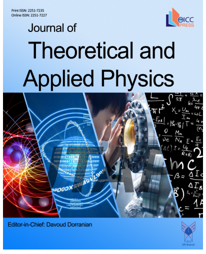

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A novel approach to fractional calculus and its applications to well-known problems

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Abstract:

This paper introduces a novel definition of fractional derivatives and fractional integrals using the conformable derivative approach. This new framework not only aligns more closely with the classical concept of derivatives but also provides a more practical and intuitive structure for fractional calculus. The proposed definition is applicable in two key ranges: $0 \leq \alpha < 1$ and $n-1 \leq \alpha < n$, where n is a positive integer. We also demonstrate that when $\alpha = 1$, our definition seamlessly corresponds to the classical first-order derivative. The advantages of this approach include improved compatibility with classical calculus and enhanced computational convenience, making it a valuable tool for both theoretical investigations and practical applications. By bridging fractional calculus with traditional derivative concepts, our definition facilitates easier analysis and interpretation of fractional differential equations and their solutions. We further explore the implications of this definition in various contexts, including its impact on stability and convergence properties in numerical methods, and provide examples to illustrate its effectiveness and applicability.

Keywords: Mittag-Leffler function; Fractional derivative; Conformable derivative

1. Introduction

Fractional calculus extends the concepts of differentiation and integration to non-integer (fractional) orders. The origins of this mathematical field trace back to the late 17th century, with a notable mention in a correspondence between Leibniz and L'Hôpital on September 30, 1695, where L'Hôpital inquired, "What does $\frac{d^n}{dz^n} f(x)$ mean when $n = 1/2$?". Over time, a substantial body of mathematical work has developed around fractional integrals and derivatives. Despite fractional calculus being a natural extension of classical calculus, it has historically had limited application in physics [1–3]. This may be due, in part, to the fact that foundational concepts were not easily accessible in the mathematical literature until more recently. The application of fractional differential equations has become increasingly relevant in accurately modeling various systems across science and engineering, including fields such as viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos, and fractals.

Lyapunov's Second or Direct Method is particularly notable for evaluating the stability of differential equations without needing explicit solutions. This approach employs a Lyapunov function to analyze the asymptotic behavior of solutions, making it especially useful for nonlinear systems. Its application to non-integer order systems is particularly intriguing, as it extends the Lyapunov function's utility to systems involving fractional derivatives. This paper investigates the application of fractional-like derivatives of the Lyapunov function for stability analysis in perturbed motion equations, presenting several theorems analogous to those of the direct Lyapunov method for specific types of motion equations. The key fractional derivatives discussed include Riemann-Liouville, Caputo, Caputo-Fabrizio [4], and Atangana-Baleanu [5], all of which are characterized by their integral representations [6, 7].

1. The Riemann-Liouville fractional derivative of order

$\alpha \in [n-1, n)$ of a function f is given by:

$$^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx,$$

where Γ denotes the Gamma function.
2. Caputo fractional derivative of order $\alpha \in [n-1, n)$ of f is

$$^{C}D_a^\alpha (f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx$$

3. Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ of f is

$$^{CF}D_a^\alpha (f)(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(x) \exp\left[-\frac{(t-x)^\alpha}{1-\alpha}\right] dx$$

where $M(\alpha)$ is known to be a normalized function such that $M(0) = M(1) = 1$.
4. Atangana-Baleanu fractional derivative of order $\alpha \in (0,1)$ of f in Caputo sense is

$$^{ABC}D_a^\alpha (f)(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(x) E_\alpha \left[-\frac{(t-x)^\alpha}{1-\alpha}\right] dx$$

where $M(\alpha)$ retains the properties observed in the Caputo-Fabrizio fractional derivative. Many of these definitions do not align with the fundamental properties of the ordinary derivative, except for linearity. In [7], a novel fractional derivative definition was introduced that adheres to the basic principles of the ordinary derivative. In this paper, we employ this definition along with the Mittag-Leffler function to propose a new fractional derivative, demonstrate its properties, and illustrate its effectiveness with various examples.

2. Definition of new fractional derivative

In this section, we begin by introducing the Mittag-Leffler function, which plays a significant role in various physical processes and often emerges in the solutions to fractional differential equations. The Mittag-Leffler function serves as a broader generalization of the exponential function, much like how the gamma function extends the factorial function to non-integer values. This function is instrumental in capturing the dynamics of systems described by fractional calculus, providing a crucial link between fractional order models and their traditional integer-order counterparts.
Definition 2.1. The one-parameter Mittag-Leffler function E_α is defined as:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad (1)$$

where α is a positive real number and Γ denotes the gamma function. This function generalizes the exponential function and is particularly useful in the context of fractional calculus and the study of anomalous diffusion and relaxation processes. In the following definition, we introduce a new fractional derivative formulated using the Mittag-Leffler function. Specifically, we consider the first two terms of the series expansion of the Mittag-Leffler function:

$$E_\alpha\left(\varepsilon t^{-\alpha}\right) = 1 + \frac{\varepsilon t^{-\alpha}}{\Gamma(\alpha+1)} + O\left(t^{-2\alpha}\right). \quad (2)$$

This approximation allows us to define a new fractional derivative that incorporates the Mittag-Leffler function's properties, providing a refined approach to modeling fractional dynamics. The term $O\left(t^{-2\alpha}\right)$ represents higher-order terms that are generally small compared to the first two terms, thus simplifying the expression while capturing the essential behavior of the Mittag-Leffler function in this context.
Definition 2.2. Consider a function $f : [0, \infty) \to \mathbb{R}$. The new fractional derivative of order α, defined in the sense of the conformable fractional derivative, is given by:

$$D^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(tE_\alpha\left(\varepsilon t^{-\alpha}\right)\right) - f(t)}{\varepsilon}, \quad (3)$$

where $t > 0$ and $\alpha \in (0,1)$. In this context, we say that the function f is α-differentiable.
If f is α-differentiable on the interval $(0,t)$, and if the limit

$$\lim_{t \to 0^+} f^{(\alpha)}(t) \quad (4)$$

exists, then we define the fractional derivative at zero as:

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t). \quad (5)$$

Theorem 2.3. [1]
Let f and g be α-differentiable functions at a point $t > 0$, with $0 < \alpha \leq 1$. Then the following properties hold: 1. Linearity: $D^\alpha(af + bg) = aD^\alpha(f) + bD^\alpha(g)$ for all $a, b \in \mathbb{R}$.
2. Power Function: $D^\alpha(t^p) = \frac{p^{p-\alpha}}{\Gamma(\alpha+1)}$ for all $p \in \mathbb{R}$.
3. Constant Function: $D^\alpha(c) = 0$ for any constant function $f(t) = c$.
4. Product Rule: $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$.
5. Quotient Rule: $D^\alpha\left(\frac{f}{g}\right) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$.
Lemma 2.4. [1]
Let f be both α-differentiable and differentiable at a point $t > 0$, with $0 < \alpha \leq 1$. Then the α-fractional derivative of f is given by:

$$D^\alpha(f)(t) = \frac{t^{1-\alpha}}{\Gamma(\alpha+1)} f'(t), \quad (6)$$

where $f'(t)$ denotes the standard derivative of f with respect to t, and $\Gamma(\alpha+1)$ is the Gamma function evaluated at $\alpha+1$.

2.0.1 New fractional integral

If a function f is α-differentiable in the interval (a,b), we define the α-fractional integral of f for $a \geq 0$ and $a < t < b$ as follows:
Definition 2.5. The new α-fractional integral of a function f that is α-differentiable is given by:

$$I_a^\alpha(f)(t) = \int_a^t \frac{\Gamma(\alpha+1)}{x^{1-\alpha}} f(x)\, dx, \quad (7)$$

where $\alpha \in (0,1)$.
One of the key results of this definition is stated in the following theorem:
Theorem 2.6. If f is continuous on the domain of I^α for $t \geq a$, then:

$$D^\alpha\left(I^\alpha(f)\right)(t) = f(t). \quad (8)$$

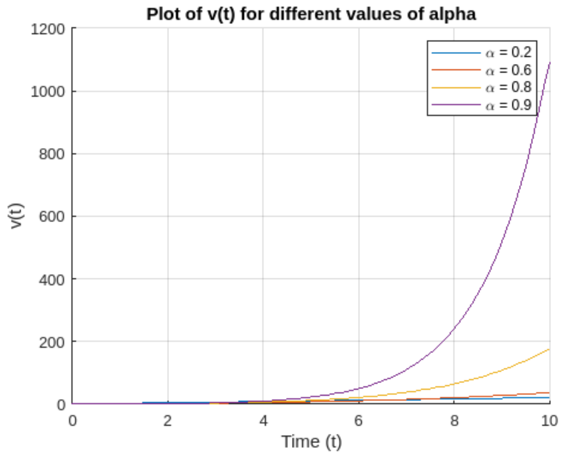

Figure 1. Comparative solutions for serval values of α.

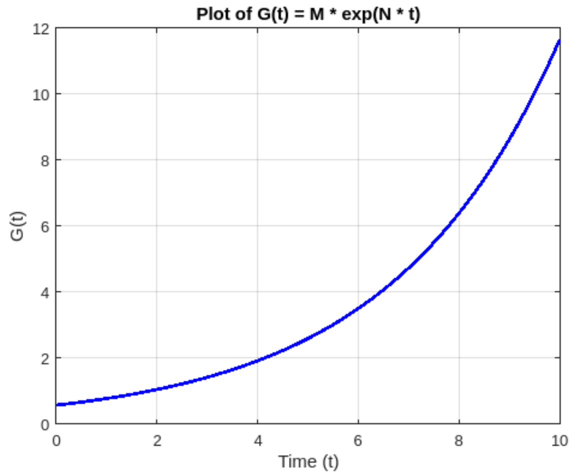

Figure 2. Comparative solutions for serval values of α.

Proof: See [6].

Theorem 2.7. [1] Let $\alpha \in (0,1]$, then:

1) $\mathbf{D}^\alpha\left(\frac{\Gamma(1+\alpha)}{\alpha}t^\alpha\right) = 1$.

2) $\mathbf{D}^\alpha\left(\sin\frac{1}{\alpha}t^\alpha\right) = \cos(\frac{\Gamma(1+\alpha)}{\alpha}t^\alpha)$.

3) $\mathbf{D}^\alpha\left(\cos\frac{1}{\alpha}t^\alpha\right) = -\sin(\frac{\Gamma(1+\alpha)}{\alpha}t^\alpha)$.

4) $\mathbf{D}^\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{\Gamma(1+\alpha)}{\alpha}t^\alpha}$.

3. Application

3.1 Fractional ordinary differential

The initial value of fractional ordinary differential equations employing the conformable fractional derivative, with $0 < \alpha < 1$,

$$T_t^\alpha v(t) = \lambda v(t), \quad t > 0, \tag{9}$$

with the initial condition

$$v(t)|_{t=0} = v_0, \tag{10}$$

has the solution

$$v(t) = v_0 e^{\Gamma(\alpha+1)\lambda\frac{t^\alpha}{\alpha}}. \tag{11}$$

3.2 Application to physics

The conformable derivative is a relatively recent concept in fractional calculus, designed to address some limitations of traditional fractional derivatives. It offers a more intuitive approach and is easier to apply in various contexts, including physics. Here's an overview of the conformable fractional derivative and its applications:

3.2.1 Model of population growth

Population growth models are crucial for analyzing and forecasting changes in population dynamics over time. These models find applications across diverse disciplines including biology, ecology, epidemiology, and sociology, where they are instrumental in predicting shifts in population size and structure. The essence of a population growth model lies in its ability to describe how populations evolve, considering factors such as birth rates, death rates, immigration, and emigration.

To formulate a population growth model, we typically start by defining the core components and parameters that influence population changes. These may include intrinsic growth rates, carrying capacities, and environmental influences, among others. By integrating these elements into mathematical frameworks, we can derive equations that represent population dynamics, enabling us to forecast future trends and understand the underlying processes driving population changes. We approach the problem of population growth modeling by

$$\frac{dG}{dt} = NG(t), \quad G(0) = M. \tag{12}$$

In this context, N represents a constant, M denotes the initial population size, and G is the function describing population dynamics. The solution to this problem can be expressed as follows:

$$G(t) = Me^{Nt}. \tag{13}$$

The solution is illustrated in the following figure: We now reformulate this problem using a new conformable fractional approach:

$$G^{(\alpha)}(t) = NG(t), \quad G(0) = M. \tag{14}$$

From equation (6), we obtain:

$$\frac{t^{1-\alpha}}{\Gamma(1+\alpha)}G'(t) = NG(t). \tag{15}$$

Therefore, we can proceed by employing the following expression:

$$G'(t) = N\Gamma(1+\alpha)t^{\alpha-1}G(t) \text{ or } \frac{dG(t)}{dt} = N\Gamma(1+\alpha)t^{\alpha-1}G(t). \tag{16}$$

By applying the method of separation of variables, we obtain:

$$\frac{dG(t)}{y(t)} = N\Gamma(1+\alpha)t^{\alpha-1}dt. \tag{17}$$

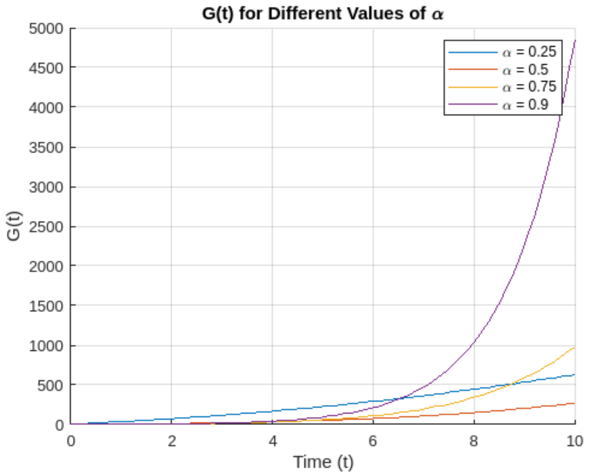

Figure 3. Comparative solutions for serval values of α.

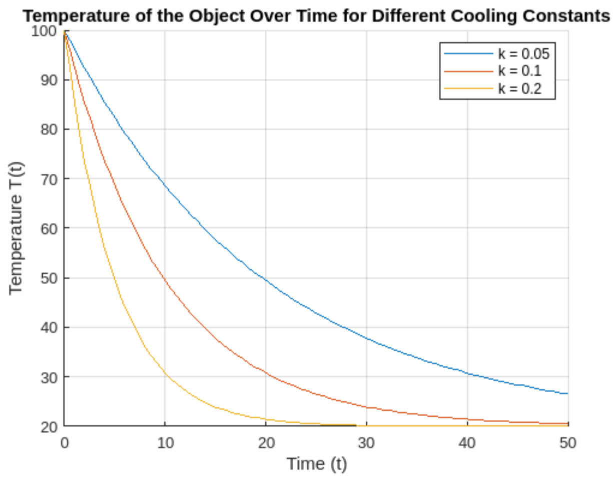

Figure 4. Comparative solutions for various values of K.

By performing the integration, we obtain:

$$\ln G(t) = \frac{N\Gamma(1+\alpha)}{\alpha} t^{\alpha} + B, \tag{18}$$

we obtain:

$$G(t) = e^{B} e^{\frac{N\Gamma(1+\alpha)}{\alpha} t^{\alpha}}, \tag{19}$$

and applying the initial condition, we get:

$$G(t) = M e^{\frac{N\Gamma(1+\alpha)}{\alpha} t^{\alpha}}. \tag{20}$$

3.2.2 Model of body cooling

Newton's law of cooling describes how the temperature of an object changes over time as it exchanges heat with its surroundings. This principle states that the rate of temperature change of the object is directly proportional to the difference between its temperature and the ambient temperature. In essence, the law provides a mathematical framework for understanding how an object's temperature evolves as it moves towards thermal equilibrium with its environment.
However, applying this concept in practice involves some nuances. For instance, the assumption that temperature differences remain constant and the method of heat transfer through materials, such as sweaters, need to be carefully considered. This is because the sweater's ability to insulate and its heat transfer characteristics can influence the cooling process. Nonetheless, for simplicity, it is often assumed that the heat transfer coefficient remains constant, simplifying the problem to a predictable model where the rate of cooling is proportional to the temperature difference between the object and its surroundings.
The problem of Newton's law of cooling can be formally expressed by the differential equation:

$$\frac{dT}{dt} = -k\left(T(t) - T_c\right), \tag{21}$$

with the initial condition:

$$T(0) = T_0. \tag{22}$$

Here, $T(t)$ denotes the temperature of the object at time t, T_c is the constant ambient temperature, T_0 is the initial temperature of the object, and k is the cooling constant. At $t = 0$, the initial temperature is T_0. The solution to the differential equation is given by:

$$T(t) = T_c + (T_0 - T_c)\, e^{-kt}. \tag{23}$$

This equation describes how the temperature of the object evolves over time, approaching the ambient temperature T_c asymptotically as t increases. The solution is illustrated in the following figure: By applying equation (6), we derive the new conformable differential formula for this specific problem:

$$\frac{d^{\alpha}T(t)}{dt^{\alpha}} = -k\left(T(t) - T_c\right), \tag{24}$$

so,

$$\frac{t^{1-\alpha}}{\Gamma(1+\alpha)} T'(t) = -k\left(T(t) - T_c\right). \tag{25}$$

Therefore, we obtain:

$$\frac{dT(t)}{dt} = -\Gamma(1+\alpha) t^{\alpha-1} k\left(T(t) - T_c\right), \tag{26}$$

and

$$\frac{dT(t)}{T(t) - T_c} = -\Gamma(1+\alpha) t^{\alpha-1} k. \tag{27}$$

Therefore,

$$\mathrm{Ln}\left(T(t) - T_c\right) = \frac{-k\Gamma(1+\alpha) t^{\alpha}}{\alpha} + C, \tag{28}$$

then,

$$T(t) = T_c + e^{C} e^{\frac{-k\Gamma(1+\alpha) t^{\alpha}}{\alpha}}, \tag{29}$$

Incorporating the initial condition, we obtain the final solution:

$$T(t) = T_c + (T_0 - T_c)\, e^{\frac{-k\Gamma(1+\alpha) t^{\alpha}}{\alpha}}. \tag{30}$$

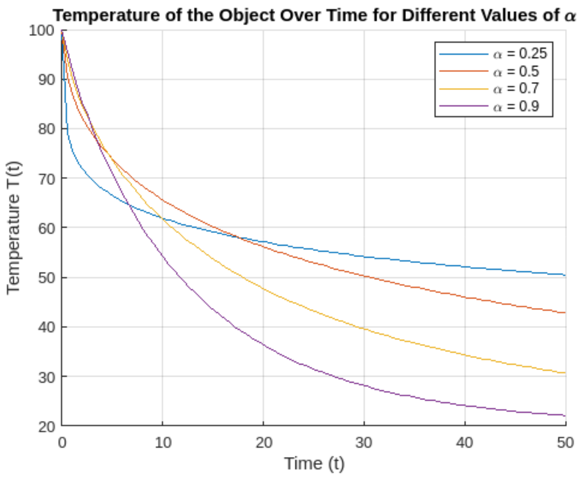

Figure 5. Comparative solutions for various values of α.

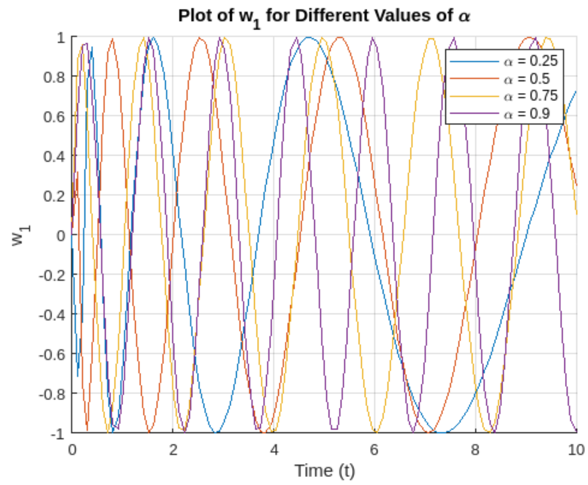

Figure 6. Comparative solutions for various values of α.

3.2.3 Fractional Heat equation

The heat equation is a cornerstone in the realms of physics and thermal engineering, encapsulating how temperature within a material evolves over time as a result of applied heat and the material's thermal characteristics. This equation finds extensive applications across various disciplines, including thermodynamics, geophysics, meteorology, and materials engineering, among others.

By solving the heat equation, one can predict how temperature changes with time within a given material. While analytical solutions are feasible for certain straightforward scenarios, most situations require numerical approaches to achieve precise results. Commonly employed numerical techniques include the finite difference method, finite element method, and finite volume method.

The practical applications of the heat equation are numerous. For instance:

- It is used to model heat transfer in buildings, aiding in the enhancement of energy efficiency.
- It helps in predicting temperature distributions during material manufacturing processes.
- It supports the study of heat movement in soils, which is crucial for geothermal energy research.
- It contributes to the understanding of meteorological phenomena, such as cloud formation and ocean currents.

The heat equation thus serves as a fundamental tool for analyzing and forecasting thermal phenomena across a wide array of scientific and technological fields. Its elegant mathematical structure underpins much of thermal modeling and drives significant advancements in both science and engineering.

Additionally, fractional calculus generalizes the classical heat equation to fractional heat equations, which are represented as:

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial x^2}, \quad (31)$$

$$w(0,t) = 0, \quad t > 0, \quad (32)$$

$$w(1,t) = 0, \quad t > 0, \quad (33)$$

$$\frac{\partial w(x,0)}{\partial t} = 0, \quad (34)$$

$$w(x,0) = F(x), \quad 0 < x < 1. \quad (35)$$

Consider the conformable fractional linear differential equations with constant coefficients, based on the results presented in [2]:

$$\frac{d^\alpha}{dy^\alpha}\frac{d^\alpha w}{dy^\alpha} \pm \mu^2 w = 0. \quad (36)$$

We start with the equation $R^2 \pm \mu^2 = 0$. From this equation, we find that R can be either $\pm\nu$ or $\pm\mu i$. Additionally, based on Theorem (2.7), we establish that

$$w = e^{\mp\frac{\mu\Gamma(1+\alpha)}{\alpha}t^\alpha}, \quad (37)$$

we obtain two distinct solutions of the equation. Furthermore, in the second case, using properties (2) and (3) outlined in Theorem (2.7), we find that

$$w_1 = \sin\left(\frac{\mu\Gamma(1+\alpha)}{\alpha}t^\alpha\right) \text{ and } w_2 = \cos\left(\frac{\mu\Gamma(1+\alpha)}{\alpha}t^\alpha\right). \quad (38)$$

Additionally, we present a novel fractional differential formula of order 2α with constant coefficients, which is expressed by

$$\frac{d^\alpha}{dx^\alpha}\frac{d^\alpha w}{dx^\alpha} + a\frac{d^\alpha w}{dx^\alpha} + bw = h(x). \quad (39)$$

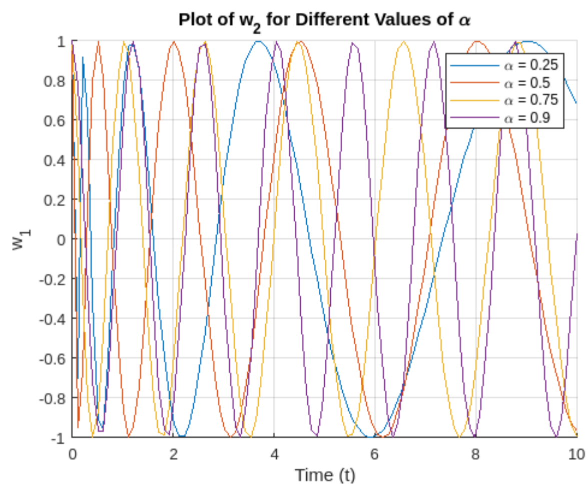

Figure 7. Comparative solutions for various values of α.

We can denote $\frac{d^\alpha}{dx^\alpha}$ as $\mathscr{D}^\alpha$, so we have:

$$\mathscr{D}^\alpha\left(\mathscr{D}^\alpha w\right)+a\mathscr{D}^\alpha w+bw=h(x). \qquad (40)$$

To find the solution, we analyze the equation $R^2+aR+b=0$. By applying the characteristics of the new conformable derivative described in [2], and using properties (2), (3), and (4) as detailed in Theorem (2.7), we develop a framework similar to the theory of conventional linear differential equations.

Now, let us examine the heat equation (31). We will use the method of separation of variables. Suppose $w(x,t)=M(x)R(t)$. Substituting this into the differential equation gives us:

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha R(t)}{dt^\alpha}M(x)=R(t)\frac{d^2M(x)}{dx^2}, \qquad (41)$$

we obtain:

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha R(t)}{dt^\alpha}/R(t)=\frac{d^2M(x)}{dx^2}/P=\beta, \qquad (42)$$

for some constant. Consequently:

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha R(t)}{dt^\alpha}-\beta R=0, \quad \text{and} \quad \frac{d^2M(x)}{dx^2}-\beta M=0. \qquad (43)$$

We consider the expression:

$$\frac{d^2M(x)}{dx^2}-\beta M=0. \qquad (44)$$

As is well known, there are three scenarios for the values of β to consider: $\beta=0$, $\beta=-\nu^2$, and $\beta=\nu^2$. Conditions (32) and (33) necessitate that

$$\nu=n\pi \text{ and } M_n(x)=c_n\sin(n\pi x). \qquad (45)$$

By applying equations (36) and (38), we obtain:

$$R(t)=b_1\cos\left(\frac{n\pi\Gamma(1+\alpha)}{\alpha}t^\alpha\right)+b_2\sin\left(\frac{n\pi\Gamma(1+\alpha)}{\alpha}t^\alpha\right). \qquad (46)$$

And the condition (34) now gives $b_2=0$, and $R(t)=b_1\cos(\frac{n\pi\Gamma(1+\alpha)}{\alpha}t^\alpha)$, then, using equation (45), we obtain :

$$w(x,t)=\sum_{n=1}^{\infty}a_n\sin(n\pi x)\cos\left(\frac{n\pi\Gamma(1+\alpha)}{\alpha}t^\alpha\right), \qquad (47)$$

using condition (35), we determine that a_n is the n-th Fourier coefficient of the function $h(x)$.

4. Conclusion

This method offers several significant advantages, including a more seamless connection to classical calculus concepts and enhanced computational efficiency. These benefits make it highly valuable for both theoretical research and practical applications. By bridging fractional calculus with traditional derivative concepts, our definition simplifies the analysis and interpretation of fractional differential equations, making them more accessible and easier to understand. This integration not only streamlines the solution process but also lays the groundwork for more intuitive and effective problem-solving approaches.

In addition, we explore the broader implications of this definition across various fields, focusing particularly on its influence on the stability and convergence of numerical methods. These factors are crucial for ensuring accuracy and reliability in computational tasks. Through a series of detailed examples, we demonstrate the versatility and applicability of our approach, showcasing its potential to address a wide range of practical challenges. Overall, by highlighting both theoretical advancements and real-world applications, we underscore the definition's significant contribution to the field of fractional calculus and its potential to drive further progress in both academic and applied contexts.

Authors Contribution
All authors equally contributed to the conception, methodology, data analysis, and writing of the manuscript. All authors read and approved the final manuscript.

Availability of data and materials
The datasets generated (or analyzed) during the current study are available from the corresponding author on reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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