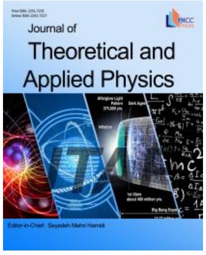

Volume 19, Issue 6, 192556 (1-6)

Journal of Theoretical and Applied Physics (JTAP)

https://doi.org/10.57647/j.jtap.2025.1906.56

Study of the Electric Quadrupole Moment of One-proton Halo Nucleus ^{8}B by Using the Two-body Model, the Collective Model and the Microscopic Single-particle Shell Model

Md. Mahbub Alam Mamun, Md. Sagor Mia, Tazul Islam*

Department of Physics, Mawlana Bhashani Science and Technology University, Santosh, Tangail, 1902, Bangladesh

***Corresponding author:** tazul@mbstu.ac.bd

Original Research

Received:
14 August 2025

Revised:
9 Septembre 2025

Accepted:
17 October-2025

Publish online:
31 December 2025

Abstract

The nuclear electric quadrupole moment of the proton-rich weakly bound Boron-8 (^{8}B) halo nucleus is investigated using three theoretical approaches: the microscopic single-particle shell model, the collective model, and the two-body model. These approaches provide a reasonable prediction of the quadrupole deformation that occurred due to the core polarization and many-body effects. These approaches also provide how the extended proton distribution influences the quadrupole moment. The experimental value of electric quadrupole moment (Q) of ^{8}B is 6.45 ± 0.14 e fm^{2}. While the calculated value using the two-body model is 6.86 e fm² and the collective model gives 5.83 e fm² and the microscopic single-particle shell model gives 4.03 e fm². The ^{8}B nucleus is a one-proton halo structure with a loosely bound valence proton outside a dense ^{7}Be core, and it is an excellent candidate for studying nuclear deformation and quadrupole effects near the proton drip-line. By comparing the predictions of these models with available experimental data, their effectiveness in describing ^{8}B's structure can be measured. These predictions reveal the structure of the halo configuration that influences the quadrupole properties of exotic nuclei.

Keywords: Nuclear electric quadrupole moment; Proton-rich; Weakly bound; Microscopic single-particle shell model; Collective model; Two-body model; Proton drip-line

1. Introduction

The study of nuclear structure provides deep information about the fundamental properties of atomic nuclei including their shapes, sizes, and electromagnetic characteristics. One such property is the electric quadrupole moment (Q), which is a crucial property of a nucleus particularly exotic nuclei such as ^{8}B which exhibits a halo structure. The electric quadrupole moment (Q) of ^{8}B, a proton-rich weakly bound halo nucleus, is treated as a fundamental property that reveals be determined using various models such as the shell model [2], [3], ab initio methods [4] (e.g., No-Core Shell Model [5] and Green's Function Monte Carlo method [6]), and cluster model [7] etc. These approaches provide critical details about its intrinsic deformation, charge distribution, and underlying nuclear structure. Therefore, ^{8}B exhibits a unique proton halo configuration where a single valence proton rotating around an inert ^{7}Be core at an unusually large distance due to its extremely low binding energy (~137 keV) [1]. This extended proton distribution leads to significant deviations from spherical symmetry, making the quadrupole moment a key parameter in characterizing the nucleus. Theoretically, the quadrupole moment arises from the contribution of the halo proton that can predictions that normally range between 3 and 10 e fm^{2}. These predictions must take into account for complex effects including core polarization, coupling to continuum states (due to the near-threshold binding of the proton) and two-body dynamics (^7Be + proton), all of which are the complicated theoretical descriptions of ^8B. Experimentally determining the electric quadrupole moment (Q) has been challenging due to the nucleus's short half-life (~770 ms) [1] and low production yields, but advanced techniques such as laser spectroscopy [8] (measuring hyperfine splitting), beta-detected nuclear magnetic resonance [9] (β-NMR), and Coulomb excitation provide predictions nearly close to 7 e fm^2 [10].

The value of the quadrupole moment has vast implications beyond structural nuclear physics, particularly in astrophysics, where ^8B plays a crucial role in the solar neutrino problem [11], [12]. As a key participant in the proton-proton chain reaction in the stars [13], its structure directly influences the high-energy solar neutrino flux making precise knowledge of its electromagnetic properties essential for accurate solar model predictions.

Now, discrepancies between different theoretical models and experimental measurements, though relatively small about 15% to 40%, highlight the need for further refinement possibly through next-generation facilities and advanced computational techniques such as continuum-coupled shell models [14] or lattice quantum chromodynamics (QCD) [15] for light nuclei. Future high-precision studies will not only enhance our understanding of ^8B's exotic structure but also improve constraints on nuclear interactions in weakly bound systems contributing to broader questions in nuclear astrophysics and the behavior of matter under extreme conditions.

Thus, the electric quadrupole moment of ^8B stands as a vital probe into the nature of halo nuclei bridging between nuclear theory, experimental works and astrophysical applications in a way that few other properties can do. The study of ^8B's quadrupole moment also offers valuable insights into the isospin dependence of nuclear forces as comparison with its mirror nucleus (^8Li) reveals small but important differences which are able to change the balance between proton and neutron interactions [16].

Such comparison tests our understanding of charge symmetry breaking in nuclear forces and provide constraints for refining nuclear interaction potentials. Furthermore, the system serves as a testing ground for exploring the limits of nuclear binding as ^8B sits very close to the proton drip line where the competition between nuclear attraction and Coulomb repulsion becomes particularly delicate. In this work, the electric quadrupole moment is computed using three theoretical approaches for studying the structure of this particular exotic nucleus near the proton drip line [17].

2. Methodology

2.1. Two-body model

The two-body model is particularly well-suited for halo nuclei like ^8B. It effectively captures the deformation due to the long-range spatial distribution of the valence nucleon, which significantly enhances the quadrupole moment. In this model, ^8B is treated as a composition of a dense ^7Be core and a valence proton,

$$^8B = {}^7Be + p \tag{1}$$

where, ^7Be is a tightly bound core (inert or slightly deformed), p is a valence proton loosely bound in a halo orbital.

This is a classic two-body (core + valence nucleon) halo nucleus system. For the calculation of the electric quadrupole moment, it's necessary to consider another term recoil quadrupole moment (Q_{recoil}), which arises from the interaction between the core and valence proton, such as polarization, recoil motion [18]. Because the valence proton orbits at a large distance, its motion induces a recoil of the ^7Be core. This recoil is not ignorable; it contributes an additional recoil quadrupole moment. Now, Quadrupole moment of ^8B is,

$$Q(^8B) = Q_{core} + Q_{valence} + Q_{recoil} \tag{2}$$

where, intrinsic quadrupole moment of ^7Be core [19], $Q_{core} = -5.3$ e fm^2 recoil quadrupole moment [18], Q_{recoil} = 18.2 e fm2. To determine $Q_{valence}$, which comes from the halo proton distribution using a microscopic formula for the quadrupole moment of a single valence nucleon in a shell [20] is,

$$Q_{valence} = -\frac{(2j-1)}{(2j+2)}\frac{j(2j-1)}{(2j+1)}\, e_p^{eff} < r^2 > \tag{3}$$

Here, the valence proton of ^8B is in the $p_{3/2}$ orbital [21], so total angular momentum, $j = \frac{3}{2}$. effective charge of proton [22], $e_p^{eff} = 1.12$ e. nuclear radius [23], $<r^2> = (4.24)^2 = 17.98$ fm^2

Putting these values in Eq. (2) results:

$$Q(^8B) = -5.3 - 6.04 + 18.2 = 6.86 \text{ e fm}^2 \tag{4}$$

Finally, the quadrupole moment is calculated from Eq. (1),

$$Q(^8B) = -5.3 - 6.04 + 18.2 = 6.86 \text{ e fm}^2 \tag{5}$$

2.2. Microscopic single-particle shell model

The microscopic single particle shell model is a fundamental framework in nuclear physics that is used to describe the structure of nuclei in terms of individual nucleons. This model assumes that the core is inert and only valence nucleons actively contribute to the properties such as magnetic and electric quadrupole moments [20].

The electric quadrupole moment operator is given by [20], $Q = \sum_{i=1}^{A} e_i\,(3z_i^2 - r_i^2)$. Then, the electric quadrupole moment for the valence proton of ^8B is,

$$Q = e < \psi_{val} | (3z^2 - r^2) | \psi_{val} > \tag{6}$$

To solve Eq. (6), consider the wave function as follows,

$$\psi_{val} = R_{nl}(r) Y_{lm}(\theta, \phi) \tag{7}$$

Here, $R_{nl}(r)$ is the radial part and $Y_{lm}(\theta,\phi)$ is the angular part. From Eq. (6), to determine the expectation value,

$$< \psi_{val}|(3z^2 - r^2)|\psi_{val} > = \int \psi^*_{val}(3z^2 - r^2)\psi_{val}\, d^3r \quad (8)$$

In Eq. (8), considering, z = rcosθ. Hence,

$$3z^2 - r^2 = 3r^2\cos^2\theta - r^2 = r^2(3\cos^2\theta - 1) \quad (9)$$

And, in spherical coordinates,

$$d^3r = r^2\sin\theta dr d\theta d\phi \quad (10)$$

Putting these values and also the value of the wave function in Eq. (8),

$$= \int_0^\infty R^2_{nl}(r)\, r^4 dr \int_0^\pi \int_0^{2\pi} Y^*_{lm}(\theta,\phi)(3\cos^2\theta - 1)Y_{lm}(\theta,\phi)\sin\theta d\theta d\phi$$

Therefore,

$$Q = e\int_0^\infty R^2_{nl}(r)\, r^4 dr \int_0^\pi \int_0^{2\pi} Y^*_{lm}(\theta,\phi)(3\cos^2\theta - 1)Y_{lm}(\theta,\phi)\sin\theta d\theta d\phi \quad (11)$$

Eq. (11) has two parts, which are the angular part and the radial part given below, respectively.

$$\int_0^\pi \int_0^{2\pi} Y^*_{lm}(\theta,\phi)(3\cos^2\theta - 1)Y_{lm}(\theta,\phi)\sin\theta d\theta d\phi \quad (12)$$

And,

$$\int_0^\infty R^2_{nl}(r)\, r^4 dr \quad (13)$$

To solve the angular part from Eq. (12),

$$\int_0^\pi \int_0^{2\pi} Y^*_{lm}(\theta,\phi)(3\cos^2\theta - 1)Y_{lm}(\theta,\phi)\sin\theta d\theta d\phi$$
$$\int_0^\pi \int_0^{2\pi} |Y_{10}(\theta,\phi)|^2(3\cos^2\theta - 1)\sin\theta d\theta d\phi \quad (14)$$

To use the explicit form of $Y_{10}(\theta, \phi)$, the spherical harmonic is given by [24],

$$Y_{lm}(\theta,\phi) = N_{lm}P_l^m(\cos\theta)e^{im\phi} \quad (15)$$

here, $P_l^m(\cos\theta)$ are associated legenre polynomials and, N_{lm} is a normalization constant.

Orbital angular momentum of valence proton of ^8B is, l = 1. magnetic quantum number of valence proton of ^8B is, m = 0.

Therefore, normalization constant [24],

$$N_{lm} = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} \quad (16)$$

So,

$$N_{10} = \sqrt{\frac{3}{4\pi}}$$

The Legendre polynomial [25], is given by,

$$P_1^0(x) = x$$
$$P_1^0(\cos\theta) = \cos\theta \quad (17)$$

And,

$$e^{im\phi} = e^0 = 1 \quad (18)$$

Put the values of Eqs. (16), (17) and (18) in Eq. (15),

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$
$$\Rightarrow |Y_{10}(\theta,\phi)|^2 = \frac{3}{4\pi}\cos^2\theta \quad (19)$$

From Eq. (14),

$$\int_0^{2\pi} d\phi \int_0^\pi (\frac{3}{4\pi}\cos^2\theta)(3\cos^2\theta - 1)\sin\theta\, d\theta$$
$$= \frac{3}{4\pi}\int_0^{2\pi} d\phi \int_0^\pi \cos^2\theta(3\cos^2\theta - 1)\sin\theta\, d\theta$$
$$= \frac{3}{4\pi} \times 2\pi \times \int_0^\pi \cos^2\theta(3\cos^2\theta - 1)\sin\theta\, d\theta \quad (20)$$
$$= \frac{3}{2}\int_0^\pi \cos^2\theta(3\cos^2\theta - 1)\sin\theta\, d\theta = \frac{4}{5}$$

Now, the radial part from Eq. (13),

$$\int_0^\infty R^2_{nl}(r)r^4\, dr = \int_0^\infty R^2_{11}(r)r^4\, dr \quad (21)$$

Since n = 1, l = 1 for the valence proton of ^8B.

Here, the quantum harmonic oscillator is given by [26]

$$R_{nl}(r) = N_{nl}.r^l.e^{\frac{-r^2}{2a^2}}$$
$$\Rightarrow R_{11}(r) = N_{11}.r^1.e^{\frac{-r^2}{2a^2}} \quad (22)$$

Put this value in Eq. (21) and then,

$$\int_0^\infty R_{11}^2(r)r^4\,dr = \int_0^\infty N_{11}^2 r^2 e^{-\frac{r^2}{a^2}} r^4\,dr \tag{23}$$

$$= \int_0^\infty N_{11}^2 r^6 e^{-\frac{r^2}{a^2}}\,dr$$

From the gamma function [24],

$$\int_0^\infty r^{2n} e^{-\lambda r^2} dr = \frac{1}{2}\lambda^{-(n+\frac{1}{2})}\Gamma(n+\frac{1}{2})$$

$$\int_0^\infty r^6 e^{-\frac{r^2}{a^2}} dr = \frac{1}{2}a^7\Gamma\left(3+\frac{1}{2}\right) \tag{24}$$

$$= \frac{1}{2}a^7\frac{15\sqrt{\pi}}{8}$$

From Eq. (23),

$$\int_0^\infty N_{11}^2 r^6 e^{-\frac{r^2}{a^2}} dr = N_{11}^2\frac{15\sqrt{\pi}}{16}a^7 \tag{25}$$

From the normalization of the radial wavefunction**,**

$$\int_0^\infty |R_{11}(r)|^2 r^2 dr = 1$$

$$\Rightarrow \int_0^\infty N_{11}^2 r^4 e^{\frac{-r^2}{a^2}} dr = 1$$

$$\Rightarrow N_{11}^2 \times \frac{1}{2}\times a^5\Gamma(2.5) = 1 \tag{26}$$

$$\Rightarrow N_{11}^2\frac{3\sqrt{\pi}}{8}a^5 = 1$$

$$\Rightarrow N_{11}^2 = \frac{8}{3\sqrt{\pi}.a^5}$$

Put this value in Eq. (25),

$$\int_0^\infty N_{11}^2 r^6 e^{-\frac{r^2}{a^2}} dr = \frac{8}{3\sqrt{\pi}.a^5}\cdot\frac{15\sqrt{\pi}}{16}a^7 = \frac{5}{2}a^2 \tag{27}$$

Since, the harmonic oscillator length scale [26],

$$a = \sqrt{\frac{\hbar}{m\omega}} \tag{28}$$

Here, the energy spacing for a 3D harmonic oscillator is given by [27],

$$\hbar\omega \approx 41A^{-\frac{1}{3}}\,\text{MeV} = 3.285\times 10^{-12}\text{ J}$$

$$\omega = \frac{\hbar\omega}{\hbar} = 3.114\times 10^{22}\text{ rad/s} \tag{29}$$

So, $a = 1.42$ fm.

From Eq. (27), $\frac{5}{2}a^2 = 5.041\text{ fm}^2$.

From Eq. (11), $Q = e \times 5.041 \times \frac{4}{5} = 4.03\text{ e fm}^2$.

2.3. Collective model

The collective model is also a theoretical framework in nuclear physics, and it's useful to describe deformation effects in many nuclei. According to this model quadrupole moment arises from the collective deformation and also from rotational and vibrational motions of the nucleus [28].

Now**,** Spectroscopic quadrupole moment [28],

$$Q = \frac{3K^2 - I(I+1)}{(I+1).(2I+3)} \times Q_0 \tag{30}$$

Here, Nuclear spin, I = 2, projection of I on the symmetry axis, K = 2

Therefore,

$$Q = \frac{6}{21}\times Q_0 \tag{31}$$

Intrinsic quadrupole moment is given by [28],

$$Q_0 = \frac{3}{\sqrt{5\pi}}ZeR^2\beta \tag{32}$$

where, deformation parameter [29], β = 0.3 and, radius of the halo nucleus [24], R=4.24 fm and, proton number, Z = 5.Putting these values in Eq. (32) and then,

$$Q_0 = 20.414\text{ efm}^2 \tag{33}$$

Now, from Eq. (31),

$$Q = \frac{6}{21}\times 20.414\text{ e fm}^2 = 5.831\text{ e fm}^2 \tag{34}$$

3. Results and Discussion

In this work, the nuclear electric quadrupole moment of ^{8}B halo nucleus is investigated using three different theoretical models: the two-body model, the collective model and the microscopic single-particle shell model. While the experimental value of electric quadrupole moment (Q) of ^{8}B is 6.45 ± 0.14 e fm^2 [1] which indicates a significant deformation in the ^{8}B nucleus and it's expected due to the proton Halo structure. The experimental and three different theoretical values of the quadrupole moment are presented in Fig. 1 to show the visual comparison. Now, in the two-body model, ^{8}B is treated as a dense ^{7}Be core and a valence proton which causes quadrupole deformation and the calculated value is 6.86 e fm^2 with a small deviation of only 6% with the experimental result.

This model effectively employed core-proton correlation which is crucial for describing quadrupole deformation in ^{8}B. In the collective model, the quadrupole moment arises from the collective deformation and rotation of the nucleus. Using this model, the calculated value of quadrupole moment of ^{8}B is 5.83 e fm^2 which is slightly lower than the experimental value with approximately 10 % deviation but still reasonably close. The collective model offers slightly lower value because it accounts for overall deformation but lack of sensitivity to the Halo structure.

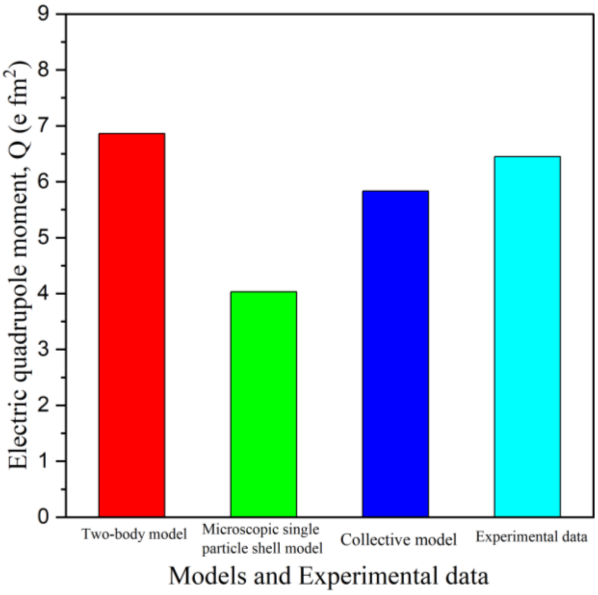

Figure. 1: Comparison of electric quadrupole moments of ^{8}B between three different models and experimental data.

On the other hand, the microscopic single-particle shell model calculation predicts the lowest value which is 4.03 e fm². This value is significantly below than the experimental result about 38%. This discrepancy arises from the model's limitation in accurately describing weakly bound systems and its inability to explain the wave function of the extended valence proton. The large deviation highlights the necessity of including continuum and halo effects. Among these three approaches, the two-body model provides the best agreement with the experimental data.

4. Conclusions

This investigation examined the electric quadrupole moment of Boron-8 (^{8}B) halo nucleus using three different theoretical frameworks, which are compared with the experimental data. Among these three models, the two-body model provided the most accurate result, nearly equal to the experimental data, and confirmed its effectiveness in describing halo structure. The result of this model reflects the model's ability to describe the extended spatial distribution and weak binding of the valence proton, which are essential characteristics of the halo structure of ^{8}B. Although the collective model is widely used for deformed nuclei, it provides a slightly lower value than the experimental value due to its limited treatment of single-particle halo dynamics. And the microscopic single-particle shell model provides a significantly lower value because this model is limited in describing halo nuclei due to its dependence on well-bound single-particle states and lack of treatment of the extended wavefunction of the valence proton, and also the lack of the coupling of the valence proton to continuum states near the proton drip line. These frameworks highlight that the shape of the nucleus of ^{8}B is prolate due to the positive value of quadrupole deformation. The predictions of these approaches highlight the importance of choosing appropriate models for the exotic nucleus. Hence, the two-body approach is particularly well suited for the system of a proton-rich weakly bound halo nucleus ^{8}B. Future work may improve these models for greater accuracy, which requires incorporating more sophisticated factors such as three-body dynamics, continuum effects etc.

Authors Contribution
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Availability of data and materials
Not applicable. In fact, all results are obtained without any software and found by manual computations. In other words, the manuscript is in the pure mathematics (mathematical analysis) category.

Conflict of interests
The author states that there is no conflict of interest.

References

[1] T. Sumikama *et al.*, "Electric quadrupole moment of the proton halo nucleus ^{8}B," *Phys. Rev. C*, vol. 74, no. 2, p. 024327, Aug. 2006, doi: 10.1103/PhysRevC.74.024327

[2] A. K. Azhibekov, V. V. Samarin, K. A. Kuterbekov, and M. A. Naumenko, "Shell model calculations for deformed Li isotopes," *Eurasian J. Phys. Funct. Mater.*, vol. 3, no. 4, pp. 307–318, Dec. 2019, doi: 10.29317/ejpfm.2019030403

[3] B. A. Brown, "The nuclear shell model towards the drip lines," *Prog. Part. Nucl. Phys.*, vol. 47, no. 2, pp. 517–599, Jan. 2001, doi: 10.1016/S0146-6410(01)00159-4

[4] N. Shimizu, "Recent Progress of Shell-Model Calculations, Monte Carlo Shell Model, and Quasi-Particle Vacua Shell Model," *Physics (College. Park. Md).*, vol. 4, no. 3, pp. 1081–1093, Sep. 2022, doi: 10.3390/physics4030071

[5] P. Navrátil, S. Quaglioni, I. Stetcu, and B. R. Barrett, "Recent developments in no-core shell-model calculations," *J. Phys. G Nucl. Part. Phys.*, vol. 36, no. 8, p. 083101, Aug. 2009, doi: 10.1088/0954-3899/36/8/083101

[6] J. W. Moskowitz, K. E. Schmidt, M. A. Lee, and M. H. Kalos, "A new look at correlation energy in atomic and molecular systems. II. The application of the Green's function Monte Carlo method to LiH," *J. Chem. Phys.*, vol. 77, no. 1, pp. 349–355, Jul. 1982 , doi: 10.1063/1.443612

[7] W. S. Hwash, "Nuclear Structure of the Heaviest Boron Isotope," *Eurasian Phys. Tech. J.*, vol. 19, no. 1 (39), pp. 113–118, Mar. 2022, doi: 10.31489/2022No1/113-118

[8] X. F. Yang, S. J. Wang, S. G. Wilkins, and R. F. G. Ruiz, "Laser spectroscopy for the study of exotic nuclei," *Prog. Part. Nucl. Phys.*, vol. 129, p. 104005, Mar. 2023, doi: 10.1016/j.ppnp.2022.104005

[9] T. Minamisono et *al.*, "Giant quadrupole moment andproton halo discovered in ^{8}B," *Hyperfine Interact.*,

vol. 78, no. 1–4, pp. 165–168, 1993, doi: 10.1007/BF00568133

[10] D. C. Kean, “Measurement of quadrupole moments through Coulomb excitation,” in *Nuclear Interactions*, Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, pp. 80–94. doi: 10.1007/3-540-09102-5_583

[11] C. A. Rouse, “Interior Structure of the Sun,” *Nature*, vol. 224, no. 5223, pp. 1009–1010, Dec. 1969, doi: 10.1038/2241009a0

[12] I. Lopes and S. Turck-Chièze, “Detecting Gravity Modes in the Solar ^{8}B Neutrino Flux,” *Astrophys. J.*, vol. 792, no. 2, p. L35, Aug. 2014, doi: 10.1088/2041-8205/792/2/L35

[13] M. Agostini *et al.*, “Comprehensive measurement of pp-chain solar neutrinos,” *Nature*, vol. 562, no. 7728, pp. 505–510, Oct. 2018, doi: 10.1038/s41586-018-0624-y

[14] H. W. Barz, I. Rotter, and J. Höhn, “Coupled channels calculations in the continuum shell model with complicated configurations,” *Nucl. Phys. A*, vol. 275, no. 1, pp. 111–140, Jan. 1977, doi: 10.1016/0375-9474(77)90279-2

[15] F. Winter *et al.*, “First lattice QCD study of the gluonic structure of light nuclei,” *Phys. Rev. D*, vol. 96, no. 9, p. 094512, Nov. 2017, doi: 10.1103/PhysRevD.96.094512

[16] T. Naito, X. Roca-Maza, G. Colò, H. Liang, and H. Sagawa, “Isospin symmetry breaking in the charge radius difference of mirror nuclei,” *Phys. Rev. C*, vol. 106, no. 6, p. L061306, Dec. 2022, doi: 10.1103/PhysRevC.106.L061306

[17] P. J. Woods and C. N. Davids, “Nuclei Beyond the Proton DRIP-LINE,” *Annu. Rev. Nucl. Part. Sci.*, vol. 47, no. 1, pp. 541–590, Dec. 1997, doi: 10.1146/annurev.nucl.47.1.541

[18] H. Esbensen and G. F. Bertsch, “Nuclear induced breakup of halo nuclei,” *Phys. Rev. C*, vol. 59, no. 6, pp. 3240–3245, Jun. 1999, doi: 10.1103/PhysRevC.59.3240

[19] S. Shen, S. Elhatisari, D. Lee, U.-G. Meißner, and Z. Ren, “Ab Initio Study of the Beryllium Isotopes Be^{7} to Be^{12},” *Phys. Rev. Lett.*, vol. 134, no. 16, p. 162503, Apr. 2025, doi: 10.1103/PhysRevLett.134.162503

[20] A. Bohr, B. R. Mottelson, and G. Breit, “Nuclear Structure, Vol. 1,” *Phys. Today*, vol. 23, no. 9, pp. 58–60, Sep. 1970, doi: 10.1063/1.3022342

[21] T. Minamisono *et al.*, “Proton halo of B8 disclosed by its giant quadrupole moment,” *Phys. Rev. Lett.*, vol. 69, no. 14, pp. 2058–2061, 1992, doi: 10.1103/PhysRevLett.69.2058

[22] Z. Dongmei *et al.*, “Quadrupole moment and a proton halo structure in ^{17}F (I $\pi \leq 5/2+$),” *J. Phys. G Nucl. Part. Phys.*, vol. 34, no. 3, pp. 523–528, 2007, doi: 10.1088/0954-3899/34/3/010

[23] G. A. Korolev *et al.*, “Halo structure of ^{8}B determined from intermediate energy proton elastic scattering in inverse kinematics,” *Phys. Lett. B*, vol. 780, pp. 200–204, May 2018, doi: 10.1016/j.physletb.2018.03.013

[24] G. Arfken and J. Mathews, *Mathematical Methods for Physicists*, vol. 40, no. 4. Elsevier, 2013. doi: 10.1016/C2009-0-30629-7

[25] M. L. Boas, “Mathematical methods in physical sciences,” *J. Symb. Log.*, vol. 57, no. 1, pp. 271–272, 1992, [Online]. Available: https://www.cambridge.org/core/journals/journal-of-symbolic-logic

[26] D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd ed. Cambridge University Press, 2018.

[27] K. S. Krane and W. G. Lynch, “Introductory Nuclear Physics,” *Phys. Today*, vol. 42, no. 1, pp. 78–78, Jan. 1989, doi: 10.1063/1.2810884

[28] A. Bohr, B. R. Mottelson, G. Breit, and G. E. Brown, “Nuclear Structure, Vol. 2: Nuclear Deformations,” *Phys. Today*, vol. 30, no. 3, pp. 59–62, Mar. 1977, doi: 10.1063/1.3037453

[29] H. Kitagawa and H. Sagawa, “Quadrupole-moments in mirror nuclei and proton halo,” *Phys. Lett. B*, vol. 299, no. 1–2, pp. 1–5, Jan. 1993, doi: 10.1016/0370-2693(93)90874-H.