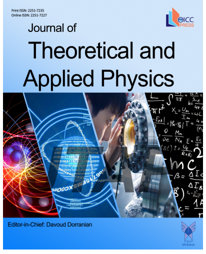

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A hypergeometric sequence of biorthogonal polynomials and its relationship with Meixner-Pollaczek polynomials

Mohammad Masjed-Jamei[1,*], Nasser Saad[2]

[1] *Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, Iran.*
[2] *School of Mathematical and Computational Sciences, University of Prince Edward Island, Charlottetown, Canada.*

[*] **Corresponding author:** mmjamei@kntu.ac.ir

Original Research

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Abstract:

Some orthogonal polynomial systems are mapped onto each other by the Fourier trans-form. In this paper, based on generalized Laguerre polynomials, we introduce a new sequence of biorthogonal hypergeometric polynomials and obtain its orthogonali-ty relation explicitly. In this sense, we also recall some similar results available in the literature as the sequence introduced in this work is the last one for classical cases.

Keywords: Generalized Laguerre polynomials; Meixner-Pollaczek polynomials; Fourier trans-form; Parseval identity; Hypergeometric series

1. Introduction

Using a suitable integral transform, many orthogonal sets can be mapped onto other orthogonal sets in the weighted -space. For example, Koornwinder in [1] has proved that the classical Jacobi polynomials can be mapped onto the Wilson polynomials by using Jacobi function transform. In this direction, Koelink in [2] has shown that the continuous Hahn polynomials are the Fourier transform of Jacobi orthogonal polynomials. Since in the Askey's scheme of hypergeometric orthogonal polynomials, Wilson polynomials are located in the top with the most degrees of freedom, all mentioned references can be generalized by the integral transform of Wilson polynomials. This line of thought has been followed by Groenevelt in [3]. See also [4, 5] in this regard.

Although in [6] Koornwinder has shown that the classical Laguerre polynomials are mapped onto the Meixner-Pollaczek polynomials by the Mellin transform, using their Fourier transform, a new sequence of hypergeometric biorthogonal polynomials can still be derived together with its explicit orthogonality relation. In this paper, we introduce such type of hypergeometric polynomials and study their general properties. To meet this goal, we begin with the monic type of the generalized Laguerre polynomials [7]

$$\bar{L}_n^{(\alpha)}(x) = (-1)^n(\alpha+1)_n {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\middle| x\right) \tag{1}$$

which are orthogonal with respect to the weight function $x^\alpha e^{-x}$ on $[0,\infty)$ as

$$\int_0^\infty x^\alpha e^{-x}\bar{L}_m^{(\alpha)}(x)\bar{L}_n^{(\alpha)}(x)dx = n!\Gamma(n+\alpha+1)\delta_{n,m} \tag{2}$$

where

$$\delta_{n,m} = \begin{cases} 0 & (n\neq m)\\ 1 & (n=m)\end{cases}$$

Note in definition (1) that ${}_1F_1(.)$ is a special case of the generalized hypergeometric series [8]

$$ {}_pF_q\left(\begin{matrix}a_1,\cdots\ a_p\\ b_1,\cdots\ b_q\end{matrix}\middle| z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} \tag{3}$$

where $(a)_b$ in (3) denotes the Pochhammer symbol given by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)} = \begin{cases} 1 & (b=0,\ a\in\mathbb{C}\backslash\{0\}) \\ a(a+1)\cdots(a+b-1) & (b\in\mathbb{N},\ a\in\mathbb{C}) \end{cases}$$

We here also add although the Askey and Racah polynomials stand at the summit of the Askey scheme of classical orthogonal polynomials, it should not be however over-looked that this framework relies fundamentally on a concept of orthogonality with respect to measures, i.e. an approach that by its nature excludes important families such as the Bessel polynomials. Since the development of the theory of topological vector spaces has offered a significantly broader perspective, the notion of orthogonal-ity can be meaningfully extended beyond the classical Hilbert space setting, see a new theory of least p-variances approximation and p-uncorrelated functions [9] as an im-provement of classical least square theory. Within this more general context, classical families of orthogonal polynomials may be therefore characterized in ways that trans-cend the limitations of measure-based formulations. See e.g. [10–12] for more details.

2. Fourier transform of generalized Laguerre polynomials

To derive the Fourier transform of monic polynomials (1), we will use the well-known Gamma integral

$$\Gamma(z) = \int_0^{+\infty} x^{z-1}e^{-x} \quad (\mathrm{Re}(z)>0)$$

satisfying the basic recurrence relation $\Gamma(z+1) = z\Gamma(z)$.
The Fourier transform of a function, say $g(x)$, is defined as

$$\mathbf{F}(s) = \mathbf{F}(g(x)) = \int_{-\infty}^{+\infty} e^{-isx}g(x)dx$$

with the inverse transform as

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{isx}\mathbf{F}(s)ds$$

The Parseval identity of Fourier theory is given by

$$\int_{-\infty}^{+\infty} g(x)h(x)dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathbf{F}(g(x))\overline{\mathbf{F}(h(x))}ds \tag{4}$$

where $g,h\in L^2(\mathbb{R})$.
Now, referring to (1) and (2) and assuming that $a,b,c,d\in\mathbb{R}$ are free parameters, let us define the two specific functions

$$g(x) = e^{ax}e^{-\frac{1}{2}e^x}\overline{L}_n^{(b)}(e^x) \text{ and } h(x) = e^{cx}e^{-\frac{1}{2}e^x}\overline{L}_m^{(d)}(e^x) \tag{5}$$

in terms of the Laguerre polynomials, to which we will apply the Fourier transform.
Clearly for both defined functions, the Fourier transform exists. However, for $g(x)$ defined in (5) we get

$$\begin{aligned}\mathbf{F}(g(x)) &= \int_{-\infty}^{+\infty} e^{isx}e^{ax}e^{-\frac{1}{2}e^x}\overline{L}_n^{(b)}(e^x)dx \\ &= \int_0^{+\infty} t^{-is-1+a}e^{-\frac{1}{2}t}\overline{L}_n^{(b)}(t)dt \\ &= (-1)^n(b+1)_n\sum_{k=0}^n\frac{(-n)_k}{(b+1)_k k!} \\ &\times\left(\int_0^{+\infty} t^{-is-1+a+k}e^{-\frac{1}{2}t}dt\right) \\ &= (-1)^n(b+1)_n\sum_{k=0}^n\frac{(-n)_k}{(b+1)_k k!}2^{a-is+k}\Gamma(a-is+k) \\ &= (-1)^n(b+1)_n\Gamma(a-is)2^{a-is}{}_2F_1\left(\left.\begin{matrix}-n,a-is\\ b+1\end{matrix}\right|2\right)\end{aligned} \tag{6}$$

where we have used the well-known identity

$$\Gamma(a+k) = \Gamma(a)(a)_k$$

By substituting (6) into the Parseval's identity (4) we obtain

$$\begin{aligned}&2\pi\int_{-\infty}^{+\infty} e^{(a+c)x}e^{-e^x}\overline{L}_n^{(b)}(e^x)\overline{L}_m^{(d)}(e^x)dx \\ &= 2\pi\int_0^{+\infty} t^{a+c-1}e^{-t}\overline{L}_n^{(b)}(t)\overline{L}_m^{(d)}(t)dt \\ &= (-1)^{n+m}(b+1)_n(d+1)_m\int_{-\infty}^{+\infty}2^{a-is}\Gamma(a-is) \\ &\times\overline{2^{c-is}\Gamma(c-is)}{}_2F_1\left(\left.\begin{matrix}-n,a-is\\ b+1\end{matrix}\right|2\right)\overline{{}_2F_1\left(\left.\begin{matrix}-m,c-is\\ d+1\end{matrix}\right|2\right)}ds \\ &= 2^{a+c}(-1)^{n+m}(b+1)_n(d+1)_m\int_{-\infty}^{+\infty}\Gamma(a-is)\overline{\Gamma(c-is)} \\ &\times{}_2F_1\left(\left.\begin{matrix}-n,a-is\\ b+1\end{matrix}\right|2\right)\overline{{}_2F_1\left(\left.\begin{matrix}-m,c-is\\ d+1\end{matrix}\right|2\right)}ds\end{aligned} \tag{7}$$

On the other hand, if in the left side of (7) we take

$$b = d = a+c-1$$

then according to the orthogonality (2), relation (7) eventually reads as

$$\begin{aligned}&\int_{-\infty}^{+\infty}\Gamma(a-is)\overline{\Gamma(c-is)}{}_2F_1\left(\left.\begin{matrix}-n,a-is\\ a+c\end{matrix}\right|2\right) \\ &\times\overline{{}_2F_1\left(\left.\begin{matrix}-m,c-is\\ a+c\end{matrix}\right|2\right)}ds = \frac{2\pi\Gamma(a+c)n!}{2^{a+c}(a+c)_n}\delta_{n,m}\end{aligned}$$

This means that the hypergeometric polynomial

$$Z_n(x;a,c) = {}_2F_1\left(\left.\begin{matrix}-n,a-x\\ a+c\end{matrix}\right|2\right) \tag{8}$$

satisfies the biorthogonality relation

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Gamma(a-ix)\Gamma(c+ix)Z_n(ix;a,c)Z_m(-ix;c,a)dx \\ = \frac{\Gamma(a+c)n!}{2^{a+c}(a+c)_n}\delta_{n,m}\end{aligned} \tag{9}$$

where $a+c>0$.
Remark 1. For $n=m=0$, (9) is equivalent to

$$\int_{-\infty}^{+\infty}\Gamma(a-ix)\Gamma(c+ix)dx=\frac{2\pi}{2^{a+c}}\Gamma(a+c) \qquad (10)$$

leading to a positive weight function when $a=c$. On the other hand, a remarkable point is that this particular case is closely related to the Meixner-Pollaczek polynomials

$$P_n^{(\lambda)}(x;\theta)=\frac{(2\lambda)_n}{n!}e^{in\theta}{}_2F_1\left(\left.\begin{matrix}-n,\lambda+ix\\2\lambda\end{matrix}\right|1-e^{2i\theta}\right)$$
$$\text{for } \lambda>0 \text{ and } 0<\theta<\pi$$

which are orthogonal on $(-\infty,\infty)$ with respect to the weight function

$$w(x;\lambda,\theta)=|\Gamma(\lambda+ix)|^2e^{(2\theta-\pi)x} \qquad (11)$$

as follows

$$\int_{-\infty}^{+\infty}w(x;\lambda,\theta)P_n^{(\lambda)}(x;\theta)P_m^{(\lambda)}(x;\theta)dx=\frac{2\pi\Gamma(n+2\lambda)}{(2\sin\theta)^{2\lambda}n!}\delta_{n,m}$$

In other words, for $\theta=\frac{\pi}{2}$, the weight function (11) becomes

$$w(x;\lambda,\frac{\pi}{2})=|\Gamma(\lambda+ix)|^2$$

and therefore we have

$$P_n^{(\lambda)}(x;\frac{\pi}{2})=\frac{(2\lambda)_n}{n!}i^n{}_2F_1\left(\left.\begin{matrix}-n,\lambda+ix\\2\lambda\end{matrix}\right|2\right)$$
$$=\frac{(2\lambda)_n}{n!}i^nZ_n(-ix;\lambda,\lambda)$$

in which we have used the clear relations

$$e^{in\frac{\pi}{2}}=(e^{i\frac{\pi}{2}})^n=i^n \text{ and } 1-e^{2i\frac{\pi}{2}}=2$$

In this sense, we add that as the Jacobi polynomials [7] are denoted by

$$P_n^{(\alpha,\beta)}(x)={}_2F_1\left(\left.\begin{matrix}-n,n+\alpha+\beta+1\\\alpha+1\end{matrix}\right|\frac{1-x}{2}\right)$$
$$=(-1)^n{}_2F_1\left(\left.\begin{matrix}-n,n+\alpha+\beta+1\\\beta+1\end{matrix}\right|\frac{1+x}{2}\right) \qquad (12)$$

comparing (8) with (12) yields

$$Z_n(x;a,c)=P_n^{(a+c-1,-x-n-c)}(-3)$$
$$=(-1)^nP_n^{(-x-n-c,a+c-1)}(3)$$

Here we recall that there are some results similar to our results derived in (8) and (9) in the literature. It is shown in [13] that the hypergeometric function

$$A_n(x;a,b,c,d)={}_3F_2\left(\left.\begin{matrix}-n,n-b-c,d-x\\a+d,-c+1-x\end{matrix}\right|1\right)$$

satisfies a finite orthogonality relation of the form

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Gamma(a+ix)\Gamma(b-ix)\Gamma(c+ix)\Gamma(d-ix)$$
$$\times A_n(ix;a,b,c,d)A_m(-ix;d,c,b,a)dx=\frac{n!}{(b+c-2n)}$$
$$\times\frac{\Gamma(b+c+1-n)\Gamma(c+d)\Gamma(a+b)\Gamma^2(a+d)}{\Gamma(a+d+n)\Gamma(a+b+c+d-n)}\delta_{n,m}$$

where $a+d>-n$, $b+c>2n$, $a+b>0$ and $c+d>0$. Moreover, it is shown in [13] that the hypergeometric sequence

$$B_n(x;a,b)={}_2F_1\left(\left.\begin{matrix}-n,n-a-b\\-a+1-x\end{matrix}\right|\frac{1}{2}\right)$$

satisfies the finite orthogonality relation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Gamma(a+ix)\Gamma(b-ix)B_n(ix;a,b)B_m(-ix;b,a)dx$$
$$=\frac{n!\Gamma(a+b+1-n)}{(a+b-2n)2^{a+b}}\delta_{n,m} \Leftrightarrow a+b>2n \qquad (13)$$

Note that replacing $n=m=0$ in (13) exactly gives the same as result (10).
It is shown in [14] that the exponential functions

$$E_n(x;q_1,p_1;q_2,p_2)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{\exp(\frac{-2q_1q_2}{q_1+q_2})}$$
$$\times\exp\left(\frac{q_1k}{q_1+q_2}\big((q_1-q_2)k+q_2-2(q_1p_1+q_2p_2)\big)\right)$$
$$\times\exp(-2q_1kx) \qquad (14)$$

with

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{bmatrix}n\\n-k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$
$$\text{where } (a;q)_k=(1-a)(1-aq)\cdots(1-aq^{k-1})$$

satisfy a biorthogonality relation as

$$\int_{-\infty}^{+\infty}\exp(q_1(ix+p_1)^2+q_2(ix+p_2)^2)E_n(ix;q_1,p_1;q_2,p_2)$$
$$\times E_m(ix;q_2,-p_2;q_1,-p_1)dx$$
$$=\left(\sqrt{\frac{\pi}{q_1+q_2}}\exp\big((2n+(p_2-p_1)^2)\frac{q_1q_2}{q_1+q_2}\big)\right.$$
$$\left.\times\big(\exp(\frac{-2q_1q_2}{q_1+q_2});\exp(\frac{-2q_1q_2}{q_1+q_2})\big)_n\right)\delta_{n,m} \qquad (15)$$

provided that $q_1,q_2>0$ and $p_1,p_2\in\mathbb{R}$.
If in (14) we suppose $q_1=q_2=q/2$ for $q>0$ and $p_1=-p_2$ for $p_1\in\mathbb{R}$, then the spe-cial sequence

$$E_n^{(q)}(x)=E_n(-x;\frac{q}{2},p_1;\frac{q}{2},-p_1)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{e^{-\frac{q}{2}}}e^{k\frac{q}{4}}e^{kqx}$$

which is independent of p_1 satisfies the orthogonality relation

$$\int_{-\infty}^{+\infty}e^{-qx^2}E_n^{(q)}(ix)E_m^{(q)}(-ix)dx$$
$$=\int_{-\infty}^{+\infty}e^{-qx^2}E_n^{(q)}(-ix)E_m^{(q)}(ix)dx$$
$$=\left(\sqrt{\frac{\pi}{q}}e^{n\frac{q}{2}}(e^{-\frac{q}{2}};e^{-\frac{q}{2}})_n\right)\delta_{n,m}$$

Moreover, it can be verified that

$$E_n^{(q)}(ix)=C_n^{(q)}(x)+iS_n^{(q)}(x) \text{ and } E_n^{(q)}(-ix)=C_n^{(q)}(x)-iS_n^{(q)}(x)$$

in which

$$C_n^{(q)}(x)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{e^{-\frac{q}{2}}}e^{k\frac{q}{4}}\cos(kqx) \tag{16}$$

and

$$S_n^{(q)}(x)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{e^{-\frac{q}{2}}}e^{k\frac{q}{4}}\sin(kqx) \tag{17}$$

Finally, if relations (16) and (17) are replaced in (15), the following result is derived: The trigonometric sequence [14]

$$\begin{aligned}M_{n,+}^{(q)}(x)&=\frac{S_n^{(q)}(x)+C_n^{(q)}(x)}{\sqrt{2}}\\&=\sum_{k=0}^{n}(-e^{q/4})^k\begin{bmatrix}n\\k\end{bmatrix}_{e^{-\frac{q}{2}}}\sin(kqx+\frac{\pi}{4})\end{aligned}$$

satisfies the real orthogonality relation

$$\begin{aligned}&\int_{-\infty}^{+\infty}e^{-qx^2}M_{n,+}^{(q)}(x)M_{m,+}^{(q)}(x)dx\\&=\left(\sqrt{\frac{\pi}{4q}}e^{n\frac{q}{2}}(e^{-\frac{q}{2}};e^{-\frac{q}{2}})_n\right)\delta_{n,m}\ \Leftrightarrow\ q>0\end{aligned}$$

It is shown in [15] that the hypergeometric sequence

$$\begin{aligned}K_n(x;p_1,p_2,p_3,p_4)=&\sum_{k=0}^{[n/2]}\frac{(-[\frac{n}{2}])_k(\frac{1}{2}-p_3-[\frac{n+1}{2}])_k}{(-p_3-p_4-n+1/2)_k k!}\\&\times{}_2F_1\left(\begin{matrix}2k-n-2p_1,p_2+\frac{x}{2}\\2p_2\end{matrix}\middle|2\right)\end{aligned}$$

satisfies the orthogonality relation

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{K_n(ix;\alpha,\beta,\alpha+l,\beta+u-1)}{W^*(x;\beta)}\\&\times\frac{K_m(-ix;l,u,\alpha+l,\beta+u-1)}{W^*(x;u)}dx\\&=\frac{(2u-1)(2\beta-1)}{\pi 2^{2n+1}}\frac{\Gamma(\alpha+l+1/2)\Gamma(\beta+u)}{\Gamma(\alpha+\beta+l+u+1/2)}\\&\times\prod_{j=1}^{n}\frac{(j+(1-(-1)^j)(\alpha+l))}{(j+\alpha+\beta+l+u-3/2)}\\&\times\frac{(j+(1-(-1)^j)(\alpha+l)+2\beta+2u-2)}{(j+\alpha+\beta+l+u-1/2)}\delta_{n,m}\end{aligned}$$

where $\beta,u>1/2$, $\alpha+l>-1/2$ and $W^*(x;.)$ is defined as

$$W^*(x;\lambda)=\int_{-\pi/2}^{\pi/2}e^{x\theta}\cos^{2\lambda-2}\theta d\theta$$

Moreover, it is shown in [15] that another hypergeometric sequence as

$$\begin{aligned}J_n(x;q_1,q_2)=x^{\frac{1-(-1)^n}{2}}&\sum_{k=0}^{[n/2]}\frac{(-[\frac{n}{2}])_k(\frac{1}{2}-q_2-[\frac{n+1}{2}])_k}{(\frac{1}{2}-q_1-[\frac{n+1}{2}])_k 2^k k!}\\&\times{}_1F_1\left(\begin{matrix}q_1+k-[\frac{n}{2}]\\1-(-1)^n/2\end{matrix}\middle|\frac{1}{2}x^2\right)\end{aligned}$$

satisfies the orthogonality relation

$$\begin{aligned}&\int_{-\infty}^{+\infty}e^{-x^2}J_n(x;a,b)J_m(x;b-a,b)dx\\&=\frac{\pi 2^{-b-2[\frac{n+1}{2}]}\Gamma([\frac{n}{2}]+1)\Gamma(b+\frac{1}{2}+[\frac{n+1}{2}])}{\Gamma(a+\frac{1}{2}+[\frac{n+1}{2}])\Gamma(b-a+\frac{1}{2}+[\frac{n+1}{2}])}\delta_{n,m}\end{aligned}$$

where $a,b>-1/2$ and $(-1)^{2b}=1$.
Finally, it is shown in [16] that the hypergeometric sequence

$$\begin{aligned}B_n(x;p_1,p_2,p_3,p_4)=&\sum_{k=0}^{n}\frac{(-n)_k(n+1-2p_3)_k}{(2-2p_2)_k}\\&\times\frac{(1-p_2+ip_1/2)_k}{(1-p_3+ip_4/2)_k k!}\\&\times{}_1F_1\left(\begin{matrix}1-p_2+k+ip_1/2\\2-2p_2+k\end{matrix}\middle|2x\right)\end{aligned}$$

satisfies a finite orthogonality relation as

$$\begin{aligned}&\int_{-\infty}^{+\infty}e^{-x}B_n(\frac{x}{2};\alpha,\beta,v,w)B_m(\frac{x}{2};\alpha-w,v-\beta,v,-w)dx\\&=\frac{n!\Gamma(2v-n)}{2v-2n-1}\frac{\Gamma(\beta+i\alpha/2)\Gamma(\beta-i\alpha/2)}{\Gamma(2\beta-1)\Gamma(2v-2\beta-1)}\\&\times\frac{\Gamma(v-\beta+i(w-\alpha)/2)\Gamma(v-\beta-i(w-\alpha)/2)}{\Gamma(v-iw/2)\Gamma(v+iw/2)}\delta_{n,m}\end{aligned}$$

if and only if $m,n=0,1,\cdots,N=\max\{m,n\}\leq v-1/2$, $\alpha,w\in\mathbb{R}$ and $1/2<\beta<v-1/2$.

3. Conclusion

In this paper, based on the well-known generalized Laguerre polynomials, we introduced a new sequence of biorthogonal hypergeometric polynomials and obtained its orthogonality relation explicitly. In this sense, we also recalled some similar results available in the literature as the sequence introduced in this work was the last one for classical cases.

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Authors Contribution
Authors have contributed equally in preparing and writing the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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