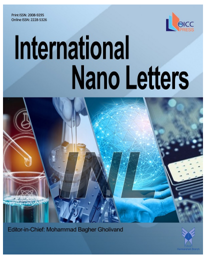

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Nonlinear Dynamical Band Structure Induced by Frequency-Dependent Nonlinearity in Duffing Oscillators

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Original Research

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Abstract:

We demonstrate that a nonlinear medium whose nonlinear susceptibility is explicitly frequency-dependent-such as graphene-exhibits nonlinear dispersive dynamical states governed by the Duffing equation. These states selectively suppress or support certain frequency components of the driving field, forming a nonlinear band structure without any spatial periodicity. This reveals a new mechanism of nonlinear frequency filtering originating solely from the dynamical response of the nonlinear oscillator. The nonlinear dispersive medium modifies the effective Floquet quasi-energy landscape of the driven Duffing oscillator. The Jacobi elliptic solutions exist only in regions where the nonlinear Floquet modes are stable, forming frequency-selective "nonlinear Floquet bands". Outside these bands the Floquet multipliers diverge, resulting in forbidden frequencies.

Keywords: Nonlinear dispersive medium; Graphene; Frequency band pass filter; Floquet states; Duffing oscillator

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1. Introduction

Surface Plasmons (SPs) are oscillating electrons that can be confined at a metal-dielectric interface or on a two-dimensional material surface yet beyond the light diffraction limit. Field enhancement of SPs' localization can be much stronger than that of the driving wave. However, SPs' decaying nature restricts their propagation length to a few millimetres [1, 2]. A spaser (SP nanolaser) is a plasmonic analogue of a laser. It consists of a gain medium (e.g., quantum dots) coupled to a nanostructure that supports SP modes. Coherent plasmon oscillations can be amplified by the spaser to overcome the loss and avoid SPs' decay [3–5]. The electric field amplitude of a nonlinear plasmonic cavity can obey a Duffing-like equation for which the analytic solution is in terms of Jacobi Elliptic Functions (JEFs) [6, 7]. In spaser arrays or coupled nanospasers with a gain-loss competition mechanism, specific issues of nonlinear dynamics including self-pulsing, periodic modulation of plasmonic fields and oscillating between gain and loss modes can be described by JEFs [8–10].

Nonlinear optical grating is a spatially periodic modulation of a material's nonlinear susceptibility (χ^2 or χ^3) that enables controlled nonlinear interactions, quasi-phase matching, and induces nonlinear diffraction or beam coupling. Nonlinear grating and periodic structures like photonic crystals play a pivotal role in frequency conversion and all-optical signal processing [11–14]. One of the main properties of a nonlinear grating is to create allowed/forbidden frequency bands. In a nonlinear resonant cavity like spaser, the frequency pass/stop-bands may arises from a feedback mechanism, a soliton-induced structure and particularly a temporal modulation of $\chi^2(t)$ or $\chi^3(t)$ yet not from spatial periodicity [15, 16]. A spaser cavity with a nonlinear medium whose nonlinear susceptibility is explicitly frequency-dependent-such as graphene-exhibits nonlinear dispersive dynamical states governed by the Duffing equation [17]. This conveys selectively suppress or support certain frequency components of the driving field, forming a nonlinear band structure without any spatial periodicity proposing a new mechanism of nonlinear frequency filtering that originates solely from the dynamical response of the nonlinear oscillator.

Floquet states are the intrinsic dynamical modes of a quantum system whose governing equations are periodic in time. For such systems, even though the Hamiltonian varies explicitly with time, the solutions can be expressed as time-periodic functions multiplied by an exponential phase factor, in direct analogy with Bloch states in spatially periodic systems. Floquet states represent steady, phase-locked dynamical responses of a system under periodic driving. They generalize the concept of stationary Eigenstates to time-dependent systems by replacing ordinary energy Eigenvalues with quasienergies, which are defined modulo the driving frequency. In nonlinear or dissipative systems, floquet states correspond to stable periodic orbits whose existence and stability are determined by floquet multipliers. Only those states with bounded multipliers are physically observable, while unstable floquet states give rise to forbidden dynamical regimes [18–20].

In this article, we reveal a previously unexplored mechanism for frequency-selective propagation in nonlinear systems. By solving the driven Duffing equation with a nonlinear coefficient explicitly dependent on the driving frequency—as is the case for graphene and several dispersive nonlinear media-we show that the system enters a nonlinear dynamical regime characterized by Jacobi elliptic solutions. These solutions exist only within specific frequency windows, resulting in intrinsic allowed and forbidden frequency bands. Remarkably, this “nonlinear band structure” emerges in the absence of any spatial periodicity, grating, or external feedback, and is instead a direct consequence of nonlinear dispersion. This establishes a tunable nonlinear dispersive states capable of dynamically filtering, suppressing, or enhancing frequency components of the driving field. The frequency-selective behaviour observed in our system can be interpreted within the framework of nonlinear Floquet theory [21]. The periodically driven Duffing oscillator with a frequency-dependent nonlinear coefficient behaves as a nonlinear Floquet system with an amplitude-dependent quasi-energy spectrum. The Jacobi elliptic solutions correspond to stable nonlinear Floquet modes and exist only within specific parameter windows. These windows define the “allowed” frequencies. Outside these windows, the Floquet multipliers exceed unity in magnitude, rendering the periodic solutions unstable and thereby creating forbidden frequencies. This establishes a direct link between the observed nonlinear dispersive frequency selection and a nonlinear floquet band structure.

2. Theory

We assume a double-quantum dot resonating model for spaser [22]. Accordingly, we can simulate the system as a chain of Duffing oscillators with nearest-neighbor coupling as given in Eq. (1).

$$\ddot{A}_n+\alpha\dot{A}_n+\omega_0^2A_n+\gamma(\omega)|A|^2A-C(A_{n+1}-2A_n+A_{n-1})=0, \tag{1}$$

where A is the plamsonic wave amplitude function; α is the damping rate; ω_0 is the natural frequency; C is the coupling strength of the two resonators; $\gamma(\omega)$ is the frequency-dependent nonlinear coefficient. Graphene as a 2d material can be assumed as the spaser gain medium. Hence, the SPs can be strongly supported by the graphene large effective area and the nonlinear coefficient γ will read the graphene frequency-dependent nonlinear optical conductivity as given in Eq. (2) [23, 24].

$$\sigma^{(3)}_{\text{intra}}(\omega)=\frac{i(e^2/4\hbar)(\hbar v_F e)^2}{48\pi(\hbar\omega)^4}\Theta\left(\frac{\hbar\omega}{2|E_F|}\right), \tag{2}$$

where e is the electric charge; $\hbar$ is the reduced Planck's constant; E_F is the Fermi energy; v_F is the Fermi velocity; $\Theta(\xi)=17\Lambda(\xi)-64\Lambda(2\xi)+45\Lambda(3\xi)$, where $\Lambda(\xi)=\ln\left|\frac{1+\xi}{1-\xi}\right|+i\pi\Pi(|\xi|-1)$ and $\Pi(\xi)$ is a step function, nonzero for $\xi>0$ and zero for $\xi<0$. One may seek a plane wave as in the form $A_n(t)=A_0e^{i(kn-\omega t)}$ where A_0 is the slowly varying amplitude and k is the wavenumber. One may balance the linear terms first to obtain Eq. (3).

$$[-\omega^2+\omega_0^2+2C(1-\cos k)A]+\gamma(\omega)|A|^2A=0. \tag{3}$$

Dividing by nonzero amplitude A and solving for $k(\omega,A)$ one may obtain the nonlinear dispersion relation as given in Eq. (4).

$$k(\omega,A)=\arccos\left(1-\frac{\omega^2-\omega_0^2-\gamma(\omega)A^2}{2C}\right). \tag{4}$$

This is a very naive model which can still indicate the presence of allowed/forbidden frequency window. The more precise approach is to consider JEFs as solutions to Eq. (1) and then to find the floquet quasi-energy spectrum or the floquet multipliers through the lumped waves. This is achievable by assuming a perturbation in solutions and then to compute floquet exponents λ from the monodromy matrix. The floquet multiplier $\mu=e^{\lambda T}$ can thus be obtained in which T is the period assigned to flouqet states [25, 26]. Jacobi elliptic solution can be given by Eq. (5).

$$A(t)=c_1\text{cn}(\Gamma t+c_2,\Delta), \tag{5}$$

where $\Gamma=(\alpha+c_1^2\gamma)^{1/2}$ and $\Delta=(c_1^2\gamma/2(\alpha+c_1^2\gamma))^{1/2}$; c_1 and c_2 can be determined by the time lattice condition.

If the nonlinear coefficient γ is much larger than the damping rate α, Eq. (5) will be reduced to Eq. (6) regarded as pulse regime which commences beyond a threshold pumping level.

$$A_{NL}(t)=c_1\frac{\text{sech}\Gamma^{(R)}t\left(\cos\Gamma^{(i)}t-i\tanh\Gamma^{(R)}t\sin\Gamma^{(i)}t\right)}{1-\text{sech}^2\Gamma^{(R)}t\sin^2\Gamma^{(i)}t}, \tag{6}$$

where (R) and (i) stand for the real and imaginary parts. Otherwise, If the nonlinear coefficient γ is much smaller than the damping rate α, Eq. (5) will be reduced to Eq. (7) known as damping regime.

$$A_{NL}(t) = c_1 \frac{\cos \Pi^{(R)} t - i \sin \Pi^{(R)} t \tanh \Pi^{(i)} t}{\text{sech} \Pi^{(i)} t}. \quad (7)$$

3. Results and discussion

Figure 1 (a) shows the dispersion relation and the static linear stability of the nonlinear steady-state amplitude in accordance with Eq. (4) for which the strong coupling case $C = 1$ is assumed. The natural frequency is scaled to $\omega_0 = 1$ while the other parameters in Eq. (1) is also scaled. The absolute value of wavenumber vs. ω shown in Fig. 1 (a) reveals that beyond a threshold frequency (at around $\omega = 1.75$), evanescent wave appears. In Fig. 1 (b), the stable zones are shown by stability flag 1 and reversely. Except for the natural frequency $\omega_0 = 1$ of which it is anticipated to be stable, the region of real wavenumber is wholly unstable. However, in evanescent wave region, extended stable zones is formed (e.g. in the vicinity of $\omega = 1.2$) indicating that the localizing nature of SPs can only occur at certain frequency regions.
We also evaluate the dynamical behaviour of amplitude as can be obtained by the approach given in Eq. (5). Figure 2 (a) shows that the amplitude evolution obeys a periodic pattern as sketched for a wide frequency range while Fig. 2 (b) unveils allowed/forbidden frequency ranges for an assumed spaser system with a frequency-dependent gain medium like graphene. The spanned forbidden range at frequency around $\omega = 2.1$ is clearly observable.
The lumped oscillation has a crucial role for this frequency filtering feature [27–29]. Figure 3 (a) shows the discrete breather and lump solutions obtained by also considering the SPs spatial variation along the propagation axis. This periodic lump provides an energy conservation within a localized oscillatory behaviour. Finally, we obtain solitary solution vs. nonlinearity in the spaser (beyond a threshold pumping level) mentioned as the pulse regime as shown in Fig. 3 (b). This discrete feature of solitary solution is indeed in consequence of nonlinear time lattice as predefined for floquet states.

4. Conclusion

The frequency-selective behaviour observed in our spaser system can be interpreted within the framework of nonlinear Floquet theory. The periodically driven Duffing oscillator with a frequency-dependent nonlinear coefficient behaves as a nonlinear Floquet system with an amplitude-dependent quasi-energy spectrum. The Jacobi elliptic solutions correspond to stable nonlinear Floquet

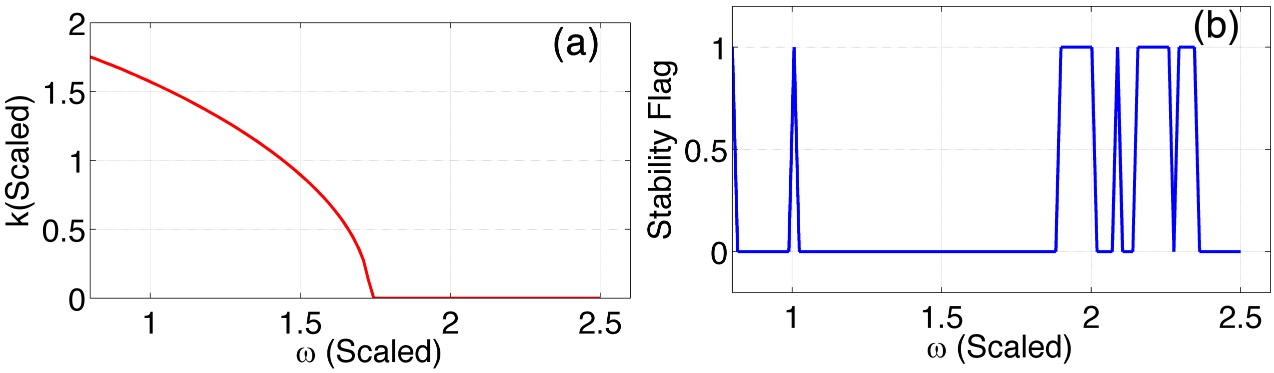

Figure 1. (a) Absolute value of wavenumber vs. frequency and (b) stability flag; 1:stable; 0:unstable

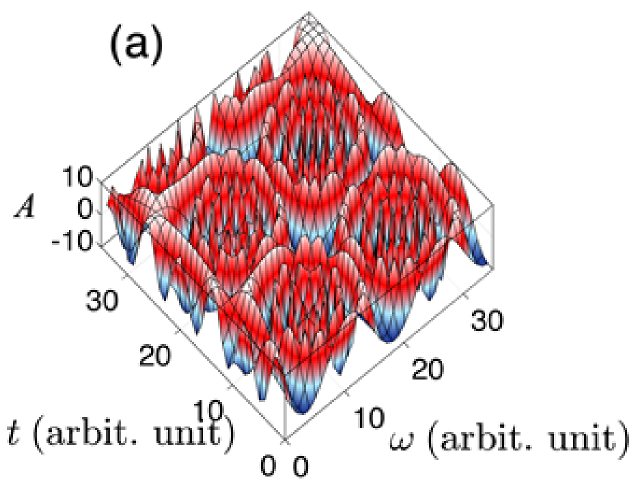

Figure 2. (a) Periodic evolution of amplitude, (b) pass/stop frequency regions of amplitude.

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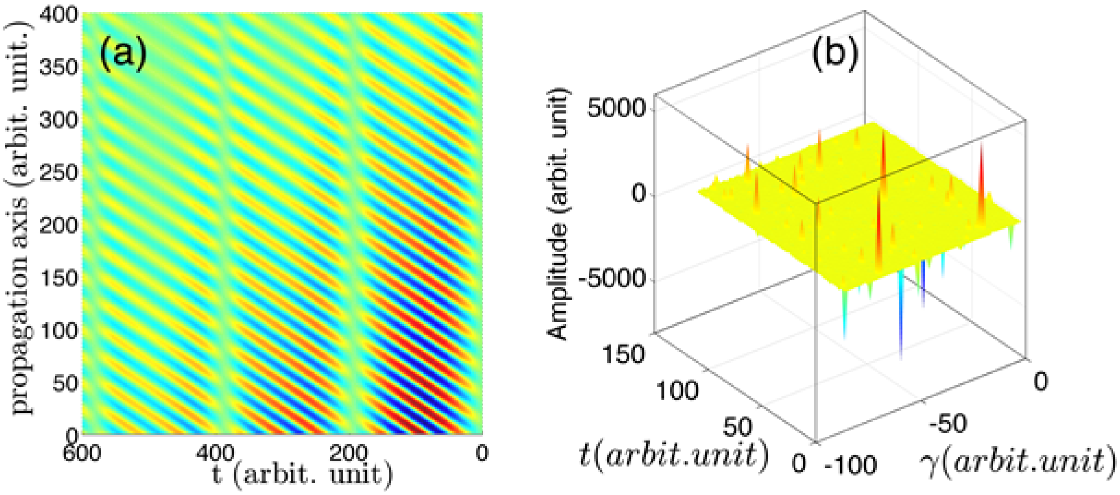

Figure 3. Spaser (a) breather wave and lump solution, (b) pulse regime.

modes and exist only within specific parameter windows. These windows define the "allowed" frequencies. Outside these windows, the Floquet multipliers exceed unity in magnitude, rendering the periodic solutions unstable and thereby producing forbidden frequencies. This establishes a direct link between the observed nonlinear dispersive frequency selection and a nonlinear Floquet band structure.

Authors Contribution
All authors contributed equally to the conception, design, execution, and writing of this work. All authors read and approved the final manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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