gluons and massless up and down quarks confined within a perturbative vacuum characterized by the bag constant *B*. This model is widely used in phenomenological studies of the deconfinement transition and provides an analytically tractable baseline for examining the influence of q-deformation on thermodynamic quantities. We note that several more sophisticated approaches exist, such as quasiparticle models, PNJL/Polyakov-loop, extended chiral models, and lattice-QCD-based parameterizations. However, incorporating these frameworks would introduce additional dynamical assumptions that may obscure the specific effects of the deformation parameter q. For this reason, the MIT bag model is used as a controlled and transparent reference framework in the present study.

In the numerical analysis, we adopt the standard MIT bag-model, $B^{1/4} = 145\ \mathrm{MeV}$ [63]. This parameter choice lies within the conventional phenomenological range used in early quark–gluon plasma studies and provides a consistent baseline for comparative analyses [64-66]. Although this value does not coincide with the modern lattice-QCD determination of the crossover temperature $T_C \approx 155 - 165\ \mathrm{MeV}$ at $\mu = 0$ [67], it remains widely employed in analytical coexistence models where the emphasis is on exploring qualitative trends-such as the effect of the deformation parameter q, rather than reproducing lattice-QCD results quantitatively. We therefore use $B^{1/4} = 145\ \mathrm{MeV}$ as a standard reference value in the present study.

The main quantity of interest is the order parameter which represents the mean value of the hadronic volume fraction $< h(T,V,q) >$:

$$< h(T,V,q) >= \frac{[-\beta V(f_{HG}-f_{QGP})-1]exp[-\beta\ \ (f_{HG}-f_{QGP})]+1}{[-\beta V(f_{HG}-f_{QGP})][exp[-\beta\ \ (f_{HG}-f_{QGP})]-1]}, \quad (30)$$

with:

$$\begin{cases} f_{HG} = -\frac{3T^4}{\pi^2} h_4^{(b)}(q) \\ f_{QGP} = -\frac{4T^4}{\pi^2}\left(6h_4^{(f)}(q) + 4h_4^{(b)}(q)\right) + B. \end{cases} \quad (31)$$

The mean values of both energy density $< \varepsilon(T,V,q) >$ and the entropy density $< s(T,V,q) >$ are related to the order parameter as follows:

$$< \varepsilon(T,V,q) >= \varepsilon_{QGP} + \left(\varepsilon_{HG} - \varepsilon_{QGP}\right) < h(T,V,q) >;$$

$$< s(T,V,q) >= s_{QGP} + (s_{HG} - s_{QGP}) < h(T,V,q) >, \quad (32)$$

where:

$$\begin{cases} \varepsilon_{HG} = \frac{9}{\pi^2} T^4 h_4^{(b)}(q) \\ \varepsilon_{QGP} = \frac{12T^4}{\pi^2}\left(6h_4^{(f)}(q) + 4h_4^{(b)}(q)\right) + B. \end{cases} \quad (33)$$

and:

$$\begin{cases} s_{HG} = \frac{12}{\pi^2} T^3 h_4^{(b)}(q) \\ s_{QGP} = \frac{16\ ^3}{\pi^2}\left(6h_4^{(f)}(q) + 4h_4^{(b)}(q)\right). \end{cases} \quad (34)$$

We obtain the expressions of the pressures of the HG and QGP phases, respectively, as follows:

$$\begin{cases} P_{HG} = \frac{3T^4}{\pi^2} h_4^{(b)}(q) \\ P_{QGP} = \frac{4T^4}{\pi^2}\left(6h_4^{(f)}(q) + 4h_4^{(b)}(q)\right) - B. \end{cases} \quad (35)$$

In addition to the previous quantities, we can also calculate other important thermodynamic quantities, such as susceptibility and specific heat density for a vanishing chemical potential $\mu = 0 \; or \; z = e^{\beta\mu} = 1$.

For the susceptibility we have:

$$\chi(T,V,q) = \frac{A_1 A_2 - A_3 A_4}{{A_1}^2}, \quad (36)$$

where the constants A_{i} $(i = 1, \ldots, 4)$ denote auxiliary coefficients obtained from the q-deformed thermodynamic functions: A_1 , A_2 from the hadronic gas sector and A_3 , A_4 from the quark–gluon plasma sector. They are dimensionless functions of q, T, and V, explicitly defined in Eq. (37).

$$A_1 = G.A_5 exp[G]$$

$$A_2 = G[exp[G] - 1]$$

$$A_3 = [G - 1]exp[G] + 1 \quad (37)$$

$$A_4 = A_5\{[1 + G]exp[G] - 1\}$$

$$A_5 = -\frac{39}{\pi^2} VT^2 h_4^{(b)}(q) - \frac{72}{\pi^2} VT^2 h_4^{(f)}(q) - \frac{BV}{T^2}$$

$$G = -\frac{V}{T}(f_{HG} - f_{QGP}).$$

and the specific heat density is given by:

$$c(T,V,q) = A_6 + A_7 < h > +(\varepsilon_{HG} - \varepsilon_{QGP})\chi. \quad (38)$$

where:

$$A_6 = \frac{288}{\pi^2} T^3 h_4^{(f)}(q) + \frac{192}{\pi^2} T^3 h_4^{(b)}(q)$$

$$A_7 = -\frac{288}{\pi^2} T^3 h_4^{(f)}(q) - \frac{156}{\pi^2} T^3 h_4^{(b)}(q). \quad (39)$$

In order to conduct a hydrodynamic study of the system under consideration near the phase transition, two important quantities can be calculated: the average value of pressure and the sound velocity squared, which are expressed for a vanishing chemical potential respectively, by the following expressions:

$$< P(T,V,q) >= P_{QGP} + \left(P_{HG} - P_{QGP}\right) < h(T,V,q) >, \quad (40)$$

$$c_s{}^2 = \frac{\partial<P>}{\partial<\varepsilon>} = \frac{\partial<P>/\partial T}{\partial<\varepsilon>/\partial T} = \frac{\partial<P>/\partial T}{c(T,V,q)}, \quad (41)$$

where:

$$\frac{\partial < P >}{\partial T} = A_8 + A_9 < h > + \left(P_{HG} - P_{QGP}\right)\chi$$

$$A_8 = \frac{96}{\pi^2} T^3 h_4^{(f)}(q) + \frac{64}{\pi^2} T^3 h_4^{(b)}(q)$$

$$A_9 = -\frac{96}{\pi^2} T^3 h_4^{(f)}(q) - \frac{52}{\pi^2} T^3 h_4^{(b)}(q). \quad (42)$$

4 Numerical Results

We mention that we have represented the variations of all relevant quantities for a vanishing chemical potential $\mu = 0$ and a value $B^{1/4} = 145MeV$ for the bag constant.

According to the Gibbs condition, equilibrium between the confinement hadronic and deconfinement plasma phases must be achieved when moving from the hadronic gas to the quark-gluon plasma; therefore, we write:

$$P_{HG}(\mu_C, T_C) = P_{QGP}(\mu_C, T_C), \quad (43)$$

where T_C is the critical temperature and μ_C is the critical chemical potential. To study the pressure behavior, we represent the pressure variations as a function of the temperature T for the hadronic and plasma phases in Figure 1 for different values of the deformed parameter q , with zero chemical potential $\mu = 0$ and $B^{1/4} = 145MeV$. This figure shows that for a temperature smaller than the transition temperature $(T < T_C(q))$, the pressure of the hadronic phase is greater. This implies that the hadronic phase is thermodynamically more stable than the plasma phase, whereas when the temperature exceeds the transition temperature $(T > T_C(q))$, it appears that the quark-gluon plasma phase is the most stable because it has a greater pressure in this temperature range, which leads to the presence of the hadronic phase in $(T < T_C(q))$ and the quark-gluon plasma phase in $(T > T_C(q))$. Figure 1 demonstrates that the plasma phase pressure increases with the deformed parameter q. Furthermore, the transition temperature $T_C(q)$ exhibits a clear decreasing dependence on q.

Figure 1 Pressure variation for hadronic and plasma phases as a function of temperature for different values of the deformed parameter q at zero chemical potential $\mu = 0$ and volume $V = 150\ MeV$.

In Figure 2, we represent the variations of the critical temperature $T_C(q)$ as a function of the deformed parameter q. Its shape confirms what we described previously, which is that the transition temperature $T_C(q)$ is related to the value of the deformed parameter q, where the value of $T_C(q)$decreases with increasing q.

Figure 2 Critical temperature $T_C(q)$ variation as a function of the deformed parameter q.

It is important to note that the numerical values of the transition temperature obtained in this work are lower than the lattice-QCD estimate $T_\mathrm{C} \approx 150 - 160\ MeV$. This difference is expected because our thermodynamic description employs a minimal ideal-hadron-gas model matched to the MIT bag-model equation of state for the QGP. Such simplified equations of state typically yield a reduced transition temperature, and the effect is amplified at finite system volume, where the rounding of the transition further broadens and shifts the location of the susceptibility peak. Consistently, when we evaluate the $q \longrightarrow 1$ limit at large volume, the model reproduces the standard MIT bag-model value of T_C. These considerations indicate that the reduced values of T_C in the present work originate from the effective model and the finite-volume treatment rather than from the q-deformation itself. The objective of this study is therefore not to predict an absolute transition temperature, but rather to analyze the relative response of thermodynamic observables to small variations in the deformation parameter q.

In Figure 3, we represent variations of order parameter h (top), the mean value of the entropy density normalized s/T^3 (middle), and the mean value of the energy density normalized ε/T^4 (bottom) with temperature, for different values of the deformed parameter q and different system volumes. We indicate that we have represented the variations of these quantities for a volume ($V = 150\ fm^3$) for all considered values of the deformation parameter, but for $q = 1$ we also have represented these variations for a volume ($V = 1000\ fm^3$) to facilitate comparison of current results with previous works, where it appears from this figure that the above mentioned quantities when approaching the thermodynamic limit ($V = 1000\ fm^3$) and at the transition temperature $T_C(q)$, show a sharp phase transition and discontinuity, which reflects the presence of latent heat accompanying the phase transition, and this indicates that the phase transition is first-order. For a small volume ($V = 150 fm^3$), the phase transition is smoothed, as evidenced by the finite width of the specific heat peak in Figure 4. The peak value of the susceptibility χ^{max}At this volume is an order of magnitude lower than its extrapolated value in the thermodynamic limit $V \longrightarrow \infty$.

The deformation parameter q significantly enhances the normalized thermodynamic quantities. As shown in Figure 4 (middle and bottom), the peak values of s/T^3 and ε/T^4 increase systematically with q. For a volume of $V = \mathbf{1}50\text{fm}^3$, the peak value of ε/T^4 increases by over 25% when q is increased from 1.00 to 3.

It is important to emphasize that the QCD phase transition at zero chemical potential is widely accepted to be second order or a crossover, as demonstrated by lattice QCD simulations and effective field theory approaches [57]. Our q-deformed phase coexistence model reveals sharper transition features driven by the statistical deformation and finite volume, manifesting as enhanced latent heat and discontinuities. These behaviors reflect modifications induced by non-extensive statistics rather than a fundamental change to the transition’s order. The smoothing and rounding of thermodynamic quantities evident in finite volume and deformed statistics align qualitatively with crossover phenomena. The consensus on the second-order character of the QCD phase transition at zero chemical potential is not intended to be challenged or replaced by our model, although it offers new insights into how q-deformation influences thermodynamic transitions and finite-size effects. These subtleties may be further clarified in future work with improved numerical precision and more precise parameter choices.

Figure 3 Variations of order parameter (top), the mean value of the entropy density normalized by T^3 (middle) and the mean value of the energy density normalized by T^4 (bottom) with temperature, for different values of the deformed parameter q and different system volumes.

In Figure 4, we represent the variations of the specific heat density c(top) and the susceptibility χ(bottom) with respect to the temperature, for different values of the deformed parameter q and at system volume $V = 150\ fm^3$. It is clear from these two figures that the expected infinite Dirac delta function at the thermodynamic limit "rounds" at the transition temperatures to peaks with a considerable width, and this is normal for our case because we chose a small volume for the system (V=150 fm^3).

The magnitudes of the peaks in both specific heat and susceptibility are strongly enhanced by the deformation parameter. As quantified in Figure 4, the maximum value of the susceptibility χ^{max} for $q = 3$ is more than 40% the value for $q = 1$, indicating a significant sharpening of the transition response with increased deformation.

Figure 4 Variations of the specific heat density (top) and the susceptibility (bottom) with temperature, for different values of the deformed parameter q and for $V = 150\ fm^3$.

As is known in hydrodynamics, the representation of thermodynamic quantities variations is often in terms of the average value of energy density $< \varepsilon >$ rather than temperature [60], so in Figure 5 we have represented the average pressure variation $< P >$ (top), the ratio $< P > / < \varepsilon >$ (middle) and the square of the speed of sound variations c_s^2 with respect to $< \varepsilon >$ in terms of this for different deformation parameter q.

Figure 5 Variations of the mean value of pressure $< P >$ (top), the ratio $< P >/< \varepsilon >$ (middle) and the speed of sound (bottom) with energy density, for different values of the deformed parameter q and for volume $V = 150 fm^3$.

We note that the pressure increases with increasing $< \varepsilon >$ in both the hadronic and plasma phases, while in the mixed HG-QGP region, the pressure increases very slowly with increasing energy density, and we also note that it increases with increasing q in the hadronic and mixed phases, but it appears that $< P >$ is not related to the deformation parameter in the plasma phase. As for the ratio $< P >/< \varepsilon >$, it increases slowly in both the hadronic and mixed phases with increasing $< \epsilon >$, and q, and also $< P >/< \varepsilon >$ contradicts the value 1/3 expected for a relativistic ideal gas, while in the mixed phase the ratio $< P > /< \varepsilon >$ takes its minimum value and this value corresponds to the softest point for the equation of state. In the high-temperature plasma phase $T > T_C(q)$, the thermodynamic quantities $< P >$, $< P >/< \varepsilon >$, and ${c_s}^2$ converge to values independent of q, as demonstrated in Figures 5 and 6. For all calculated values of q, ${c_s}^2$ asymptotically approaches 1/3, the value for a relativistic ideal gas.

The last observations deduced from the lower graph of Figure 5 can also be supported by representing in Figure 6 the variations of the square of the speed with respect to temperature for different values of the deformation parameter q and for a volume $V = 150fm^3$, as this graph also shows that the values of ${c_s}^2$ contradict the value 1/3 in the hadronic phase, and this is explained according to the works [68-72] by the small size of the system and not by the effect of the deformation parameter q.

Figure 6 Variations of the speed of sound with temperature, for different values of the deformed parameter q and for volume $V = 150fm^3$.

Before discussing the effect of the deformation parameter q it is useful to verify the undeformed, limit $q \longrightarrow 1$. Setting $q = 1$ in our expressions restores the standard Bose-Einstein and Fermi-Dirac distributions and reproduces the familiar MIT bag-model thermodynamics. We plot in Figure 7 the temperature dependence of the order parameter h for three values of the bag constant, $B^{1/4} = 145, 200, 220\ MeV$, at fixed deformation parameter $q = 1$ and volume $V\ = 1000\ \mathrm{fm}^3$ For each choice of B, the order parameter stays close to unity at low temperature, indicating a dominantly confined phase, and then drops very sharply to nearly zero in a narrow temperature interval, signaling the deconfinement transition. As $B^{1/4}$ increases from 145 to $220\ MeV$, the position of the steep fall of $\langle h \rangle$ moves systematically to higher temperatures, from around $T{\sim}104\mathrm{MeV}$ for $B^{1/4} = 145\ MeV$, which agree with the result found in [68] to about $T{\sim}160\mathrm{MeV}$ for $B^{1/4} = 220\mathrm{MeV}$ in agreement with the known MIT bag-model prediction and consistent with lattice-QCD crossover values at $\mu = 0$ [69]. This behavior reflects the fact that a larger bag constant makes the quark phase energetically more costly, so that a higher temperature is required for deconfinement to become favorable in the model.

Figure 7 The temperature dependence of the order parameter h for three values of the bag constant, $B^{1/4} = 145, 200, 220\ \mathrm{MeV}$, at fixed deformation parameter $q = 1$and volume $V\ = 1000\ \mathrm{fm}^3$.

To quantify the influence of finite system volume on the character of the transition, we compute the susceptibility $\chi(T;V)$ for several volumes and extract two finite-size indicators: (i) the transition width $\delta T(V)$, defined as the full width at half maximum of $\chi(T;V)$, and (ii) the susceptibility peak height $\chi_{\max}(V)$. The results show that the rounding of the transition follows the expected finite-size scaling behaviour for a first-order transition, namely $\delta T(V) \propto V^{-1}$ and $\chi_{\max}(V) \propto V$. This trend is fully consistent with finite-volume analyses in phenomenological models such as those of Spieles et al. (1998) and Ladrem et al. (2005) [44, 45], and with finite-volume expectations from lattice QCD. Figures 8 and 9 display the evolution of the transition profile with V, Figure 8 shows the plot of δT versus $1/V$ and Figure 9 presents the linear plot of $\chi_{\max}(V)$, both of which confirm the anticipated scaling and illustrate the continuous sharpening of the transition as the thermodynamic limit is approached.

Figure 8 The transition width δT variations with the volume inverse $1/V$.

Figure 9 Variations of the maxima of the specific heat density $\chi_{\max}(V)$with the volume.

The transition's sharpening with increasing q can be measured using a variety of thermodynamic indicators. First, the order parameter shows a larger discontinuity at the transition point for larger values of q, suggesting a more pronounced phase separation. Second, the energy density at T_C shows a larger jump, indicating an increase in latent heat. This is a common sign that first-order behavior has improved. Third, stronger fluctuations at the transition point are confirmed by the peaks in the specific heat and susceptibility becoming narrower and higher as q increases. These findings collectively show quantitatively that the first-order like characteristics of the deconfinement transition within the coexistence model are amplified by q-deformation.

Although increasing the deformation parameter q enhances the first-order–like features of the transition (larger jumps in energy density, stronger discontinuities, and higher thermodynamic peaks), this behavior should not be confused with genuine critical-point phenomena. In particular, q-deformation does not generate divergent correlation lengths or scale-invariant fluctuations, which are necessary for altering universality classes. Instead, q acts here as an effective parameter that modifies the strength of quantum-statistical correlations and the stiffness of the equation of state. As a consequence, the sharpening induced by larger q values mimics aspects of a stronger first-order transition but does not correspond to the emergence of a critical point or to a change in universality class within our model.

The results of the phase transition in our model display notable sensitivity to variations in both the bag constant B and the system volume V.

- **Sensitivity to B:** The critical temperature T_C is significantly influenced by the bag constant B, which establishes the vacuum pressure differential between hadronic and quark-gluon plasma phases. The deconfinement transition becomes more challenging to accomplish thermodynamically as B tends to increase the vacuum pressure, which typically shifts T_C higher. Conversely, smaller B values lower T_C and can soften the phase transition. This dependence is well-known and consistent with MIT bag model predictions and other phenomenological approaches.
- **Sensitivity to volume V:** Because of finite-size effects, the system volume has a significant impact on how sharp the phase transition is. As thermal fluctuations become more significant, smaller volumes result in smoothing or rounding of the phase transition signatures, such as latent heat and peaks in specific heat or susceptibility. The transition becomes sharper as volume approaches the thermodynamic limit, getting closer to an ideal first-order transition with discontinuities in thermodynamic observables. This inverse relation between transition sharpness and volume is quantitatively confirmed in our results, with peak widths scaling approximately as $1/V$.

Together, variations in B set the baseline thermodynamic conditions for the transition, while volume effects control the manifestation and observability of sharp transition features. Hence, both parameters must be carefully chosen and analyzed to accurately model and interpret QCD phase transition phenomena in finite systems.

While the q-deformation parameter q introduces subtle modifications to the thermodynamic behavior near the phase transition, the magnitude of these effects is relatively small. Consequently, accurately determining q solely based on phase transition observables requires extremely precise calculations, a level of accuracy currently achievable only through lattice QCD simulations. Our study adopts a complementary theoretical approach, using the q-deformed statistics framework to qualitatively and quantitatively explore how non-extensive effects could influence phase transition features and finite-size behavior. Although our model does not provide a definitive extraction of q, it offers valuable insights into the possible signatures and impacts of statistical deformation, potentially guiding future high-precision numerical investigations.

Figure 10 illustrates the order parameter $h(T, V, q)$ as a function of temperature for different system volumes and for two values of the deformation parameter, $q = 1$ and $q = 2$, at fixed bag constant $B^{1/4} = 145 MeV$.

Figure 10 Variations of the order parameter h as a function of temperature for different system volumes and for two values of the deformation parameter, $q = 1$ (dashed lines) and $q = 2$ (solid lines), at fixed bag constant $B^{1/4} = 145 MeV$.

At low temperature the order parameter remains close to unity, indicating a dominantly confined (hadronic) phase, while it rapidly decreases toward zero in a narrow temperature interval around $T = 100 - 110\ MeV$, corresponding to the onset of the deconfined quark–gluon phase.

For a given value of q, the figure shows that increasing the system volume from $V = 25\ \mathrm{to}\ 1000\ \mathrm{fm}^3$ systematically sharpens the temperature dependence of h. In small volumes the transition is significantly rounded: the order parameter drops smoothly over a broad temperature range, reflecting enhanced thermal fluctuations and the absence of a true non-analyticity in finite systems. As the volume grows, the curves become increasingly step-like and the width of the transition region shrinks, consistent with the expected approach to the thermodynamic limit where a first-order or very rapid crossover transition is recovered.

Comparing dashed $q = 1$ and solid $q = 2$ curves at fixed volume shows that the deformation parameter significantly modifies both the location and sharpness of the transition. For $q = 2$, the system tends to stay longer in the ordered phase (larger h) as temperature increases, and then undergoes a more abrupt drop, implying a higher pseudo-critical temperature and a stronger first-order–like character relative to the undeformed case $q = 1$. This behavior is consistent with the general role of q-deformation in many-body systems, where non-linear higher-order correlations effectively stiffen the ordered phase and enhance discontinuities in the relevant order parameter.

5 Conclusion and outlook

In this work, we investigated the influence of the q-deformation parameter on the thermal deconfinement phase transition between a hadronic gas and a quark–gluon plasma within the phase coexistence model. By incorporating the algebraic q-deformed oscillator framework into the thermodynamics of both phases, we derived modified expressions for key quantities including pressure, entropy and energy densities, susceptibility, and specific heat, and analyzed their behavior through detailed numerical evaluations.

Our results show that increasing the deformation parameter q lowers the critical temperature T_C, strengthens quantum-statistical correlations, and systematically enhances thermodynamic quantities near the transition. The transition becomes sharper with larger q, as reflected in the amplified discontinuities of the order parameter and energy density, and the increased peak heights of susceptibility and specific heat. These features mimic the behavior of a stronger first-order transition, although they do not indicate a genuine change in universality class. Instead, q-deformation acts effectively to encode non-extensive and correlation-induced modifications to the microscopic quantum structure of the system.

Finite-size effects were also analyzed, showing the expected rounding of the transition for small volumes and a recovery of sharp first-order–like features as the volume increases. The transition width and susceptibility peak height follow the anticipated finite-size scaling relations, confirming consistency with earlier phenomenological approaches.

While the absolute values of T_C differ from modern lattice-QCD determinations due to the simplified MIT bag-model equation of state used here, the purpose of the study is to reveal how deformation alters thermodynamic trends, rather than to reproduce physical transition temperatures. Our findings demonstrate that even modest deviations from standard quantum statistics can significantly influence the thermodynamics of strongly interacting matter.

In order to better constrain the physically relevant range of the deformation parameter, future work might incorporate more realistic equations of state, incorporate interactions beyond the ideal-gas approximation, or use lattice-based numerical inputs. All things considered, this work demonstrates the usefulness of q-deformed statistical mechanics as an additional framework for examining finite-size behavior, microscopic correlations, and non-extensive effects close to the QCD phase transition.

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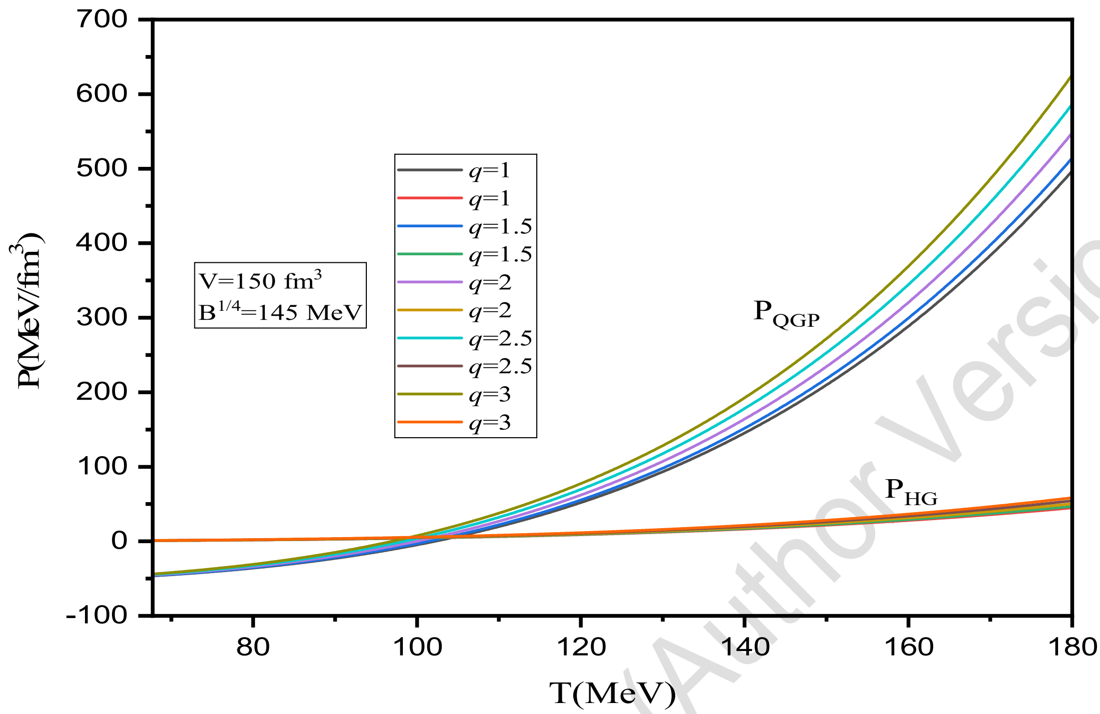


Figure 1 Pressure variation for hadronic and plasma phases as a function of temperature for different values of the deformed parameter q at zero chemical potential $\mu = 0$ and volume $V = 150\ MeV$.

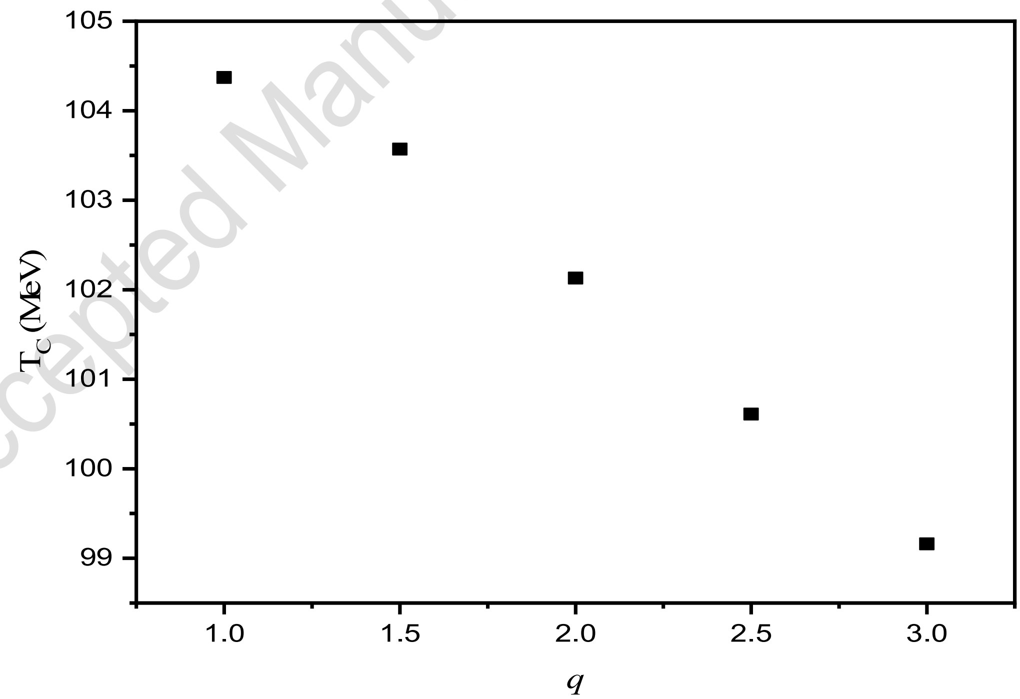
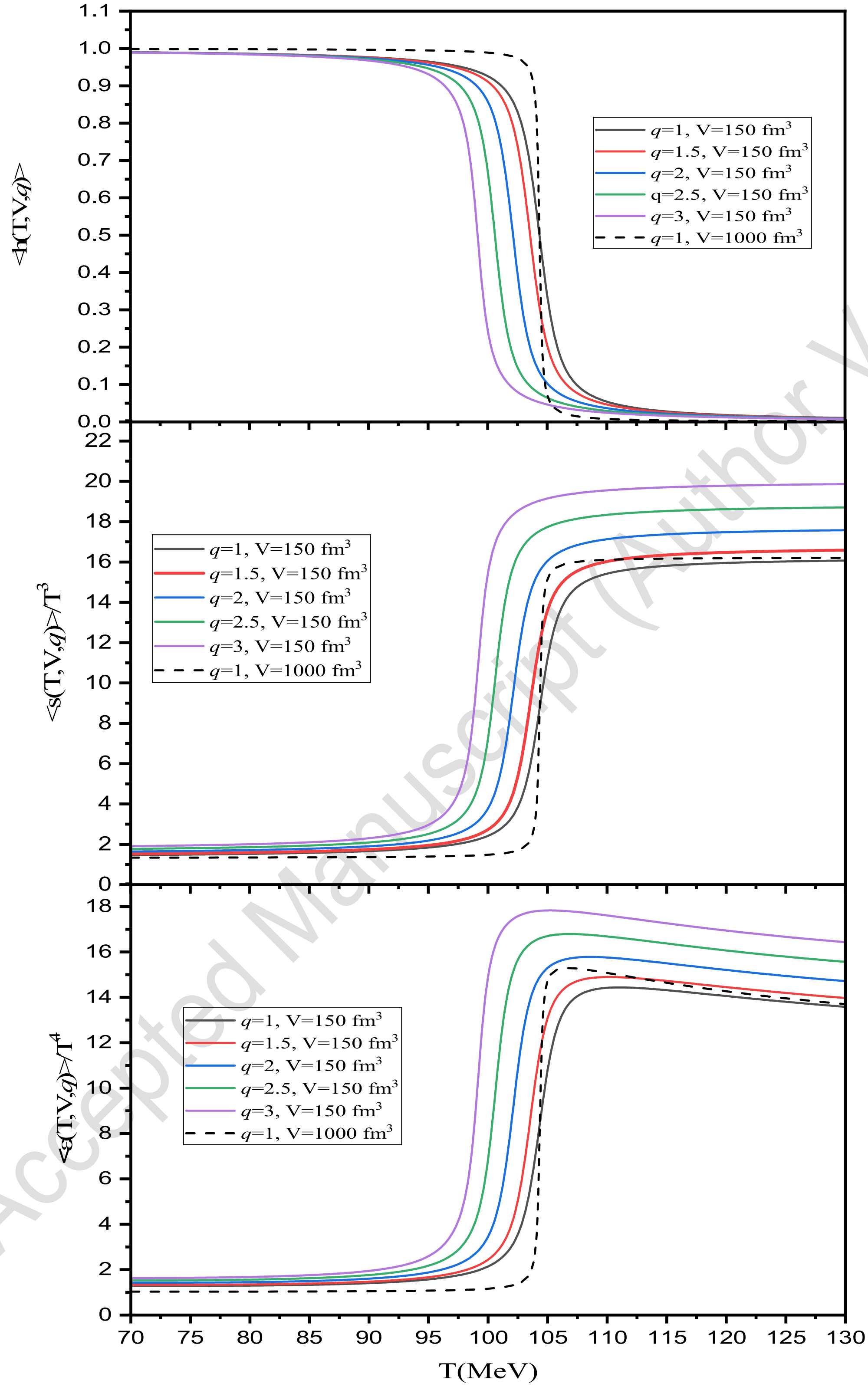


Figure 2 Critical temperature $T_C(q)$ variation as a function of the deformed parameter q.



<h(T,V,q)>
<s(T,V,q)>/T^3
<ε(T,V,q)>/T^4
q=1, V=150 fm^3
q=1.5, V=150 fm^3
q=2, V=150 fm^3
q=2.5, V=150 fm^3
q=3, V=150 fm^3
q=1, V=1000 fm^3
T(MeV)

Figure 3 Variations of order parameter (top), the mean value of the entropy density normalized by T^3 (middle) and the mean value of the energy density normalized by T^4 (bottom) with temperature, for different values of the deformed parameter q and different system volumes.

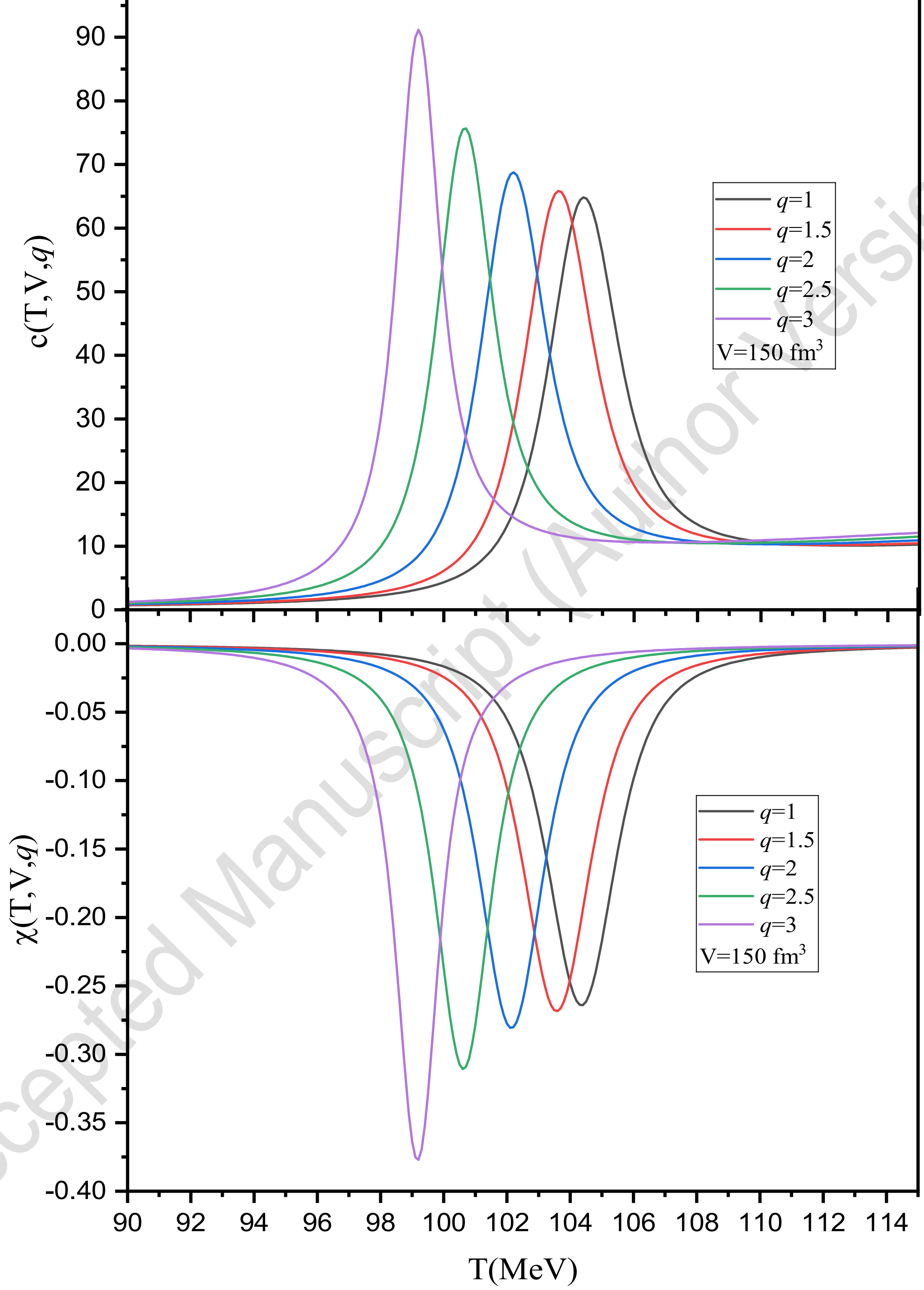


Figure 4 Variations of the specific heat density (top) and the susceptibility (bottom) with temperature, for different values of the deformed parameter q and for $V = 150\ fm^3$.

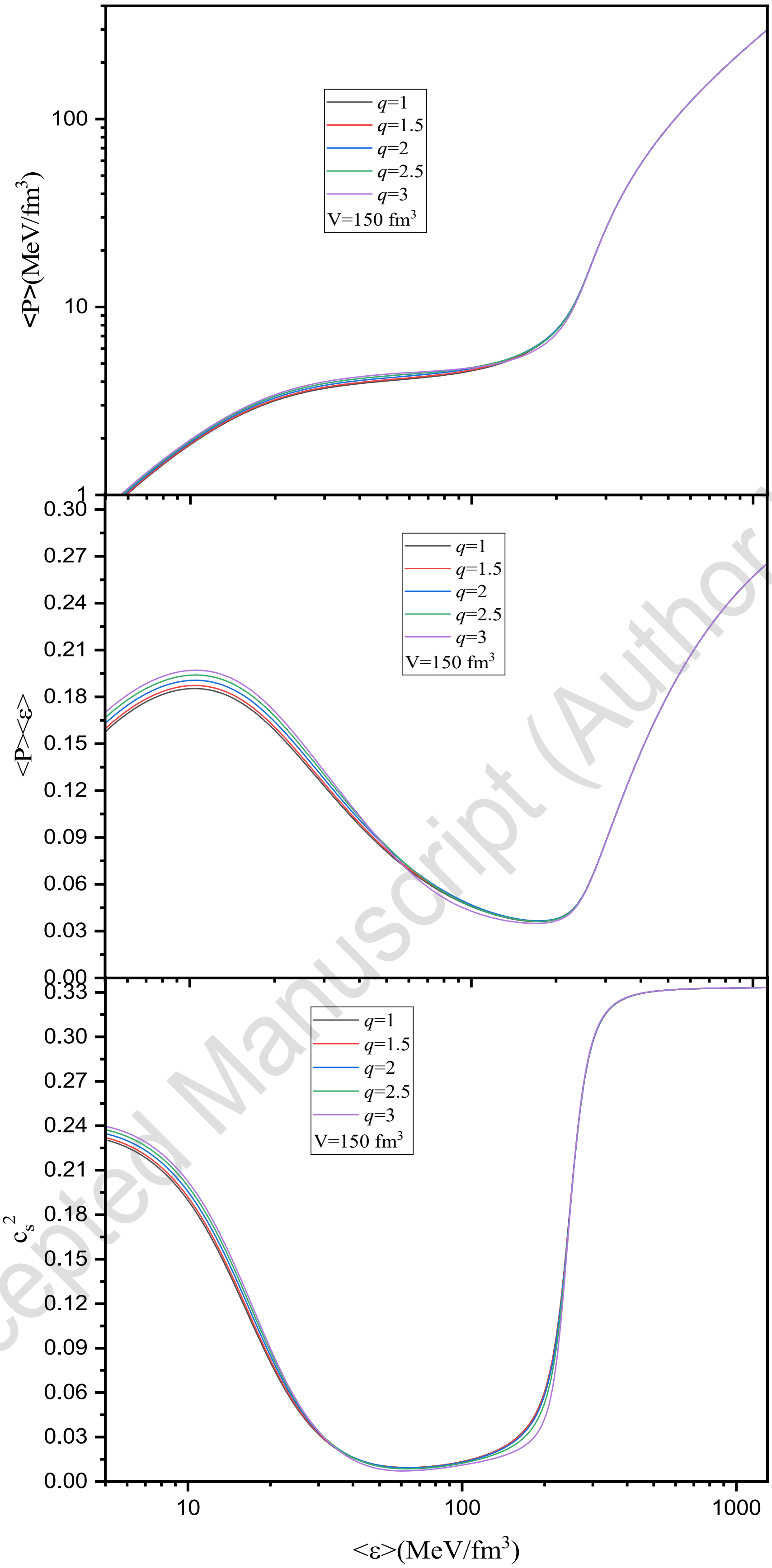


Figure 5 Variations of the mean value of pressure $< P >$ (top), the ratio $< P >/< \varepsilon >$ (middle) and the speed of sound (bottom) with energy density, for different values of the deformed parameter q and for volume $V = 150 fm^3$.

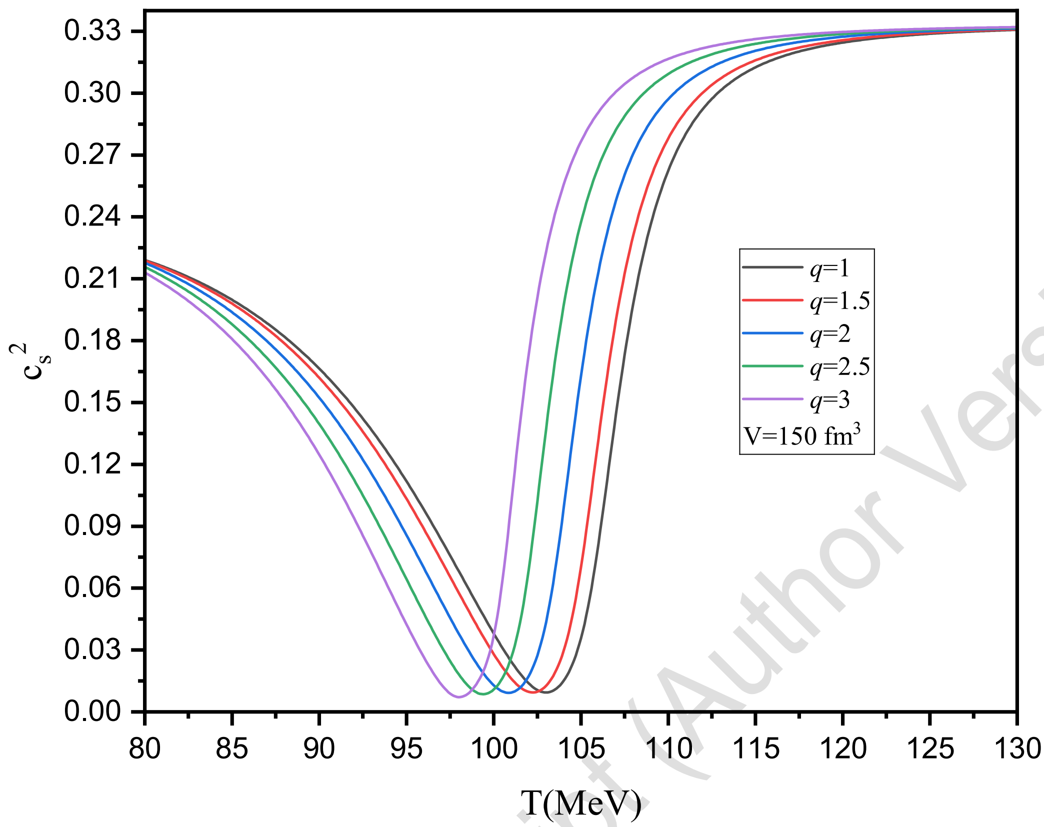


Figure 6 Variations of the speed of sound with temperature, for different values of the deformed parameter q and for volume $V = 150 fm^3$.

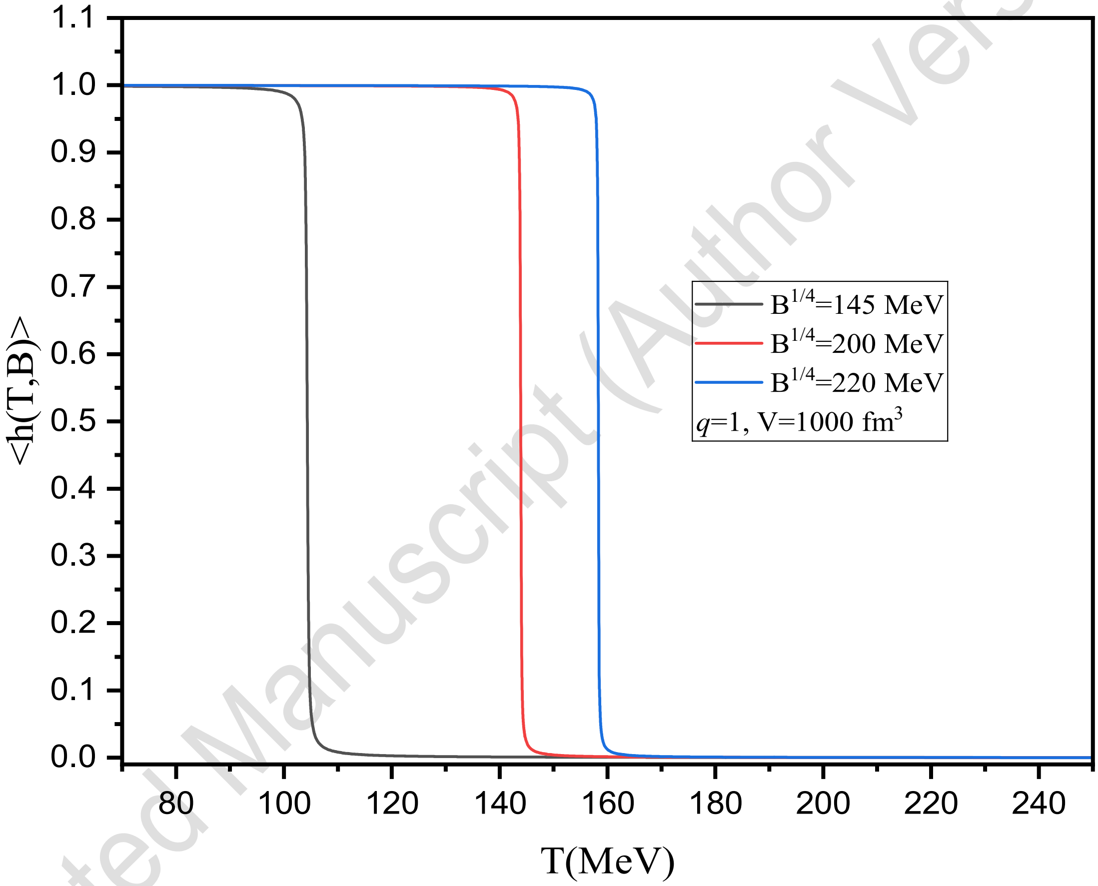


Figure 7 The temperature dependence of the order parameter h for three values of the bag constant, $B^{1/4} = 145, 200, 220\ MeV$, at fixed deformation parameter $q = 1$and volume $V\ = 1000\ \mathrm{fm}^3$

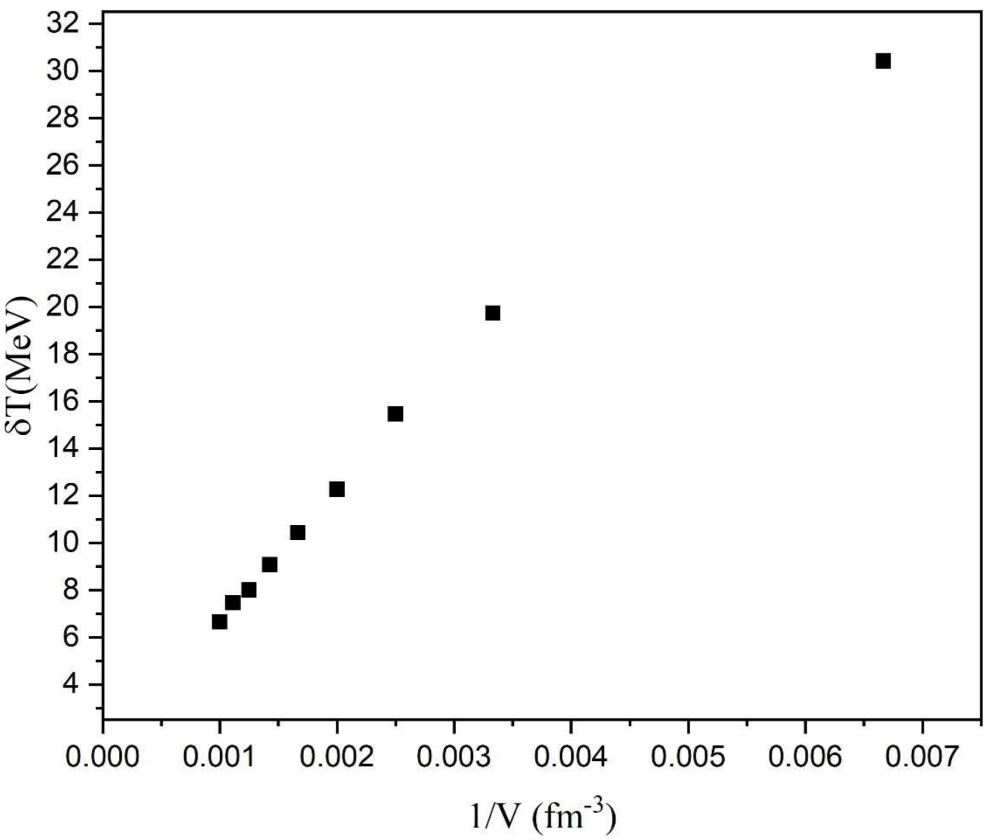


Figure 8 The transition width δT variations with the volume inverse $1/V$.

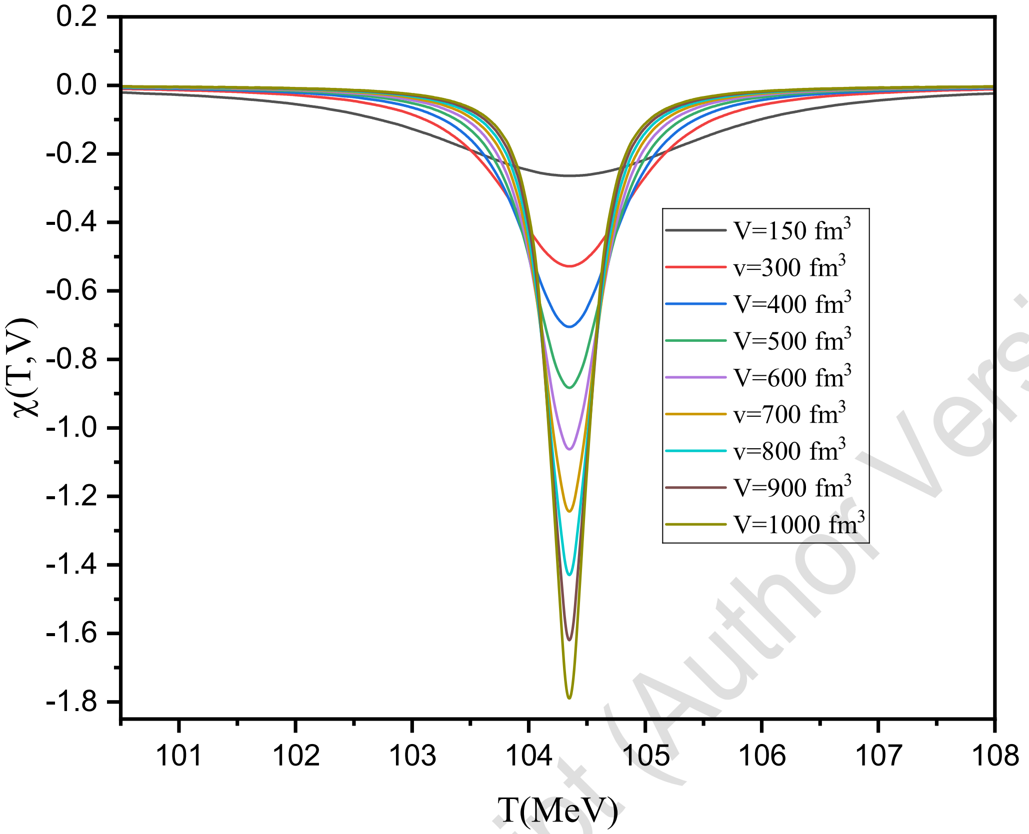


Figure 9 Variations of the maxima of the specific heat density $\chi_{\max}(V)$with the volume.

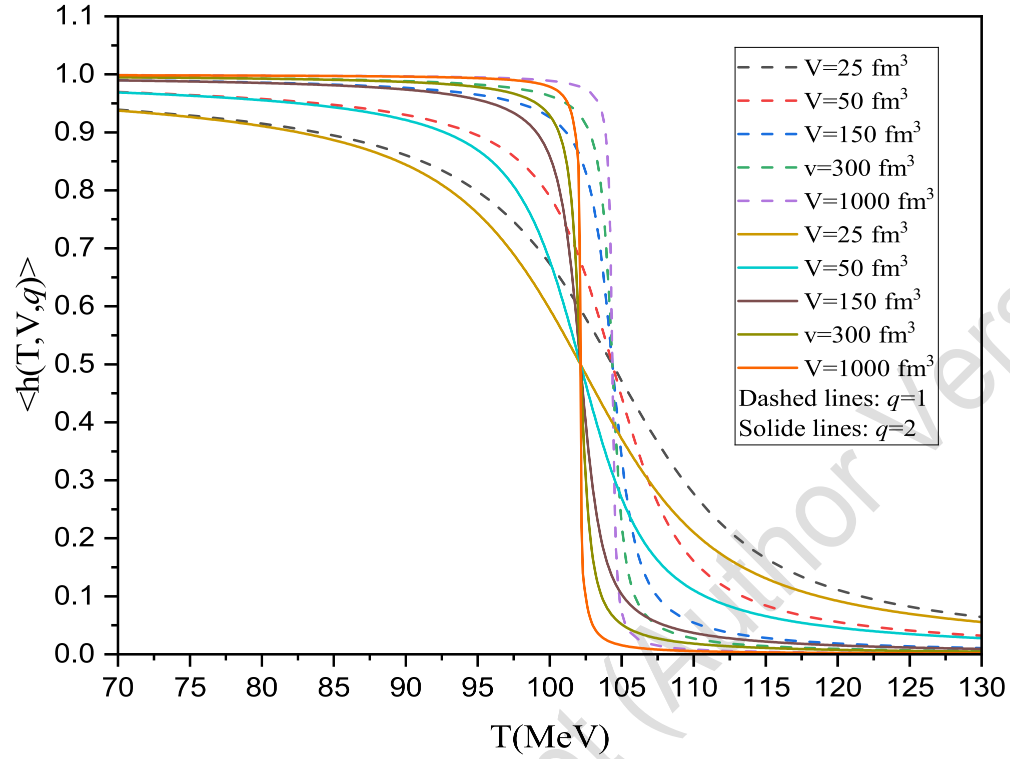


Figure 10 Variations of the order parameter h as a function of temperature for different system volumes and for two values of the deformation parameter, $q = 1$ (dashed lines) and $q = 2$ (solid lines), at fixed bag constant $B^{1/4} = 145MeV$

Declarations

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Competing Interests

"*The authors have no relevant financial or non-financial interests to disclose."*