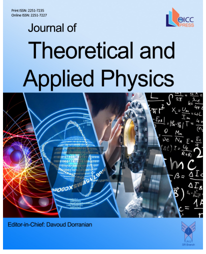

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Energy spectra of Deng Fan potential within the framework of relativistic KG equation using extended Nikiforov-Uvarov approach

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Original Research

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Abstract:

In this paper, we study the quantum dynamic of the Klein-Gordon particles/antiparticles with generalized Deng-Fan potential. We determine the energy spectra and the corresponding wave function expressed in terms of confluent Heun function using the extended Nikiforov Uvarov method. The influence of the potential parameters on the energy spectra is discussed in details.

Keywords: Extended Nikiforov Uvarov method, Deng-Fan potential, Confluent Heun function

1. Introduction

Solutions of Klein Gordon equation (KGE) has many useful applications in the field of nuclear and high energy physics since it can be used to describe the quantum dynamic of the system at the relativistic regime [1, 2]. The solutions of the KGE are restricted to the nature of the potential models. This implies that the relativistic KGE contains four vector momentum operator and a scalar rest mass [3, 4]. The restricted potential coupling in the KGE comprising of the four vector potential $V(r)$ and the space-time scalar potential $S(r)$. The KGE in three dimensional spaces is defined as [4–6],

$$\left\{-\left(i\frac{\partial}{\partial t}-V(r)\right)^2-\nabla^2+(S(r)+m_0)\right\}\psi(\mathbf{r})=0 \quad (1)$$

where ∇^2 is the Laplacian operator, $i\partial/\partial t$ is the energy operator, $\psi(\mathbf{r})$ is the wave function, m_0 is the rest mass and the natural unit notation ($\hbar=c=1$) has been used. As it is well known that the KGE is used in describing spin-zero particles due to its square terms and in some special cases contain some known solutions already known in the solutions of non-relativistic Schrodinger equation. The KGE becomes very complicated because of the square term in it especially for some potentials with unequal scalar and vector potentials. In recent time, quit a number of researchers have shown interest in finding the solutions of the Schrodinger, Klein-Gordon and Dirac equations for different varieties of potential. The most important efforts put by researchers in obtaining either the exact or approximate solutions of the wave equation is because of the information that can be extract of it which describe quantum dynamical system of the system [7, 8]. Nevertheless, once the solutions of the system have been obtained then the solutions can be used in different areas of physics, chemistry and chemical physics [9–12].

In obtaining the analytical solutions of the Schrodinger-like equation, many authors in recent years have devoted their interest in proposing different analytical techniques such as factorization [13], Nikiforov-Uvarov (NU) [14], asymptotic iteration method (AIM) [15], supersuymmetric quantum mechanics [16], exact quantization rule [17], Nikiforov-Uvarov-functional-analysis (NUFA) method [18, 19] among others. A great number of research works have been reported on the KGE with different potential models [20–23]. Okorie et al. [24] and his co-workers studied the bound and scattering state KGE with Mobius square potential. Ikot et al. [25] studied the bound and scattering state of Deng Fan potential in higher dimensions. The unequal vector and scalar Deng Fan potential takes the form [26],

$$V(r)=V_0+\frac{bV_1}{(e^{\alpha r}-1)}+\frac{b^2V_2}{(e^{\alpha r}-1)^2},\quad S(r)=S_0+\frac{bS_1}{(e^{\alpha r}-1)}+\frac{b^2S_2}{(e^{\alpha r}-1)^2} \tag{2}$$

where V_i, $S_i =$, $i=0,\ 1,\ 2$ are the potential parameters, $b=e^{\alpha r_e}-1$, α is the screening parameter and r_e is the equilibrium bond length. Deng Fan in some literature are usually referred to as Morse-like potential and it can be used in the study of nucleon motion and for studying interaction of nuclei. Another usefulness of the Deng Fan potential is in diatomic molecule spectra analysis [27].

Recently, Nagiyev and Ahmadov [28] studied the motion of a charged relativistic particle in three dimensions with noncentral coulomb plus ring-shaped potential, using finite difference method. Here, the dynamic symmetry group of the radial part of the equation of motion was obtained. By using the supersymmetric quantum mechanics formalism, the modified Klein-Fock-Gordon equation for the Hulthen plus Yukawa potential was solved [29]. Its relativistic energy eigenvalues and wave functions were obtained. These energies were seen to be very sensitive to the potential parameters for the quantum states considered. With the help of an improved approximation scheme, the bound state solutions of KGE with Manning-Rosen and a class of Yukawa potentials were studied [30]. The energy levels obtained were seen to be sensitive to both potential parameters and quantum numbers involved. In another development, Ahmadov and his collaboartors [31] solved the KGE for a combined Hulthen-Yukawa potential model using the NU and traditional approaches. The energies obtained were inversely proportional to the quantum numbers at certain values of screening parameter.

In this present article, we consider the KGE of Equation 1 in the presence of the generalized Deng-Fan potential of Equation 2 and solve it analytically using ENU [32–34] method, since other known method cannot solve it because of the higher power of polynomial of order four in the resulting Schrodinger-like equation. The organization of the paper is as follows. The solution of the KGE is given in section 2. Results and Discussion are given in section 3. Finally, a brief conclusion is presented in section 4.

2. Solutions of the Radial KGE via ENU method

The radial KGE in D-dimensions for unequal vector and scalar potential is defined as [26],

$$\left\{\frac{d^2}{dr^2}+E_{nl}^2+V^2(r)-S^2(r)-m_0^2-2E_{nl}V(r)-2m_0S(r)-\frac{(D+2l-1)(D+2l-3)}{4r^2}\right\}R_{nl}(r)=0 \tag{3}$$

Substituting Equation (2) into Equation (3) yields

$$\left\{\frac{d^2}{dr^2}+E_{nl}^2+\left(V_0+\frac{bV_1}{(e^{\alpha r}-1)}+\frac{b^2V_2}{(e^{\alpha r}-1)^2}\right)^2-\left(S_0+\frac{bS_1}{(e^{\alpha r}-1)}+\frac{b^2S_2}{(e^{\alpha r}-1)^2}\right)-m_0-2E_{nl}\left(V_0+\frac{bV_1}{(e^{\alpha r}-1)}+\frac{b^2V_2}{(e^{\alpha r}-1)^2}\right)-2m_0\left(S_0+\frac{bS_1}{(e^{\alpha r}-1)}+\frac{b^2S_2}{(e^{\alpha r}-1)^2}\right)-\frac{(D+2l-1)(D+2l-3)}{4r^2}\right\}R_{nl}(r)=0 \tag{4}$$

Now using the improved Greene-Aldrich approximation scheme for the centrifugal barrier [35]

$$\frac{1}{r^2}\approx\alpha^2\left(c_0+\frac{1}{(e^{\alpha r}-1)}+\frac{1}{(e^{\alpha r}-1)^2}\right), \tag{5}$$

and using the new coordinate transformation of the form $\xi=(e^{\alpha r}-1)$, Equation (4) becomes,

$$\frac{d^2R_{nl}(\xi)}{d\xi^2}+\frac{\xi^2}{\xi^2(1+\xi)}\frac{dR_{nl}(\xi)}{d\xi}+\frac{1}{\xi^4(1+\xi)^2}\times(\varepsilon_n^2\xi^4+\lambda_1\xi^3+\lambda_2\xi^2+\lambda_3\xi+\lambda_4)R_{nl}(\xi)=0 \tag{6}$$

where

$$\begin{aligned}
\varepsilon_n^2&=\frac{1}{\alpha^2}\{E_{nl}^2+V_0-2E_{nl}V_0-m_0^2-S_0^2-2m_0S_0-\alpha^2c_0\gamma\},\\
\lambda_1&=\frac{1}{\alpha^2}\{2bV_0V_1-2bV_1E_{nl}-2bS_0S_1-2bm_0S_1-\alpha^2\gamma\},\\
\lambda_2&=\frac{1}{\alpha^2}\{b^2V_1^2+2b^2V_0V_2-2E_{nl}b^2V_2-b^2S_1^2-2b^2S_0S_2-2m_0b^2S_2-\alpha^2\gamma\}\\
\lambda_3&=\frac{b^3}{\alpha^2}(2V_1V_2-2S_1S_2),\ \ \lambda_4=\frac{b^4}{\alpha^2}(V_2^2-S_2^2),\\
\gamma&=\frac{(D+2l-1)(D+2l-3)}{4},\ \ c_0=\frac{1}{12}
\end{aligned} \tag{7}$$

As the name implies, the ENU method [28–30] is an extension of the standard Nikiforov-Uvarov (NU) [14] and its NUFA methods [18, 19]. This method was developed and proposed by Karayer et al. [27] and her co-workers. The ENU takes the form [32–34],

$$\psi''(\xi)+\frac{\tilde{\tau}_e(\xi)}{\sigma_e(\xi)}\psi'(\xi)+\frac{\tilde{\sigma}_e(\xi)}{\sigma_e^2(\xi)}\psi(\xi)=0 \tag{8}$$

From Equation (6) and Equation (8), we obtain the following polynomials

$$\begin{aligned}&\tilde{\tau}_e=\xi^2,\ \ \sigma_e(\xi)=\xi^2(1+\xi)\\ &\tilde{\sigma}_e(\xi)=\varepsilon_n^2\xi^4+\lambda_1\xi^3+\lambda_2\xi^2+\lambda_3\xi+\lambda_4\end{aligned} \tag{9}$$

By using the ENU method, we obtain the $\pi_e(\xi)$ as follows

$$\pi_e(\xi)=\xi^2+\xi\pm\Big((1+P-\varepsilon^2)\xi^4+(2+P+Q-\lambda_1)\xi^3+(Q-\lambda_2+1)\xi^2-\lambda_3\xi-\lambda_4\Big)^{1/2} \tag{10}$$

Here, $G(\xi)=P\xi+Q$ and $\pi_e(\xi)$ must be a second degree polynomial and the terms under the square root sign must be equal to a square of a polynomial of degree two of the form $(A+B\xi+C\xi^2)^2$. Therefore, the polynomial in Equation (10) becomes,

$$\pi_e(\xi)=\xi^2+\xi\pm(A+B\xi+C\xi^2) \tag{11}$$

The parameters A, B, C, P, Q are obtained as follows

$$\begin{aligned}&A=\pm\sqrt{-\lambda_4},\ \ B=-\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}},\\ &C=-\left(\pm\sqrt{-\lambda_4}+\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}}\right)\pm\sqrt{(\lambda_1-\lambda_2+\lambda_3-\lambda_4-\varepsilon^2)}\\ &Q=\lambda_1-\varepsilon^2+2BC-C^2\\ &P=-B^2-2AC+2BC-1+(\lambda_1-\lambda_2)\end{aligned} \tag{12}$$

The two possible values of $G(\xi)$ are given as,

$$\begin{aligned}G_1(\xi)=&(-B^2-2AC+2BC-1+(\lambda_1-\lambda_2))\xi+\\ &(\lambda_1-\varepsilon^2+2BC-C^2)\end{aligned} \tag{13}$$

$$\begin{aligned}G_2(\xi)=&(-B^2-2AC+2BC-1(\lambda_1-\lambda_2))\xi-\\ &(\lambda_1-\varepsilon^2+2BC-C^2)\end{aligned} \tag{14}$$

The four corresponding polynomials for $\pi_e(\xi)$ with different values of $G_1(\xi)$ and $G_2(\xi)$ are obtained as follows:

$$\pi_e(\xi)=\xi^2+\xi+\left(\pm\sqrt{-\lambda_4}+\left(-\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}}\right)\xi+\left(-\left(\pm\sqrt{-\lambda_4}+\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}}\right)\sqrt{(\lambda_1-\lambda_2+\lambda_3-\lambda_4-\varepsilon^2)}\right)\xi^2\right) \tag{15a}$$

$$\pi_e(\xi)=\xi^2+\xi+\left(\pm\sqrt{-\lambda_4}+\left(-\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}}\right)\xi+\left(-\left(\pm\sqrt{-\lambda_4}+\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}}\right)\sqrt{(\lambda_1-\lambda_2+\lambda_3-\lambda_4-\varepsilon^2)}\right)\xi^2\right) \tag{15b}$$

$$\pi_e(\xi)=\xi^2+\xi+\left(\pm\sqrt{-\lambda_4}+\left(-\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}}\right)\xi+\left(-\left(\pm\sqrt{-\lambda_4}+\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}}\right)\sqrt{(\lambda_1-\lambda_2+\lambda_3-\lambda_4-\varepsilon^2)}\right)\xi^2\right) \tag{15c}$$

$$\pi_e(\xi)=\xi^2+\xi+\left(\pm\sqrt{-\lambda_4}+\left(-\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}}\right)\xi+\left(-\left(\pm\sqrt{-\lambda_4}+\frac{\lambda_3}{\pm2\sqrt{-\lambda_4}}\right)\sqrt{(\lambda_1-\lambda_2+\lambda_3-\lambda_4-\varepsilon^2)}\right)\xi^2\right) \tag{15d}$$

The following four distinct choices ++, –, +-, -+ had been used in obtaining the four values of $\pi_e(\xi)$ polynomials. The polynomials $h(\xi)$, $\tau(\xi)$ and $h_n(\xi)$ are obtained as follows:

$$h(\xi)=(p+2\pm C)\xi+(Q+1\pm B) \tag{16}$$

$$\tau(\xi)=6\xi^2+2\xi\pm2(A+B\xi+C\xi^2) \tag{17}$$

$$h_n(\xi)=-n-3n\xi\mp n(B+2C\xi)-\frac{n(n-1)}{6}-n(n-1)\xi+C_n \tag{18}$$

with C_n being the integration constant.
Now equating equations $h(\xi)=h_n(\xi)$ yields the following equation,

$$(p+2\pm C)\xi+(Q+1\pm B)=-n-3n\xi\mp n(B+2C\xi)-\frac{n(n-1)}{6}-n(n-1)\xi+C_n \tag{19}$$

By comparing the coefficients of "ξ" and constant in Equation (19) yield the eigenvalue equation and the integration constant as follows,

$$P+2+3n+n(n-1)=(2n+2)C \tag{20}$$

$$Q+1+B=-n-\frac{n(n-1)}{6}-nB+C_n \tag{21}$$

where the positive (++) option is used in obtaining Equations (20) and (21), respectively. Solving Equation (20) explicitly gives the energy eignvalues for the relativistic spinless particles with generalized Deng-Fan potential in D-dimensions as,

$$\begin{aligned}&\{E_{nl}^2+V_0^2-2E_{nl}V_0-m_0^2-S_0^2-2m_0S_0-\alpha^2c_0\gamma\}=\\ &\alpha^2(-\lambda_4+2\lambda_3+\lambda_1-\lambda_2)\\ &-\alpha^2\left[-\sqrt{-\lambda_4}+\frac{\lambda_3}{2\sqrt{-\lambda_4}}+\frac{1}{2}\times\left(\frac{\lambda_1-\lambda_2+3n+n(n-1)-(\frac{\lambda_3}{2\sqrt{-\lambda_4}})^2-1}{n+1+\sqrt{-\lambda_4}+(\frac{\lambda_5}{2\sqrt{-\lambda_4}})}\right)\right]^2\end{aligned} \tag{22}$$

The four eigen functions can be determine for all values of π_{e_1}, π_{e_2}, π_{e_3} and π_{e_4} using Equations (15)a-(15)d. However,

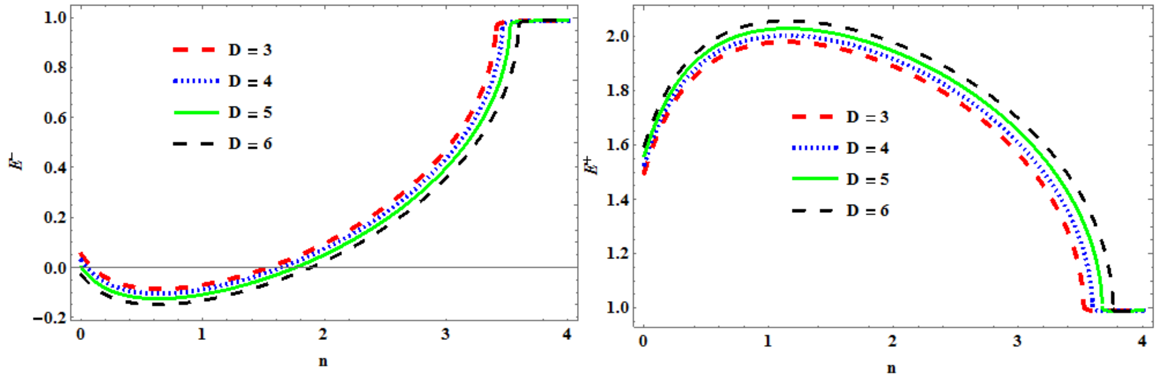

Figure 1. Variation of energies spectra with principal quantum number n for various dimensions.

we will determine one which is applicable to others. We find that the function $\varphi(\xi)$ is obtained as,

$$\varphi_e(\xi) = z^{(\frac{1}{2}+A+B)}(1-z)^{\frac{1}{2}(A+B+C)}e^{-\frac{A}{\xi}} \tag{23}$$

Using the condition $h(\xi) = h_n(\xi)$, we have that

$$\chi_n''(\xi)+\left[1+\frac{1+2A+2B}{\xi}+\frac{2(A+B+C)}{1-\xi}\right]\xi'(\xi)+ \left[\frac{P+Q+B+2C-\frac{1}{2}}{\xi}+\frac{P+Q+B+2C-\frac{1}{2}}{1-\xi}\right]\chi(\xi)=0 \tag{24}$$

The confluent Heun differential equation (CHDE) which is a special case of Heun differential equation is defined as [36, 37],

$$\chi_n''(\xi)+\left[\alpha+\frac{\beta+1}{\xi}+\frac{\gamma+1}{\xi-1}\right]\chi'(\xi)+\left[\frac{\mu}{\xi}+\frac{\nu}{\xi-1}\right]\chi(\xi)=0 \tag{25}$$

By comparing Equations (24) and (25), the parameters α, β, γ, δ are obtained as follows,

$$\begin{aligned} \alpha &= 1,\ \ \beta = 2(A+B),\ \ \gamma+1 = -2(A+B+C) \\ \mu &= -\nu = -(P+Q+B+2C-\frac{1}{2}) \end{aligned} \tag{26}$$

The solution of Confluent Heun differential equation of Equation (25) can only be a polynomial $H_c(a,b,\gamma,\delta,x)$ of order n if and only if $\mu+\nu = -n\alpha$ [36, 37], where $\delta = \mu+\nu-\alpha/2(\beta+\gamma+2)$ and $\eta = \alpha/2(\beta+1)-\mu-1/2(\beta+\gamma+\beta\gamma)$. Thus the solution of Equation (24) becomes,

$$\chi(\xi) = Heun(\alpha,\beta,\gamma,\delta,\eta,\xi) \tag{27}$$

Thus, the total wave function for the Klein Gordon equation becomes

$$\psi_{nl}(r) = N_n r^{-(\frac{D-1}{2})}(e^{-\alpha r})^{(\frac{1}{2}+A+B)}(1-e^{-\alpha r})^{\frac{1}{2}-(A+B+C)} e^{-Ae^{\alpha r}}H_c(\alpha,\beta,\gamma,\delta,\eta,r) \tag{28}$$

where N_n is the normalization constant. The normalization can be determined and details analysis on how to evaluate the normalization can be seen in Refs. [36, 37].

3. Results and discussion

In this work, we considered the following parameters $\alpha = 0.4$, $V_0 = 1$, $V_1 = 0.8$, $V_2 = 0.7$; $m_0 = 1$, $S_0 = S_1 = S_2 = 0.1$, $b = 0.1$ in our analysis.

In Figs. 1-11, we plot the variations of the energy spectrum of the KG particles versus the quantum numbers and potential parameters. The left panel represents the negative energy (particles) and the right panel is the positive energy (antiparticles) for different dimensions. Fig. 1 shows the plot of the energy spectrum of the KG particles/antiparticles versus the principal quantum number n. It can be seen in Fig. 1 that the negative energy spectrum first started to decrease and later started increasing as the quantum number

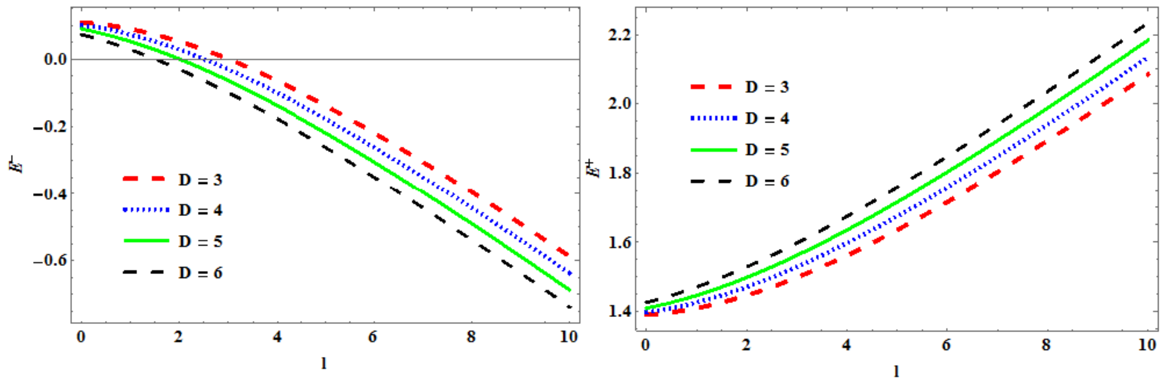

Figure 2. Variation of energies spectra with angular momentum quantum number l for various dimensions.

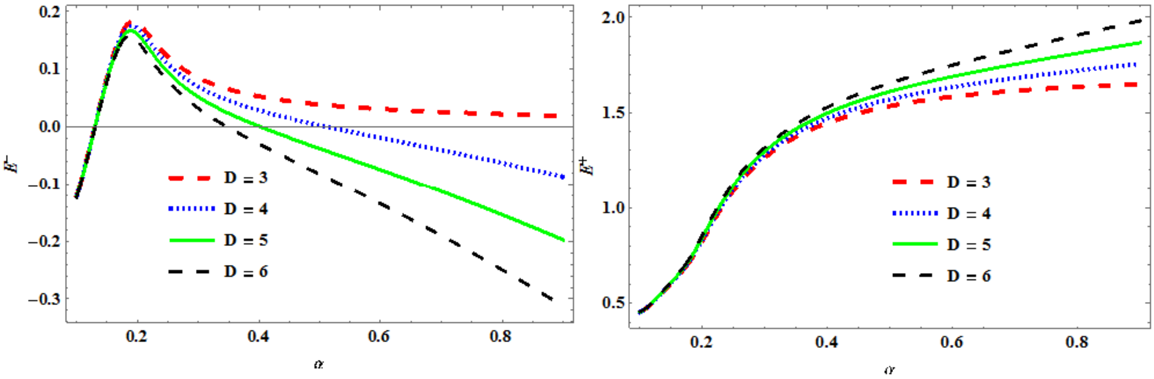

Figure 3. Variation of energies spectra with screening parameter α for various dimensions.

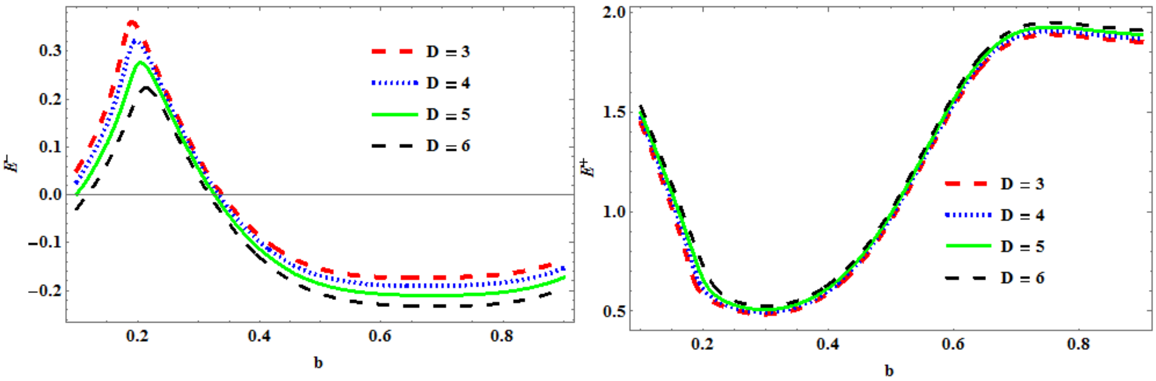

Figure 4. Variation of energies spectra with potential parameter b for various dimensions.

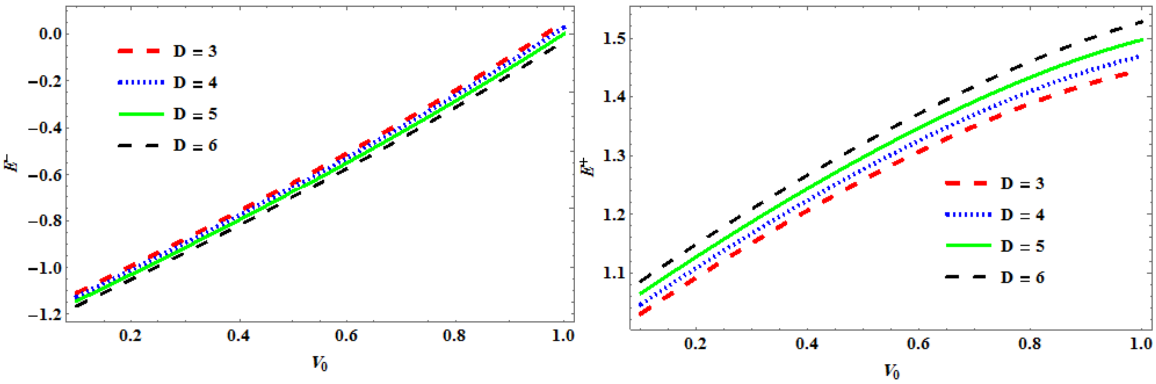

Figure 5. Variation of energies spectra with potential parameter V_0 for various dimensions.

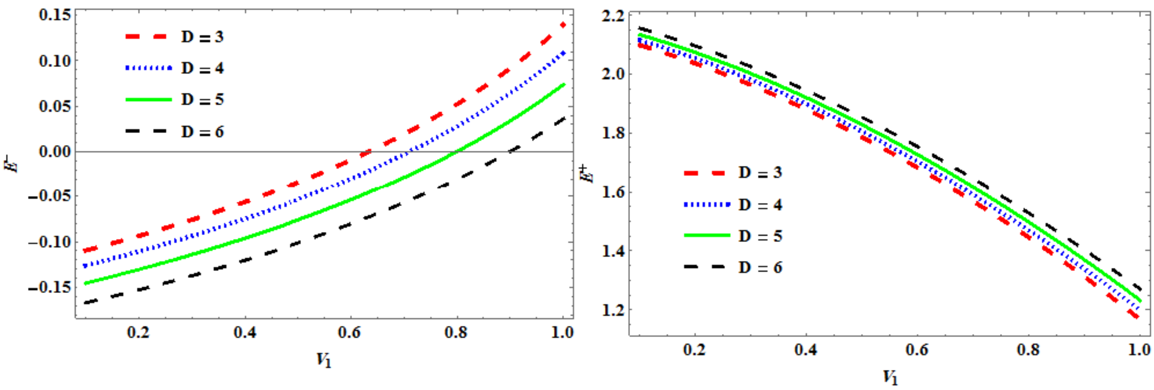

Figure 6. Variation of energies spectra with potential parameter V_1 for various dimensions.

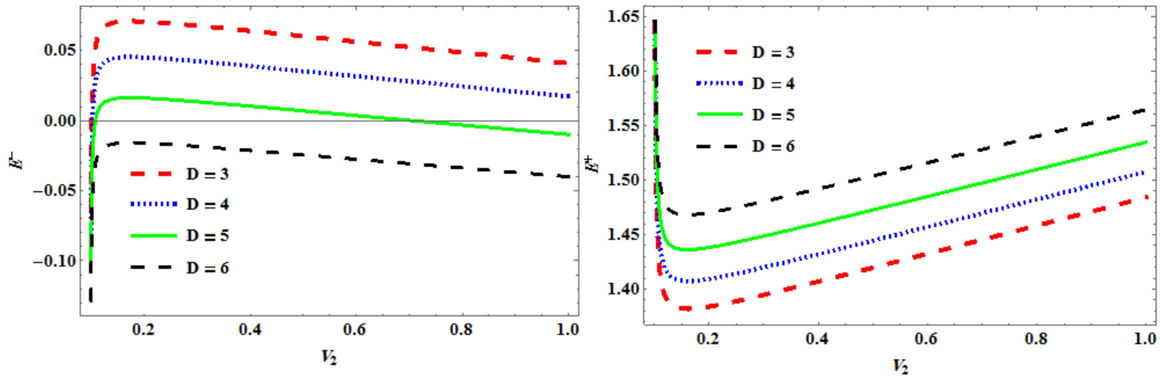

Figure 7. Variation of energies spectra with potential parameter V_2 for various dimensions.

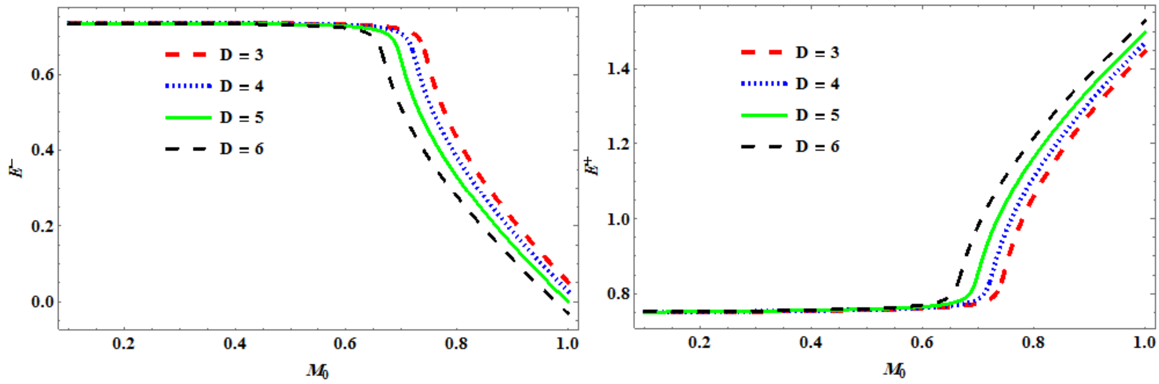

Figure 8. Variation of energies spectra with rest mass m_0 for various dimensions.

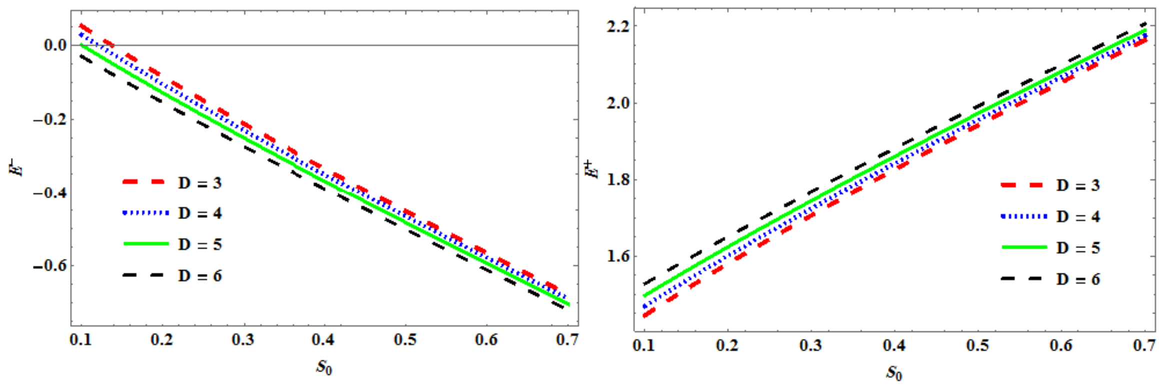

Figure 9. Variation of energies spectra with potential parameter S_0 for various dimensions.

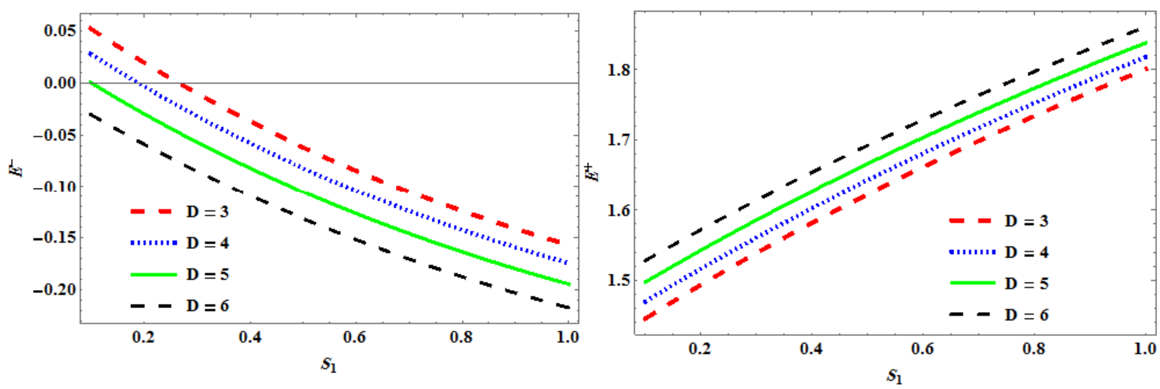

Figure 10. Variation of energies spectra with potential parameter S_1 for various dimensions.

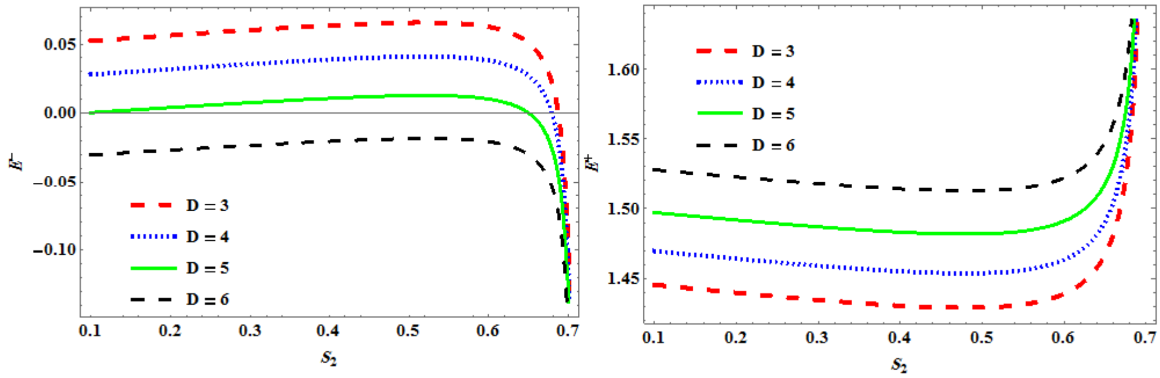

Figure 11. Variation of energies spectra with potential parameter S_2 for various dimensions.

is increased while the positive energy first increases and later decreases as the quantum number n is increased for different dimensions $D = 3, 4, 5$ and 6. The variation of the energy spectra versus the orbital angular momentum quantum number l is illustrated in Fig. 2. It is observed in Fig. 2 that the energy of the particles decreases as the orbital quantum number is increased while the energy of the antiparticles increases as the orbital quantum number is increased for different dimensions. The variations of the energy spectra against the screening parameter are displayed in Fig. 3 for the particles and antiparticles. As seen in Fig. 3, the energy of the particle increases to a peaked value and started decreasing as the screening parameter is increased for different dimensions while the energy spectrum of the antiparticle increased as the screening parameter is increased for different dimensions. The behaviour of the energy spectra of the KG particles/antiparticles versus the potential parameter b are shown in Fig. 4. It is seen clearly in Fig. 4 that the energy spectrum of the KG particle increases to a peaked value of 0.4 MeV and become bounded as the potential parameter b is increased for different dimensions. Similarly, the energy spectrum of the antiparticles first decreases to a minimum value of 0.5 MeV and started increases as the potential parameter b is further increased. The plots of the energy spectra versus potential parameter V_0 are shown in Fig. 5. The variations of the energy spectra are linearly and that of the particles gives negative energy while the antiparticles spectrum are positive as illustrated in Fig. 5 for different dimensions. Fig. 6 shows the variation of the energy spectra of the Klein-Gordon particles/antipar-

Table 1. Energy spectrum of Deng-Fan potential at different quantum states and dimensions.

n	l	$D = 3$	$D = 4$	$D = 5$	$D = 6$
0	0	1.38919	1.39647	1.40842	1.42480
1	0	1.89791	1.90300	1.91142	1.92309
	1	1.91142	1.92309	1.93789	1.95570
	0	1.79694	1.80490	1.81476	1.82836
2	1	1.80490	1.82836	1.84554	1.86609
	2	1.82836	1.86609	1.88979	1.91639
	0	1.41100	1.90300	1.91142	1.92309
	1	1.44099	1.92309	1.93789	1.95570
3	2	1.49572	1.95570	1.97634	1.99967
	3	1.56831	1.99967	2.02551	2.05368
	0	0.990227	0.990236	0.990253	0.990278
	1	0.990253	0.990278	0.990313	0.990361
4	2	0.990313	0.990361	0.990428	0.990521
	3	0.990428	0.990521	0.990660	0.990885
	4	0.990660	0.990885	0.991331	0.992940
	0	0.996064	0.996077	0.996099	0.996131
	1	0.996099	0.996131	0.996172	0.996225
	2	0.996172	0.996225	0.996290	0.996370
5	3	0.996290	0.996370	0.996466	0.996584
	4	0.996466	0.996584	0.996726	0.996901
	5	0.996726	0.996901	0.997119	0.997394

ticles versus the potential parameter V_1. The energy level of the particles is bounded for small values of the potential parameter V_1 and becomes positive energy as the potential parameter is further increased and the energy spectrum of the antiparticles decreases as the potential parameter is increased as shown in Fig. 6. The behaviour of the energy spectra of the Klein-Gordon particles/antiparticles against the potential parameter V_2 are displayed in Fig. 7. It is observed here that the energy spectrum of the particles increases as the potential parameter is increased and the energy spectrum for the antiparticles decreases as the potential parameter is increased as shown in Fig. 7. The variation of the energy spectra of the KG particles/antiparticles versus the rest mass of the particles are displayed in Fig. 8. The energy spectrum for the KG particles varied linearly and later decreases as the rest mass is increased while the energy spectrum of the KG antiparticles also varied linearly and then increases as the rest mass of the particles is increased. Fig. 9 shows the variation of the energy spectra versus the scalar potential parameter S_0. The energy spectrum of the particles decreases while the energy of the antiparticles decreases as the potential parameter S_0 is increased for different dimensions. The energy spectra of the KG particles versus potential parameters S_1 and S_2 are illustrated in Figs. 10-11. In each case, the energy of the particles decreases and the energy of antiparticles increases as the potential parameters are increased.
It has been reported that the bound state energies in D-dimensions are invariant under a transformation of an increase in the higher dimension by two ($D \rightarrow D+2$) and a decrease in the rotational quantum number by one ($l \rightarrow l-1$) [9] which implies that there exists an inter-dimensional degeneracy symmetry for the D-dimensional relativistic energy spectra of the Deng–Fan potential. Here, we show the energy spectrum of the KG particles in Table 1. We reported bound state energies of the Deng–Fan potential at different quantum states as seen presented in Tables 1 at different dimensions. The energy spectrum of the KG particles increases with increase in the principal and orbital momentum quantum numbers for different dimensions.

4. Conclusion

In this work, we obtained the approximate bound state solution of the KGE for generalized Deng potential in higher dimensions using the ENU method. The coordinate transformation carried out led to second order differential with a polynomial of at most fourth order and the resulting KGE was solved using the ENU to obtain the energy level for the system. Under the conditions $h(\xi) = h_n(\xi)$, the hypergeometric equation of Equation (8) was transformed into Heun differential equation and special case of the Heun differential equation called Confluent differential equation was obtained and from there, the wave function was obtained. One can see that the wave function of the KGE with the Deng Fan potential yields confluent Heun function.
The behaviour of the bound state energies of the KG particles versus quantum numbers and all the potential parameters of the Deng-Fan are illustrated in Figs. 1-11. It has been clearly shown that the bound state energies of the KG particles are greatly affected by the Deng-Fan potential parameters and quantum states considered. Also, there is an inter-dimensional degeneracy symmetry that occurs in the various bound state energy at higher dimensions, as shown in Table 1. The obtained wave functions are expressed in terms in terms of confluent Heun function. It can be concluded that the ENU is an efficient and power method can be used in findings solutions to the Schrodinger, Klein-Gordon and Dirac equations with higher polynomials of at most power four. These solutions can find many applications in high energy physics, nuclear physics and molecular physics.

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Ethical approval
This manuscript does not report on or involve the use of any animal or human data or tissue. So the ethical approval is not applicable.

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All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials
This manuscript has no associated data or the data will not be deposited. Authors Comments: All data included in this manuscript are available on request by contacting the corresponding author. This manuscript has associated data in a data repository.

Conflict of Interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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