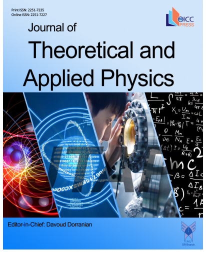

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Exact solutions to the Bernoulli and Riccati equations with conformable derivatives: An application to liquid flow in reservoirs, tanks, and funnels

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Abstract:

This study presents an explicit representation of the solution for a linear conformable differential system with variable coefficients, utilizing the method of variation of constants combined with the state-transition approach. To tackle the exact solutions of nonlinear fractional Bernoulli-type and Riccati-type differential equations involving conformable derivatives-as well as separable fractional differential equations-these are skillfully transformed into an equivalent linear conformable system through appropriate variable substitutions. Theoretical results are further substantiated by detailed numerical and simulated examples. Moreover, the practical applicability of the proposed method is demonstrated through modeling liquid flow in engineering structures such as reservoirs, tanks, and funnels.

Keywords: Conformable fractional derivative; Riccati type differential equation; Bernoulli type differential equation; Representation of solutions; State-transition operator

1. Introduction

Fractional calculus may be regarded as an extension of integer calculus. This gives fractional calculus various privileges that integer calculus does not have. For instance, it is observed by most of researchers who study in this subject that real-world problems and real-life social issues are more appropriately represented by fractional-order systems rather than integer-order systems. Today, fractional-order systems have been used in almost all of areas such as image and signal possessing, engineering, biophysics, models of neurons, thermodynamics, mathematical physics; see [1–7]. When having a look at the literature, we have observed that there are so many definitions about fractional derivatives such as Euler, Fourier, Abel, Liouville, Riemann, Grünwald, Hadamard, Weyl, Erdélyi-Kober, Caputo [8–14], conformable [15–18], etc, fractional derivatives. The conformable fractional derivative is of particular importance over the others because one of the most likely reasons is that it satisfies the corresponding quotient rule, the corresponding product rule, the corresponding chain rule, the corresponding mean value theorem, the corresponding Rolle theorem, generally the corresponding semigroup property compared to the classical 1st derivative while the others do not. Also, it begins to attract everyone's attention, whether they are mathematicians or not, because its definition is so simple and so close to the well-known 1st derivative. In addition, the importance of conformable fractional derivatives in the medical world is undeniable based on the available studies [19–25]. Moreover, Dazhi and Maokang [26] in 2017 firstly gave the physical and geometrical interpretations of the conformable fractional derivatives.

Recent advances in the analysis of conformable fractional partial differential equations have demonstrated the effectiveness of various analytical methods in obtaining exact and bifurcating solutions, particularly in nonlinear models. For instance, Zhang et al. [27, 28] and Xu et al. [29] explored exact and traveling wave solutions of time-fractional equations using techniques such as the extended tanh-function method and bifurcation analysis, revealing rich dynamical behaviors like solitons and periodic structures. These studies emphasize the versatility of conformable fractional derivatives in capturing complex physical phenomena. In contrast, the present work focuses on a linear conformable system framework and demonstrates how certain nonlinear conformable equations (e.g., Bernoulli-type and Riccati-type) can be reduced to such linear systems via variable transformations. This approach not only offers analytical tractability but also broadens the applicability of conformable fractional modeling, as exemplified through fluid flow simulations in practical engineering contexts.

A range of social and scientific phenomena are expressed through linear fractional differential systems with variable coefficients, such as linearized aircraft systems, linearized population growth, linearized battery diffusion, and linearized parameter distribution in charge transfer. Despite the abundance of literature on linear fractional differential systems with constant coefficients, only a few studies focus on those with variable coefficients and their explicit solutions. In this context, it needs development and further research.

Bernoulli equation, which is a nonlinear differential equation, has many applications. It is used to describe how a plane generates lift, to compute the velocity of fluid flow, to explain why ships have to run away from each other as they pass, to all incompressible fluid flow problems, to study the unstable potential flow used in the theory of ocean surface waves and acoustics, to describe a locate localized pressure decrease produced by high flow rate near blockages, etc. The ordinary Bernoulli differential equation is a differential equation of the following representation

$$\nu^{'}(\varsigma)+\Delta(\varsigma)\nu(\varsigma)=\Omega(\varsigma)\nu^{n}(\varsigma),\ n\in\mathbb{R},\ n\neq 0,1,$$

where $\Delta(\varsigma)$ and $\Omega(\varsigma)$ are integrable functions. Even though this equation was first thrown out for consideration in 1695 by Jacob Bernoulli, Gottfried Leibniz presented the earliest exact solution which was published in the same year. In order to obtain this solution, Leibniz employed a substitution to transform the ordinary Bernoulli equations to a linear differential equations as Kline stated in [30]. It seems that this method (approach) is still exploited today. Moreover, the Bernoulli equation contains the equation known as the logic differential equation.

The ordinary Riccati differential equation is also a nonlinear differential equation whose exact solution can be easily obtained under the existence of a particular solution of it. It has various application in engineering and science problems, such as robust stabilization, stochastic theory, optimal control, financial mathematics.

According to our observation, no one has considered the fractional version of the ordinary differential equations in the sense of obtaining exact solutions even though fractional calculus has been quite improved in the last decades. That is why it is either so difficult or impossible to solve it based on all fractional approaches. Thus we will use the classical approach to solve the conformable fractional Bernoulli-type differential equation because the comfortable fractional derivative adopts the classical method.

Nonlinear differential equations are necessary equipments to model many physical and social phenomena. As can be observed in the literature, obtaining the exact solution of a non-linear equation is either so difficult or impossible. Although the Bernoulli and Riccati equations are nonlinear differential equations, their exact solution can be easily acquired.

When having look at works related to the Bernoulli and Riccati fractional equations in the last years, Johansyah et al. in [31] the Bernoulli fractional differential equation using adomian decomposition method. Cang et al. in [32] obtained series solution to the Riccati differential equation with fractional order. Momani et al. in [33] settled out the fractional differential equations via decomposition method. Their numerical solutions were investigated in the references [34–36].

To the best of my knowledge, no one has considered the fractional Bernoulli type differential equations and the fractional Riccati type differential equations in the conformable sense so far. This and the cited-above works inspire us to consider the comfortable fractional Bernoulli-type and Riccati type differential equation, respectively, as noted below

$$\mathbb{D}^{\beta}\nu(\varsigma)+\Delta(\varsigma)\nu(\varsigma)=\Omega(\varsigma)\nu^{n}(\varsigma) \tag{1}$$

where $\mathscr{D}^{\beta}$ is the conformable fractional derivative, $\Delta(\varsigma)$ and $\Omega(\varsigma)$ are β-integrable functions on $(0,\infty)$, and $n\neq 0,1$, and

$$\mathbb{D}^{\beta}\nu(\varsigma)=\mho(\varsigma)+\Delta(\varsigma)\nu(\varsigma)+\Omega(\varsigma)\nu^{2}(\varsigma), \tag{2}$$

where $\mho(\varsigma)$ is also β-integrable with $\Delta(\varsigma)\neq 0$ and $\Omega(\varsigma)\neq 0$.

2. Preliminaries

Definition 1 [37] If a function $\nu:[0,\infty)\to\mathbb{R}$ guarantees the existence of

$$\mathbb{I}^{\beta}\nu(\varsigma)=\int_{0}^{\varsigma}\nu(s)\,s^{\beta-1}ds,\ \ \varsigma>0,$$

then ν is β-(fractional) integrable.

Definition 2 [37] For a function $\nu:[0,\infty)\to\mathbb{R}$, the following fractional expression

$$\mathbb{D}^{\beta}\nu(\varsigma)=\lim_{\varepsilon\to 0}\frac{\nu\left(\varsigma+\varepsilon\varsigma^{1-\alpha}\right)-\nu(\varsigma)}{\varepsilon},\ \ \varsigma>0,\ \ 0<\beta\leq 1,$$

is said to be the β-ordered conformable derivative of ν. In addition,

$$\mathbb{D}^{\beta}\nu(0)=\lim_{\varsigma\to 0^{+}}\mathbb{D}^{\beta}\nu(\varsigma),$$

when $\nu(.)$ is differentiable and

$$\lim_{\varsigma\to 0^{+}}\mathbb{D}^{\beta}\nu(\varsigma),$$

exists.

Lemma 1 [38] For $0<\beta\leq 1$, the β-ordered conformable fractional derivative of a function $\nu:[0,\infty)\to\mathbb{R}$ exists if and only if it is differentiable at a point ς, and also

$$\mathbb{D}^{\beta}\nu(\varsigma)=\varsigma^{1-\beta}\nu^{'}(\varsigma).$$

Theorem 1 [18] Let $\nu(\varsigma)$ be continuous on the domain of $\mathbb{I}^{\beta}$, then $\mathbb{D}^{\beta}\mathbb{I}^{\beta}\nu(\varsigma)=\nu(\varsigma)$ and $\mathbb{I}^{\beta}\mathbb{D}^{\beta}\nu(\varsigma)=\nu(\varsigma)$.

3. Solution to linear conformable system

In this section, we introduce a linear conformable system with variable coefficients and look for its explicit solution to use it in coming section.
A linear conformable system with variable coefficients is introduced by

$$\mathbb{D}^{\beta}\nu(\varsigma)=\Delta(\varsigma)\nu(\varsigma)+\Omega(\varsigma),\ \Delta(\varsigma),\nu(\varsigma),\Omega(\varsigma)\in\mathbb{R}, \quad (3)$$

having an initial condition $\nu(0)=\nu_0$.
Theorem 2 An explicit solution of the linear conformable system with variable coefficients (3) is given by

$$\nu(\varsigma)=e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\left[\int_0^{\varsigma}e^{-\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\Omega(\varsigma)\varsigma^{\beta-1}d\varsigma+C\right],$$

where $C=\nu_0$.
In order to prove this theorem, we mainly use constant variation method along with the state-transition approach, which can be seen as a fractional approach because it is frequently used to get a solution to a linear conformable system with variable coefficients. To apply the constant variation method, we must get a solution of the homogeneous linear conformable system with variable coefficients. To do this, we utilize the state-transition operator.
Definition 3 The state-transition operator of system (3) is defined as noted below

$$\mathbb{P}(\varsigma,\eta)=\sum_{k=0}^{\infty}\mathbb{J}_{\eta}^{k\circ\beta}\Delta(\varsigma),$$

where

$$\mathbb{J}_{\eta}^{k\circ\beta}\Delta(\varsigma)=1,$$

and

$$\mathbb{J}_{\eta}^{(k+1)\circ\beta}\Delta(\varsigma)=\mathbb{I}^{\beta}\left(\Delta(\varsigma)\mathbb{J}_{\eta}^{k\circ\beta}\Delta(\varsigma)\right),\ k=0,1,2,\ldots.$$

Proof of Theorem 3: We claim that the series $\mathbb{P}(\varsigma,0)$ is a solution to the homogeneous linear conformable system with variable coefficients, that is,

$$\mathbb{D}^{\beta}\mathbb{P}(\varsigma,0)=\Delta(\varsigma)\mathbb{P}(\varsigma,0).$$

Keeping Theorem 2 and Definition 3 in mind, one has

$$\begin{aligned}\mathbb{D}^{\beta}\mathbb{P}(\varsigma,0)&=\mathbb{D}^{\beta}\sum_{k=0}^{\infty}\mathbb{J}_{\eta}^{k\circ\beta}\Delta(\varsigma)\\&=\sum_{k=1}^{\infty}\mathbb{D}^{\beta}\mathbb{J}_{\eta}^{k\circ\beta}\Delta(\varsigma)\\&=\sum_{k=1}^{\infty}\mathbb{D}^{\beta}\mathbb{I}^{\beta}\left(\Delta(\varsigma)\mathbb{J}_{\eta}^{(k-1)\circ\beta}\Delta(\varsigma)\right)\\&=\sum_{k=1}^{\infty}\Delta(\varsigma)\mathbb{J}_{\eta}^{(k-1)\circ\beta}\Delta(\varsigma)\\&=\Delta(\varsigma)\sum_{k=1}^{\infty}\mathbb{J}_{\eta}^{(k-1)\circ\beta}\Delta(\varsigma)\\&=\Delta(\varsigma)\sum_{k=0}^{\infty}\mathbb{J}_{\eta}^{k\circ\beta}\Delta(\varsigma)\\&=\Delta(\varsigma)\mathbb{P}(\varsigma,0).\end{aligned}$$

Before continuing the proof, let's expand the series $\mathbb{P}(\varsigma,0)$ as follows:

$$\begin{aligned}\mathbb{P}(\varsigma,0)&=\sum_{k=0}^{\infty}\mathbb{J}_{0}^{k\circ\beta}\Delta(\varsigma)\\&=\mathbb{J}_{0}^{0\circ\beta}\Delta(\varsigma)+\mathbb{J}_{0}^{1\circ\beta}\Delta(\varsigma)+\ldots+\mathbb{J}_{0}^{k\circ\beta}\Delta(\varsigma)+\ldots.\end{aligned}\quad (4)$$

This time, our claim is as noted below:

$$\mathbb{J}_{0}^{k\circ\beta}\Delta(\varsigma)=\frac{1}{k!}\left(\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma\right)^{k},\ k=0,1,2,\ldots.$$

Let's apply the mathematical induction on k. For $k=0$, it is satisfactory. Assume that it is hold for $k=n$, that is,

$$\mathbb{J}_{0}^{n\circ\beta}\Delta(\varsigma)=\frac{1}{n!}\left(\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma\right)^{n}.$$

For $k=n+1$, consider

$$\begin{aligned}\mathbb{J}_{0}^{(n+1)\circ\beta}\Delta(\varsigma)&=\mathbb{I}^{\beta}\left(\Delta(\varsigma)\mathbb{J}_{0}^{n\circ\beta}\Delta(\varsigma)\right)\\&=\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}\mathbb{J}_{0}^{n\circ\beta}\Delta(\varsigma)d\varsigma\\&=\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}\left[\frac{1}{n!}\left(\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma\right)^{n}\right]d\varsigma,\end{aligned}$$

implementing the integration by parts, one can easily get the following result:

$$\mathbb{J}_{0}^{(n+1)\circ\beta}\Delta(\varsigma)=\frac{1}{(n+1)!}\left(\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma\right)^{n+1},$$

which confirms our second claim. From (4), it can be written by

$$\mathbb{P}(\varsigma,0)=1+\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma+\ldots+\frac{1}{k!}\left(\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma\right)^{k}+\cdots=e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}.$$

Then, the general solution to the homogeneous linear conformable system having variable coefficients is given by

$$\nu_h(\varsigma)=Ce^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}$$

where $C\in\mathbb{R}$. Based on the variation of constant method, we look for a special solution as noted below:

$$\nu_s(\varsigma)=C(\varsigma)e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma},$$

which has to satisfy the linear conformable system having variable coefficients, that is,

$$\mathbb{D}^{\beta}\left(C(\varsigma)e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\right)=\Delta(\varsigma)C(\varsigma)e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}+\Omega(\varsigma).$$

It is equal to

$$C(\varsigma)\mathbb{D}^{\beta}e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}+e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\mathbb{D}^{\beta}C(\varsigma)=\Delta(\varsigma)C(\varsigma)e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}+\Omega(\varsigma).$$

If the above equation is simplified, one gets the below one

$$e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\mathbb{D}^{\beta}C(\varsigma)=\Omega(\varsigma).$$

It follows that

$$C(\varsigma)=\int_0^{\varsigma}e^{-\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\Omega(\varsigma)\varsigma^{\beta-1}d\varsigma.$$

The whole solution which is the sum of $v_h(\varsigma)$ and $v_s(\varsigma)$ is given as follows

$$v(\varsigma)=Ce^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}+e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\int_0^{\varsigma}e^{-\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\Omega(\varsigma)\varsigma^{\beta-1}d\varsigma.$$

For $\varsigma=0$, the desired result is obtained from the just above equation, that is,

$$v(\varsigma)=e^{\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\left[\int_0^{\varsigma}e^{-\int_0^{\varsigma}\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\Omega(\varsigma)\varsigma^{\beta-1}d\varsigma+C\right],$$

where $C=v_0$.
The proof of Theorem 3 explains why the two different solutions presented in [39] must be the same.

4. Solution to conformable Bernoulli equation

In this section, we look for an exact solution to the nonlinear conformable fractional Bernoulli type differential equations introduced in (1). To do this, we employ a substitution to transform nonlinear conformable fractional Bernoulli type differential equations to the conformable fractional differential equations (or the conformable systems with variable coefficients), which was investigated in Section 3.
Theorem 3 A representation of an explicit solution to the nonlinear conformable fractional Bernoulli type differential equation given in (1) can be expressed by

$$v(\varsigma)=\left[e^{(n-1)\int\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\left((1-n)\int e^{(1-n)\int\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\Omega(\varsigma)\varsigma^{\beta-1}d\varsigma+C\right)\right]^{1-n}, \tag{5}$$

where e^{ς} is the known exponential function and C is an integration constant.
Proof If both sides of the equation (1) are divided by the factor of $\Omega(\varsigma)$, it is equivalent to the following equation

$$\frac{\mathbb{D}^{\beta}v(\varsigma)}{v^{n}(\varsigma)}+\Delta(\varsigma)\frac{v(\varsigma)}{v^{n}(\varsigma)}=\Omega(\varsigma).$$

In terms of Lemma 2, the above equation transforms to the below form

$$\varsigma^{\beta}\frac{v'(\varsigma)}{v^{n}(\varsigma)}+\Delta(\varsigma)\frac{1}{v^{n-1}(\varsigma)}=\Omega(\varsigma). \tag{6}$$

If one applies the substitution $z(\varsigma)=\frac{1}{v^{n-1}(\varsigma)}$ with $\frac{z'(\varsigma)}{1-n}=\frac{v'(\varsigma)}{v^{n}(\varsigma)}$, (6) turns into the following equation

$$\varsigma^{\beta}\frac{z'(\varsigma)}{1-n}+\Delta(\varsigma)z(\varsigma)=\Omega(\varsigma).$$

In other words,

$$\mathbb{D}^{\beta}z(\varsigma)=-(1-n)\Delta(\varsigma)z(\varsigma)+(1-n)\Omega(\varsigma), \tag{7}$$

which is the linear conformable system with variable coefficients with respect to the function $z(\varsigma)$ discussed in Section 3.
According to this, an explicit solution of (7) is obtained in the following structure

$$z(\varsigma)=e^{(n-1)\int\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\left((1-n)\int e^{(1-n)\int\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\Omega(\varsigma)\varsigma^{\beta-1}d\varsigma+C\right).$$

Based on the backward substitution $z(\varsigma)=\frac{1}{v^{n-1}(\varsigma)}$, a representation of the desired exact solution to the equation (1) can be noted as follows

$$v(\varsigma)=\left[e^{(n-1)\int\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\left((1-n)\int e^{(1-n)\int\Delta(\varsigma)\varsigma^{\beta-1}d\varsigma}\Omega(\varsigma)\varsigma^{\beta-1}d\varsigma+C\right)\right]^{1-n}.$$

The proof is completed.
Example 1 We consider the following comfortable fractional Bernoulli-type differential equation as noted below

$$\begin{cases}\mathbb{D}^{\beta}v(\varsigma)+5v(\varsigma)=10v^{2}(\varsigma),\\ v(0)=\frac{1}{3}.\end{cases}$$

From (5), the corresponding solution is given by

$$\begin{aligned}v(\varsigma)&=\left[e^{5\int\varsigma^{\beta-1}d\varsigma}\left(-10\int e^{-5\int\varsigma^{\beta-1}d\varsigma}\varsigma^{\beta-1}d\varsigma+C\right)\right]^{-1}\\&=\left[2+Ce^{5\frac{\varsigma^{\beta}}{\beta}}\right]^{-1}.\end{aligned} \tag{8}$$

Now, let us make some calculation to verify the obtained general solution as follows:

$$\mathbb{D}^{\beta}\left(2+Ce^{5\frac{\varsigma^{\beta}}{\beta}}\right)^{-1}+5\left(2+Ce^{5\frac{\varsigma^{\beta}}{\beta}}\right)^{-1}=10\left(2+Ce^{5\frac{\varsigma^{\beta}}{\beta}}\right)^{-2}$$

$$\Rightarrow\frac{-5Ce^{5\frac{\varsigma^{\beta}}{\beta}}}{\left(2+Ce^{5\frac{\varsigma^{\beta}}{\beta}}\right)^{2}}+5\frac{\left(2+Ce^{5\frac{\varsigma^{\beta}}{\beta}}\right)}{\left(2+Ce^{5\frac{\varsigma^{\beta}}{\beta}}\right)^{2}}-10\frac{1}{\left(2+Ce^{5\frac{\varsigma^{\beta}}{\beta}}\right)^{2}}=0.$$

When considering the initial condition $v(0)=\frac{1}{3}$, it is easy to get $C=1$. The explicit solution to the nonlinear Bernoulli-type differential equation with the comfortable fractional derivative of order $\beta=\frac{1}{5}$ is given by

$$v(\varsigma)=\frac{1}{2+e^{25\varsigma^{\frac{1}{5}}}}.$$

In figure 1, graphs of the solutions (8) are drawn in the cases of $\beta=0.5,\ \beta=0.6,\ \beta=0.7,\ \beta=0.9,\ \beta=1$ with the initial conditions $v(0)=\frac{1}{3}$ with $C=1$ to show the change of the solutions depending on the changing parameter β.
Remark 2

(i) If $n=0$, then the equation (1) reduces to that of Section 3.

(ii) Let us consider the case of $n=1$. Then the equation (1) turns into the following structure

$$\mathbb{D}^{\beta}v(\varsigma)+\Delta(\varsigma)v(\varsigma)=\Omega(\varsigma)v(\varsigma).$$

One can write it as follows

$$\frac{v'(\varsigma)}{v(\varsigma)}=\left[\Omega(\varsigma)-\Delta(\varsigma)\right]\varsigma^{\beta-1}d\varsigma, \qquad (9)$$

which is called the separable (conformable) fractional differential equation. It is known that the separable differential equation is a differential equation in which the variables can be separated from each other. So, the separable (conformable) fractional differential equation is the separable differential equation with the fractional order. In order to solve, integrating both sides of (9), one has

$$v(\varsigma)=e^{\int[\Omega(\varsigma)-\Delta(\varsigma)]\varsigma^{\beta-1}d\varsigma}+C, \qquad (10)$$

which is different from (5) in the case of $n=1$.
Example 2 We consider the following comfortable fractional Bernoulli-type differential equation in the special case of $n=1$, which is also called the separable (conformable) fractional differential equation,

$$\begin{cases} \varsigma^{-1}\mathbb{D}^{\frac{2}{3}}v(\varsigma)-v(\varsigma)=2v(\varsigma), \\ v(0)=2. \end{cases}$$

From (10), the corresponding solution is given by

$$v(\varsigma)=e^{\int[2\varsigma+\varsigma]\varsigma^{\frac{2}{3}-1}d\varsigma}+C=e^{\int 3\varsigma^{\frac{2}{3}}d\varsigma}+C=e^{\frac{9}{5}\varsigma^{\frac{5}{3}}}+C.$$

When considering the initial condition, it is easy to get $C=1$. The explicit solution to the he separable (conformable) fractional differential equation is given by

$$v(\varsigma)=1+e^{\frac{9}{5}\varsigma^{\frac{5}{3}}}.$$

Example 3 We consider the following comfortable fractional Bernoulli-type differential equation without any initial conditions

$$\mathbb{D}^{\beta}v(\varsigma)-v(\varsigma)=\varsigma v^{\frac{1}{2}}(\varsigma).$$

Dividing the equation by $v^{\frac{1}{2}}(\varsigma)$ and applying $\eta=\frac{1}{v^{\frac{1}{2}}}$, one obtains

$$\mathbb{D}^{\beta}\eta(\varsigma)=\frac{1}{2}\eta(\varsigma)+\frac{\varsigma}{2}.$$

which is the linear conformable system with variable coefficients with respect to the function η investigated in Section 3.

$$\begin{aligned}\eta(\varsigma)&=e^{-\frac{1}{4}\int\varsigma^{\beta-1}d\varsigma}\left(\frac{1}{4}\int e^{-\frac{1}{4}\int\varsigma^{\beta-1}d\varsigma}\varsigma^{\beta}d\varsigma+C\right)\\&=e^{-\frac{1}{4}\frac{\varsigma^{\beta}}{\beta}}\left(-4^{\frac{1}{\beta}}\beta^{\frac{1}{\beta}}\Gamma\left(\frac{\beta+1}{\beta},\frac{\varsigma^{\beta}}{4\beta}\right)+C\right)\end{aligned}$$

where $\Gamma(.)$ is the known incomplete gamma function. Based on the backward substitution $v=\eta^{-2}$, a representation of the desired exact solution to the equation can be noted as follows

$$v(\varsigma)=\left[e^{-\frac{1}{4}\frac{\varsigma^{\beta}}{\beta}}\left(-4^{\frac{1}{\beta}}\beta^{\frac{1}{\beta}}\Gamma\left(\frac{\beta+1}{\beta},\frac{\varsigma^{\beta}}{4\beta}\right)+C\right)\right]^{-2}.$$

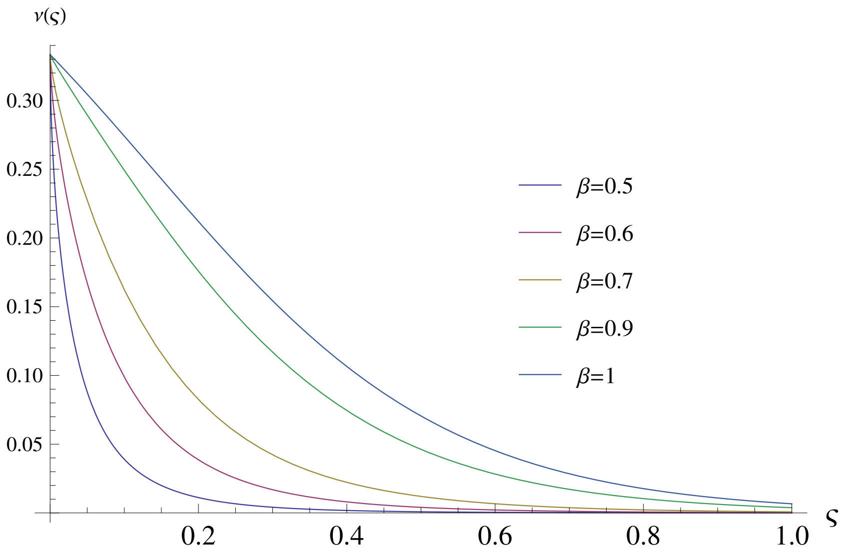

Figure 1. Graphs of the solutions (8) with $v(0)=1/3$ and $C=1$ for each $\beta=0.5$, $\beta=0.6$, $\beta=0.7$, $\beta=0.9$, and $\beta=1$

5. Solution to conformable Riccati equation

In this section, we mainly investigate an exact solution to the nonlinear conformable fractional Riccati type differential equations introduced in (2). In the literature, without any conditions, the ordinary Riccati differential equation can be solved by transforming to a second order linear equation. This approach is so difficult to adapt the fractional sense and also extra arguments will be necessary. So we research for an exact solution to the nonlinear conformable fractional Riccati type differential equations under the assumption of the existence of a particular solution of it.
Remark 3

- (i) The equation (2) with $\Delta(\varsigma)=0$ reduces to the conformable fractional type differential equation which is introduced in (1).
- (ii) The equation (2) with $\Omega(\varsigma)=0$ becomes the linear conformable fractional equation with variable coefficients discussed in Section 3.

Theorem 4 An explicit solution to the equation (2) is given by

$$v(\varsigma)=v_1(\varsigma)+\left[e^{-\int R(\varsigma)\varsigma^{\beta-1}d\varsigma}\left(-\int e^{\int R(\varsigma)\varsigma^{\beta-1}d\varsigma}\Omega(\varsigma)\,\varsigma^{\beta-1}d\varsigma+C\right)\right]^{-1}, \tag{11}$$

where $v_1(\varsigma)$ is a particular solution and $R(\varsigma)=\Delta(\varsigma)+2\Omega(\varsigma)v_1(\varsigma)$.
Proof Let $v_1(\varsigma)$ be a particular solution of the equation (2), i.e.,

$$\mathbb{D}^{\beta}v_1(\varsigma)=\mho(\varsigma)+\Delta(\varsigma)v_1(\varsigma)+\Omega(\varsigma)v_1^2(\varsigma),$$

Based on the substitution $v(\varsigma)=v_1(\varsigma)+\frac{1}{\eta(\varsigma)}$ with $v'(\varsigma)=v_1'(\varsigma)-\frac{\eta'(\varsigma)}{\eta^2(\varsigma)}$, one gets

$$\varsigma^{1-\beta}\left(v_1'(\varsigma)-\frac{\eta'(\varsigma)}{\eta^2(\varsigma)}\right)=\mho(\varsigma)+\Delta(\varsigma)\left(v_1(\varsigma)+\frac{1}{\eta(\varsigma)}\right)\Omega(\varsigma)\left(v_1(\varsigma)+\frac{1}{\eta(\varsigma)}\right)^2.$$

Under simple calculation, the following is obtained

$$\mathbb{D}^{\beta}\eta(\varsigma)=-\left[\Delta(\varsigma)+2\Omega(\varsigma)v_1(\varsigma)\right]\eta(\varsigma)-\Omega(\varsigma),$$

which is the linear conformable fractional equation with variable coefficients depending on $\eta(\varsigma)$. In the light of this work, one gets

$$\eta(\varsigma)=\left[e^{-\int R(\varsigma)\varsigma^{\beta-1}d\varsigma}\left(-\int e^{\int R(\varsigma)\varsigma^{\beta-1}d\varsigma}\Omega(\varsigma)\varsigma^{\beta-1}d\varsigma+C\right)\right],$$

where $R(\varsigma)=\Delta(\varsigma)+2\Omega(\varsigma)v_1(\varsigma)$. Based on the backward substitution, the desired exact solution of the system (2) can be easily obtained.
Example 4 We take the comfortable fractional Riccati type differential equation into consideration

$$\begin{cases}\mathbb{D}^{\beta}v(\varsigma)+v^2(\varsigma)-1=0,\\ \quad v(0)=3.\end{cases}$$

Based on the Riccati structure given in (2), it can be rearranged as follows

$$\begin{cases}\mathbb{D}^{\beta}v(\varsigma)=1-v^2(\varsigma),\\ \quad v(0)=3.\end{cases}$$

From (11), the corresponding solution is given by

$$\begin{aligned}v(\varsigma)&=1+\left[e^{2\int\varsigma^{\beta-1}d\varsigma}\left(\int e^{-2\int\varsigma^{\beta-1}d\varsigma}\varsigma^{\beta-1}d\varsigma+C\right)\right]^{-1}\\&=1+\left(-\frac{1}{2}+Ce^{2\frac{\varsigma^{\beta}}{\beta}}\right)^{-1}\\&=1+\frac{2}{2Ce^{2\frac{\varsigma^{\beta}}{\beta}}-1}\\&=\frac{2Ce^{2\frac{\varsigma^{\beta}}{\beta}}+1}{2Ce^{2\frac{\varsigma^{\beta}}{\beta}}-1}.\end{aligned} \tag{12}$$

Now, let us make some calculation to verify the obtained general solution as follows:

$$\begin{aligned}&\mathbb{D}^{\beta}\left(1+\frac{2}{2Ce^{2\frac{\varsigma^{\beta}}{\beta}}-1}\right)+\left(1+\frac{2}{2Ce^{2\frac{\varsigma^{\beta}}{\beta}}-1}\right)^2-1\\&=\frac{-8Ce^{2\frac{\varsigma^{\beta}}{\beta}}}{\left(2Ce^{2\frac{\varsigma^{\beta}}{\beta}}-1\right)^2}+\frac{4}{\left(2Ce^{2\frac{\varsigma^{\beta}}{\beta}}-1\right)^2}+\\&\frac{4\left(2Ce^{2\frac{\varsigma^{\beta}}{\beta}}-1\right)}{\left(2Ce^{2\frac{\varsigma^{\beta}}{\beta}}-1\right)^2}=0.\end{aligned}$$

When considering the initial condition $v(0)=3$, it is easy to get $C=1$. The explicit solution to the nonlinear Riccati type differential equation with the comfortable fractional derivative of order $\beta=\frac{2}{3}$ is given by

$$v(\varsigma)=\frac{2e^{3\varsigma^{\frac{2}{3}}}+1}{2e^{3\varsigma^{\frac{2}{3}}}-1}.$$

In figure 2, graphs of the solutions (12) are drawn in the cases of $\beta=0.3$, $\beta=0.5$, $\beta=0.6$, $\beta=0.7$, $\beta=0.9$, $\beta=1$ with the initial conditions $v(0)=3$ with $C=1$ to show the change of the solutions depending on the changing parameter β.

6. An application to liquid flow in reservoirs, tanks, and funnels

We recognize that water exits the tank in “volume”, and the departing water volume is related to “velocity” as expressed

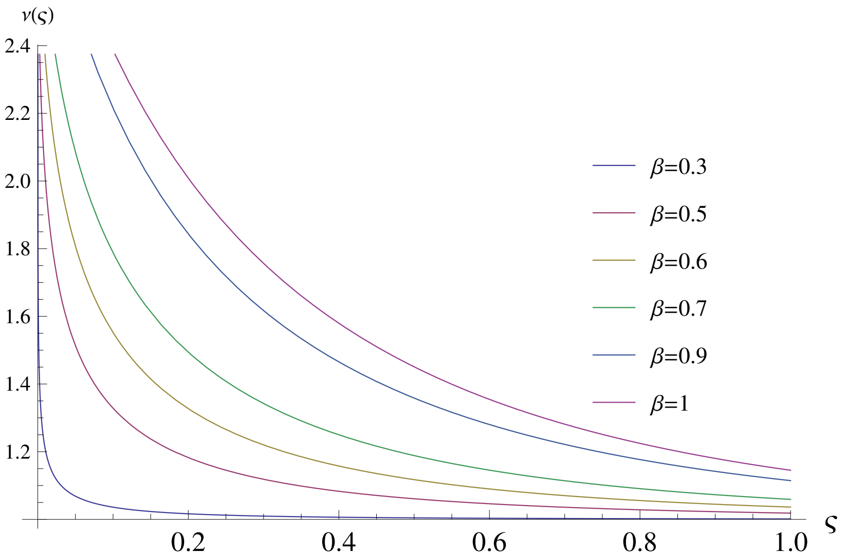

Figure 2. Graphs of the solutions (12) with $v(0) = 3$ and $C = 1$ for each $\beta = 0.3$, $\beta = 0.5$, $\beta = 0.6$, $\beta = 0.7$, $\beta = 0.9$, and $\beta = 1$.

in Equation $\dot{V} = \frac{\dot{Q}}{\rho} = Av \quad (\mathrm{m}^3/\mathrm{s})$. ρ is the mass density of the fluid kg/m^3, A is the cross-sectional area m^2, v is the velocity m/s, Δt is the duration of fluid flow s. The base units associated with the above quantities are kilograms kg, meters m, and seconds s.

T he amount of water that leaves the tank during the time increment or interval Δt can thus be described in words by the physical expression:

(linear distance traveled by the volume of water)
× (the cross-sectional area of the passage)

Here 'the cross-sectional area of the passage' means 'the cross-sectional area of the drainage pipe'. With the assumption of negligible friction between the moving water and the container wall, the total volume leaving the tank during time Δt can thus be modeled mathematically as in m^3, as given in $\Delta V = Av(t)\Delta t$, which is the change in volumetric flow if the velocity in a moving fluid varies with time-that is, $v = v(t)$ where ΔV represents the rate of change of volumetric flow, A is the cross-sectional area, and Δt is the time derivative of velocity, with the duration of water flow Δt in seconds.

As presented in $v(t) = \sqrt{2gh}$ which express the exit velocity of the liquid from the reservoir(this derivation includes an assumption that the friction %between the moving liquid and the container wall is negligible), we have the exit velocity $v(t)$ from the tank:

$$v(t) = \sqrt{2gh(t)},$$

where g is the gravitational acceleration. The volume of water leaving the tank through the exit tube during Δt is thus:

$$\Delta V = v(t) \cdot A \cdot \Delta t.$$

We realize at this point that whatever amount of water is leaving the tank through the drainage tube must be equal to the same amount of water supplied by the tank in relation to the drop of water level in the tank.

Now, let us examine the reduction of water level in the tank with $\Delta h(t)$ during the same time interval Δt, as illustrated in figure 3.

The equivalent volume reduction in the tank during the time period Δt is equal to the volume of the "disk" of water shown in gray in figure 3, which has a diameter D and is

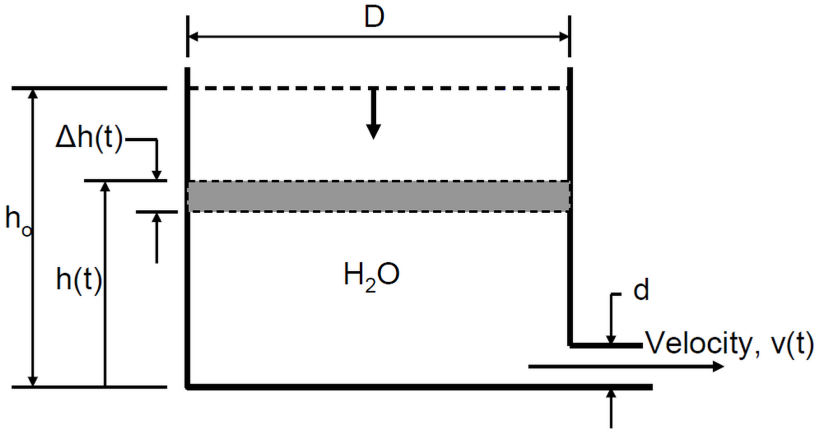

Figure 3. Drop of water level in the tank during drainage.

infinitesimally thin with a thickness of $\Delta h(t)$. Consequently, the volume of water supplied by the tank can be expressed as:

$$\Delta V = -\frac{\pi D^2}{4}\Delta h(t).$$

It must be remembered to apply a negative $(-)$ sign in the above expression for ΔV because the water level $\Delta h(t)$ in the tank reduces (thus the reduction of volume of water in the tank) with the increase of the time variable t.

The law of conservation of mass requires the total volume of water leaving the tank during the time increment Δt to be equal to the total volume of water supplied by the tank. We thus arrive at the equality:

$$-\frac{\pi D^2}{4}\Delta h(t) = \frac{\pi d^2}{4}\sqrt{2gh(t)}\Delta t,$$

or in a different form:

$$\frac{\Delta h(t)}{\Delta t} = -\left[h(t)\right]^{1/2}\left(\frac{d^2}{D^2}\right)\sqrt{2g}.$$

Because the drainage of the tank is a continuous process, which implies that $\Delta \to 0$, we may replace the finite increment denoted by Δ in the above equation by the infinitesimally small increment designated by d as $\Delta \to 0$. We will then have the following differential equation describing the drainage process:

$$\frac{dh(t)}{dt} = -\sqrt{2g}\left(\frac{d^2}{D^2}\right)\sqrt{h(t)}, \tag{13}$$

where d is the diameter of a small drainage tube attached at the bottom of the tank. Equation (13) can be used to determine the instantaneous water level $h(t)$ in the water tank at any given time t during the drainage process. For more details, readers can review the book [40].

The initial value problem (IVP) associated with the conformable Bernoulli fractional equations can be restructured by substituting the ordinary derivative for the fractional derivative , where $0 < \beta \leq 1$, and incorporating an initial condition. Within this framework, the requisite initial value problem can be redefined as delineated below:

$$\begin{cases} \mathbb{D}^{\beta} h(t) = -\sqrt{2g}\left(\frac{d^2}{D^2}\right)\sqrt{h(t)}, & t \in (0,T], \\ h(0) = h_0, \end{cases} \tag{14}$$

where

$$\lim_{\beta \to 1} \mathbb{D}^{\beta} h(t) = \frac{dh}{dt}.$$

Based on (5), the solution to the application system (14) can be expressed as follows

$$h(t) = \left[-\sqrt{2g}\left(\frac{d^2}{D^2}\right)\int t^{\beta-1}dt + \sqrt{h_0}\right]^{1-\frac{1}{2}}.$$

For tank diameter $D = 30.48cm$, drain pipe diameter $d = 2.54$, Initial water level in the tank $h_0 = 30.48cm$, gravitational acceleration $g = 981.86\ cm/s^2$, then the time (T) required to empty the tank is for $\beta = 34$ is 37.14 seconds. In figure 4, one can observe the effect of β on T. In general, it can be said that as the β value increases, the emptying time of the tank decreases. Thus, we can observe how the β parameter affects the process solution to the system.

7. Conclusion

In this paper, the Bernoulli type and Riccati type differential equations in the conformable sense are introduced in addition to the separable (conformable) fractional differential equation. By reducing them to the linear conformable systems with variable coefficients, their explicit solutions are investigated via ordinary approach. Numerical and simulated examples are offered to illustrate our theoretical findings. An application to liquid flow in reservoirs, tanks, and funnels is presented.

As future work, different kinds of stability and controllability of the Riccati type differential equations with any initial conditions can be discussed. Since difficult integrals may be encountered in the solutions of the equations, numerical solutions of the equations can be developed.

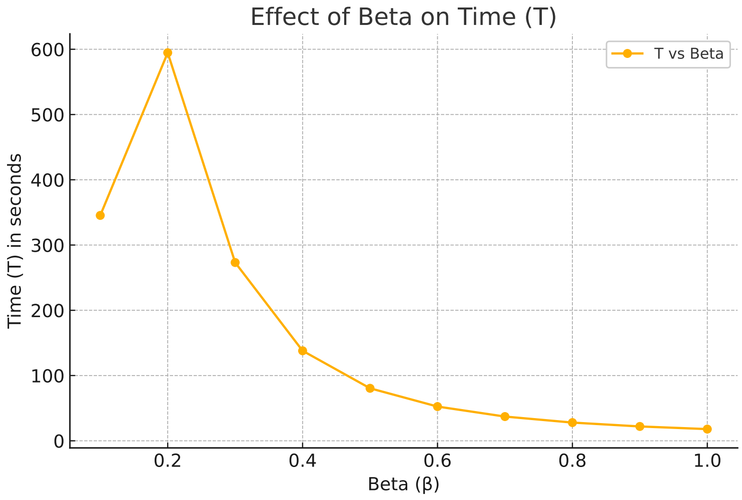

Figure 4. Effect of β on T for the solution to system (14).

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Availability of data and materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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