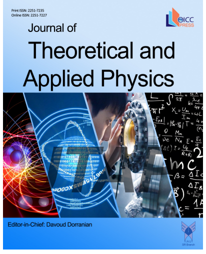

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Approximate solutions of the Dirac equation with screened Kratzer-Hellmann problem including generalized tensor interaction

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Original Research

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Abstract:

The Dirac equation presents better perspective of understanding the motion of particles in region of relativistic quantum mechanics. In this regard, we examine the approximate bound state solutions of the Dirac equation under the spin and pseudospin symmetries for the screened Kratzer-Hellmann potential including generalized tensor interaction. By using the Nikiforov-Uvarov functional analysis method and an approximation scheme, the analytical, numerical and graphical energies of the combined potential were obtained for both symmetries, for different quantum numbers. Degeneracies were observed in the energy values in the absence of the generalized tensor interaction and these degeneracies were removed with the help of the generalized tensor interaction, which is made up of the Coulomb, the Yukawa and the Hulthen potentials. The variations of the energy eigenvalues with screening parameter for spin and pseudospin symmetries were studied for various values of the quantum numbers. The increase and decrease of the energy eigenvalues are observed for both symmetries, indicating tightly bound and loosely bound states, respectively. Our study shows that the obtained energies are very sensitive to the screening parameter and quantum numbers.

Keywords: Dirac equation; Energy eigenvalues; Generalized tensor interaction; Degeneracy; Nikiforov-Uvarov functional analysis method

1. Introduction

Dirac equation (DE) is known to be a wave equation widely applied to relativistic quantum mechanics. It describes the spin origin, relativistic behaviours of atoms, molecules and fundamental particles and antiparticles [1, 2]. DE also offers a significant understanding of high energy particles existing within a strong potential field [3]. Dirac particles mostly interact with electromagnetic fields in many branches of science including photonic devices, modern instruments and particle accelerators [4, 5].

Various potential models have been used to study DE [6–27], with the help of different analytical methods and approximation schemes [28–32]. DE involves two major constituents, being the spin symmetry and the pseudospin symmetry. Spin symmetry is made up of equal scalar and vector potentials. In the case of pseudospin symmetry, the scalar potential is equal to the negative of the vector potential [33]. Also, two degeneracies of states involving quantum numbers are produced by spin symmetry. The pseudospin symmetry produces quasi-degeneracy with two-unit difference in orbital angular momentum [34]. Recent studies show that the Dirac equation under spin and pseudospin symmetries with a Hellmann-like tensor potential for a class of Yukawa potential has been investigated using supersymmetric quantum mechanics formalism [35]. Recently, Tas [36] studied the DE with the ultra- generalized exponential hyperbolic potential. The energy equations for the Schrödinger and Klein–Gordon particles in the spin and pseudospin limit were obtained. Karayer et al. [37] investigated the DE with spin and pseudospin symmetries within a combined Manning-Rosen and Yukawa potential, enhanced by a Coulomb tensor interaction. The tensor coupling effects on the eigenstate degeneracies of the Dirac doublet were conspicuously established. In addition, Onate et al. [38] studied the DE for spin and pseudospin symmetries with a coshine Yukawa potential model. It was observed that the coshine Yukawa potential and the real Yukawa potential have the same trend of variation as the angular momentum number. In contrast, the variation of the screening parameter with the energy for the two potential models differs in their trend. Also, the energy eigenvalues of the coshine Yukawa potential model are more bound, as compared to the energies of the real Yukawa potential model. Other recent studies of potential energy curves and combined potential in different relativistic regimes are enumerated in the following references [39–44].

In the present study, we examine the effect of generalized tensor interaction on the screened Kratzer-Hellmann potential, with the help of the bound state solutions of the DE with spin and pseudospin symmetries. The Screened Kratzer-Hellmann potential is proposed as [45]

$$V(r)=\left(\frac{V_0}{r}+\frac{V_1 e^{\delta r}}{r}+\frac{V_2}{r^2}\right)e^{-\delta r} \tag{1}$$

where r is the distance of separation of the potential, δ is the screening parameter and V_0, V_1, V_2 are the potential strengths. The generalized tensor interaction is made up of the Coulomb, the Yukawa and the Hulthen potentials, given as [46]

$$T(r)=-\left[\frac{1}{r}(H_C+H_Y e^{-\delta r})+H_H\frac{e^{-\delta r}}{(1-e^{-\delta r})}\right] \tag{2}$$

where H_C, H_Y, H_H are components of the Coulomb, Yukawa and Hulthen potentials, respectively. It has been established that combined tensor potential is better, as compared to a single tensor potential [47]. The tensor interaction is used to remove the degeneracy between two states in the combined potentials with a tensor interaction. In addition, it plays a critical role in expounding a complete solution to the problem of squaring the DE [48, 49]. In addition, these studies promise to be applicable in quantum mechanical system with fermions, nucleon-nucleon interactions, nuclear binding energies, quark-gluon plasma, etc.

Our major contributions in this work are highlighted as follows:

- √ Derivation of the relativistic and nonrelativistic energy expressions of screened Kratzer-Hellmann potential with generalized tensor interaction in both spin and pseudospin symmetries condition, using the Nikiforov-Uvarov functional analysis (NUFA) method [50].
- √ Presentation of numerical solutions of the DE for the screened Kratzer-Hellmann potential with generalized tensor interaction in both spin and pseudospin symmetries condition at various quantum states.
- √ Graphical variations of the energy eigenvalues of the screened Kratzer-Hellmann potential with generalized tensor interaction with respect to screening parameter for both spin and pseudospin symmetries condition at various quantum states.

2. Review of DE with tensor coupling

The DE for fermionic massive spin$-1/2$ particles moving under an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$ and a tensor potential $T(r)$ ($\hbar=c=1$) is given as

$$[\mathbf{x}\cdot\mathbf{p}+y(M+S(r))-ib\mathbf{x}\cdot\hat{\mathbf{r}}T(r)]\psi(r)=[E-V(r)]\psi(r) \tag{3}$$

Here, E is the relativistic energy of the system, $\mathbf{p}=-i\nabla$ is the 3-dimensional momentum operator, M is the mass of the fermionic particle, $\mathbf{x},y$ are the 4×4 Dirac matrices defined as

$$\mathbf{x}=\begin{pmatrix}0 & \alpha_i\\ \alpha_i & 0\end{pmatrix}, y=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \tag{4}$$

where I is 2×2 unitary matrix and α_i being the three- vector Pauli spin matrices given as

$$\alpha_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \alpha_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \alpha_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{5}$$

The eigenvalues of the spin-orbit coupling operator are known to be $\kappa=(j+1/2)>0$, $\kappa=-(j+1/2)<0$; for unaligned spin $j=l-1/2$ and the aligned spin $j=l+1/2$, respectively. The set (H^2,K,J^2,J_z) forms the complete set of conservative quantities with $\mathbf{J}$ being the total angular momentum operator and $\hat{\mathbf{K}}=-y(\alpha\cdot\mathbf{L}+1)$ is the spin-orbit where $\mathbf{L}$ is orbit angular momentum. The spinors can be classified according to their angular momentum j, the spin-orbit quantum number κ and the radial quantum number n. The spinors can be written as

$$\psi_{n\kappa}(r)=\frac{1}{r}\begin{pmatrix}R_{n\kappa}(r) & Y^{l}_{jm}(\theta,\varphi)\\ iQ_{n\kappa}(r) & Y^{\tilde{l}}_{jm}(\theta,\varphi)\end{pmatrix} \tag{6}$$

Here, $R_{n\kappa}(r)$, $Q_{n\kappa}(r)$ represent the upper and lower components of the Dirac spinors; $Y^{l}_{jm}(\theta,\varphi)$, $Y^{\tilde{l}}_{jm}(\theta,\varphi)$ represent the spin and pseudospin spherical harmonics and m is the projection on the z-axis.

Using the following identities [51];

$$\begin{aligned}
&(\alpha\cdot\mathbf{A})(\alpha\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}+i\alpha\cdot(\mathbf{A}\times\mathbf{B}),\\
&\alpha\cdot\mathbf{p}=\alpha\cdot\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}+i\frac{\alpha\cdot\mathbf{L}}{r}\right),\\
&(\alpha\cdot\mathbf{L})Y^{\tilde{l}}_{jm}(\theta,\varphi)=(\kappa-1)Y^{\tilde{l}}_{jm}(\theta,\varphi),\\
&(\alpha\cdot\mathbf{L})Y^{l}_{jm}(\theta,\varphi)=-(\kappa+1)Y^{l}_{jm}(\theta,\varphi),\\
&(\alpha\cdot\hat{\mathbf{r}})Y^{l}_{jm}(\theta,\varphi)=-Y^{\tilde{l}}_{jm}(\theta,\varphi),\\
&(\alpha\cdot\hat{\mathbf{r}})Y^{\tilde{l}}_{jm}(\theta,\varphi)=-Y^{l}_{jm}(\theta,\varphi)
\end{aligned} \tag{7}$$

Eq. (3) becomes the two coupled first-order Dirac equations given as:

$$\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)R_{n\kappa}(r)=(M+E_{n\kappa}-\Delta(r))Q_{n\kappa}(r) \tag{8}$$

$$\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)Q_{n\kappa}(r)=(M-E_{n\kappa}+\Sigma(r))R_{n\kappa}(r) \tag{9}$$

where $\Delta(r) = V(r) - S(r)$; $\Sigma(r) = V(r) + S(r)$. Also, $\Delta(r)$ and $\Sigma(r)$ are the difference and sum potentials, respectively. By eliminating $R_{n\kappa}(r)$ and $Q_{n\kappa}(r)$ in Eqs. (8) and (9), the following second-order Schrodinger-like equations are obtained:

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} - \frac{dU(r)}{dr} - U^2(r) - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) + \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)}{(M + E_{n\kappa} - \Delta(r))}\right\} R_{n\kappa}(r) = 0 \quad (10)$$

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} + \frac{dU(r)}{dr} - U^2(r) - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) - \frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)}{(M - E_{n\kappa} + \Sigma(r))}\right\} Q_{n\kappa}(r) = 0 \quad (11)$$

where $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$, $\kappa(\kappa+1) = l(l+1)$.
For spin symmetry to occur, $d\Delta(r)/dr = 0$ and $\Delta(r)$ becomes a constant, C_S [52]. Hence, Eq. (10) becomes

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} - \frac{dU(r)}{dr} - U^2(r) - (M + E_{n\kappa} - C_S)\Sigma(r) + E_{n\kappa}^2 - M^2 + C_S(M - E_{n\kappa})\right\} R_{n\kappa}(r) = 0 \quad (12)$$

Here, $\kappa = l$ for $\kappa > 0$ and $\kappa = -(l+1)$ for $\kappa < 0$. The lower spinor component can be obtained from Eq. (8) as

$$Q_{n\kappa}(r) = \frac{1}{(M + E_{n\kappa} - C_S)}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right) R_{n\kappa}(r) \quad (13)$$

There exists only real positive energy spectrum for exact spin symmetry where $E_{n\kappa} \neq -M$ for $C_S = 0$. In addition, pseudospin symmetry occurs when $d\Sigma(r)/dr = 0$ and $\Sigma(r)$ becomes a constant, C_{PS} [52]. Hence, Eq. (11) becomes

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} + \frac{dU(r)}{dr} - U^2(r) - (M - E_{n\kappa} + C_{PS})\Delta(r) - (M^2 - E_{n\kappa}^2 + C_{PS}(M + E_{n\kappa}))\right\} Q_{n\kappa}(r) = 0 \quad (14)$$

Here, $\kappa = -\tilde{l}$ for $\kappa < 0$ and $\kappa = \tilde{l} + 1$ for $\kappa > 0$. The $SU(2)$ pseudospin symmetry can be obtained when $\tilde{l} \neq 0$, in which degenerate states are produced with $j = \tilde{l} \pm 1/2$. The upper spinor component can then be obtained from Eq. (9) as

$$R_{n\kappa}(r) = \frac{1}{(M - E_{n\kappa} + C_{PS})}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right) Q_{n\kappa}(r) \quad (15)$$

Here, there exist only real negative energy spectrum for exact pseudospin symmetry where $E_{n\kappa} \neq M$ for $C_{PS} = 0$.

3. Analytical solutions of DE with screened Kratzer-Hellmann potential including generalized tensor interaction using NUFA

3.1 Spin symmetry solution

In this subsection, we substitute the screened Kratzer-Hellmann potential of Eq. (1) and the generalized tensor interaction of Eq. (2) into Eq. (12) to obtain

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)^2}{r} - \frac{2\kappa H_C}{r} - \frac{2\kappa H_Y e^{-\delta r}}{r^2} - \frac{2\kappa H_H e^{-\delta r}}{r(1-e^{-\delta r})} - \frac{\delta H_Y e^{-\delta r}}{r} - \frac{H_C}{r^2} - \frac{H_Y e^{-\delta r}}{r^2} - \frac{\delta H_H e^{-\alpha r}}{(1-e^{-\delta r})} - \frac{\delta H_H e^{-2\delta r}}{(1-e^{-\delta r})} - \frac{H_C^2}{r^2} - \frac{2H_C H_Y e^{-\delta r}}{r^2} - \frac{H_Y^2 e^{-2\delta r}}{r^2} - \frac{2H_C H_Y e^{-\delta r}}{r(1-e^{-\delta r})} - \frac{2H_H H_Y e^{-2\delta r}}{r(1-e^{-\delta r})} - \frac{H_H^2 e^{-2\delta r}}{(1-e^{-\delta r})^2} - \frac{UV_0 e^{-\delta r}}{r} - \frac{UV_1}{r} - \frac{UV_2 e^{-\delta r}}{r^2} - \varepsilon_{n\kappa}^2\right\} R_{n\kappa}(r) = 0 \quad (16)$$

Here, the sum potential $\Sigma(r)$ is taken as the screened Kratzer-Hellmann potential, the difference potential $\Delta(r)$ taken as constant C_S, the tensor potential $U(r)$ taken as the generalized tensor interaction and the following parameters Q and $\varepsilon_{n\kappa}^2$ are also defined as

$$U = (M + E_{n\kappa} - C_S);\ \ \varepsilon_{n\kappa}^2 = -(E_{n\kappa}^2 - M^2 + C_S(M - E_{n\kappa})) \quad (17)$$

Due to the presence of the centrifugal terms in Eq. (16), we employ the following approximation scheme [53]:

$$\frac{1}{r^2} \equiv \frac{\delta^2}{(1-e^{-\delta r})^2};\ \ \frac{1}{r} \equiv \frac{\delta}{(1-e^{-\delta r})} \quad (18)$$

By substituting Eq. (18) and the coordinate transformation $s = e^{-\delta r}$ into Eq. (16), we have

$$\frac{d^2 R_{n\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dR_{n\kappa}(s)}{ds} + \frac{(-B_1 s^2 + B_2 s - B_3)}{(s(1-s))^2} R_{n\kappa}(s) = 0 \quad (19)$$

Here, the following parameters are defined:

$$B_1 = \frac{\varepsilon_{n\kappa}^2}{\delta^2} - \frac{UV_0}{\delta} + \frac{2H_H H_Y}{\delta} + \frac{H_H^2}{\delta} + H_Y(H_Y - 1);$$
$$B_2 = \frac{2\varepsilon_{n\kappa}^2}{\delta^2} - \frac{U}{\delta}(V_0 - V_1) - UV_2 - 2H_Y(\beta_{\kappa C} + 1) - \frac{H_H}{\delta}(2\beta_{\kappa C} + 1);$$
$$B_3 = \frac{\varepsilon_{n\kappa}^2}{\delta^2} + \frac{UV_1}{\delta} + \beta_{\kappa C}(\beta_{\kappa C} + 1);$$
$$\beta_{\kappa C} = \kappa + H_C \quad (20)$$

By employing the NUFA method as presented in "Appendix A" and proposing a wave function of the form

$$R_{n\kappa}(s) = s^{\sigma_1}(1-s)^{w_1} f_{n\kappa}(s) \quad (21)$$

where

$$\sigma_1 = \sqrt{\frac{(M^2 - E_{n\kappa}^2 - C_s(M - E_{n\kappa}))}{\delta^2} + \frac{UV_1}{\delta} + \beta_{\kappa C}(\beta_{\kappa C}+1)};$$
$$w_1 = \frac{1}{2}\left[1 + \sqrt{1 + 4\left(\frac{H_H}{\delta}(2\beta_{\kappa C} + 2H_Y + H_H + 1)\right.} \right.$$
$$\left.\overline{+(\beta_{\kappa C}+1)(2H_Y + \beta_{\kappa C}) + UV_2 + H_Y(H_Y - 1))}\right] \quad (22)$$

the approximate energy spectra of the screened Kratzer-Hellmann potential with generalized tensor interaction for the spin symmetry limit in closed form is obtained as

$$M^2 - E_{n\kappa}^2 + C_S(M - E_{n\kappa}) = \delta^2\left\{\left(\frac{t}{2(n+w_1)} - \frac{(n+w_1)}{2}\right)^2 - \left(\beta_{\kappa C}(\beta_{\kappa C}+1) + \frac{UV_1}{\delta}\right)\right\} \quad (23)$$

where

$$t = \frac{H_H}{\delta}(2H_Y + H_H) - \beta_{\kappa C}(\beta_{\kappa C}+1) - \frac{U}{\delta}(V_0 + V_1) + H_Y(H_Y - 1) \quad (24)$$

By using Eq. (21), the wave component of the screened Kratzer-Hellmann potential with the generalized tensor interaction for the spin symmetry limit becomes

$$R_{n\kappa}(s) = s^{\sigma_1}(1-s)^{w_1}\,{}_2F_1(-n, n+2(\sigma_1+w_1), 2\sigma_1+1, s) \quad (25)$$

3.2 Pseudospin symmetry solution

In this subsection, the screened Kratzer-Hellmann potential of Eq. (1) and the generalized tensor interaction of Eq. (2) are substituted into Eq. (14) to obtain

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa H_C}{r} - \frac{2\kappa H_Y e^{-\delta r}}{r^2} - \frac{2\kappa H_H e^{-\delta r}}{r(1-e^{-\delta r})} + \frac{\delta H_Y e^{-\delta r}}{r} + \frac{H_C}{r^2} + \frac{H_Y e^{-\delta r}}{r^2} + \frac{\delta H_H e^{-\delta r}}{(1-e^{-\delta r})} + \frac{\delta H_H e^{-2\delta r}}{(1-e^{-\delta r})} - \frac{H_C^2}{r^2} - \frac{2H_C H_Y e^{-\delta r}}{r^2} - \frac{H_Y^2 e^{-2\delta r}}{r^2} - \frac{2H_C H_Y e^{-\delta r}}{r(1-e^{-\delta r})} - \frac{2H_H H_Y e^{-2\delta r}}{r(1-e^{-\delta r})} - \frac{H_H^2 e^{-2\delta r}}{(1-e^{-\delta r})^2} + \frac{U'V_0 e^{-\delta r}}{r} + \frac{U'V_1}{r} + \frac{U'V_2 e^{-\delta r}}{r^2} + \zeta_{n\kappa}^2\right\}Q_{n\kappa}(r) = 0 \quad (26)$$

Here, the difference potential $\Delta(r)$ is taken as the screened Kratzer-Hellmann potential, the sum potential $\Sigma(r)$ taken as constant C_{PS}, the tensor potential $T(r)$ taken as the generalized tensor interaction and the following parameters Q' and $\zeta_{n\kappa}^2$ are also defined as

$$U' = (M - E_{n\kappa} + C_{PS});\ \ \zeta_{n\kappa}^2 = -(M^2 - E_{n\kappa}^2 + C_{PS}(M + E_{n\kappa})) \quad (27)$$

By employing the approximation scheme of Eq. (18) and the coordinate transformation $s = -qe^{-\delta r}$, we obtain

$$\frac{d^2 Q_{n\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dQ_{n\kappa}(s)}{ds} + \frac{(-D_1 s^2 + D_2 s - D_3)}{(s(1-s))^2}Q_{n\kappa}(z) = 0 \quad (28)$$

where

$$D_1 = \frac{U'V_0}{\delta} + \frac{2H_H H_Y}{\delta} + \frac{H_H^2}{\delta} + H_Y(H_Y+1) - \frac{\zeta_{n\kappa}^2}{\delta^2};$$
$$D_2 = \frac{U'}{\delta(V_0 - V_1)} - U'V_2 - 2H_Y(\beta_{\kappa C} - 1) - \frac{H_H}{\delta}(2\beta_{\kappa C} - 1) - \frac{2\zeta_{n\kappa}^2}{\delta^2};$$
$$D_3 = \beta_{\kappa C}(\beta_{\kappa C} - 1) - \frac{U'V_1}{\delta} - \frac{\zeta_{n\kappa}^2}{\delta^2};$$
$$\beta_{\kappa C} = \kappa + H_C \quad (29)$$

With the help of the NUFA method as presented in "Appendix A" and proposing a wave function of the form

$$Q_{n\kappa}(s) = s^{\sigma_2}(1-s)^{w_2} f_{n\kappa}(s) \quad (30)$$

where

$$\sigma_2 = \sqrt{\beta_{\kappa C}(\beta_{\kappa C}-1) - \frac{U'V_1}{\delta} + \frac{M^2 - E_{n\kappa}^2 + C_{PS}(M - E_{n\kappa})}{\delta^2}};$$
$$w_2 = \frac{1}{2}\left[1 + \left(\frac{H_H}{\delta}(2\beta_{\kappa C} + 2H_Y + H_H - 1) + (\beta_{\kappa C} - 1)(2H_Y + \beta_{\kappa C}) - U'V_2 + H_Y(H_Y+1)\right)\right] \quad (31)$$

the approximate energy spectra of the screened Kratzer-Hellmann potential with generalized tensor interaction for the pseudospin symmetry limit in closed form is obtained as

$$M^2 - E_{n\kappa}^2 + C_S(M - E_{n\kappa}) = \delta^2\left\{\left(\frac{t'}{2(n+w_2)} - \frac{(n+w_2)}{2}\right)^2 - \left(\beta_{\kappa C}(\beta_{\kappa C}-1) - \frac{(U'V_1)}{\delta}\right)\right\} \quad (32)$$

where

$$t' = \frac{H_H}{\delta}(2H_Y + H_H) + \frac{U'}{\delta}(V_0 + V_1) + H_Y(H_Y+1) - \beta_{\kappa C}(\beta_{\kappa C} - 1) \quad (33)$$

In addition, the wave component of the screened Kratzer-Hellmann potential with generalized tensor interaction for the pseudospin symmetry limit becomes

$$Q_{n\kappa}(s) = s^{\sigma_2}(1-s)^{w_2}\,{}_2F_1(-n, n+2(\sigma_2+w_2), 2\sigma_2+1, s) \quad (34)$$

3.3 Spin and nonrelativistic limit for the Screened Kratzer-Hellmann Potential (SKHP) model

We obtained the above solution by employing the following transformations to the solution of the spin symmetry limit of Eq. (23):

$$C_S = H_H = H_Y = H_C = 0;\ \ \kappa = l;\ \ R_{n\kappa}(r) \to R_{nl}(r);$$
$$M + E_{n\kappa} \to \frac{2\mu}{\hbar^2};\ \ M - E_{n\kappa} \to E_{nl} \quad (35)$$

Hence, the resulting nonrelativistic solution of SKHP model becomes:

$$E_{nl} = \delta V_1 + \frac{\hbar^2\delta^2 l(l+1)}{2\mu} - \frac{\hbar^2\delta^2}{2\mu}\left(\frac{\left(\frac{2\mu}{\hbar^2\delta}(V_2+V_1)+l(l+1)\right)}{2\left[n+\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)+\frac{2\mu V_2}{\hbar^2}}\right]} - \frac{2\left[n+\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)+\frac{2\mu V_2}{\hbar^2}}\right]}{2\left[n+\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)+\frac{2\mu V_2}{\hbar^2}}\right]}\right)^2 \quad (36)$$

This result is very consistent with the result obtained in Ref. [45].

4. Results and discussion

In our present study, the numerical analysis of the energies obtained are carried out in the absence $(H_c = H_H = H_Y = 0)$ and the presence $(H_c = H_H = H_Y = 0.05)$ of the generalized tensor interaction potential for various values of the quantum numbers n, l and κ. The following parameters were employed in this work:

$$V_1 = 3 \text{ fm}^{-1},\ V_2 = -5 \text{ fm}^{-1},\ V_3 = -10 \text{ fm}^{-1},$$
$$C_s = 10 \text{ fm}^{-1},\ C_{ps} = -10 \text{ fm}^{-1},\ \alpha = 0.01 \text{ fm}^{-1},$$
$$M = 4.76 \text{ fm}^{-1}$$

Different energy eigenvalues of the screened Kratzer-Hellmann potential with and without generalized tensor interaction for spin symmetry and pseudospin symmetry are presented in Tables 1 and 2, respectively. Eqns. (23) and (32) give the expression for the eigenvalues of the screened Kratzer-Hellmann potential with generalized tensor interaction respectively, for the spin and pseudospin symmetries. It is observed that there is a slight decrease and increase in energy values in both symmetries as the quantum numbers n, l and $|\kappa|$ increases, as seen in Tables 1 and 2, respectively. In the absence of the generalized tensor, degeneracy is seen to occur as a results of multiple quantum states sharing the same energy. In the case of the spin symmetry, Dirac spin-doublet eigenstates exist with the same n and κ states. As the generalized tensor interaction occurs, the degeneracies disappear in the spin symmetry. In the case of pseudospin symmetry, the Dirac spin-doublets eigenstates are observed when n and l are different. The degeneracies also disappear in the presence of the generalized tensor interaction. In addition, we observe that $(np_{\frac{3}{2}}, np_{\frac{1}{2}})$, $(nd_{\frac{5}{2}}, nd_{\frac{3}{2}})$, $(nf_{\frac{7}{2}}, nf_{\frac{5}{2}})$, etc pair states degenerate in the case of spin symmetry. Conversely, $(ns_{\frac{1}{2}}, nd_{\frac{3}{2}})$, $(np_{\frac{3}{2}}, nf_{\frac{5}{2}})$, $(nd_{\frac{5}{2}}, ng_{\frac{7}{2}})$, etc pair states degenerate in the case of pseudospin symmetry. The trend of our results is consistent with the results obtained in literatures [18]. The variation of energy eigenvalues with screening parameter for selected values of n and l is presented in Fig. 1 for both spin and pseudospin symmetry case. Here, the energy eigenvalues decrease slightly first and later increase monotonously for the spin symmetry case. In the case of the pseudospin symmetry, the energy eigenvalues increase gradually first and later decrease monotonously. The increase and decrease of the energy eigenvalues obtained is an indication of tightly bound and loosely bound states, respectively.

5. Conclusion

In this paper, we have solved the DE involving the screened Kratzer-Hellmann potential with generalized tensor interaction using the NUFA method and an approximation scheme to handle the centrifugal term. Implicit approximate solutions of the DE under spin and pseudospin symmetries are obtained analytically and numerically, for various values of the quantum numbers. In the absence of the generalized tensor interaction, the energies obtained for both spin and pseudospin symmetries are seen to be the same with Dirac spin-doublet eigenstates, hence degeneracy occurring. As the generalized tensor interaction occurs, the degeneracies are eliminated and there exists variance in energies at various quantum states. A plot of energy eigenvalues with screening parameter for selected values of quantum states is also presented for both spin and pseudospin symmetry case. It can be deduced that the energy eigenvalues of the screened Kratzer-Hellmann potential including generalized tensor interaction are sensitive to the screening parameter and quantum numbers for spin symmetry and pseudospin symmetry cases. Our results are seen to agree with results in literatures, and they promise to be applicable to a wide scope of physical systems [54–57]. Our study gives insight information about the quantum mechanical system like fermions, nucleon-nucleon interactions, nuclear binding energies, quark-gluon plasma, etc [58–64].

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Authors Contribution
The intellectual substance, idea, and design of this study, or the analysis and interpretation of the data (if applicable), as well as the manuscript's writing, were all sufficiently contributed to by each author.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Table 1. Bound State Energies ("in" fm^{-1}) of the screened Kratzer-Hellmann potential with and without generalized tensor interaction of the spin symmetry with various quantum states at $\delta = 0.01$.

l	$(n, \kappa < 0)$	$(l, j = l + 1/2)$	$E_{n,\kappa<0} \begin{cases} H_C = 0, \\ H_H = 0, \\ H_Y = 0 \end{cases}$	$E_{n,\kappa<0} \begin{cases} H_C = 0.05, \\ H_H = 0.05, \\ H_Y = 0.05 \end{cases}$	$(n, \kappa > 0)$	$(l, j = l - 1/2)$	$E_{n,\kappa>0} \begin{cases} H_C = 0, \\ H_H = 0, \\ H_Y = 0 \end{cases}$	$E_{n,\kappa>0} \begin{cases} H_C = 0.05, \\ H_H = 0.05, \\ H_Y = 0.05 \end{cases}$
1	0, - 2	$0p_{3/2}$	4.760259177	4.757422013	0, 1	$0p_{1/2}$	4.760259177	4.759469105
2	0, - 3	$0d_{5/2}$	4.759930961	4.758033960	0, 2	$0d_{3/2}$	4.759930961	4.759353326
3	0, - 4	$0f_{7/2}$	4.759710261	4.758145655	0, 3	$0f_{5/2}$	4.759710261	4.759301912
4	0, - 5	$0g_{9/2}$	4.759572520	4.758099523	0, 4	$0g_{7/2}$	4.759572520	4.759276857
1	1, - 2	$1p_{3/2}$	4.759745852	4.758331655	1, 1	$1p_{1/2}$	4.759745852	4.759361054
2	1, - 3	$1d_{5/2}$	4.759603322	4.758379796	1, 2	$1d_{3/2}$	4.759603322	4.759298391
3	1, - 4	$1f_{7/2}$	4.759497414	4.758384127	1, 3	$1f_{5/2}$	4.759497414	4.759271762
4	1, - 5	$1g_{9/2}$	4.759426179	4.758321567	1, 4	$1g_{7/2}$	4.759426179	4.759260786
1	2, - 2	$2p_{3/2}$	4.759507895	4.758960220	2, 1	$2p_{1/2}$	4.759507895	4.759302184
2	2, - 3	$2d_{5/2}$	4.759436885	4.758748917	2, 2	$2d_{3/2}$	4.759436885	4.759269916
3	2, - 4	$2f_{7/2}$	4.759381297	4.758675500	2, 3	$2f_{5/2}$	4.759381297	4.759259064
4	2, - 5	$2g_{9/2}$	4.759342552	4.758606548	2, 4	$2g_{7/2}$	4.759342552	4.759257798
1	3, - 2	$3p_{3/2}$	4.759383885	4.759189241	3, 1	$3p_{1/2}$	4.759383885	4.759271621
2	3, - 3	$3d_{5/2}$	4.759346104	4.758978750	3, 2	$3d_{3/2}$	4.759346104	4.759258542
3	3, - 4	$3f_{7/2}$	4.759316043	4.758890825	3, 3	$3f_{5/2}$	4.759316043	4.759258599
4	3, - 5	$3g_{9/2}$	4.759295139	4.758830929	3, 4	$3g_{7/2}$	4.759295139	4.759264542

Table 2. Bound State Energies ("in" fm^{-1}) of the screened Kratzer-Hellmann potential with and without generalized tensor interaction of the pseudospin symmetry with various quantum states at $\delta = 0.01$.

$\tilde{l}$	$(n, \kappa < 0)$	(l, j)	$E_{n,\kappa<0} \begin{cases} H_C = 0, \\ H_H = 0, \\ H_Y = 0 \end{cases}$	$E_{n,\kappa<0} \begin{cases} H_C = 0.05, \\ H_H = 0.05, \\ H_Y = 0.05 \end{cases}$	$(n-1, \kappa > 0)$	$(l+2, j+1)$	$E_{n,\kappa>0} \begin{cases} H_C = 0, \\ H_H = 0, \\ H_Y = 0 \end{cases}$	$E_{n,\kappa>0} \begin{cases} H_C = 0.05, \\ H_H = 0.05, \\ H_Y = 0.05 \end{cases}$
1	1, - 1	$1s_{1/2}$	-4.759183116	-4.759508445	0, 2	$0d_{3/2}$	-4.762332772	-4.758354997
2	1, - 2	$1p_{3/2}$	-4.757572378	-4.759407135	0, 3	$0f_{5/2}$	-4.755283943	-4.758675522
3	1, - 3	$1d_{5/2}$	-4.758511662	-4.759348219	0, 4	$0g_{7/2}$	-4.758153899	-4.758802825
4	1, - 4	$1f_{7/2}$	-4.758774688	-4.759310676	0, 5	$0h_{9/2}$	-4.758659871	-4.758871702
1	2, - 1	$2s_{1/2}$	-4.758478875	-4.759265208	1, 2	$1d_{3/2}$	-4.759183116	-4.758565505
2	2, - 2	$2p_{3/2}$	-4.758245507	-4.759262060	1, 3	$1f_{5/2}$	-4.757572378	-4.758744730
3	2, - 3	$2d_{5/2}$	-4.758676494	-4.759248382	1, 4	$1g_{7/2}$	-4.758511662	-4.758824431
4	2, - 4	$2f_{7/2}$	-4.758832151	-4.759236562	1, 5	$1h_{9/2}$	-4.758774688	-4.758870225
1	3, - 1	$3s_{1/2}$	-4.758504991	-4.759087972	2, 2	$2d_{3/2}$	-4.758478875	-4.758675066
2	3, - 2	$3p_{3/2}$	-4.758527267	-4.759136698	2, 3	$2f_{5/2}$	-4.758245507	-4.758780684
3	3, - 3	$3d_{5/2}$	-4.758760429	-4.759154843	2, 4	$2g_{7/2}$	-4.758676494	-4.758830700
4	3, - 4	$3f_{7/2}$	-4.758859696	-4.759164316	2, 5	$2h_{9/2}$	-4.758832151	-4.758860314
1	4, - 1	$4s_{1/2}$	-4.758609585	-4.758985551	3, 2	$3d_{3/2}$	-4.758504991	-4.758734147
2	4, - 2	$4p_{3/2}$	-4.758665996	-4.759049319	3, 3	$3f_{5/2}$	-4.758527267	-4.758796504
3	4, - 3	$4d_{5/2}$	-4.758803713	-4.759083331	3, 4	$3g_{7/2}$	-4.758760429	-4.758826404
4	4, - 4	$4f_{7/2}$	-4.758869507	-4.759106290	3, 5	$3h_{9/2}$	-4.758859696	-4.758843932

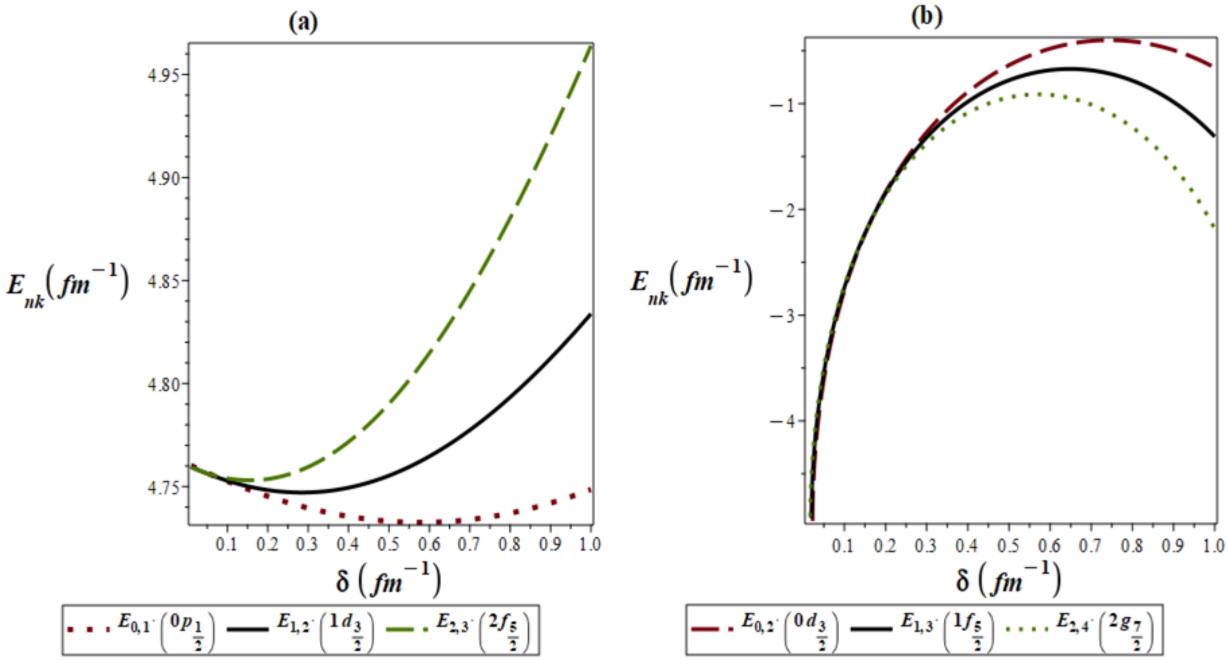

Figure 1. Variation of energy eigenvalues of the screened Kratzer-Hellmann potential with generalized tensor interaction with respect to screening parameter for (a) spin symmetry and (b) pseudospin symmetry cases, at various quantum states.

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Appendix A

Review of Nikiforov-Uvarov-Functional Analysis (NUFA) method

Using the concepts of NU method [31], parametric NU method [65] and the functional analysis method [66], we proposed a simple and elegant method for solving a second order differential equation of the hypergeometric type called Nikiforov-Uvarov-Functional Analysis (NUFA) method. As it is well-known, the NU is used to solve a second-order differential equation of the form [67]

$$\frac{d^2\psi(s)}{ds^2}+\frac{\tilde{\tau}(s)}{\sigma(s)}\frac{d\psi(s)}{ds}+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \qquad (A.1)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. Tezcan and Sever [58] latter introduced the parametric form of NU method in the form

$$\frac{d^2\psi(s)}{ds^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d\psi(s)}{ds}+\frac{1}{s^2(1-\alpha_3 s)^2}[-\xi_1 s^2+\xi_2 s-\xi_3]\psi(s)=0 \qquad (A.2)$$

where α_i and ξ_i $(i=1,2,3)$ are all parameters. It can be observed in Eq. (A.2) that the differential equation has two singularities at $s\to 0$ and $s\to\frac{1}{\alpha_3}$ thus we take the wave function in the form,

$$\psi(s)=s^{\lambda}(1-\alpha_3 s)^{\nu}f(s) \qquad (A.3)$$

Substituting Eq. (A.3) into Eq. (A.2) leads to the following equation,

$$\begin{aligned}
&s(1-\alpha_3 s)\frac{d^2 f(s)}{ds^2}+[\alpha_1+2\lambda-(2\lambda\alpha_3+2\nu\alpha_3\\
&+\alpha_2)s]\frac{df(s)}{ds}-\alpha_3(\lambda+\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3}-1)\\
&+\sqrt{\frac{1}{4}(\frac{\alpha_2}{\alpha_3}-1)^2+\frac{\xi_1}{\alpha_3^2}})(\lambda+\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3}-1)\\
&-\sqrt{\frac{1}{4}(\frac{\alpha_2}{\alpha_3}-1)^2+\frac{\xi_1}{\alpha_3^2}})+[\frac{\lambda(\lambda-1)+\alpha_1\lambda-\xi_3}{s}\\
&+\frac{\nu(\nu-1)\alpha_3+\alpha_2\nu-\alpha_1\alpha_3\nu-\frac{\xi_1}{\alpha_3}+\xi_2-\xi_3\alpha_3}{(1-\alpha_3 s)}]f(s)=0
\end{aligned} \qquad (A.4)$$

Eq. (A.4) can be reduced to a Gauss hypergeometric equation if and only if the following functions vanished,

$$\lambda(\lambda-1)+\alpha_1\lambda-\xi_3=0 \qquad (A.5)$$

$$\nu(\nu-1)\alpha_3+\alpha_2\nu-\alpha_1\alpha_3\nu-\frac{\xi_1}{\alpha_3}+\xi_2-\xi_3\alpha_3=0 \qquad (A.6)$$

Thus, Eq. (A.4) now becomes

$$\begin{aligned}
&s(1-\alpha_3 s)\frac{d^2 f(s)}{ds^2}+[\alpha_1+2\lambda-(2\lambda\alpha_3+2\nu\alpha_3\\
&+\alpha_2)s]\frac{df(s)}{ds}-\alpha_3(\lambda+\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3-1})\\
&+\sqrt{\frac{1}{4}(\frac{\alpha_2}{\alpha_3-1})^2+\frac{\xi_1}{\alpha_3^2}})(\lambda+\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3-1})\\
&-\sqrt{\frac{1}{4}(\frac{\alpha_2}{\alpha_3-1})^2+\frac{\xi_1}{\alpha_3^2}})f(s)=0
\end{aligned} \qquad (A.7)$$

Solving Eqs. (A.5) and (A.6) completely give,

$$\lambda=\frac{1}{2}((1-\alpha_1)\pm\sqrt{(1-\alpha_1)^2+4\xi_3}) \qquad (A.8)$$

$$\begin{aligned}
\nu=\frac{1}{2\alpha_3}\Bigg(&(\alpha_3+\alpha_1\alpha_3-\alpha_2)\\
&\pm\sqrt{(\alpha_3+\alpha_1\alpha_3-\alpha_2)^2+4(\frac{\xi_1}{\alpha_3}+\alpha_3\xi_3-\xi_2)}\Bigg)
\end{aligned} \qquad (A.9)$$

Eq. (A.7) is the hypergeometric equation type of the form,

$$x(1-x)\frac{d^2 f(x)}{dx^2}+[c+(a+b+1)x]\frac{df(x)}{dx}-[ab]f(x)=0 \qquad (A.10)$$

where a,b,c are given as follows,

$$a=\sqrt{\alpha_3}(\lambda+\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3}-1)+\sqrt{\frac{1}{4}(\frac{\alpha_2}{\alpha_3}-1)^2+\frac{\xi_1}{\alpha_3^2}}) \qquad (A.11)$$

$$\begin{aligned}
&b=\sqrt{\alpha_3}(\lambda+\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3}-1)-\sqrt{1/4(\frac{\alpha_2}{\alpha_3}-1)^2+\frac{\xi_1}{\alpha_3^2}})\\
&c=\alpha_1+2\lambda
\end{aligned} \qquad (A.12)$$

Setting either a or b equal to a negative integer $-n$, the hypergeometric function $f(s)$ turns to a polynomial of degree n. Hence, the hypergeometric function $f(s)$ approaches finite in the following quantum condition i.e. $a=-n$, where $n=0,1,2,3,\cdots,n_{\max}$.
Using the above quantum condition,

$$\sqrt{\alpha_3}(\lambda+\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3}-1)+\sqrt{\frac{1}{4}(\frac{\alpha_2}{\alpha_3}-1)^2+\frac{\xi_1}{\alpha_3^2}})=-n \qquad (A.13)$$

$$\lambda+\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3}-1)+\frac{n}{\sqrt{\alpha_3}}=-\sqrt{\frac{1}{4}(\frac{\alpha_2}{\alpha_3}-1)^2+\frac{\xi_1}{\alpha_3^2}} \qquad (A.14)$$

Squaring both sides of Eq. (A.14) and rearranging, we obtain the energy equation for the NUFA method as

$$\begin{aligned}
&\lambda^2+2\lambda(\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3}-1)+\frac{n}{\sqrt{\alpha_3}})+(\nu+\frac{1}{2}(\frac{\alpha_2}{\alpha_3}-1)+\frac{n}{\sqrt{\alpha_3}})^2\\
&-\frac{1}{4}(\frac{\alpha_2}{\alpha_3-1})^2-\frac{\xi_1}{\alpha_3^2}=0
\end{aligned} \qquad (A.15)$$

By substituting Eqs. (A.8) and (A.9) into Eq. (A.3), we obtain the corresponding wave equation for the NUFA method as

$$\begin{aligned}
\psi(s)=&N_{nl}s^{\frac{(1-\alpha_1)+\sqrt{(\alpha_1-1)^2+4\xi_3}}{2}}\\
&\times(1-\alpha_3 s)^{\frac{(\alpha_3+\alpha_1\alpha_3-\alpha_2)+\sqrt{(\alpha_3+\alpha_1\alpha_3-\alpha_2)^2+4(\frac{\xi_1}{\alpha_3^2}+\alpha_3\xi_3-\xi_2)}}{2\alpha_3}}\\
&\times {}_2F_1(a,b,c;s)
\end{aligned} \qquad (A.16)$$

where N_{nl} is the normalization constant.