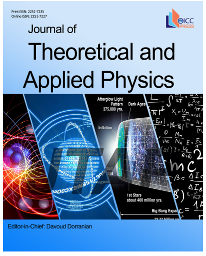

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Application of a finite class of orthogonal polynomials related to T-student distribution in spectral methods

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Original Research

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Abstract:

Classical orthogonal polynomials play an important role in science and engineering. These polynomials are divided into two parts; the infinite and finite sequences of orthogonal polynomials. In this paper, we present a sequence of orthogonal polynomials $(I_n^{(p)}(x))$ which is finitely orthogonal with respect to T-student distribution on infinite interval $(-\infty,+\infty)$. In doing so, in the first part of the paper, general properties of this sequence such as orthogonality relation, Rodrigues type formula, recurrence relations and also some of its applications such as Gauss quadrature formulas and so on are indicated. In addition, in the second part, we show how one can apply $I_n^{(p)}(x)$ in approximations and particularly in spectral methods. Error analysis and convergence of the method are thoroughly investigated. At the end, two numerical examples are given for the efficiency and accuracy of the proposed method. In conclusion, the finite class of orthogonal polynomials related to T-student distribution provides an efficient spectral method on unbounded domain.

Keywords: Spectral methods; Finite class of orthogonal polynomials; T-student distribution

1. Introduction

The history of orthogonal polynomials and special functions goes back to the 19th century. Orthogonal polynomials have applications in both mathematics and physics (combination, harmonic analysis, statistics, number theory). The applications are not limited only to mathematics and physics. Scientists have also utilized these studies in the fields of biology, chemistry and computer science [1–6].

Classical orthogonal polynomials are the solutions of the second order Sturm-Liouville equation

$$\sigma(x)y_n''(x)+\tau(x)y_n'(x)-\lambda_n y_n(x)=0, \tag{1}$$

where $\sigma(x)=ax^2+bx+c$ and $\tau(x)=dx+e$ are polynomials independent of n and $\lambda_n=n(n-1)a+nd$ is the eigenvalue parameters depending on $n=0,1,\cdots$ [4–6]. To be exact, six special classes of orthogonal polynomials can be obtained from (1) and they are the Jacobi, Laguerre and Hermite polynomials and polynomials that are respectively orthogonal with respect to the generalized T, inverse Gamma and F distributions [4, 7]. The first three of them are known as the infinite and the rest of them are known as the finite classical orthogonal polynomials. Table 1 shows these six orthogonal polynomials in details.

It is worth to point out that for the infinite classical orthogonal polynomials, the weight function of Jacobi polynomials, Laguerre polynomials and Hermite polynomials are related to probability density function of Beta distribution, Gumma distribution and Standard Normal distribution, respectively. Also, for the finite classical orthogonal polynomials, as is mentioned, the weight function of $M_n^{(p,q)}$, $I_n^{(p)}$ and $N_n^{(p)}$ are related to probability density F distribution, Student's T-distribution and Inverse Gama distribution.

In this paper, we study spectral approximations by orthogonal polynomials $I_n^{(p)}(x)$ with the weight function $w^{(p)}=\left(1+x^2\right)^{-\left(p-\frac{1}{2}\right)}$ on unbounded interval $(-\infty,+\infty)$. In the last two decades, a considerable progress has been made in using

Table 1. Table of all six classical orthogonal polynomials.

COPs	Polynomial	$\sigma(x)$	$\tau(x)$	Weight function	Interval
Infinite	Jacobi	$1-x^2$	$-(\alpha+\beta+2)x+(\beta-\alpha)$	$(1-x)^{\alpha}(1+x)^{\beta};\alpha,\beta>-1$	$[-1,1]$
Infinite	Laguerre	x	$\alpha+1-x$	$x^{\alpha}e^{-x};\alpha>-1$	$[0,\infty)$
Infinite	Hermite	1	$-2x$	e^{-x^2}	$(-\infty,\infty)$
Finite	$M_n^{(p,q)}$	x^2+x	$(2-p)x+(1+q)$	$x^q(1+x)^{-(p+q)}$	$[0,\infty)$
Finite	$I_n^{(p)}$	x^2+1	$(3-2p)x$	$(1+x^2)^{-(p-\frac{1}{2})}$	$(-\infty,\infty)$
Finite	$N_n^{(p)}$	x^2	$(2-p)x+1$	$x^{-p}e^{-\frac{1}{x}}$	$[0,\infty)$

orthogonal polynomials such as Laguerre and Hermite polynomials for solving PDEs in unbounded domains. In general, spectral methods for unbounded domains can be essentially classified into four categories (see [8] and references therein):

- Truncating the domain and solving PDEs on bounded domains supplemented with artificial or transparent boundary conditions;
- Approximation by classical orthogonal systems on unbounded domains, e.g., Laguerre or Hermite polynomials;
- Approximation by non-classical or mapped orthogonal systems, e.g., image of classical Jacobi polynomials through a suitable mapping;
- Mapping which means one maps unbounded domains to bounded domains and uses standard spectral methods to solve the mapped PDEs in bounded domains.

In general, while numerically solving a problem initially formulated on an unbounded domain, one truncates this domain, which necessitates setting the artificial boundary conditions at the newly formed external boundary [9]. Most of the currently used techniques for setting the artificial boundary conditions can be divided into two groups. The first group provides high accuracy and robustness of the numerical procedure but often appear to be fairly cumbersome and (computationally) expensive. The second one is algorithmically simple, numerically cheap, and geometrically universal; however, they usually lack accuracy of computations [9]. On the other hand, with a suitable choice of mappings and/or scaling parameters, the other three approaches can all be effectively applied to a variety of problems with decaying (or even growing) solutions. Each of these categories has advantages and disadvantages, and in this paper we will restrict ourselves to the second category and we will proceed based on this category.

The outline of this paper is as follows. In section 2, we give some properties of orthogonal polynomials $I_n^{(p)}(x)$. In section 3, we give the analysis of approximations by $I_n^{(p)}(x)$. These results will be useful for error analysis of spectral methods for unbounded domains. In section 4, we consider spectral-Galerkin methods and provide an error analysis by the obtained results. Finally, we give two numerical examples and show the validation of proposed method.

2. Classical orthogonal polynomials related to T-student distribution

2.1 Orthogonality and its consequences

Consider the following second order Sturm-Liouville equation

$$(1+x^2)y''+(3-2p)xy'-n(n+2-2p)y=0. \tag{2}$$

An explicit polynomial solution of Sturm-Liouville equation (2) can be obtained by applying the Frobenius method as follows

$$I_n^{(p)}(x)=n!\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{p-1}{n-k}\binom{n-k}{k}(2x)^{n-2k}. \tag{3}$$

These polynomials are finitely orthogonal with respect to the weight function $w^{(p)}=\left(1+x^2\right)^{-\left(p-\frac{1}{2}\right)}$ on $(-\infty,+\infty)$, i.e.,

$$\int_{-\infty}^{+\infty}\left(1+x^2\right)^{-\left(p-\frac{1}{2}\right)}I_n^{(p)}(x)I_m^{(p)}(x)dx=0 \;\Leftrightarrow\; \begin{cases} m\neq n, \quad p>\ell+1 \\ \ell=\max\{m,n\}, \end{cases}$$

and

$$\gamma_n^{(p)}:=\int_{-\infty}^{+\infty}\left(1+x^2\right)^{-\left(p-\frac{1}{2}\right)}\left(I_n^{(p)}(x)\right)^2dx=\frac{n!2^{2n-1}\sqrt{\pi}\,\Gamma^2(p)\Gamma(2p-2n)}{(p-n-1)\Gamma(p-n)\Gamma(p-n+\frac{1}{2})\Gamma(2p-n-1)}. \tag{4}$$

The Rodrigues' formula for this class of polynomials is

$$I_n^{(p)}(x) = \frac{(-2)^n (p-n)_n}{(2p-2n-1)_n} \left(1+x^2\right)^{p-\frac{1}{2}} \frac{d^n}{dx^n} \left(\left(1+x^2\right)^{n-\left(p-\frac{1}{2}\right)} \right),$$

where $(n)_k = \frac{\Gamma(n+k)}{\Gamma(n)}$. For example, if $n = 0,1,2$ and 3, we have

$$\begin{aligned}
I_0^{(p)}(x) &= 1,\\
I_1^{(p)}(x) &= 2(p-1)x,\\
I_2^{(p)}(x) &= 4(p-2)(p-1)x^2 - 2(p-1),\\
I_3^{(p)}(x) &= 8(p-3)(p-2)(p-1)x^3 - 12(p-2)(p-1)x.
\end{aligned}$$

Clearly, the leading coefficient $k_n^{(p)}$ of $I_n^{(p)}(x)$ is

$$k_0^{(p)} = 1, \qquad k_n^{(p)} = 2^n \prod_{i=1}^{n} (p-i). \tag{5}$$

Using the explicit formula (3), one can obtain the three-term recurrence formula $I_n^{(p)}(x)$ in the following

$$I_{n+1}^{(p)}(x) = 2\left(p-(n+1)\right) x I_n^{(p)}(x) - n\left(2p-(n+1)\right) I_{n-1}^{(p)}(x). \tag{6}$$

In addition, it is not difficult to verify that

$$\frac{d}{dx} I_n^{(p)}(x) = 2n(p-1) I_{n-1}^{(p-1)}(x). \tag{7}$$

This means that the finite set $\left\{ \frac{d}{dx} I_n^{(p>\ell+1)}(x) \right\}_{n=1}^{n=\ell}$ is also orthogonal with respect to $w^{(p-1)}(x)$ (see [10]). Moreover, replacing (7) in (2) leads to

$$2p(1+x^2)\frac{d}{dx} I_n^{(p)}(x) - 2p(2p-1)x I_n^{(p)}(x) = (n+1-2p) I_{n+1}^{(p+1)}(x).$$

2.2 Gauss quadrature formulas

It is well-known that $(n+1)$-point Gauss quadrature rule

$$\int_a^b f(x)w(x)dx = \sum_{j=0}^{n} f(x_j)w_j + E_n[f], \tag{8}$$

is analytically exact for polynomials of degree at most $2n+1$. This rule has the highest possible precision degree among all integration rules with $n+1$ points. Here nodes x_j are zeros of an orthogonal polynomial, w_j's are corresponding weights and $E_n[f]$ is the quadrature error. If $f(x) \in C^{n+1}[a,b]$, quadrature error will be

$$E_n[f] = \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x-x_i) dx, \qquad \xi_x \in [a,b]. \tag{9}$$

In next theorem, we obtain the $(n+1)$-point Gauss quadrature rule associated with $I_{n+1}^{(p)}(x)$.
Theorem 2.1. (Gauss quadrature)
Let $\{x_j\}_{j=0}^n$ be the set of zeros of $I_{n+1}^{(p)}(x)$, $(p > n+2)$. Then there exists a unique set of quadrature weights $\{w_j\}_{j=0}^n$ such that

$$\int_{-\infty}^{+\infty} q(x) w^{(p)}(x) dx = \sum_{j=0}^{n} q(x_j) w_j, \qquad \forall q \in P_{2n+1},$$

where the quadrature weights are all positive and given by

$$w_j = \frac{n! 2^{2n-1} \sqrt{\pi}\, \Gamma^2(p) \Gamma(2p-2n)}{I_n^{(p)}(x_j) I_n^{(p-1)}(x_j)(n+1)(p-1)\Gamma(p-n)\Gamma(p-n+\frac{1}{2})\Gamma(2p-n-1)}. \tag{10}$$

Proof. In view of Theorem 3.5 in [8], it suffices to derive the second formula for the quadrature weights in (10). Let L_j be the Lagrange basis polynomials related to $\{x_j\}_{j=0}^{n}$, i.e.,

$$L_j(x) = \prod_{i=0, i\neq j}^{n} \frac{x-x_i}{x_j-x_i},$$

so taking $f(x) = L_j(x)$ in (8) and using (9), we obtain the quadrature weights

$$w_j = \int_{-\infty}^{+\infty} L_j(x) w^{(p)}(x)dx, \quad 0 \leqslant j \leqslant n. \tag{11}$$

Applying Christoff-Darboux formula, we have

$$w_j = \int_{-\infty}^{+\infty} L_j(x) w^{(p)}(x)dx = \frac{k_{n+1}^{(p)}}{k_n^{(p)}} \frac{\left\|I_n^{(p)}\right\|_{w^{(p)}}^2}{I_n^{(p)}(x_j)\frac{d}{dx}I_{n+1}^{(p)}(x_j)}, \quad 0 \leqslant j \leqslant n.$$

We note that from (7), we have

$$\frac{d}{dx} I_{n+1}^{(p)}(x_j) = 2(n+1)(p-1) I_n^{(p-1)}(x_j). \tag{12}$$

Therefore, using (4), (5) and (12) leads to

$$\begin{aligned} w_j &= \frac{k_{n+1}^{(p)}}{k_n^{(p)}} \frac{\left\|I_n^{(p)}\right\|_{w^{(p)}}^2}{I_n^{(p)}(x_j)\frac{d}{dx}I_{n+1}^{(p)}(x_j)} \\ &= \frac{2^{n+1}\prod_{i=1}^{n+1}(p-i)}{2^n \prod_{i=1}^{n}(p-i)} \frac{\left\|I_n^{(p)}\right\|_{w^{(p)}}^2}{2 I_n^{(p)}(x_j) I_n^{(p-1)}(x_j)(n+1)(p-1)} \\ &= \frac{n! 2^{2n-1}\sqrt{\pi}\,\Gamma^2(p)\Gamma(2p-2n)}{I_n^{(p)}(x_j) I_n^{(p-1)}(x_j)(n+1)(p-1)\Gamma(p-n)\Gamma(p-n+\frac{1}{2})\Gamma(2p-n-1)}. \end{aligned}$$

2.3 Computation of nodes and weights

In this section an efficient algorithm, which is called Eigenvalue Method, is given for computing zeros of orthogonal polynomials.

Theorem 2.2. For $p > n+2$, the zeros $\{x_j\}_{j=0}^{n}$ of the orthogonal polynomial $I_{n+1}^{(p)}(x)$ are eigenvalues of the following symmetric tridiagonal matrix

$$A_{n+1} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2(p-2)}} & & & & \\ \frac{1}{\sqrt{2(p-2)}} & 0 & \sqrt{\frac{2p-3}{2(p-2)(p-3)}} & & & \\ & \sqrt{\frac{2p-3}{2(p-2)(p-3)}} & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \sqrt{\frac{n(2p-(n+1))}{4(p-n)p-(n+1)}} \\ & & & & \sqrt{\frac{n(2p-(n+1))}{4(p-n)p-(n+1)}} & 0 \end{bmatrix}.$$

Proof. It is well-known that the zeros $\{x_j\}_{j=0}^{n}$ of the orthogonal polynomials are the eigenvalues of the following matrix (see [4–6, 8]):

$$A_{n+1} = \begin{bmatrix} \alpha_0 & \beta_1 & & & & \\ \beta_1 & \alpha_1 & \beta_2 & & & \\ & \beta_2 & \alpha_2 & & & \\ & & & \ddots & \beta_{n-1} & \\ & & & \beta_{n-1} & \alpha_{n-1} & \beta_n \\ & & & & \beta_n & \alpha_n \end{bmatrix},$$

where

$$\alpha_j = \frac{b_j}{a_j}, \quad j \geqslant 0; \quad \beta_j = \frac{1}{a_{j-1}}\sqrt{\frac{a_{j-1}c_j}{a_j}}, \quad j \geqslant 1,$$

with $\{a_j, b_j, c_j\}$ being the coefficients of the three-term recurrence relation, namely,

$$p_{j+1}(x) = (a_j x - b_j)p_j(x) - c_j p_{j-1}(x), \quad j \geqslant 1,$$

with $p_{-1} := 0$. Hence, in order to obtain the zeros of $I_{n+1}^{(p)}(x)$ where $p > n+2$, we use Theorem 2.2 and three-term recurrence relation (6), that is

$$\alpha_j = 0,$$
$$\beta_j = \frac{1}{2(p-j)}\sqrt{\frac{j(p-j)(2p-(j+1))}{p-(j+1)}},$$

which implies the proof.

According to Gauss integration theory, by having zeros of orthogonal polynomials, one can approximate the integral $\int_{-\infty}^{+\infty} f(x)w^{(p)}(x)dx$ with precision degree $2n+1$ provided to be convergence. Due to this, the following results are given in Tables 2, 3 and 4 for $n = 1, 2$ and 3, respectively.

Here we give two examples and check the accuracy and efficiency of the proposed results.

Table 2. 2-point integration formula $n = 1$.

p	(x_0, w_0)	(x_1, w_1)
4	$(-0.5, 0.5333)$	$(0.5, 0.5333)$
5	$(-0.4082, 0.4571)$	$(0.4082, 0.4571)$
6	$(-0.3535, 0.4063)$	$(0.3535, 0.4063)$
7	$(-0.3162, 0.3694)$	$(0.2816, 756.36)$

Table 3. 3-point integration formula $n = 2$.

p	(x_0, w_0)	(x_1, w_1)	(x_2, w_2)
5	$(-0.8660, 0.1015)$	$(0, 0.7111)$	$(0.8660, 0.1015)$
6	$(-0.7071, 0.1015)$	$(0, 0.6095)$	$(0.7071, 0.1015)$
7	$(-0.6123, 0.0985)$	$(0., 0.5417)$	$(0.6123, 0.0985)$
8	$(-0.5477, 0.0947)$	$(0., 0.4925)$	$(0.5477, 0.0947)$

Table 4. 4-point integration formula $n = 3$.

p	(x_0, w_0)	(x_1, w_1)	(x_2, w_2)	(x_3, w_3)
6	$(-1.1880, 0.0111)$	$(-0.2975, 0.3951)$	$(0.2975, 0.3951)$	$(1.1880, 0.0111)$
7	$(-0.9659, 0.0140)$	$(-0.2588, 0.3553)$	$(0.2588, 0.3553)$	$(0.9659, 0.0140)$
8	$(-0.8343, 0.0156)$	$(-0.2320, 0.3253)$	$(0.2320, 0.3253)$	$(0.8343, 0.0156)$
9	$(-0.7449, 0.0164)$	$(-0.2122, 0.3017)$	$(0.2122, 0.3017)$	$(0.7449, 0.0164)$

Example 2.3. Assume that $f(x) = x^7 + 6x^5 + x^2 + 1$, $p = 9$ and $n = 3$. So we have

$$\int_{-\infty}^{+\infty} (x^7 + 6x^5 + x^2 + 1)\left(1 + x^2\right)^{-\frac{17}{2}} dx = 0.681985$$
$$\sum_{j=0}^{n} f(x_j)w_j \simeq 0.01f(-0.74) + 0.30f(-0.21) + 0.30f(0.21) + 0.016f(0.74) = 0.681985.$$

Example 2.4. Assume that $f(x) = e^{\sin(x)-\cos(x)}$, $p = 9$ and $n = 3$. So we have

$$\int_{-\infty}^{+\infty} e^{\sin(x)-\cos(x)}\left(1 + x^2\right)^{-\frac{17}{2}} dx = 0.251699$$
$$\sum_{j=0}^{n} f(x_j)w_j \simeq 0.01f(-0.74) + 0.30f(-0.21) + 0.30f(0.21) + 0.016f(0.74) = 0.251702.$$

2.4 Interpolation and discrete transforms

Suppose that $\{x_j, w_j\}_{j=0}^{n}$ is a set of Gauss quadrature nodes and weights. Define the associated discrete inner product and discrete norm as

$$\langle u,v\rangle_{n,w^{(p)}} := \sum_{j=0}^{n} u(x_j)v(x_j)w_j, \quad \|u\|_{n,w^{(p)}} := \sqrt{\langle u,u\rangle_{n,w^{(p)}}}.$$

Note that $\langle .,.\rangle_{n,w^{(p)}}$ is an approximation of the continuous inner product $(.,.)_{w^{(p)}}$ and the exactness of Gauss-type quadrature formulas leads to

$$\langle u,v\rangle_{n,w^{(p)}} = (u,v)_{w^{(p)}}, \qquad \forall u,v \in P_{2n+1}.$$

For any $u \in C(-\infty,+\infty)$, the interpolation operator $\varpi_n : C(-\infty,+\infty) \to P_n$ is defined by

$$(\varpi_n u)(x_j) = u(x_j), \qquad 0 \leqslant j \leqslant n,$$

which can be expressed by

$$(\varpi_n u)(x) = \sum_{j=0}^{n} \tilde{u}_j I_j^{(p)}(x) \in P_n.$$

Given the physical values $\{u(x_j)\}_{j=0}^{n}$, the coefficients $\{\tilde{u}_j\}_{j=0}^{n}$ can be obtained by

$$\tilde{u}_j = \frac{1}{\gamma_j^{(p)}} \sum_{i=0}^{n} u(x_i) I_j^{(p)}(x_i) w_i \quad 0 \leqslant j \leqslant n,$$

where $\gamma_j^{(p)}$ is defined in (4).

2.5 Differentiation in the physical space

For the Gauss points $\{x_j\}_{j=0}^{n}$, let $\{L_j\}_{j=0}^{n}$ be the Lagrange basis polynomials. Clearly, we have

$$u(x) = \sum_{j=0}^{n} u(x_j) L_j(x),$$

for any $u \in P_n$. Hence, by differentiating m times we obtain

$$u^{(m)}(x_k) = \sum_{j=0}^{n} u(x_j) L_j^{(m)}(x_k), \qquad 0 \leqslant k \leqslant n.$$

These derivative values can be evaluated by the general formula

$$\mathbf{u}^{(m)} = D^m \mathbf{u}, \quad (D^m = DD\ldots D, \quad m \geqslant 1),$$

where

$$D = \left(d_{kj}\right)_{0\leqslant j,k\leqslant n} = \left(L'_j(x_k)\right)_{0\leqslant j,k\leqslant n},$$
$$\mathbf{u}^{(m)} = \left(u^{(m)}(x_0), u^{(m)}(x_1), \ldots, u^{(m)}(x_n)\right)^T, \ \mathbf{u}^{(0)} = \mathbf{u}.$$

Therefore, it is necessary to compute the first-order differentiation matrix D.
Theorem 2.5. The entries of D are computed for Gauss points $\{x_j\}_{j=0}^{n}$ by

$$d_{kj} = L'_j(x_k) = \begin{cases} \frac{I_n^{(p-1)}(x_k)}{I_n^{(p-1)}(x_j)} \frac{1}{(x_k - x_j)}, & k \neq j, \\ \frac{n(p-1) I_{n-1}^{(p-2)}(x_k)}{I_n^{(p-1)}(x_k)} & k = j. \end{cases}$$

Proof. The Lagrange basis polynomials are

$$L_j(x) = \frac{I_{n+1}^{(p)}(x)}{\frac{d}{dx}\left(I_{n+1}^{(p)}(x)\right)|_{x=x_j}(x - x_j)}, \qquad p > n+2, \quad 0 \leqslant j \leqslant n.$$

So, we have

$$d_{kj} = L'_j(x_k) = \frac{\frac{d}{dx}\left(I_{n+1}^{(p)}(x)\right)|_{x=x_k}}{\frac{d}{dx}\left(I_{n+1}^{(p)}(x)\right)|_{x=x_j}} \frac{1}{(x_k - x_j)} = \frac{2(n+1)(p-1)I_n^{(p-1)}(x_k)}{2(n+1)(p-1)I_n^{(p-1)}(x_j)} \frac{1}{(x_k - x_j)} = \frac{I_n^{(p-1)}(x_k)}{I_n^{(p-1)}(x_j)} \frac{1}{(x_k - x_j)}. \qquad \forall k \neq j.$$

Also, for $k = j$, we get

$$\begin{aligned} d_{kk} = \lim_{x\to x_k} L'_k(x) &= \frac{1}{\frac{d}{dx}\left(I_{n+1}^{(p)}(x)\right)|_{x=x_k}} \lim_{x\to x_k} \frac{\frac{d}{dx}\left(I_{n+1}^{(p)}(x)\right)(x-x_k) - I_{n+1}^{(p)}(x)}{(x-x_k)^2} \\ &= \frac{\frac{d^2}{dx^2}\left(I_{n+1}^{(p)}(x)\right)|_{x=x_k}}{2\frac{d}{dx}\left(I_{n+1}^{(p)}(x)\right)|_{x=x_k}} = \frac{4n(n+1)(p-1)^2 I_{n-1}^{(p-2)}(x_k)}{4(n+1)(p-1)I_n^{(p-1)}(x_k)} = \frac{n(p-1)I_{n-1}^{(p-2)}(x_k)}{I_n^{(p-1)}(x_k)}. \end{aligned}$$

2.6 Differentiation in the frequency space

In this subsection, by differentiation in the frequency space we want to express the expansion coefficients of the derivatives of a function according to expansion coefficients of the function itself. To be exact, given $u \in P_n$, instead of using the Lagrange basis polynomials, we expand u in terms of the orthogonal polynomials

$$u(x) = \sum_{j=0}^{n} \tilde{u}_j I_j^{(p)}(x), \quad \text{with} \quad \tilde{u}_j = \frac{1}{\gamma_j^{(p)}} \int_{-\infty}^{+\infty} u(x) I_j^{(p)}(x) w^{(p)}(x) dx,$$

and

$$u'(x) = \sum_{j=1}^{n} \tilde{u}_j \frac{d}{dx}\left(I_j^{(p)}(x)\right) = \sum_{j=0}^{n} \tilde{u}_j^{(1)} I_j^{(p)}(x) \in P_{n-1} \text{ with } \tilde{u}_n^{(1)} = 0. \tag{13}$$

In order to express $\{\tilde{u}_j^{(1)}\}_{j=0}^n$ in terms of $\{\tilde{u}_j\}_{j=0}^n$, we know that $\left\{\frac{d}{dx}\left(I_j^{(p)}(x)\right)\right\}$ are also orthogonal [10, 11]. Indeed, this property holds for the classical orthogonal polynomials. In doing so, consider equation (2). Then by differentiating it with respect to x and writing $z(x) = y'(x)$, we obtain

$$x^2 z'' + (5+2p)xz' - ((n-2p)(n+1) + n - 3)\, z = 0. \tag{14}$$

Again by applying the Frobenius method, we get the explicit solution

$$\frac{d}{dx}\left(I_n^{(p)}(x)\right) = 2(p-1)n! \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{p-2}{n-1-k} \binom{n-(k+1)}{k} (2x)^{n-(2k+1)}.$$

It is easy to see that, the finite set $\left\{\frac{d}{dx}\left(I_j^{(p>n+1)}(x)\right)\right\}_{j=1}^n$ is orthogonal with respect to the weight function $w^{(p-1)} = \left(1+x^2\right)^{-\left(p-\frac{3}{2}\right)}$ on $(-\infty, +\infty)$, i.e.,

$$\int_{-\infty}^{+\infty} \left(1+x^2\right)^{-\left(p-\frac{3}{2}\right)} \frac{d}{dx}\left(I_n^{(p)}(x)\right) \frac{d}{dx}\left(I_m^{(p)}(x)\right) dx = 0\,, \;\Leftrightarrow\; \begin{cases} m \neq n,\; p > 2\ell + 1, \\ \ell = \max\{m, n\}, \end{cases} \tag{15}$$

and

$$\eta_n^{(p,1)} := \int_{-\infty}^{+\infty} \left(1+x^2\right)^{-\left(p-\frac{3}{2}\right)} \left(\frac{d}{dx}\left(I_n^{(p)}(x)\right)\right)^2 dx = \frac{(n-1)!\,2^{2n-3}\sqrt{\pi}\,\Gamma^2(p-1)\Gamma(2p-2n)}{(p-n-1)\Gamma(p-n)\Gamma(p-n+\frac{1}{2})\Gamma(2p-n-2)}. \tag{16}$$

Also, equation (14) leads to the following three-term recurrence relation

$$\frac{d}{dx}\left(I_{n+1}^{(p)}(x)\right) = \left(a_n^{(1)} x - b_n^{(1)}\right) \frac{d}{dx}\left(I_n^{(p)}(x)\right) - c_n^{(1)} \frac{d}{dx}\left(I_{n-1}^{(p)}(x)\right), \tag{17}$$

where

$$\begin{aligned} a_n^{(1)} &= \frac{2(p-(n+1))(n+1)}{n}, \\ b_n^{(1)} &= 0, \end{aligned}$$

$$c_n^{(1)} = (n+1)(2p-(n+2)).$$

Remark 2.6. Using the above procedure, one can obtain

$$\int_{-\infty}^{+\infty} (1+x^2)^{-\left(p-\frac{2k+1}{2}\right)} \frac{d^k}{dx^k}\left(I_n^{(p)}(x)\right) \frac{d^k}{dx^k}\left(I_m^{(p)}(x)\right) dx = 0, \Leftrightarrow \begin{cases} m \neq n,\ p > 2\ell+1, \\ \ell = \max\{m,n\}, \end{cases} \tag{18}$$

and

$$\begin{aligned} \eta_n^{(p,k)} &= \int_{-\infty}^{+\infty} (1+x^2)^{-\left(p-\frac{2k+1}{2}\right)} \left(\frac{d^k}{dx^k}\left(I_n^{(p)}(x)\right)\right)^2 dx \\ &= \frac{(n-k)!2^{2n-(2k+1)}\sqrt{\pi}\Gamma^2(p-k)\Gamma(2p-2n)}{(p-n-1)\Gamma(p-n)\Gamma(p-n+\frac{1}{2})\Gamma(2p-n-k-1)}. \end{aligned} \tag{19}$$

In addition, it is worth to point out that

$$\gamma_n^{(p)} = 2^{2k} \prod_{i=1}^{k} \frac{(n-i+1)(p-i)^2}{(2p-n-1-i)} \eta_n^{(p,k)}.$$

Now, by differentiating the three-term recurrence relation (6) and using (17), we get

$$I_n^{(p)}(x) = \tilde{a}_n^{(p)} \frac{d}{dx}\left(I_{n-1}^{(p)}(x)\right) + \tilde{b}_n^{(p)} \frac{d}{dx}\left(I_n^{(p)}(x)\right) + \tilde{c}_n^{(p)} \frac{d}{dx}\left(I_{n+1}^{(p)}(x)\right), \tag{20}$$

where

$$\tilde{a}_n^{(p)} = \frac{c_n}{a_n} - \frac{c_n^{(1)}}{a_n^{(1)}} = \frac{n}{2(p-n-1)}, \quad \tilde{b}_n^{(p)} = \frac{b_n}{a_n} - \frac{b_n^{(1)}}{a_n^{(1)}} = 0, \quad \tilde{c}_n^{(p)} = \frac{1}{a_n} - \frac{1}{a_n^{(1)}} = \frac{1}{2(n+1)(p-n-1)}.$$

Hence, using (13) and (20), the coefficients $\{\tilde{u}_j^{(1)}\}$ can be computed in terms of $\{\tilde{u}_j\}_{j=0}^{n}$ as follows:

$$\begin{aligned} u'(x) &= \sum_{j=0}^{n-1} \tilde{u}_j^{(1)} I_n^{(p)}(x) = \sum_{j=0}^{n-1} \tilde{u}_j^{(1)} \left(\tilde{a}_j^{(p)} \frac{d}{dx}\left(I_{j-1}^{(p)}(x)\right) + \tilde{b}_j^{(p)} \frac{d}{dx}\left(I_j^{(p)}(x)\right) + \tilde{c}_j^{(p)} \frac{d}{dx}\left(I_{j+1}^{(p)}(x)\right)\right) \\ &= \sum_{j=1}^{n-1} \left(\tilde{a}_{j+1}^{(p)} \tilde{u}_{j+1}^{(1)} + \tilde{b}_j^{(p)} \tilde{u}_j^{(1)} + \tilde{c}_{j-1}^{(p)} \tilde{u}_{j-1}^{(1)}\right) \frac{d}{dx}\left(I_j^{(p)}(x)\right) + \tilde{c}_{n-1}^{(p)} \tilde{u}_{n-1}^{(1)} \frac{d}{dx}\left(I_n^{(p)}(x)\right) \\ &= \sum_{j=1}^{n-1} \left(\frac{j+1}{2(p-j-2)} \tilde{u}_{j+1}^{(1)} + \frac{1}{2j(p-j)} \tilde{u}_{j-1}^{(1)}\right) \frac{d}{dx}\left(N_j^{(p)}(x)\right) + \frac{1}{2n(p-n)} \tilde{u}_{n-1}^{(1)} \frac{d}{dx}\left(N_n^{(p)}(x)\right). \end{aligned}$$

So, we conclude

$$\begin{aligned} &\tilde{u}_{j-1}^{(1)} = 2j(p-j)\left(\tilde{u}_j - \frac{j+1}{2(p-j-2)} \tilde{u}_{j+1}^{(1)}\right), \qquad j = n-1, \ldots, 1, \\ &\tilde{u}_n^{(1)} = 0, \quad \tilde{u}_{n-1}^{(1)} = 2n(p-n)\tilde{u}_n. \end{aligned}$$

3. Approximation by $I_n^{(p)}(x)$ polynomials

This section is devoted to the analysis of approximations by $I_n^{(p)}(x)$ polynomials. These results will be useful for error analysis of spectral methods for unbounded domains.

3.1 Inverse inequalities

We first present two inverse inequalities associated with $I_n^{(p)}(x)$ polynomials.
Theorem 3.4. For $p > n+1$ and any $\phi \in P_n$,

$$\|\phi\|_{w^{(p)}} \leqslant (2n(p-1))^{1/m} \left\| \frac{d^m}{dx^m} \phi \right\|_{w^{(p-m)}}, \qquad m \geq 1.$$

Proof. For any $\phi \in P_n$, we have

$$\phi(x) = \sum_{j=0}^{n} \tilde{\phi}_j^{(p)} I_j^{(p)}(x), \quad \text{with} \quad \tilde{\phi}_j^{(p)} = \frac{1}{\gamma_j^{(p)}} \int_{-\infty}^{+\infty} \phi(x) I_j^{(p)}(x) w^{(p)}(x) dx. \tag{21}$$

Hence, by the orthogonality of $\{I_j^{(p)}(x)\}$,

$$\|\phi\|^2_{w^{(p)}} = \sum_{j=0}^{n} \gamma_j^{(p)} \left|\tilde{\phi}_j^{(p)}\right|^2.$$

Differentiating (21) and using the orthogonality (15)-(16), we obtain

$$\phi'(x) = \sum_{j=1}^{n} \tilde{\phi}_j^{(p)} \frac{d}{dx}\left(I_j^{(p)}(x)\right),$$

and

$$\left\|\phi'\right\|^2_{w^{(p-1)}} = \sum_{j=1}^{n} \eta_j^{(p,1)} \left|\tilde{\phi}_j^{(p)}\right|^2.$$

Since $\gamma_j^{(p)} = \frac{4j(p-1)^2}{2p-j-2}\eta_j^{(p,1)}$, we obtain

$$\|\phi\|^2_{w^{(p)}} = \sum_{j=0}^{n} \gamma_j^{(p)} \left|\tilde{\phi}_j^{(p)}\right|^2 = 4\sum_{j=0}^{n} \frac{j(p-1)^2}{2p-j-2}\eta_j^{(p,1)} \left|\tilde{\phi}_j^{(p)}\right|^2 \leqslant 2n(p-1)\left\|\phi'\right\|^2_{w^{(p-1)}},$$

and

$$\|\phi\|_{w^{(p)}} \leqslant \sqrt{2n(p-1)}\left\|\phi'\right\|_{w^{(p-1)}}.$$

Using the above inequality recursively leads to

$$\|\phi\|_{w^{(p)}} \leqslant (2n(p-1))^{1/m}\left\|\frac{d^m}{dx^m}\phi\right\|_{w^{(p-m)}}.$$

Next, we obtain an inverse inequality involving the same weight function for derivatives of different order.

Theorem 3.2. For $p > n+1$ and any $\phi \in P_n$,

$$\left\|\frac{d^m}{dx^m}\phi\right\|_{w^{(p)}} \leqslant Cn^{\frac{1}{m}}\|\phi\|_{w^{(p)}},$$

where C is a positive constant. *Proof.*For any $\phi \in P_n$, we have

$$\phi(x) = \sum_{j=0}^{n} \tilde{\phi}_j^{(p)} I_j^{(p)}(x), \quad \text{with} \quad \tilde{\phi}_j^{(p)} = \frac{1}{\gamma_j^{(p)}}\int_{-\infty}^{+\infty} \phi(x) I_j^{(p)}(x) w^{(p)}(x) dx.$$

Hence, using differentiation in the frequency space, we get

$$\phi'(x) = \sum_{j=1}^{n} \tilde{\phi}_j^{(p)} \frac{d}{dx}\left(I_j^{(p)}(x)\right) = \sum_{j=1}^{n} \tilde{\phi}_j^{(p)}\left(\sum_{k=0}^{j-1} c_k I_j^{(p)}(x)\right) = \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n} \tilde{\phi}_j^{(p)}\right) c_k I_k^{(p)}(x),$$

and

$$\left\|\phi'\right\|^2_{w^{(p)}} = \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n} \tilde{\phi}_j^{(p)}\right)^2 c_k^2 \gamma_k^{(p)}.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{j=k+1}^{n} \tilde{\phi}_j^{(p)}\right)^2 \leqslant \left(\sum_{j=k+1}^{n} \gamma_j^{(p)}\left|\tilde{\phi}_j^{(p)}\right|^2\right)\left(\sum_{j=k+1}^{n}\left(\gamma_j^{(p)}\right)^{-1}\right),$$

we conclude that

$$\begin{aligned}
\left\|\phi'\right\|^2_{w^{(p)}} &= \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n} \tilde{\phi}_j^{(p)}\right)^2 c_k^2 \gamma_k^{(p)} \\
&\leqslant \|\phi\|^2_{w^{(p)}} \sum_{k=0}^{n-1} c_k^2 \gamma_k^{(p)} \left(\sum_{j=k+1}^{n}\left(\gamma_j^{(p)}\right)^{-1}\right) \leqslant C\|\phi\|^2_{w^{(p)}} \sum_{k=0}^{n-1} \frac{\gamma_k^{(p)}}{\min_k\{\gamma_{k+1}^{(p)}, \gamma_{k+2}^{(p)}, \dots, \gamma_n^{(p)}\}} \\
&= C\|\phi\|^2_{w^{(p)}} \sum_{k=0}^{n-1} \frac{\gamma_k^{(p)}}{\gamma_{k+1}^{(p)}} \leq Cn\|\phi\|^2_{w^{(p)}},
\end{aligned}$$

and

$$\left\|\phi'\right\|_{w^{(p)}} \leqslant C\sqrt{n}\|\phi\|_{w^{(p)}}.$$

Using the above inequality recursively leads to

$$\left\|\frac{d^m}{dx^m}\phi\right\|_{w^{(p)}} \leqslant Cn^{\frac{1}{m}}\|\phi\|_{w^{(p)}}.$$

3.2 Orthogonal projections

In error analysis, usually the numerical solution u_n is compared to a suitable orthogonal projection $\Pi_n u$ (or interpolation $\varpi_n u$) of the exact solution u in some appropriate Sobolev space with the norm $\|.\|_S$. Next the following the triangle inequality is used

$$\|u-u_n\|_S \leqslant \|u-\Pi_n u\|_S + \|\Pi_n u - u_n\|_S.$$

To this end, it is necessary to approximate the errors $\|u-\Pi_n u\|_S$ and $\|\Pi_n u - u_n\|_S$. Such approximation involving $I_n^{(p)}(x)$ polynomials will be the main concern of this subsection.
Consider the $L^2_{w^{(p)}}$-orthogonal projection $\Pi_{n,p} : L^2_{w^{(p)}}(-\infty,+\infty) \to P_n$, defined by

$$(\Pi_{n,p}u - u, v_n)_{w^{(p)}} = 0, \qquad \forall v_n \in P_n,$$

in which

$$\Pi_{n,p}u(x) = \sum_{j=0}^{n} \tilde{u}_j^{(p)} I_j^{(p)}(x) \quad \text{with} \quad \tilde{u}_j^{(p)} = \frac{1}{\gamma_j^{(p)}} \int_{-\infty}^{+\infty} u(x) I_j^{(p)}(x) w^{(p)}(x) dx.$$

Introduce the space

$$B_p^m(-\infty,+\infty) = \left\{ u : \frac{d^k}{dx^k}(u) \in L^2_{w^{(p-k)}}(-\infty,+\infty),\ 0 \leqslant k \leqslant m \right\},$$

equipped with the norm and semi-norm

$$\|u\|_{B_p^m} = \left(\sum_{k=0}^{m} \left\| \frac{d^k}{dx^k}(u) \right\|^2_{L^2_{w^{(p-k)}}} \right)^{1/2}, \quad |u|_{B_p^m} = \left\| \frac{d^k}{dx^k}(u) \right\|_{L^2_{w^{(p-k)}}}.$$

The space $B_p^m(-\infty,+\infty)$ distinguishes itself from the usual weighted Sobolev space $H^m_{w^{(p)}}(-\infty,+\infty)$ by involving different weight functions for derivatives of different orders [8, 12, 13]. It is obvious that $H^m_{w^{(p)}}(-\infty,+\infty)$ is a subspace of $B_p^m(-\infty,+\infty)$, that is, for any $m \geq 0$,

$$\|u\|_{B_p^m(-\infty,+\infty)} \leqslant c\|u\|_{H^m_{w^{(p)}}(-\infty,+\infty)}.$$

Now, we are ready to give the first fundamental result.
Theorem 3.3. Let $0 \leq \ell \leq m \leq n+1 < p$. If $u \in B_p^m(-\infty,+\infty)$, we have

$$\left\| \frac{d^\ell}{dx^\ell}(\Pi_{n,p}u - u) \right\|_{w^{(p-\ell)}} \leqslant 2^{m-\ell} \sqrt{\max_{j \geqslant n+1} \left\{ \frac{(j-\ell)!\Gamma^2(p-\ell)\Gamma(2p-2\ell)}{(j-m)!\Gamma^2(p-m)\Gamma(2p-2m)} \right\}} \left\| \frac{d^m}{dx^m}(u) \right\|_{w^{(p-m)}}.$$

Proof. Due to the orthogonality (18), we have

$$\left\| \frac{d^k}{dx^k}(u) \right\|^2_{w^{(p-k)}} = \sum_{j=k}^{\infty} \eta_j^{(p,k)} \left| \tilde{u}_j^{(p)} \right|^2, \quad k \geqslant 0.$$

So, we get

$$\begin{aligned}
\left\| \frac{d^\ell}{dx^\ell}(\Pi_{n,p}u - u) \right\|^2_{w^{(p-\ell)}} &= \sum_{j=n+1}^{\infty} \eta_j^{(p,\ell)} \left| \tilde{u}_j^{(p)} \right|^2 = \sum_{j=n+1}^{\infty} \eta_j^{(p,m)} \frac{\eta_j^{(p,\ell)}}{\eta_j^{(p,m)}} \left| \tilde{u}_j^{(p)} \right|^2 \\
&\leqslant \max_{j \geqslant n+1} \left\{ \frac{\eta_j^{(p,\ell)}}{\eta_j^{(p,m)}} \right\} \sum_{j=n+1}^{\infty} \eta_j^{(p,m)} \left| \tilde{u}_j^{(p)} \right|^2 \\
&= \max_{j \geqslant n+1} \left\{ \frac{(j-\ell)!2^{2j-(2\ell+1)}\Gamma^2(p-\ell)\Gamma(2p-2\ell)\Gamma(2p-j-m-1)}{(j-m)!2^{2j-(2m+1)}\Gamma^2(p-m)\Gamma(2p-2m)\Gamma(2p-j-\ell-1)} \right\} \left\| \frac{d^m}{dx^m}(u) \right\|^2_{w^{(p-m)}} \\
&= \max_{j \geqslant n+1} \left\{ \frac{(j-\ell)!2^{2(m-\ell)}\Gamma^2(p-\ell)\Gamma(2p-2\ell)}{(j-m)!\Gamma^2(p-m)\Gamma(2p-2m)} \right\} \left\| \frac{d^m}{dx^m}(u) \right\|^2_{w^{(p-m)}},
\end{aligned}$$

and

$$\left\| \frac{d^\ell}{dx^\ell}(\Pi_{n,p}u - u) \right\|_{w^{(p-\ell)}} \leqslant 2^{m-\ell} \sqrt{\max_{j \geqslant n+1} \left\{ \frac{(j-\ell)!\Gamma^2(p-\ell)\Gamma(2p-2\ell)}{(j-m)!\Gamma^2(p-m)\Gamma(2p-2m)} \right\}} \left\| \frac{d^m}{dx^m}(u) \right\|_{w^{(p-m)}}.$$

Since $H^m_{w^{(p)}}(-\infty,+\infty)$ is a Hilbert space, the best approximation polynomial for u is the orthogonal projection of u upon P_n under the inner product

$$(u,v)_{m,w^{(p)}} = \sum_{k=0}^{m} \left(\frac{d^k}{dx^k}u, \frac{d^k}{dx^k}v \right)_{w^{(p)}},$$

which induces the norm $\|.\|_{m,w^{(p)}}$ of $H^m_{w^{(p)}}(-\infty,+\infty)$. In fact, this type of approximation results are often needed in analysis of spectral methods for second-order elliptic PDEs [8, 13–17]. Therefore, we consider below the $H^1_{w^{(p)}}$-orthogonal projection. Define the orthogonal projection $\Pi^1_{n,p} : H^1_{w^{(p)}}(-\infty,+\infty) \to P_n$ by

$$\left(\Pi^1_{n,p}u - u, v_n\right)_{1,w^{(p)}} = 0, \qquad \forall v_n \in P_n. \tag{22}$$

By definition, $\Pi^1_{n,p}u$ is the best approximation of u in the sense that

$$\left\|\Pi^1_{n,p}u - u\right\|_{1,w^{(p)}} = \inf_{\phi \in P_n} \|\phi - u\|_{1,w^{(p)}}. \tag{23}$$

Using Theorem 3.3, we can derive the following estimate.
Theorem 3.4. Let $1 \le m \le n+1 < p$. If $\frac{d}{dx}(u) \in B^{m-1}_p(-\infty,+\infty)$, we have

$$\left\|\Pi^1_{n,p}u - u\right\|_{1,w^{(p)}} \leqslant C2^{m-1} \sqrt{\max_{j \geqslant n+1} \left\{ \frac{(j-1)!\Gamma^2(p-1)\Gamma(2p-2)}{(j-m)!\Gamma^2(p-m)\Gamma(2p-2m)} \right\}} \left\| \frac{d^m}{dx^m}(u) \right\|_{w^{(p-m-1)}},$$

where C is a positive constant independent of m, n, p and u.
Proof. Let $\Pi_{n-1,p}$ be the $L^2_{w^{(p)}}$ -orthogonal projection upon P_{n-1}. Set

$$\phi(x) = \int_{-\infty}^{x} \Pi_{n-1,p}u'(y)dy.$$

In view of (23), we derive from the Poincare inequality and Theorem 3.3 that

$$\begin{aligned} \left\|\pi^1_{n,p}u - u\right\|_{1,w^{(p)}} &\leqslant \|\phi - u\|_{1,w^{(p)}} \leqslant c\left\|(\phi - u)'\right\|_{w^{(p)}} \leqslant C\left\|\pi_{n-1,p}u' - u'\right\|_{w^{(p)}} \\ &\leqslant C2^{m-1} \sqrt{\max_{j \geqslant n+1} \left\{ \frac{(j-1)!\Gamma^2(p-1)\Gamma(2p-2)}{(j-m)!\Gamma^2(p-m)\Gamma(2p-2m)} \right\}} \left\| \frac{d^m}{dx^m}(u) \right\|_{w^{(p-m-1)}}. \end{aligned}$$

4. Spectral methods using $I^{(p)}_n(x)$ polynomials

In this section, we consider spectral-Galerkin methods using $I^{(p)}_n(x)$ polynomials. An advantage of using $I^{(p)}_n(x)$ polynomials is that they are mutually orthogonal, so we can work with the usual variational formulation.

4.1 $I^{(p)}_n(x)$-Galerkin method

Consider the model equation:

$$\begin{cases} -u_{xx} + \gamma u = f, \;\; x \in (-\infty,+\infty), \gamma > 0 \\ \lim\limits_{|x| \to \infty} u(x) = 0. \end{cases} \tag{24}$$

Let $H^1_{w^{(p)}}(-\infty,+\infty)$ and P_n be the spaces as defined before. Then, a weak formulation of (24) is

$$\left(u', v'\right) + \gamma(u,v) = (f,v), \;\; \forall v \in H^1_{w^{(p)}}(-\infty,+\infty),$$

where $(\cdot,\cdot)$ is the usual (non-weighted) inner product in L_2-space. The $I^{(p)}_n(x)$-Galerkin approximation to (24) is

$$\left(u'_n, v'\right) + \gamma(u_n,v) = (f,v), \;\; \forall v \in P_n.$$

As in the Laguerre and Hermite cases, the $I^{(p)}_n(x)$ polynomials are not very useful in practice due to its wild behavior at infinity. Therefore, we consider the polynomials functions defined by

$$\varphi^{(p)}_k(x) = \frac{1}{\sqrt{2^k \prod\limits_{i=1}^{k}(p-i)}} I^{(p)}_k(x)\left(1+x^2\right)^{-\left(\frac{p}{2}-\frac{1}{4}\right)}, \quad p > k+1.$$

So one verifies that

$$P_n = \operatorname{span}\left\{\varphi_0^{(p)}, \varphi_1^{(p)}, \ldots, \varphi_{n-1}^{(p)}\right\}.$$

Hence, by setting $u_n(x) = \sum_{k=0}^{n-1} u_k \varphi_k^{(p)}(x)$ and $v = \varphi_j^{(p)}(x)$ for $j = 0, 1, \cdots, n-1$, we get

$$\sum_{k=0}^{n-1} u_k \left(\left(\left(\varphi_k^{(p)}\right)', \left(\varphi_j^{(p)}\right)' \right) + \gamma\left(\varphi_k^{(p)}, \varphi_j^{(p)}\right) \right) = \left(f, \varphi_j^{(p)}\right).$$

and in matrix form

$$AU = F,$$

where

$$A = \left[a_{k,j}\right]_{n\times n}, \; U = [u_j]_{n\times 1}, \; F = [f_j]_{n\times 1},$$

in which

$$\begin{aligned} a_{k,j} &= \left(\left(\varphi_k^{(p)}\right)', \left(\varphi_j^{(p)}\right)' \right) + \gamma\left(\varphi_k^{(p)}, \varphi_j^{(p)}\right), \\ f_j &= \left(f, \varphi_j^{(p)}\right). \end{aligned} \tag{25}$$

Note that all integrals in (25), are solved by Gauss quadrature formulas (see Theorem 2.1).

4.2 Numerical results

Now, we present two numerical results to illustrate the convergence behavior of the proposed schemes. We consider (24) with two sets of exact solutions having decay properties.

Example 4.1. In this example, we suppose that the exact solution is:

$$u(x) = \frac{1}{1+x^4}, \quad x \in (-\infty, +\infty),$$

and $\gamma = 2$. The approximate solution $u_n(x)$ for different values of n and $p = n+2$ are demonstrated in Figures 1 to 4. Also, The L_2-norm errors of the proposed method for different values of n and $p = n+2$ are listed in Table 5. In these figures and tables, we clearly observe that the desired solution is provided by the approximate solution as well.

Example 4.2. In this example, we suppose that the exact solution is:

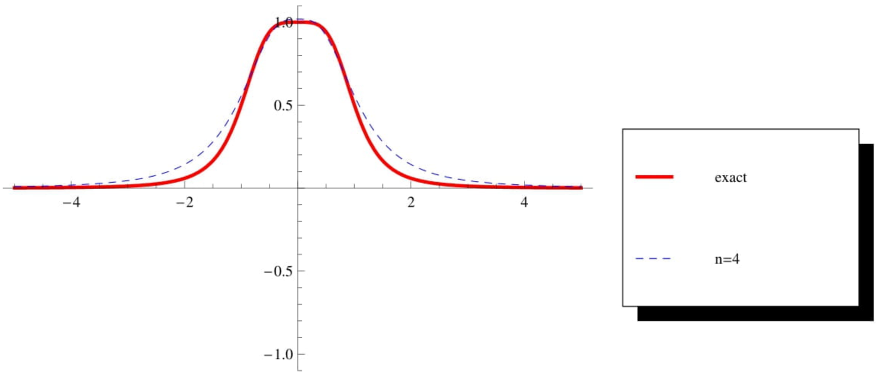

Figure 1. The plots of exact solution $u(x) = \frac{1}{1+x^4}$ and the approximate solution $u_n(x)$ for $n = 4, p = 6$.

Table 5. The L_2-norm errors of the proposed spectral method based on a finite class of orthogonal polynomials related to T-student distribution.

γ	n	p	Error
2	4	6	2.98×10^{-2}
2	8	10	3.43×10^{-3}
2	16	18	8.14×10^{-5}
2	32	34	7.95×10^{-6}

$$u(x) = \frac{\sin(x)}{1+x^2}, \quad x \in (-\infty, +\infty),$$

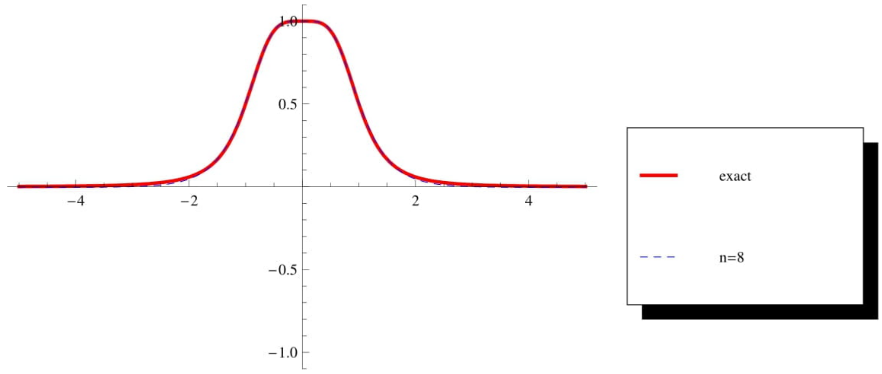

Figure 2. The plots of exact solution $u(x) = \frac{1}{1+x^4}$ and the approximate solution $u_n(x)$ for $n = 8, p = 10$.

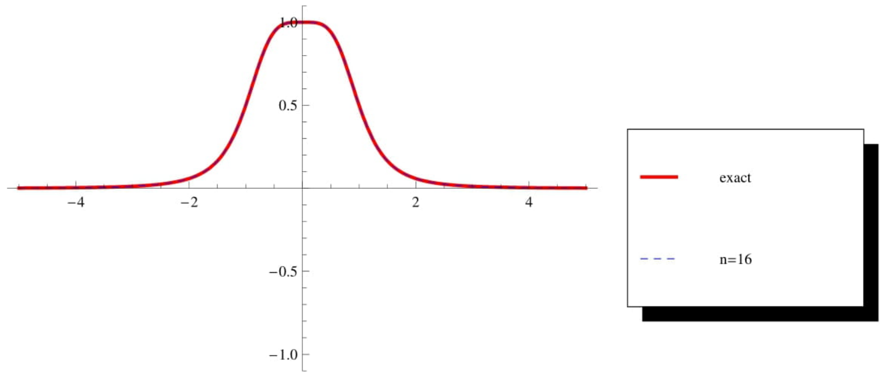

Figure 3. The plots of exact solution $u(x) = \frac{1}{1+x^4}$ and the approximate solution $u_n(x)$ for $n = 16, p = 18$.

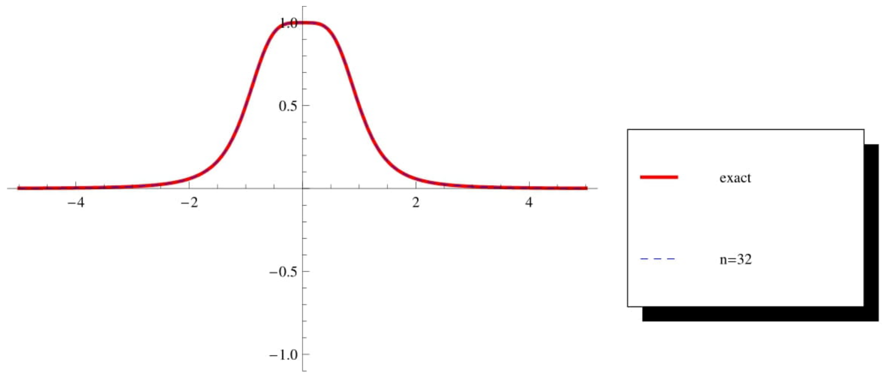

Figure 4. The plots of exact solution $u(x) = \frac{1}{1+x^4}$ and the approximate solution $u_n(x)$ for $n = 32, p = 34$.

and $\gamma = 2$. The approximate solution $u_n(x)$ for different values of n and $p = n+2$ are demonstrated in Figures 5 to 8. Also, The L_2-norm errors of the proposed method for different values of n and $p = n+2$ are listed in Table 6. In these figures and tables, we clearly observe that the desired solution is provided by the approximate solution as well.

Table 6. The L_2-norm errors of the proposed spectral method based on a finite class of orthogonal polynomials related to T-student distribution.

γ	n	p	Error
2	4	6	2.83×10^{-1}
2	8	10	9.13×10^{-3}
2	16	18	3.83×10^{-4}
2	32	34	9.63×10^{-5}

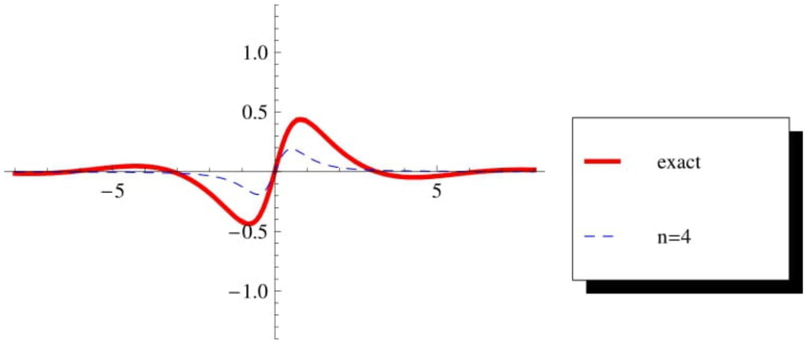

Figure 5. The plots of exact solution $u(x) = \frac{\sin(x)}{1+x^2}$ and the approximate solution $u_n(x)$ for $n = 4, p = 6$.

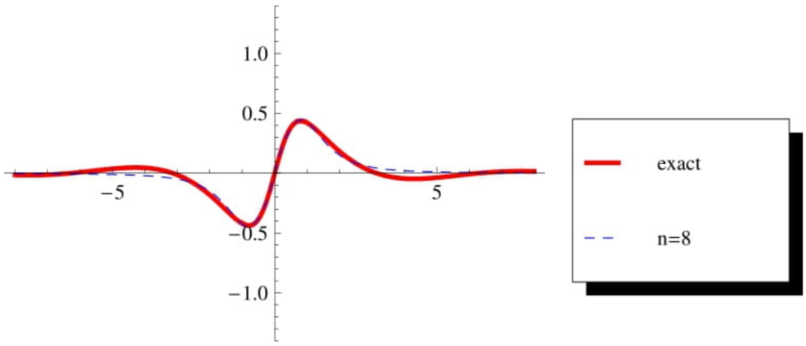

Figure 6. The plots of exact solution $u(x) = \frac{\sin(x)}{1+x^2}$ and the approximate solution $u_n(x)$ for $n = 8, p = 10$.

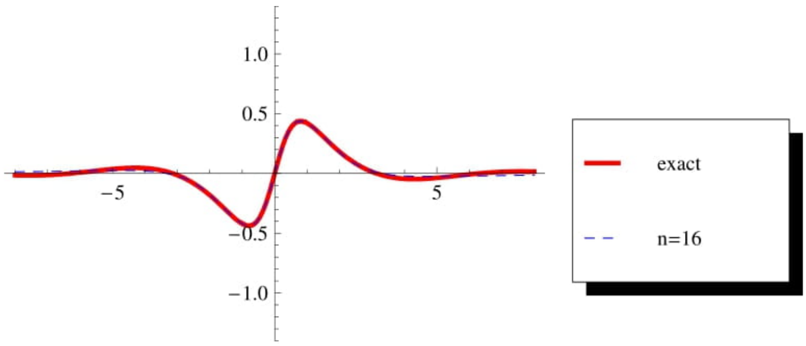

Figure 7. The plots of exact solution $u(x) = \frac{\sin(x)}{1+x^2}$ and the approximate solution $u_n(x)$ for $n = 16, p = 18$.

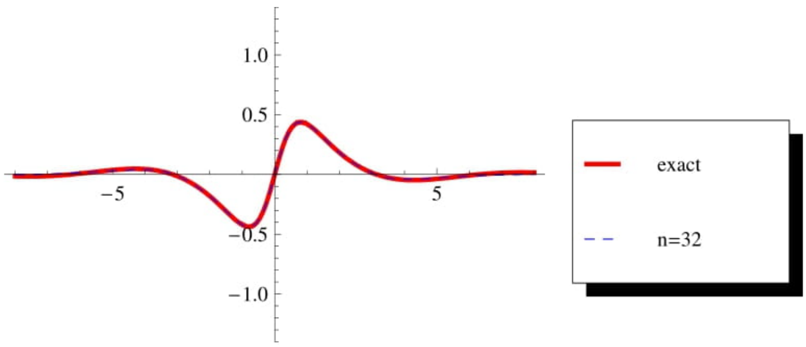

Figure 8. The plots of exact solution $u(x) = \frac{\sin(x)}{1+x^2}$ and the approximate solution $u_n(x)$ for $n = 32, p = 34$.

5. Conclusion

In this paper, we studied an application of a finite class of orthogonal polynomials related to T-student distribution in spectral methods. Almost all properties of this class of orthogonal polynomials were studied. In addition, error analysis and convergence of the proposed spectral method were given. Finally, the numerical results based on $I_n^{(p)}(x)$-Galerkin method were given and showed the exactness and convergence of the method numerically.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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