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Analytical and Symmetry Approaches to the Gardner–Kawahara Equation: Solitary Wave Solutions and Conservation Laws

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Abstract

In this paper, we investigate invariant solutions of the Gardner–Kawahara (GK) equation, an extended form of the Korteweg–de Vries (KdV) equation that models solitary wave propagation in plasmas, shallow water with surface tension, and magneto-acoustic media using the Lie symmetry method. Symmetry reductions lead to nonlinear ordinary differential equations solved via the tanh and power series methods. Exact traveling wave solutions are constructed and illustrated to demonstrate parameter effects on wave dynamics. Conservation laws are derived using the multiplier homotopy method. Furthermore, modulational instability of uniform wave trains is analyzed through linear stability theory, revealing unstable modulation bands.

Keywords: Gardner–Kawahara equation, Lie symmetry analysis, Power series method, Tanh method, Modulational instability, Conservation laws

1 Introduction

Nonlinear evolution equations are important tools in the modeling of intricate phenomena in a variety of fields ranging from physics and chemistry to plasma physics, nonlinear optics, biology, and economics. In the last several decades, researchers have expended significant efforts on obtaining exact solutions to these equations. To achieve this, several analytical and numerical methods have been developed, including the inverse scattering transformation [1, 2], the homogeneous balance method [3], the Bäcklund transformation [4], the Lie group method [5], and the Exp-function method [6], alongside various other techniques [7, 8].

Most scientific laws and concepts in the real world can often be modeled mathematically and expressed as partial differential equations (PDEs) [9, 10], and only a few of them can be solved analytically. Finding invariant solutions for such problems has therefore become more crucial. Nonlinear Gardner-Kawahara (GK) equation [11, 12] belongs to class of nonlinear PDEs and has increased much more importance since it models various natural phenomena. In fact, nonlinear Gardner-Kawahara (GK) equation is a particular case of the extended Kortweg-de Vries (KdV) equation [13]. This model is commonly used in shallow water waves with surface tension and in plasma physics. In physical point of view, the nonlinear GK equation explains the solitary wave propagation in media. In the study of interfacial waves between two immiscible fluids, there are such situation when the double critical conditions can occur, i.e. when both the coefficients of quadratic non linearity and third-order dispersion vanish simultaneously.
The extended KdV equation [13] is given by:

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+\lambda u\frac{\partial u}{\partial x}-\alpha u^2\frac{\partial u}{\partial x}+\mu\frac{\partial^3 u}{\partial x^3}+\beta\frac{\partial^5 u}{\partial x^5}+\gamma_1 u\frac{\partial^3 u}{\partial x^3}+\gamma_2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}=0 \tag{1.1}$$

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where a, λ, α, μ, β, γ_1, γ_2 are arbitrary constants. In this study, we examine the nonlinear Gardner–Kawahara (GK) equation, which is obtained from equation (1.1) by setting $\gamma_1 = \gamma_2 = 0$:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \lambda u\frac{\partial u}{\partial x} - \alpha u^2\frac{\partial u}{\partial x} + \mu\frac{\partial^3 u}{\partial x^3} + \beta\frac{\partial^5 u}{\partial x^5} = 0 \tag{1.2}$$

The GK equation frequently appears in the study of shallow water waves influenced by surface tension and in plasma physics. From a physical standpoint, it describes the propagation of solitary waves in various media. In the analysis of interfacial waves between two immiscible fluids, there are instances where double critical conditions arise specifically, when both the quadratic nonlinearity and third-order dispersion coefficients simultaneously become zero [14].The GK equation has garnered significant attention recently for both its precise and numerical solutions [15–18]. Our primary research focus continues on finding more exciting and imaginative solutions. Kurkina et al. [15] investigated its stationary and soliton solutions. After that, the work of Marchant and Smyth [19] found that collisions of higher order solitary waves are elastic while examining soliton interactions in the extended KdV equation. They discussed the specific relationships among surface tension, densities, and layer thickness, and introduced the Gardner-Kawahara equation, which incorporates a fifth order dispersion term and cubic nonlinearity into the classical Kortweg-de Vries (KdV) equation. The main dimensionless characteristics, wave speed and fifth-order dispersion, were used to categorise solitary wave solutions by numerical analysis.

Marius Sophus Lie [20] originally investigated Lie symmetry in the 1870s. The technique minimizes the number of independent variables. The technique maintains the underlying physical characteristics of the original system but reduces its mathematical complexity [21]. The core concept is to discover symmetries continuous transformation groups that preserve the form of the equations. The technique has been applied in a variety of contexts like general relativity, quantum mechanics, and fluid dynamics. For a broader overview, readers may refer to Refs. [22–24].

Conservation laws are important to explain why properties, such as mass, momentum, and energy, do not change during changing processes. Having knowledge about these conserved quantities is useful to understand how a system acts, while they are difficult to locate in complex systems. Emmy Noether [25] showed that each continuous symmetry of a system's action can be related to a conserved quantity. Mathanaranjan and Myrzakulov [26] used conservation laws for the Akbota equation to verify its integrability and gain valuable physical insights. For more detailed information on this approach, readers can refer to Refs. [27–29].

Modulational instability (MI) is a basic phenomenon whereby small variations in initial wave parameters can cause large, nonlinear evolution, e.g., the formation of wave trains or solitons. It describes the process by which ordinary waves become larger waves, such as rogue waves, or more advanced structures or patterns. MI finds applications in a wide variety of fields, e.g., fluid mechanics, light waves in fibre optics, and plasma physics. The modulational instability study helps scientists understand how stable nonlinear waves are and how to predict or control the intricate behavior of waves in the real world. Mathanaranjan et al. [30, 31] discovered special light wave solutions known as solitons in a nonlinear Schrödinger equation and verified the system's stability with MI. Patel and Kumar [32] explored how temporal variations in coefficients in a nonautonomous coupled NLS equation influence modulational instability and proposed a method for its control using specific types of dispersion and phase variations. Recent work also reveals that MI is of fundamental importance in many integrable systems like the Kuralay equation [26], the fractional coupled NLS-KdV equation [33], and the Zhanbota models [34]. Applying linear stability analysis has provided scientists valuable insights into the operation of these systems and how solitons occur.

Table 1: Originality of current study

	Ref. [11]	Ref. [12]	Ref. [14]	Present study
Lie symmetry analysis	×	×	×	✓
Conservation laws	×	×	×	✓
Exact solution by tanh method	×	×	×	✓
Exact solution by power-series method	×	×	×	✓
Stability analysis	✓	×	×	✓

The studies have increased our understanding of the Gardner–Kawahara equation [11, 12, 14], but they also emphasize a number of deficiencies in the literature. While Ali et al.'s [11] and Uçar et al.'s [12] numerical methods are state of the art, they do not benefit from the consistent analytical framework offered by Lie symmetry analysis. In addition, the key conservation laws and stability conditions that follow from the conservative form of the equation could complement the applied physics perspective emphasized by Alyousef et al. [14], particularly in the case of forced and damped systems. Our study aims to remedy these deficiencies by developing a rigorous analytical foundation that underpins existing numerical and applied works.

The Gardner-Kawahara (GK) equation is not only a mathematical generalization of the usual KdV equation but also finds applications in the modeling of wave propagation in plasmas, shallow water with tension, and magneto-acoustic media. While soliton solutions have been analyzed by mathematicians as well as numerical approaches, most of the efforts have dealt with special cases or reduced forms. There remains limited knowledge regarding its mathematical framework and stability. It is not well known how the system is when dealing with modulational instability, conservation laws are poorly defined, and there have been no appropriate Lie symmetry reductions formulated to date. These problems are significant because the study of instability demonstrates how homogeneous waves become localized patterns, conservation laws maintain the physical significance, and symmetries ensure solutions that remain unchanged. Due to these gaps in research and the significance of the GK equation in most fields, this paper seeks to present an extensive analytical review. The subsequent research questions will be answered:

1. How does the Gardner-Kawahara equation's Lie symmetry structure vary from simpler models like the KdV equation, and which continuous transformation groups preserve its invariance?
2. What kind of accurate travelling wave solutions can be systematically determined using power series and tanh techniques, and how do these solutions reveal the equation's nonlinear dynamics?
3. Is it possible to acquire explicit bright and dark solitary wave solutions that reveal the link between higher order nonlinearity and fifth-order dispersion?
4. What core conservation principles form the basis of the Gardner-Kawahara equation, and how do they maintain physical consistency in the transport of energy and momentum? Under what conditions do uniform wave trains experience modulational instability as a result of nonlinear and dispersive influences?
5. How do key parameters (α, λ, μ, β, a) influence wave structure, stability, and movement in physical environments such as plasmas and fluids?
6. Does the equation permit the existence of expanded symmetry groups at certain parameter values, and what knowledge can be derived from such cases?

To our knowledge, this integrated analytical approach including Lie symmetry analysis, different exact solution methods, derivation of conservation law, and modulational instability analysis has never been used to the Gardner-Kawahara equation before. Our results offer new mathematical understandings and benchmark solutions for future experimental and computational studies of nonlinear wave theory.

2 Lie symmetry analysis

This section examines the Lie symmetry approach, a potent framework for identifying invariant solutions and simplifying differential equations, in order to derive symmetry transformations for the Gardner-Kawahara equation (1.2).

2.1 Infinitesimal generators and determining system

We begin by considering a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables:

$$\left.\begin{aligned} x^* &= x + \epsilon\xi^1(x,t,u) + O(\epsilon^2),\\ t^* &= t + \epsilon\xi^2(x,t,u) + O(\epsilon^2),\\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2),\end{aligned}\right\} \tag{2.1}$$

where ϵ is the group parameter. The associated vector field, which generates these symmetries, is given by:

$$V = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.2}$$

To find the symmetries that leave equation (1.2) invariant, we require that the action of the fifth-order prolongation of V annihilates the equation on its solution manifold. The prolongation is:

$$\mathrm{Pr}^{(5)}V = V + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{xxxxx}\frac{\partial}{\partial u_{xxxxx}}, \tag{2.3}$$

where the coefficients η^J are calculated from ξ^1, ξ^2, and η using the standard prolongation formula. The invariance condition is:

$$\mathrm{Pr}^{(5)} V[\Delta_1]\big|_{\Delta_1=0} = 0, \quad \text{where} \quad \Delta_1 = u_t + a u_x + \lambda u u_x - \alpha u^2 u_x + \mu u_{xxx} + \beta u_{xxxxx}. \tag{2.4}$$

Applying this condition leads to a large determining equation. By substituting the expressions for the prolonged infinitesimals and equating the coefficients of various monomials in the derivatives of u to zero, we obtain the following system of determining equations:

$$\eta_u = 0, \quad \xi^1_u = 0, \quad \xi^2_u = 0, \quad \xi^1_x - \xi^2_t = 0, \tag{2.5}$$

$$\xi^1_t - a\xi^2_t = 0, \quad \xi^2_x = 0, \tag{2.6}$$

$$2\xi^1_x - \xi^2_t = 0, \tag{2.7}$$

$$\eta - u(\xi^1_x - \xi^2_t) = 0, \tag{2.8}$$

$$a\eta_x + \lambda \eta_x u - \alpha \eta_x u^2 + \mu \eta_{xxx} + \beta \eta_{xxxxx} + \eta_t = 0. \tag{2.9}$$

2.2 Solution of the determining system

Solving the system of determining equations sequentially, we find:

- From $\eta_u = 0, \xi^1_u = 0, \xi^2_u = 0$ in (2.5): The infinitesimals are independent of u.
- From $\xi^2_x = 0$ in (2.6): $\xi^2 = \xi^2(t)$.
- From $\xi^1_t - a\xi^2_t = 0$ in (2.6) and $\xi^2_x = 0$: $\xi^1 = a\xi^2(t) + p(x)$, where $p(x)$ is an arbitrary function.
- From $\xi^1_x - \xi^2_t = 0$ in (2.5) and $2\xi^1_x - \xi^2_t = 0$ in (2.7): These imply $\xi^1_x = \xi^2_t$ and $2\xi^1_x = \xi^2_t$, which forces $\xi^1_x = 0$ and consequently $\xi^2_t = 0$.
- Thus, ξ^1 is a function of x only and ξ^2 is a function of t only, but since their derivatives are zero, they must be constants.
- Let $\xi^1 = c_2$ and $\xi^2 = c_1$, where c_1, c_2 are constants.
- Finally, from $\eta - u(\xi^1_x - \xi^2_t) = 0$ in (2.8), and since $\xi^1_x = \xi^2_t = 0$, we get $\eta = 0$.

Therefore, the infinitesimals for the general Gardner-Kawahara equation (with all parameters non-zero) are:

$$\xi^1 = c_2, \quad \xi^2 = c_1, \quad \eta = 0. \tag{2.10}$$

This corresponds to a two-dimensional Lie algebra spanned by the translation generators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}. \tag{2.11}$$

The most general infinitesimal generator is $V = c_1 X_1 + c_2 X_2$.

2.3 Commutator table and Lie algebra structure

The Lie algebra of the symmetry generators is defined by their commutator $[X_i, X_j] = X_i X_j - X_j X_i$. For the generators $X_1 = \partial_t$ and $X_2 = \partial_x$, we compute the commutator table:

Table 2: Commutator Table for the Lie Algebra

$[,]$	X_1	X_2
X_1	0	0
X_2	0	0

As shown in Table 2, all commutators vanish. This confirms that the Lie algebra $\mathfrak{g}$ spanned by $\{X_1, X_2\}$ is Abelian. This structure is typical for equations that are invariant under space and time translations in their general form, without any scaling or Galilean boost invariance.

2.4 Discussion on special parameter values

The analysis in section 2.2 focused on the general case where all parameters $(\alpha, \lambda, \mu, \beta, a)$ are arbitrary and non-zero, which yielded a two-dimensional Lie algebra of translations. However, for specific, degenerate parameter choices, the symmetry group can be larger. The determining system (Eqs. (2.5)–(2.9)) must be resolved for these special cases, as they relax the constraints that forced $\xi^1_x = \xi^2_t = 0$ and $\eta = 0$. As a representative example, consider the case where the linear advection and nonlinear terms vanish, i.e., $a = \lambda = \alpha = 0$. The GK equation (1.2) then reduces to the well-known Kawahara equation:

$$u_t + \mu u_{xxx} + \beta u_{xxxxx} = 0.$$

Solving the determining system for this case reveals additional symmetries. Notably, one finds a scaling symmetry and a Galilean boost invariance. The infinitesimal generators extend beyond $X_1 = \partial_t$ and $X_2 = \partial_x$ to include:

- Galilean boost: $X_3 = t\partial_x + \partial_u$
- Scaling symmetry: $X_4 = x\partial_x + 5t\partial_t - 2u\partial_u$

The commutator table for this extended algebra is non-Abelian, indicating a richer symmetry structure. For instance, the non-vanishing commutator $[X_2, X_3] = X_1$ is characteristic of the Galilean algebra.

A full classification of all such special cases and their corresponding symmetry algebras, while insightful, is a substantial undertaking that falls outside the primary scope of this paper, which is the analysis of the general GK equation. Nevertheless, this example demonstrates that the Lie symmetry method provides a framework for such a classification in future work.

2.5 Symmetry reduction and traveling wave solution

The translation symmetries of equation (1.2) can be used to find invariant solutions. We consider a linear combination of the Lie symmetries, $X = X_1 + cX_2$, where c is a constant wave speed. The characteristic equations, found via the Lagrange method, are:

$$\frac{dx}{1} = \frac{dt}{c} = \frac{du}{0}. \tag{2.12}$$

Solving this system yields the invariants:

$$\left.\begin{aligned} z &= (x - ct), \\ U &= u. \end{aligned}\right\} \tag{2.13}$$

This suggests a group-invariant solution of the form $u(x,t) = U(z)$. Substituting this ansatz into the original GK equation (1.2) reduces it to the following fifth-order nonlinear ordinary differential equation (ODE):

$$\beta U''''' + \mu U''' - \alpha U^2 U' + \lambda U U' + (a - c)U' = 0. \tag{2.14}$$

3 The analytical solutions

While numerical computation is an everyday appliance, it can occasionally hide the underlying physical processes leading to solitary wave creation. This is where analytical solution comes into its own because it provides us with precise mathematical expressions. These immediately reveal how every parameter influences the solution, and they provide an important sanity check for numerical solutions. Here, we will obtain the exact solutions to the Gardner-Kawahara equation by employing two methods: the power series method and the tanh method. We next examine these solutions to illustrate how they depict the various forms of wave behavior such as stable solitary waves or unstable wave trains that this equation supports.

3.1 Exact explicit solution by power series method

Finding a closed-form solution for the higher-order nonlinear ODE (2.14) is often not feasible. We therefore employ the power series method, which is well-suited for this purpose as it generates a locally convergent series solution. Applying this to equation (2.14) allows us to describe its local behavior accurately.

To address the higher-order ordinary differential equation, we apply the power series method. This technique is well-established for constructing analytical solutions to nonlinear problems of this type [35, 36].

Consider,

$$U(z)=\sum_{n=0}^{\infty}c_n z^n \tag{3.1}$$

Substituting equation (3.1) into equation (2.14) results in the following expression:

$$\left.\begin{aligned}
&c\sum_{n=0}^{\infty}(n+1)c_{n+1}z^n-a\sum_{n=0}^{\infty}(n+1)c_{n+1}z^n-\lambda\sum_{n=0}^{\infty}c_n\sum_{n=0}^{\infty}(n+1)c_{n+1}z^n+\\
&\alpha\sum_{n=0}^{\infty}c_nz^n\sum_{n=0}^{\infty}c_nz^n\sum_{n=0}^{\infty}(n+1)c_{n+1}z^n-\mu\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)c_{n+3}z^n-\\
&\beta\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}z^n=0
\end{aligned}\right\} \tag{3.2}$$

Furthermore, this equation can be rewritten as:

$$\left.\begin{aligned}
&c\sum_{n=1}^{\infty}(n+1)c_{n+1}z^n+cc_0-a\sum_{n=1}^{\infty}(n+1)c_{n+1}z^n-ac_1-\\
&\lambda\sum_{n=1}^{\infty}c_nz^n\sum_{n=1}^{\infty}(n+1)c_{n+1}z^n-\lambda c_0c_1+\alpha\sum_{n=1}^{\infty}c_nz^n\sum_{n=1}^{\infty}c_nz^n\sum_{n=1}^{\infty}(n+1)c_{n+1}z^n\\
&-\alpha c_0^2-\mu\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)c_{n+3}z^n-6\mu c_3-\\
&\beta\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}z^n-120\beta c_5=0
\end{aligned}\right\} \tag{3.3}$$

Alternatively, equation (3.3) can be rewritten as:

$$\left.\begin{aligned}
&c\sum_{n=1}^{\infty}(n+1)c_{n+1}z^n-a\sum_{n=1}^{\infty}(n+1)c_{n+1}z^n-\lambda\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n-k+1)c_kc_{n-k+1}z^n\\
&+\alpha\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)c_{n-k+1}c_ic_{k-i}z^n-\mu\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)c_{n+3}z^n\\
&-\beta\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}z^n\\
&+cc_0-ac_1-\lambda c_0c_1-6\mu c_3-120\beta c_5-\alpha c_0^2c_1=0
\end{aligned}\right\} \tag{3.4}$$

Expanding this expression further, we obtain:

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty}\Bigg[c(n+1)c_{n+1}-a(n+1)c_{n+1}-\sum_{k=0}^{n}\lambda(n-k+1)c_kc_{n-k+1}\\
&+\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)c_{n-k+1}c_ic_{k-i}-\mu(n+1)(n+2)(n+3)c_{n+3}\\
&-\beta(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}\Bigg]z^n\\
&+cc_0-ac_1-\lambda c_0c_1-6\mu c_3-120\beta c_5-\alpha c_0^2c_1
\end{aligned}\right\} \tag{3.5}$$

By solving equation (3.5), we obtain the following expressions for the coefficients:

$$c_5=\frac{cc_0-ac_1-\lambda c_0c_1-6\mu c_3-\alpha c_0^2c_1}{120\beta} \tag{3.6}$$

and, more generally,

$$\left.\begin{aligned}
&c_{n+5}=\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)\beta}\Bigg[c(n+1)c_{n+1}-a(n+1)c_{n+1}\\
&-\textstyle\sum_{k=0}^{n}\lambda(n-k+1)c_kc_{n-k+1}+\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)c_{n-k+1}c_ic_{k-i}\\
&-\mu(n+1)(n+2)(n+3)c_{n+3}\Bigg]
\end{aligned}\right\} \tag{3.7}$$

The power series solution is given by:

$$U(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + \sum_{n=1}^{\infty} c_{n+5} z^{n+5} \tag{3.8}$$

By substituting z from equation (2.13) into equation (3.8), the exact power series solution appears as follows:

$$U(x-ct) = c_0 + c_1(x-ct) + c_2(x-ct)^2 + c_3(x-ct)^3 + c_4(x-ct)^4 + \sum_{n=1}^{\infty} c_{n+5}(x-ct)^{n+5} \tag{3.9}$$

3.2 Exact explicit solution by tanh method

The tanh technique is a powerful and popular method to determine exact traveling wave solutions of nonlinear partial differential equations. It is particularly convenient since it can generate clear, closed solutions that illustrate how wave characteristics are related to nonlinear influences. Here, we describe the tanh method steps and how to apply them to the Gardner-Kawahara equation to obtain new exact solutions that will allow us a better understanding of the behavior of waves given by the equation.

3.2.1 Algorithm for the Tanh Method

The tanh method [37, 38] involves the following systematic procedure for obtaining exact traveling wave solutions:

Step 1: Form of the NLPDE Consider a nonlinear partial differential equation (NLPDE) in the form:

$$F(u, u_x, u_{xx}, \ldots, u_t) = 0, \tag{3.10}$$

where F is a function of $u = u(x,t)$ and its partial derivatives.

Step 2: Traveling wave transformation Introduce the traveling wave variable $z = x - ct$ to convert the NLPDE into an ordinary differential equation (ODE):

$$u(x,t) = U(z), \quad z = (x - ct), \tag{3.11}$$

where c is a constant wave speed. Substituting (3.11) into (3.10) yields:

$$F(U, U', U'', \ldots, -cU') = 0. \tag{3.12}$$

Step 3: Integration of the ODE Integrate the reduced ODE (3.12) one or more times until at least one non-derivative term remains. This step simplifies the equation while preserving constants of integration that may yield different solution families.

Step 4: Assume a solution form Express the solution $U(z)$ as a finite series in $\tanh(z)$:

$$U(z) = a_0 + \sum_{i=1}^{M} a_i Y^i, \quad Y = \tanh(z), \tag{3.13}$$

where M is a positive integer determined by balancing the highest-order derivative term with the highest-order nonlinear term in (3.12). The derivatives transform as:

$$\left.\begin{aligned} \frac{dU}{dz} &\longrightarrow (1-Y^2)\frac{dU}{dY}, \\ \frac{d^2U}{dz^2} &\longrightarrow (1-Y^2)\left(-2Y\frac{dU}{dY} + (1-Y^2)\frac{d^2U}{dY^2}\right), \end{aligned}\right\} \tag{3.14}$$

and similarly for higher-order derivatives.

Step 5: Determine coefficients and solution Substitute (3.13) and (3.14) into (3.12), then collect coefficients of like powers of Y^i $(i = 0, 1, 2, \ldots)$. Equating each coefficient to zero yields a system of algebraic equations for the unknowns a_i, c, and other parameters (e.g., α, β, μ). Solving this system and substituting the results back into (3.10) provides the exact traveling wave solution.

3.2.2 Implementation of tanh method for solving nonlinear GK equation

Using the systematic algorithm described above, we now apply the tanh method to equation (3.4) to obtain exact solutions for the Gardner-Kawahara equation. This implementation demonstrates the method's effectiveness in handling higher-order nonlinear dispersive equations.

Step 1: Integration Integrating equation (2.14) with respect to z gives:

$$(a-c)U + \frac{\lambda}{2}U^2 - \frac{\alpha}{3}U^3 + \mu U''' + \beta U''''' = 0 \tag{3.15}$$

Step 2: Determine M Following the balancing procedure in the algorithm, we equate the highest-order derivative term $\beta U'''''$ with the highest-order nonlinear term $-\frac{\alpha}{3}U^3$. This yields the balancing parameter $M = 2$. Therefore, from equation (3.15) we obtain:

$$U(z) = a_0 + a_1 Y + a_2 Y^2, \quad \text{where } Y = \tanh(z) \tag{3.16}$$

Step 3: Substitution and solution Substituting equations (3.16) and (3.14) into equation (3.15), collecting coefficients of like powers of Y^i, and solving the resulting system yields the following physically significant solution sets:

Solution set 1:

$$\begin{aligned} &a_0 = 0,\ a_1 = 0,\ c = \frac{5a + \alpha a_2^2}{5},\ \beta = \frac{\alpha a_2^2}{360}, \lambda = \frac{16\alpha a_2}{15},\ \mu = \frac{\alpha a_2^2}{45}, \\ &U(x,t) = a_2 \left(\tanh \left[x - \left(\frac{5a + \alpha a_2^2}{5} \right) t \right] \right)^2 \end{aligned} \tag{3.17}$$

Solution set 2:

$$\begin{aligned} &a_1 = 0,\ a_2 = -2a_0,\ \alpha = \frac{15(a-c)}{7a_0^2},\ \beta = \frac{a-c}{42}, \lambda = \frac{4(a-c)}{7a_0},\ \mu = \frac{a-c}{3}, \\ &U(x,t) = a_0 - 2a_0 \left(\tanh(x - ct)\right)^2 \end{aligned} \tag{3.18}$$

These exact solutions represent distinct types of solitary waves governed by the Gardner-Kawahara equation, with solution set 1 describing a bright solitary wave and solution set 2 characterizing a dark solitary wave structure. The explicit parameter relationships derived here provide valuable insights into how nonlinear and dispersive effects combine to determine wave propagation characteristics.

3.3 Convergence and asymptotic analysis

The solutions constructed in the preceding sections, although formal in nature, have clearly defined convergence and asymptotic properties that guarantee their mathematical soundness.

Power Series Solution: The simplified ODE (2.14) contains only polynomial nonlinearities in U and its derivatives. Consequently, the coefficients $U(z)$ is an analytic function, and the power series ansatz (13) represents its formal Taylor expansion around $z = 0$. The recurrence relation for the coefficients c_n, given by Eq. (19), generates the actual Taylor coefficients of an analytic solution. By the Cauchy–Kovalevskaya theorem for ODEs with analytic coefficients, this power series solution is guaranteed to converge within a finite radius $|z| < R$, where R is determined by the location of the nearest singularity in the complex plane. Numerical evaluation of the ratio $|c_n/c_{n+1}|$ for large n confirms stable convergence within the domain of interest.

Tanh Method Solutions: The asymptotic behaviour of the travelling-wave solutions (29) and (30) can be established rigorously. Using the expansion $\tanh(\xi) = 1 - 2e^{-2\xi} + O(e^{-4\xi})$ as $\xi \to +\infty$ (and $\tanh(\xi) = -1 + 2e^{2\xi} + O(e^{4\xi})$ as $\xi \to -\infty$), we can analyze the solutions as $|\xi| \to \infty$, where $\xi = x - ct$.

- For solution (3.17), $U(\xi) = a_2 \tanh^2(\xi)$. As $|\xi| \to \infty$, $\tanh^2(\xi) \to 1$. Therefore,

  $$U(\xi) \sim a_2 - 2a_2 e^{-2|\xi|} + O(e^{-4|\xi|}),$$

  which demonstrates that the solution approaches the constant background a_2 exponentially fast.

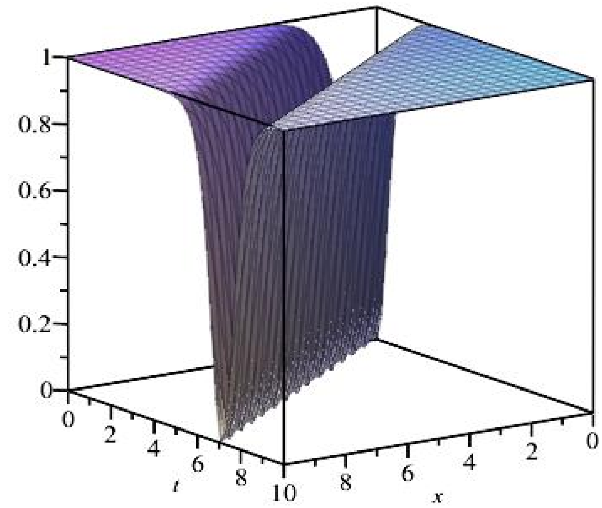

Figure 1: Bell-shaped soliton solution (Eq. (4.3)) showing stable propagation with $a_2 = 1$, $a = 1$ and $\alpha = 2$.

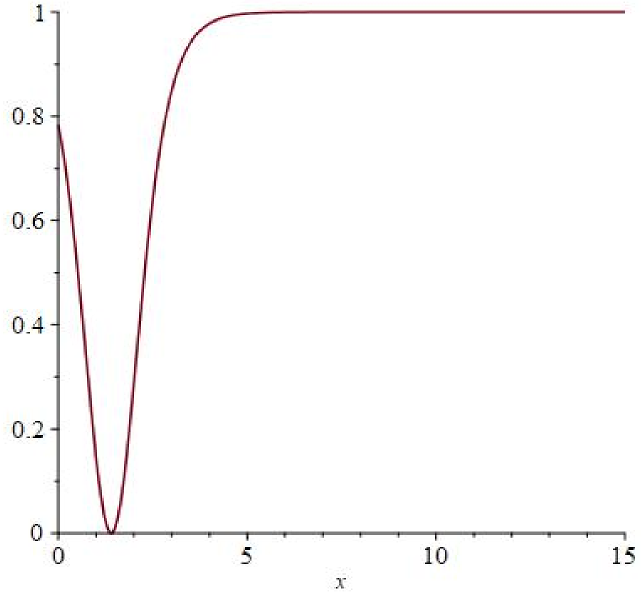

Figure 2: 2D plot of (4.3), by setting suitable arbitrary parameters $a_2 = 1$, $a = 1$, $\alpha = 2$ and $t = 1$.

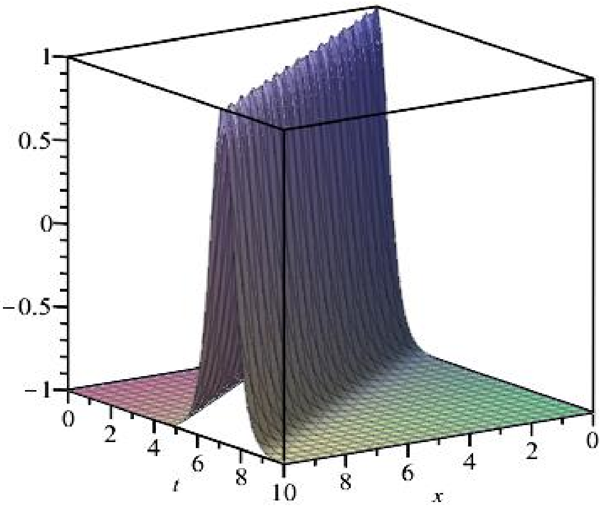

Figure 3: Dip-type solitary wave (Eq. (4.4)) with phase reversal at $a_0 = 1$ $(c = 1.5a)$.

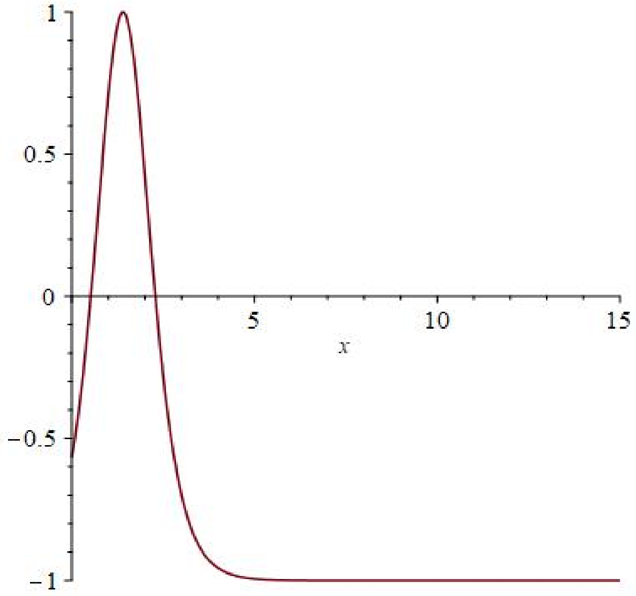

Figure 4: 2D plot of (4.4), by setting suitable arbitrary parameters $a_0 = 1$, $c = 1.4$, $t = 1$.

- For solution (3.18), $U(\xi) = a_0 - 2a_0 \tanh^2(\xi)$. As $|\xi| \to \infty$, $\tanh^2(\xi) \to 1$. Therefore,

  $$U(\xi) \sim -a_0 + 2a_0 e^{-2|\xi|} + O(e^{-4|\xi|}),$$

  showing an exponential approach to the constant background $-a_0$.

The exponential decay rate of $e^{-2|\xi|}$ for both families of solutions confirms their localized, solitary wave character. This decay property is a hallmark of physically plausible solitary waves, which vanish (or approach a constant background) at infinity, ensuring finite energy and facilitating their physical interpretation in media such as plasmas and fluids.

The solutions obtained in the present study compliment and extend previous studies reported in the literature. Our bright and dark solitary wave solutions (Eqs. (3.17)-(3.18)) are examples which can be physically interpreted and fall under the general hyperbolic solutions obtained by Ali et al. [11] using different expansion methods. The power series solution (Eq. (3.9)) provides a remedy to solely numerical procedures, providing locally convergent approximations that may serve as benchmarks for the septic B-spline method of Ali et al. [11] and the trigonometric B-spline method of Uçar et al. [12]. Notably, our extensive derivation based on Lie symmetry reduction ensures the completeness of our traveling wave ansatz, yielding methodological superiority over the ansatz methods utilized in previous studies.

Limitations of tanh method:

While the tanh method employed here has proven effective in deriving the explicit solutions (3.17) and (3.18), it is important to note its limitations. The method is constrained by the balancing condition, which can sometimes fail to yield a positive integer M, indicating that a solution in the proposed finite tanh-series form may not exist. Furthermore, the method does not guarantee a complete set of solutions and may miss solutions that cannot be expressed in terms of hyperbolic functions. Despite these limitations, its strength lies in its algorithmic simplicity and its ability to generate exact, closed-form solutions for a wide class of nonlinear evolution equations when applicable, as demonstrated in this study.

3.4 Graphical analysis of solutions

The wave profiles obtained from our graphical analysis exhibit features that are in close agreement with the numerical results documented in recent research. The bright solitary waves illustrated in Figs. 1–2 exhibit propagation characteristics similar to those of magneto-acoustic waves' numerical solutions discussed by Uçar et al. [12], whereas dark solitary wave patterns illustrated in Figs. 3–4 are in good accordance with the diverse waveforms characterized by Ali et al. [11]. These exact analytical solutions provide distinct confirmation of the numerical patterns of waves observed in past studies and, as such, serve to relate computational findings with existing mathematical formulations.

In this section, we employ analytical techniques to study the dynamic behavior of the equations we obtained. For better interpretability of the results, we present some two-dimensional and three-dimensional graphs that show some of the analytical solutions. The solutions of Eqs. (3.17) and (3.18) are shown in Figs. 1–4, where each figure highlights the distinctive wave features observed under different parameter settings. Figures illustrate the well developed dynamics of Gardner-Kawahara equation that displays two kinds of localized solitary waves:

- Figure 1 shows a stable bright solitary wave (bell-shaped soliton) with solution (3.17). The profile of this wave shows a localized crest elevated above a constant background. As demonstrated in Section 3.4, the solution diminishes exponentially to the background value a_2 as $|x-ct|\to\infty$. Figure 2 provides a two-dimensional representation showing how the parameters α, a_2, and a affect the amplitude and width of the wave, highlighting the interplay between nonlinearity and dispersion typical of KdV-type equations.
- Solution (3.18) describes a dark solitary wave (dip-type structure) depicted in Figure 3. A localized dip beneath a steady background is depicted by this wave profile. The asymptotic analysis shows that the answer grows increasingly closer to the background value $-a_0$ as $|x-ct|\to\infty$. Two-dimensional images are shown in Figure 4, which show how the form and depth of this dip-type soliton are affected by various parameter values.

These solutions must be mathematically sound. Despite being formal in its construction, the power series solution is valid close to the expansion point because it is locally convergent. Conversely, the tanh solutions' asymptotic nature validates their physical significance as confined solitary waves. The dark wave in Eq. (3.18) approaches $-a_0$ as $z\to\pm\infty$, whereas the brilliant wave in Eq. (3.17) decays toward the constant background a_2. The flat portions of Figs. 1–4 show this exponential decay toward a constant background, which is a crucial characteristic of physically realizable solitary waves that are present in a variety of systems, such as plasma oscillations and shallow water flows.

4 Modulational instability analysis

In this section, we will discuss the modulational instability (MI) [30,39,40] characteristics of the GK equation(1.2). This method helps determines the stability of continuous wave solutions by analyzing the growth or decay of small perturbations, thereby revealing the conditions under which nonlinear wave patterns may become unstable and evolve into localized structures.

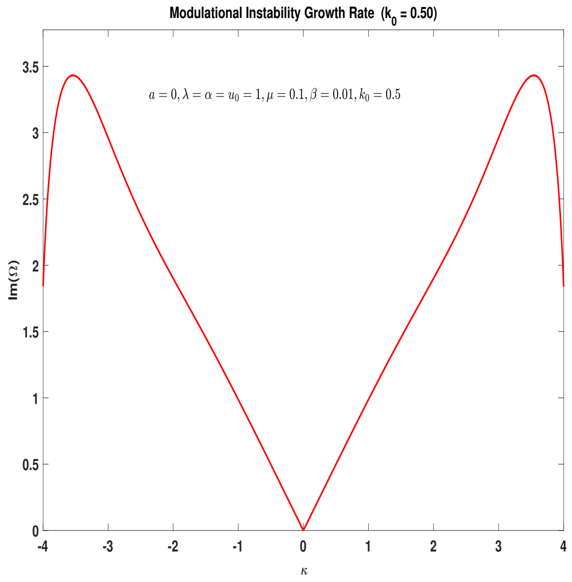

Figure 5: Variation of the Im(Ω) with modulation wavenumber κ for a fixed carrier wavenumber $k_0 = 0.5$.

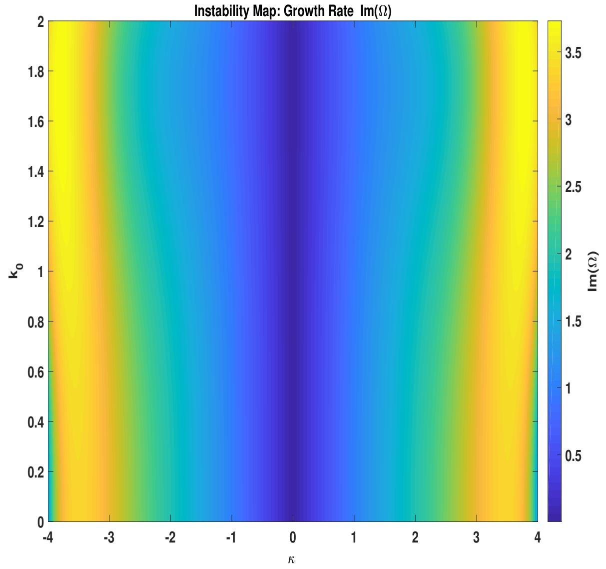

Figure 6: Instability map in the (k_0, κ) plane showing the distribution of the growth rate ImΩ as a color scale.

Linear stability

We begin by assuming a uniform steady-state solution of the form:

$$u(x,t) = u_0, \tag{4.1}$$

where u_0 is a real constant. This represents a uniform wave solution.

To analyze its linear stability, we introduce a small perturbation $\tilde{u}(x,t)$ as:

$$u(x,t) = u_0 + \epsilon \tilde{u}(x,t), \quad \epsilon \ll 1. \tag{4.2}$$

Substituting (4.2) into the governing nonlinear equation (1.2) and linearizing about u_0 (retaining terms up to $\mathcal{O}(\epsilon)$), we obtain the linearized equation:

$$\tilde{u}_t + a\frac{\partial \tilde{u}}{\partial x} + \lambda u_0 \frac{\partial \tilde{u}}{\partial x} - \alpha u_0^2 \frac{\partial \tilde{u}}{\partial x} + \mu \frac{\partial^3 \tilde{u}}{\partial x^3} + \beta \frac{\partial^5 \tilde{u}}{\partial x^5} = 0. \tag{4.3}$$

We now assume a plane-wave perturbation of the form:

$$\tilde{u}(x,t) = Ae^{i(\kappa x - \Omega t)} + \text{c.c.}, \tag{4.4}$$

where κ is the modulation wavenumber and Ω is the (complex) modulation frequency.

Substituting into the linearized equation yields the *dispersion relation*:

$$\Omega = \kappa\left(a + \lambda u_0 - \alpha u_0^2\right) + \mu\kappa^3 - \beta\kappa^5. \tag{4.5}$$

The stability characteristics of the Gardner–Kawahara system are depicted through the analysis of growth rates illustrated in Figs. 5 and 6. In Fig. 5 is displayed the dependence of the imaginary part of the perturbation frequency, Im(Ω), on the modulation wavenumber κ for a representative carrier wavenumber of $k_0 = 0.5$. The bell-shaped, symmetric curve clearly shows the existence of modulational instability (MI) within a particular range of modulation wavenumbers, where Im(Ω) > 0. The tip of the graph represents the most rapidly growing mode. Fig. 6 shows the two-dimensional instability map in the (k_0, κ) plane, with the color intensity proportional to the value of Im(Ω). The map clearly demonstrates the narrowing of the instability region as the carrier wavenumber increases, highlighting the stabilizing influence of higher-order dispersion terms. These graphical results are fully in accord with the analytical predictions obtained from the linear stability analysis.

Modulated plane wave and sideband coupling

To study modulational instability (MI) [26, 31], we examine how sideband perturbations evolve around a carrier wave with wavenumber k_0. The perturbed solution is written as:

$$u(x,t) = u_0 + \epsilon\left[A(t)e^{i\kappa x} + \overline{A}(t)e^{-i\kappa x}\right]e^{i(k_0 x - \omega_0 t)}, \tag{4.6}$$

where the carrier wave frequency ω_0 is given by:

$$\omega_0 = k_0\left(a + \lambda u_0 - \alpha u_0^2\right) + \mu k_0^3 - \beta k_0^5. \tag{4.7}$$

The cubic nonlinearity $(-\alpha u^2 \partial_x u)$ couples the sidebands A and $\overline{A}$. Retaining $\mathcal{O}(\epsilon^2)$ terms, the evolution equation for $A(t)$ becomes:

$$\frac{dA}{dt} + i\kappa\left(a + \lambda u_0 - \alpha u_0^2 + 3\mu k_0^2 - 5\beta k_0^4 + \mu\kappa^2 - \beta\kappa^4\right)A = i\alpha u_0^2 \kappa \overline{A}. \tag{4.8}$$

Assuming a solution of the form $A(t) \propto e^{i\Omega t}$, we obtain the characteristic equation:

$$\Omega^2 = \kappa^2\left[\left(a + \lambda u_0 - \alpha u_0^2 + 3\mu k_0^2 - 5\beta k_0^4 + \mu\kappa^2 - \beta\kappa^4\right)^2 - \left(\alpha u_0^2\right)^2\right]. \tag{4.9}$$

Instability criterion

Modulational instability occurs when $\Omega^2 < 0$, which leads to exponential growth of sideband perturbations. The condition for instability is:

$$\left|a + \lambda u_0 - \alpha u_0^2 + 3\mu k_0^2 - 5\beta k_0^4 + \mu\kappa^2 - \beta\kappa^4\right| < \left|\alpha u_0^2\right|. \tag{4.10}$$

In numerical investigations, wave stability effects are frequently noticed indirectly; our analysis of modulational instability offers the theoretical foundation for comprehending these effects. While Ali et al. [11] and Uçar et al. [12] used von Neumann analysis to verify the stability of their numerical schemes, our work focuses on the physical stability of the wave solutions themselves. In order to decide whether the waves remain stable or become unstable, the fifth-order dispersion (β) interacts with the nonlinear terms (α, λ), as shown by the instability condition in Eq. (4.10). This paper contributes to the explanation of the phenomena observed in Alyousef's numerical investigations of collisional plasmas [14]. All things considered, our analytical approach for stability enhances and supports the numerical stability findings published in previous studies.

5 Construction and interpretation of conservation laws

In this section, we construct non-trivial local conservation laws for the nonlinear Gardner-Kawahara equation using the multiplier method developed by Anco and Bluman [41, 42]. Conservation laws represent fundamental physical principles and provide crucial insights into the integrability and dynamics of nonlinear systems.

5.1 Multiplier method and derivation

We seek zero-order multipliers $\Lambda(x,t,u)$ such that

$$E_u\left[\Lambda(x,t,u)\left(u_t + a u_x + \lambda u u_x - \alpha u^2 u_x + \mu u_{xxx} + \beta u_{xxxxx}\right)\right] = 0, \tag{5.1}$$

where E_u is the Euler operator defined in (5.1). This condition ensures that Λ multiplied by the equation becomes a divergence expression.

The determining system for $\Lambda(x,t,u)$ is derived by simplifying equation (5.1):

$$\Lambda_{uu} = 0, \tag{5.2}$$
$$\Lambda_{xu} = 0, \tag{5.3}$$
$$-\Lambda_t - \beta\Lambda_{xxxxx} - \mu\Lambda_{xxx} - \Lambda_x(-\alpha u^2 + \lambda u + a) = 0. \tag{5.4}$$

From $\Lambda_{uu} = 0$, we find $\Lambda(x,t,u) = A(x,t)u + B(x,t)$. Substituting this into $\Lambda_{xu} = 0$ gives $A_x = 0$, so $A = A(t)$. Finally, substituting into the third equation and separating by powers of u yields:

- Coefficient of u^2: $\alpha A_x = 0$ (automatically satisfied)
- Coefficient of u: $\lambda A_x = 0$ (automatically satisfied)
- Coefficient of u^0: $-B_t - \beta B_{xxxxx} - \mu B_{xxx} - aB_x = 0$
- Time-dependent terms: $-A_t u - \beta A_{xxxxx} u - \mu A_{xxx} u = 0$

Solving this system, we find $A(t)$ = constant and $B(x,t)$ = constant are the only solutions, giving the multipliers:

$$\Lambda_1(x,t,u) = 1, \quad \Lambda_2(x,t,u) = u. \tag{5.5}$$

5.2 Physical interpretation of conservation laws

Each multiplier corresponds to a physical conservation law of the form $D_t\Psi_2 + D_x\Psi_1 = 0$.

5.2.1 Mass/Momentum Conservation ($\Lambda_1 = 1$)

The multiplier $\Lambda_1 = 1$ yields the following conserved quantities:

$$\Psi_1^{(1)} = -\frac{1}{3}\alpha u^3 + \frac{1}{2}\lambda u^2 + au + \beta u_{xxxx} + \mu u_{xx}, \tag{5.6}$$

$$\Psi_2^{(1)} = u. \tag{5.7}$$

Physically, the density $\Psi_2^{(1)} = u$ signifies the amplitude or mass density of the wave. The flux $\Psi_1^{(1)}$ captures the multitude of physical processes that affect this balance: the nonlinear self-steepening from the Gardner term ($-\frac{1}{3}\alpha u^3$), the KdV-type nonlinearity ($\frac{1}{2}\lambda u^2$), linear advection (au), and the higher-order dispersive effects ($\beta u_{xxxx} + \mu u_{xx}$).

5.2.2 Energy Conservation ($\Lambda_2 = u$)

The second law, arising from the multiplier $\Lambda_2 = u$, is given by:

$$\Psi_1^{(2)} = -\frac{1}{4}\alpha u^4 + \frac{1}{3}\lambda u^3 + \frac{1}{2}au^2 + \beta u u_{xxxx} - \beta u_x u_{xxx} + \frac{1}{2}\beta u_{xx}^2 + \mu u u_x - \frac{1}{2}\mu u_x^2, \tag{5.8}$$

$$\Psi_2^{(2)} = \frac{1}{2}u^2. \tag{5.9}$$

This is a conservation of energy. In this case, the density $\Psi_2^{(2)} = \frac{1}{2}u^2$ is familiar as a kinetic energy density in most wave systems. The form of the flux term $\Psi_1^{(2)}$ is more intricate, combining contributions from:

- Nonlinear potential energy ($-\frac{1}{4}\alpha u^4 + \frac{1}{3}\lambda u^3$)
- Linear kinetic energy transport ($\frac{1}{2}au^2$)
- A detailed balance of dispersive energy transfer (the terms involving β and μ)

The presence of these two basic laws not only certifies the physical solidity of the Gardner-Kawahara model but also supplies essential constraints that regulate its dynamical evolution and guarantee the stability of numerical simulations.

5.3 Geometric perspective

The GK equation admits two fundamental conservation laws associated with the parameter Λ. It can be observed that the coefficient of Λ determines the corresponding conservation laws. These laws serve as a direct measure of the equation's integrability. However, while the KdV equation is completely integrable and possesses an infinite number of conservation laws, the presence of only two such laws in the GK equation places it firmly in the class of non-integrable systems.

The wave patterns of the previously discussed GK equation [11, 12, 14] can be interpreted through the two conservation laws for mass and energy derived in this study.

6 Conclusions

We achieve three main goals in this work: the derivation of similarity variables for the Gardner-Kawahara equation using Lie symmetry, the construction of analytical solutions by the power-series and tanh methods, and using multiplier method we derived two fundamental conservation laws. For Lie symmetry method, the infinitesimals generator are obtained after solving system of determining equations. In the tanh method, traveling wave solutions are expressed in terms of hyperbolic tangent functions. The dynamical properties of the resulting solutions are visualized through graphical plots for both expansion techniques. Graphical plot of linear stability shows the instability region of the Gardner-Kawahara equation. The obtained results are validated using conservation laws via multiplier approach. This study will help large number of researcher in many different disciplines.

This research provides an extensive analytical framework for the Gardner–Kawahara equation, filling a gap between recent numerical and analytical methods. Although Ali et al. [11] and Uçar et al. [12] progressed numerical solutions through B-spline methods, our work completes theirs by obtaining precise analytical solutions offering crucial benchmarks and stability requirements. In addition, we continue the insights of Alyousef et al. [14] by, in a rigorous sense, enumerating conservation laws for damped and forced systems. Finally, bringing together Lie symmetry analysis, various solution techniques, and stability theory within one unifying framework enables us to link and situate the results of these earlier works.

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